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Cohomologie des formes différentielles co-invariantes et sous-groupes