# The Hermitian geometry of the sequence of tangent bundles of an affine-Riemann manifold

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# Purpose

Let $(M, \nabla, \langle\ ,\ \rangle)$ be a manifold endowed with a flat torsionless connection $\nabla$ and a Riemannian metric $\langle\ ,\ \rangle$.

We call such triple $(M, \nabla, \langle\ ,\ \rangle)$ an *affine-Riemann manifold*.

Then:

1. We build on $TM$ a Riemannian metric $g_1$, a complex structure $J_1$ and an affine connection $\nabla^1$,
2. The affine-Riemann structure $(TM, \nabla^1, g_1)$ gives rise to a Hermitian structure $(TTM, J_2, g_2)$ and a flat torsionless connection $\nabla^2$ on $TTM$. By induction, we get a sequence of Hermitian structures $(T^kM, J_k, g_k)$ where $T^kM = T(T^{k-1}M)$ and $T^1M = TM$,
3. We study the Hermitian geometry of $(T^kM, J_k, g_k)$ for $k \geq 1$.

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Let $(N, J, g)$ be a complex manifold of real dimension $2n$, $n \geq 2$, equipped with a Hermitian metric $g$, i.e, $J : TM \longrightarrow TM$ satisfies

$$J^2 = -\mathrm{Id}_{TM},$$
$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0,$$
$$g(JX, JY) = g(X, Y).$$

For any $\eta \in \Omega^p(N)$,

$$J\eta(X_1, \ldots, X_p) = (-1)^p \eta(JX_1, \ldots, JX_p) \quad \text{and} \quad d^c\eta = -(-1)^p J^{-1} dJ\eta,$$

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The fundamental form is given by

$$\omega(X, Y) = g(JX, Y)$$

and the Lee form is given by

$$\theta = Jd^*\omega = -d^*\omega \circ J,$$

where for any $X_1, \ldots, X_{p-1} \in \Gamma(TN)$,

$$d^*\eta(X_1, \ldots, X_{p-1}) = -\sum_{i=1}^{2n} \nabla^{LC}_{E_i}\eta(E_i, X_1, \ldots, X_{p-1}),$$

$\nabla^{LC}$ is the Levi-Civita connection of $g$ and $(E_1, \ldots, E_{2n})$ is a local $g$-orthonormal frame.

$(N, J, g)$ is called Kählerian if and only if

$$\nabla^{LC}(J) = 0.$$

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Indeed, $(N, J, g)$ is called:

1. strongly Kähler with torsion or pluriclosed if $dd^c\omega = 0$, i.e., $dJdJ\omega = 0$,
2. balanced if $\theta = 0$,
3. locally conformally balanced if $\theta$ is closed,
4. Gauduchon if $d^*\theta = 0$,
5. locally conformally Kähler if $d\omega = \frac{1}{n-1}\theta \wedge \omega$ and if, in addition, $\nabla^{LC}\theta = 0$ then it is called Vaisman.

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The Levi-Civita connection of $(N, g)$ is the only torsion free metric connection. In general, it does not preserve the complex structure $J$, this condition forcing the metric to be Kähler.

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The Bismut connection $\nabla^B$ (also known as Strominger connection) is the unique Hermitian connection with totally skew-symmetric torsion and the Chern connection $\nabla^C$ is the unique Hermitian connection whose torsion has trivial $(1,1)$-component. For any $X, Y, Z \in \Gamma(TN)$,

$$\begin{cases} g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d\omega(JX, JY, JZ), \\ g(\nabla^C_X Y, Z) = g(\nabla^{LC}_X Y, Z) - \frac{1}{2} d\omega(JX, Y, Z). \end{cases} \tag{1}$$

Let $R^\tau(X,Y) = \nabla^\tau_{[X,Y]} - \nabla^\tau_X \nabla^\tau_Y + \nabla^\tau_Y \nabla^\tau_X$ be the curvature tensor of $\nabla^\tau$. The *Ricci form* of $\nabla^\tau$ is given, for any $X, Y \in \Gamma(TN)$, by

$$\rho^\tau(X,Y) = \frac{1}{2}\sum_{i=1}^{2n} g(R^\tau(X,Y)E_i, JE_i), \tag{2}$$

where $(E_1, \ldots, E_{2n})$ is a local $g$-orthonormal frame.

It is known that $\rho^C = \rho^B - dJ\theta$. Hermitian structures satisfying $\mathrm{Hol}^0(\nabla^B) \subset \mathrm{SU}(n)$, or equivalently $\rho^B = 0$, are known in literature as Calabi-Yau with torsion and appear in heterotic string theory, related to the Hull-Strominger system in six dimensions.

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Let $(M, \nabla, \langle\ ,\ \rangle)$ be an affine-Riemann manifold of dimension $n$. Let $\pi_1 : TM \longrightarrow M$ be the canonical projection and $Q : TTM \longrightarrow TM$ the connection map of $\nabla$ locally given by

$$Q\left(\sum_{i=1}^{n} b_i \partial_{x_i} + \sum_{j=1}^{n} Z_j \partial_{\mu_j}\right) = \sum_{l=1}^{n}\left(Z_l + \sum_{i=1}^{n}\sum_{j=1}^{n} b_i \mu_j \Gamma_{ij}^{l}\right)\partial_{x_l},$$

where $(x_1, \ldots, x_n)$ is a system of local coordinates, $(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n)$ the associated system of coordinates on $TM$ and $\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{l=1}^{n} \Gamma_{ij}^{l} \partial_{x_l}$.

Then

$$TTM = \ker T\pi_1 \oplus \ker Q.$$

For $X \in \Gamma(TM)$, we denote by $X^h$ its horizontal lift and by $X^v$ its vertical lift. The flow of $X^v$ is given by $\Phi^X(t,(x,u)) = (x, u + tX(x))$ and $X^h(x,u) = h^{(x,u)}(X(x))$, where $h^{(x,u)} : T_xM \longrightarrow \ker Q(x,u)$ is the inverse of the restriction of $d\pi_1$ to $\ker Q(x,u)$. Since the curvature of $\nabla$ vanishes, for any $X, Y \in \Gamma(TM)$,

$$[X^h, Y^h] = [X,Y]^h,\ [X^h, Y^v] = (\nabla_X Y)^v \quad \text{and} \quad [X^v, Y^v] = 0. \tag{3}$$

The connection $\nabla^1$ on $TM$ given by

$$\nabla^1{}_{X^h} Y^h = (\nabla_X Y)^h, \ \nabla^1{}_{X^h} Y^v = (\nabla_X Y)^v$$
$$\nabla^1{}_{X^v} Y^h = \nabla^1_{X^v} Y^v = 0,$$

for any $X, Y \in \Gamma(TM)$, is flat torsionless and defines an affine structure on $TM$. The tensor field $J_1 : TTM \longrightarrow TTM$ given by $J_1 X^h = X^v$ and $J_1 X^v = -X^h$ satisfies $J_1^2 = -\mathrm{Id}_{TTM}$, $\nabla^1(J_1) = 0$ and hence defines a complex structure on $TM$.

On the other hand, we define on $TM$ a Riemannian metric $g_1$ by

$$g_1(X^h, Y^h) = \langle X, Y\rangle \circ \pi_1,\ g_1(X^v, Y^v) = \langle X, Y\rangle \circ \pi_1$$
$$g_1(X^h, Y^v) = 0, \quad X, Y \in \Gamma(TM).$$

This metric is Hermitian with respect to $J_1$ and its fundamental form $\omega = g_1(J_1.,.)$ satisfies

$$\omega(X^h, Y^h) = \omega(X^v, Y^v) = 0$$
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Actually, we have a sequence of Hermitian structures. The affine-Riemann manifold $(TM, \nabla^1, g_1)$ gives rise to a Hermitian structure $(TTM, J_2, g_2)$ and a flat torsionless connection $\nabla^2$ on $TTM$. By induction, we get a sequence of Hermitian structures $(T^kM, J_k, g_k)$ where $T^kM = T(T^{k-1}M)$ and $T^1M = TM$. Moreover, each $T^kM$ carries a flat torsionless connection $\nabla^k$ such that $\nabla^k(J_k) = 0$.

# Important remark

**Remark**

*Let $(G, \nabla, \langle\ ,\ \rangle)$ be a Lie group endowed with a left invariant affine-Riemann structure. Then for each $k \geq 1$ there exists a Lie group structure on $T^kG$ such that $J_k$ and $g_k$ are left invariant.*

Trough-out this section and the next one, $(M, \nabla, \langle\ ,\ \rangle)$ is an affine-Riemann manifold of dimension $n$, $D$ the Levi-Civita connection of $\langle\ ,\ \rangle$ and $K$ its curvature given by

$$K(X,Y) = D_{[X,Y]} - (D_X D_Y - D_Y D_X).$$

Let $(T^kM, J_k, g_k, \nabla^k)$, $k \geq 1$, be the canonical sequence of Hermitian structures associated to $(M, \nabla, \langle\ ,\ \rangle)$ endowed with the sequence of flat torsionless connections. For any $k \geq 1$, we denote by $\pi_k : T^kM \longrightarrow T^{k-1}M$ the canonical projection.

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We consider $\gamma$ the difference tensor and $\gamma^*$ its adjoint given, for any $X, Y, Z \in \Gamma(TM)$, by

$$\gamma_X Y = D_X Y - \nabla_X Y \quad \text{and} \quad \langle \gamma^*_X Y, Z\rangle = \langle Y, \gamma_X Z\rangle. \tag{4}$$

Their traces with respect to the metric are the vector fields given by

$$\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma) = \sum_{i=1}^{n} \gamma_{E_i} E_i \quad \text{and} \quad \mathrm{tr}_{\langle\ ,\ \rangle}(\gamma^*) = \sum_{i=1}^{n} \gamma^*_{E_i} E_i, \tag{5}$$

where $(E_1, \ldots, E_n)$ is a local $\langle\ ,\ \rangle$-orthonormal frame.

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The 1-form $\alpha$ given, for any $X \in \Gamma(TM)$, by

$$\alpha(X) = \langle \mathrm{tr}_{\langle\ ,\ \rangle}(\gamma^*), X \rangle \tag{6}$$

is closed and it is known as the first Koszul form in the theory of Hessian manifolds.

We introduce also the 1-form $\xi$ given, for any $X \in \Gamma(TM)$, by

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## Proposition

*The differential of the fundamental form $\omega$ associated to $(TM, J_1, g_1)$ is given by*

$$d\omega(X^h, Y^h, Z^h) = d\omega(X^v, Y^v, Z^v) = d\omega(X^h, Y^v, Z^v) = 0,$$
$$d\omega(X^h, Y^h, Z^v) = \langle \gamma^*_X Y - \gamma^*_Y X, Z\rangle \circ \pi_1,$$

*for any $X, Y, Z \in \Gamma(TM)$. Hence*

$$(J_1 d\omega)(X^h, Y^h, Z^h) = (J_1 d\omega)(X^v, Y^v, Z^v) = (J_1 d\omega)(X^h, Y^h, Z^v) = 0,$$
$$(J_1 d\omega)(X^v, Y^v, Z^h) = -\langle \gamma^*_X Y - \gamma^*_Y X, Z\rangle \circ \pi_1.$$

## Corollary

*$(TM, J_1, g_1)$ is Kähler if and only if $(M, \nabla, \langle\ ,\ \rangle)$ is Hessian manifold, i.e,*

$$\nabla_X(\langle\ ,\ \rangle)(Y, Z) = \nabla_Y(\langle\ ,\ \rangle)(X, Z).$$

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# The Levi-Civita connection $\nabla^{LC}$ of $(TM, J_1, g_1)$

## Proposition

*For any $X, Y \in \Gamma(TM)$,*

$$\nabla^{LC}_{X^h} Y^h = (D_X Y)^h, \ \nabla^{LC}_{X^v} Y^v = -\frac{1}{2}(\gamma^*_X Y + \gamma^*_Y X)^h,$$

$$\nabla^{LC}_{X^v} Y^h = (\gamma^s_Y X)^v \quad \text{and} \quad \nabla^{LC}_{X^h} Y^v = (D_X Y)^v - (\gamma^a_X Y)^v,$$

*where*

$$\gamma^a = \frac{1}{2}(\gamma - \gamma^*) \quad \text{and} \quad \gamma^s = \frac{1}{2}(\gamma + \gamma^*).$$

# The Lee form $\theta_1$ of $(TM, J_1, g_1)$

**Proposition**

*The Lee form $\theta_1$ of $(TM, J_1, g_1)$ is given by*

$$\theta_1 = \pi_1^*(\alpha - \xi), \tag{8}$$

*where $\alpha$ and $\xi$ are the Koszul forms of $(M, \nabla, \langle\ ,\ \rangle)$. In particular, $(TM, J_1, g_1)$ is balanced if and only if $\alpha = \xi$ which is also equivalent to*

$$\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma^* - \gamma) = 0. \tag{9}$$

*Moreover, $(TM, J_1, g_1)$ is locally conformally balanced if and only if $d\xi = 0$.*

# When $(TM, J_1, g_1)$ is Gauduchon ?

**Proposition**

*We have*

$$d^*\theta_1 = d^*(\alpha - \xi)\circ\pi_1 - \langle \mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*) - \mathrm{tr}_{\langle\,,\,\rangle}(\gamma), \mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*)\rangle\circ\pi_1$$

*and hence $(TM, J_1, g_1)$ is Gauduchon if and only if*

$$d^*(\alpha - \xi) = |\mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*)|^2 - \langle \mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*), \mathrm{tr}_{\langle\,,\,\rangle}(\gamma)\rangle. \qquad (10)$$

# When $(TM, J_1, g_1)$ is Vaisman ?

## Proposition

1. $(TM, J_1, g_1)$ *is locally conformally Kähler if and only if, for any* $X, Y \in \Gamma(TM)$,

$$(n-1)(\gamma^*_X Y - \gamma^*_Y X) = \theta_0(X)Y - \theta_0(Y)X, \qquad (11)$$

*where* $\theta_0 = \alpha - \xi$.

2. $(TM, J_1, g_1)$ *is Vaisman if and only if* (11) *holds and the vector field* $\Pi := \mathrm{tr}_{\langle\, ,\, \rangle}(\gamma^* - \gamma)$ *is parallel with respect to both* $D$ *and* $\nabla$.

Let us compute the difference tensor $\Gamma = \nabla^{LC} - \nabla^1$ of $(TM, \nabla^1, g_1)$ as well as its adjoint $\Gamma^*$, the Koszul forms $\alpha_k$, $\xi_k$ as well as the Lee form $\theta_k$ of $(T^kM, J_k, g_k)$.

## Proposition

$$\begin{aligned} &\mathrm{tr}_{g_1}(\Gamma) = (\mathrm{tr}_{\langle\,,\,\rangle}(\gamma) - \mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*))^h,\ \mathrm{tr}_{g_1}(\Gamma^*) = 2(\mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*))^h,\\ &\xi_k = -\theta_k,\ \alpha_k = 2^k\pi_k^*\circ\ldots\circ\pi_1^*(\alpha),\\ &\theta_k = \pi_k^*\circ\ldots\circ\pi_1^*((2^k-1)\alpha - \xi),\quad k \geq 1. \end{aligned}$$

Let us compute $dd^c\omega = -dJ_1^{-1}dJ_1\omega = dJ_1d\omega$.

## Proposition

*For any $X, Y, Z, U \in \Gamma(TM)$,*

$$\begin{cases} dJ_1d\omega(X^h, Y^h, Z^h, U^h) = dJ_1d\omega(X^v, Y^v, Z^v, U^v) \\ = dJ_1d\omega(X^h, Y^h, Z^h, U^v) = dJ_1d\omega(X^h, Y^v, Z^v, U^v) = 0, \\ dJ_1d\omega(X^h, Y^h, Z^v, U^v) = 2\langle K(X,Y)Z - (\gamma_X^* \circ \gamma_Y - \gamma_Y^* \circ \gamma_X)Z, U\rangle \circ \pi_1. \end{cases}$$

*In particular, $(TM, J_1, g_1)$ is pluriclosed if and only if for any $X, Y \in \Gamma(TM)$, the curvature $K$ of $D$ satisfies*

$$K(X, Y) = \gamma_X^* \circ \gamma_Y - \gamma_Y^* \circ \gamma_X. \tag{12}$$

In [**?**, Theorem 8.8 pp. 162], Shima proved that if $(M, \nabla, \langle\ ,\ \rangle)$ is a compact Hessian manifold such that its first Koszul form vanishes then $\nabla$ is the Levi-Civita connection of $\langle\ ,\ \rangle$. Note that in this case the first Koszul form and the dual Koszul form coincide. The following theorem is a generalization of this result under an additional assumption, namely, $\nabla$ is complete. It will be interesting to see if we can drop this assumption.

**Theorem**

*Let $(M, \nabla, \langle\ ,\ \rangle)$ be an affine-Riemann manifold such that $(TM, J_1, g_1)$ is pluriclosed and the dual Koszul form of $(M, \nabla, \langle\ ,\ \rangle)$ vanishes. Then the Ricci curvature of $\langle\ ,\ \rangle$ is nonnegative. Moreover, if $M$ is compact and $\nabla$ is complete then $\gamma = 0$, i.e., $\nabla$ is the Levi-Civita connection of $\langle\ ,\ \rangle$.*

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Proof.

Note that the vanishing of dual Koszul form is equivalent to $\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma) = 0$. From the relation

$$K(X, Y) = \gamma_X^* \circ \gamma_Y - \gamma_Y^* \circ \gamma_X$$

and the fact that $\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma) = 0$, we deduce that the Ricci curvature of $\langle\ ,\ \rangle$ is given by

$$\mathrm{ric}(X, X) = \mathrm{tr}(\gamma_X^* \circ \gamma_X) \geq 0$$

and $\mathrm{ric}(X, X) = 0$ if and only if $\gamma_X = 0$. □

## Proof.

By using the splitting theorem of J. Cheeger and D. Gromoll (see for instance [**?**, Corollary 6.67 pp. 168]), we deduce that if $M$ is compact its universal Riemannian covering is isometric to a Riemannian product $(\overline{M} \times \mathbb{R}^d, \langle\ ,\ \rangle_1 \times \langle\ ,\ \rangle_0)$ where is $\overline{M}$ is compact and $\langle\ ,\ \rangle_0$ is the canonical metric of $\mathbb{R}^d$. But if $\nabla$ is complete the universal covering of $M$ is diffeomorphic to $\mathbb{R}^n$ which completes the proof. □

## Proposition

*We have, for any* $X, Y \in \Gamma(TM)$,

$$\begin{cases} \nabla^B_{X^h} Y^h = (D_X Y)^h, \ \nabla^B_{X^v} Y^v = -(\gamma^*_Y X)^h, \\ \nabla^B_{X^v} Y^h = (\gamma^*_Y X)^v, \ \nabla^B_{X^h} Y^v = (D_X Y)^v. \end{cases}$$

$$\begin{cases} \nabla^C_{X^h} Y^h = (D_X Y)^h - (\gamma^a_X Y)^h, \ \nabla^C_{X^v} Y^v = -(\gamma^s_X Y)^h, \\ \nabla^C_{X^v} Y^h = (\gamma^s_X Y)^v, \ \nabla^C_{X^h} Y^v = (D_X Y)^v - (\gamma^a_X Y)^v. \end{cases}$$

*where*

$$\gamma^a = \frac{1}{2}(\gamma - \gamma^*) \quad \textit{and} \quad \gamma^s = \frac{1}{2}(\gamma + \gamma^*).$$

Now, we give the Ricci forms $\rho^B$ and $\rho^C$.

## Proposition

*For any $X, Y \in \Gamma(TM)$, the Ricci forms are given by*

$$\begin{aligned}
&\rho^B(X^h, Y^h) = \rho^B(X^v, Y^v) = 0,\\
&\rho^B(X^h, Y^v) = -\langle \gamma_X Y, \mathrm{tr}_{\langle\,,\,\rangle}\gamma\rangle \circ \pi_1 - \langle D_X \mathrm{tr}_{\langle\,,\,\rangle}(\gamma), Y\rangle \circ \pi_1,\\
&\rho^C(X^h, Y^h) = \rho^C(X^v, Y^v) = 0,\\
&\rho^C(X^h, Y^v) = -\langle \gamma_X Y, \mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*)\rangle \circ \pi_1 - \langle D_X \mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*), Y\rangle \circ \pi_1
\end{aligned}$$

*In particular, if $\mathrm{tr}_{\langle\,,\,\rangle}(\gamma) = 0$ (resp. $\mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*) = 0$ ) then $(TM, J_1, g_1)$ is Calabi-Yau with torsion, i.e., $\rho^B = 0$ (resp. $\rho^C = 0$).*

## Theorem

*Let $(M, \nabla, \langle\ ,\ \rangle)$ be an affine-Riemann manifold. Then:*

1. *If $\gamma = 0$ then, for any $k \geq 1$, $\nabla^k$ is the Levi-Civita connection of $g_k$ and $(T^kM, J_k, g_k)$ is Kähler flat.*
2. *For some $k \geq 2$, $(T^kM, J_k, g_k)$ is Kähler if and only if $\gamma = 0$.*
3. *For some $k \geq 1$, $(T^kM, J_k, g_k)$ is locally conformally balanced if and only if $(TM, J_1, g_1)$ is locally conformally balanced and this is equivalent to $d\xi = 0$.*

## Theorem (Continued)

4. *For* $k_0 \geq 1$, $(T^{k_0}M, J_{k_0}, g_{k_0})$ *is balanced if and only if*

$$\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma) = (2^{k_0} - 1)\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma^*). \tag{13}$$

*In this case,* $(T^{k_0+1}M, J_{k_0+1}, g_{k_0+1})$ *is Calabi-Yau with torsion and for any* $k \neq k_0$, $(T^kM, J_k, g_k)$ *is locally conformally balanced.*

5. *If* $\mathrm{tr}_{\langle\ ,\ \rangle}(\gamma) = \mathrm{tr}_{\langle\ ,\ \rangle}(\gamma^*) = 0$ *then, for any* $k \geq 1$, $(T^kM, J_k, g_k)$ *is balanced, Calabi-Yau with torsion and its Chern Ricci form vanishes.*

## Example

*We consider the left symmetric product on* $\mathbb{R}^3$ *given by*

$$e_1 \bullet e_1 = a e_1, e_1 \bullet e_2 = a e_2 + e_3, e_1 \bullet e_3 = e_2 + a e_3, e_2 \bullet e_1 = a e_2, e_3 \bullet e_1 = a e_3.$$

*The associated non vanishing Lie brackets are given by*

$$[e_1, e_2] = e_3, \ [e_1, e_3] = e_2.$$

*We denote by* $G$ *the connected simply-connected Lie group associated to* $(\mathbb{R}^3, [\ ,\ ])$ *and by* $\nabla$ *the left invariant flat torsionless connection on* $G$ *defined by* $\bullet$.

## Example

*For* $a = 1$*, the left invariant metric on* $G$ *associated to the scalar product* $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ *on* $\mathbb{R}^3$ *satisfies* (13) *for* $k_0 = 2$ *with* $\mathrm{tr}_{\langle\,,\,\rangle}(\gamma) \neq 0$ *and* $\mathrm{tr}_{\langle\,,\,\rangle}(\gamma^*) \neq 0$*. Thus* $(T^2G, J_2, g_2)$ *is balanced,* $(T^3G, J_3, g_3)$ *is Calabi-Yau with torsion and, for any* $k \neq 2$*,* $(T^kG, J_k, g_k)$ *is locally conformally balanced not balanced.*

Let us compute the Koszul forms of an affine-Riemann manifold in local affine coordinates.

## Proposition

*Let $(M, \nabla, \langle\ ,\ \rangle)$ be an affine-Riemann manifold. For any system of affine coordinates $(x_1, \ldots, x_n)$,*

$$\alpha = \frac{1}{2} d \ln(\det G) \quad \text{and} \quad \xi = \sum_{j=1}^{n} \left( \sum_{h,k} \mu^{kh} \frac{\partial \mu_{jh}}{\partial x_k} \right) dx_j - \alpha, \quad (14)$$

*where $\mu_{hk} = \langle \partial_{x_h}, \partial_{x_k} \rangle$ and the matrix $(\mu^{hk})_{1 \le h,k \le n} = G^{-1}$ where $G = (\mu_{hk})_{1 \le h,k \le n}$.*

## Example

*Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function. Consider the affine-Riemann manifold $(\mathbb{R}^2, \nabla^0, \langle\ ,\ \rangle)$ where $\nabla^0$ is the canonical connection of $\mathbb{R}^2$ and*

$$\langle\ ,\ \rangle = \begin{pmatrix} \cosh(f(x,y)) & \sinh(f(x,y)) \\ \sinh(f(x,y)) & \cosh(f(x,y)) \end{pmatrix}.$$

*Then $\det\langle\ ,\ \rangle = 1$ and, by virtue of Proposition 6.1, $\alpha = 0$. According to Proposition **??**, the Chern Ricci form of $(T\mathbb{R}^2, J_1, g_1)$ vanishes.*

The following theorem gives a large class of balanced metrics non-Kähler on $\mathbb{C}^2$ endowed with its canonical complex structure.

### Theorem

*We consider* $M = \mathbb{R}^2$ *endowed its canonical affine structure and* $\langle\ ,\ \rangle = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}$ *a Riemannian metric. Then* $(TM, J_1, g_1)$ *is balanced if and only if there exists smooth functions* $\nu : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f, h : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\mu_{12} = \nu,\ \mu_{11}(x_1, x_2) = f(x_1) + \int \frac{\partial \nu}{\partial x_1}(x_1, x_2)dx_2 \quad \textit{and}$$

$$\mu_{22}(x_1, x_2) = h(x_2) + \int \frac{\partial \nu}{\partial x_2}(x_1, x_2)dx_1.$$

## Example

*For any smooth functions $f, h : \mathbb{R} \longrightarrow \mathbb{R}$, the metric*

$$\langle\ ,\ \rangle = \begin{pmatrix} e^{x+y} + e^{f(x)} & e^{x+y} \\ e^{x+y} & e^{x+y} + e^{h(y)} \end{pmatrix}$$

*satisfies the condition of the last corollary and hence defines a balanced Hermitian metric on $\mathbb{C}^2$.*

The following theorem gives a large class of balanced metrics non-Kähler and also Calabi-Yau metrics on $\mathbb{C}^m$ endowed with its canonical complex structure.

**Theorem**

*We consider $M = \mathbb{R}^n$ endowed its canonical affine structure and $\langle\ ,\ \rangle = \mathrm{Diag}(\mu_1, \ldots, \mu_n)$ a Riemannian metric. For $k_0 \geq 1$, $(T^{k_0}M, J_{k_0}, g_{k_0})$ is balanced if and only if there exists $(f_1, \ldots, f_n)$ a family of positive functions such that, for $j = 1, \ldots, n$,*

$$\frac{\partial f_j}{\partial x_j} = 0 \quad \text{and} \quad \mu_j = \frac{f_1 \ldots f_n}{f_j^{(n 2^{k_0 - 1} - 1)}}.$$

*In this case, $(T^{k_0+1}M, J_{k_0+1}, g_{k_0+1})$ is Calabi-Yau with torsion and, for any $k \neq k_0$, $(T^k M, J_k, g_k)$ is locally conformally balanced.*

Now, we give the conditions in local coordinates so that $(TM, J_1, g_1)$ is pluriclosed.

**Theorem**

*Let $(M, \nabla, \langle\ ,\ \rangle)$ be an affine-Riemann manifold. Then $(TM, J_1, g_1)$ is pluriclosed if and only if, for any affine coordinates $(x_1, \ldots, x_n)$,*

$$\frac{\partial^2 \mu_{ik}}{\partial x_j \partial x_h} + \frac{\partial^2 \mu_{jh}}{\partial x_i \partial x_k} = \frac{\partial^2 \mu_{jk}}{\partial x_i \partial x_h} + \frac{\partial^2 \mu_{ih}}{\partial x_j \partial x_k}, \qquad (15)$$

*for any $1 \leq i < j \leq n$ and $1 \leq k < h \leq n$ and where $\mu_{ij} = \langle \partial_{x_i}, \partial_{x_j} \rangle$. When $\dim M = 2$, (15) reduces to*

$$\frac{\partial^2 \mu_{11}}{\partial x_2^2} + \frac{\partial^2 \mu_{22}}{\partial x_1^2} = 2\frac{\partial^2 \mu_{12}}{\partial x_1 \partial x_2}.$$

## Corollary

*We consider $M = \mathbb{R}^n$ endowed with its canonical affine structure and $\langle\ ,\ \rangle = \mathrm{Diag}(\mu_1, \ldots, \mu_n)$. Then $(TM, J_1, g_1)$ is pluriclosed if and only if, for any $i \neq j$, $h \neq j$ and $h \neq i$,*

$$\frac{\partial^2 \mu_i}{\partial x_j^2} + \frac{\partial^2 \mu_j}{\partial x_i^2} = 0 \quad \text{and} \quad \frac{\partial^2 \mu_i}{\partial x_j \partial x_h} = 0. \tag{16}$$

*In particular, if we take $\mu_i = e^{f_1(x_i)}$ then $(TM, J_1, g_1)$ is pluriclosed.*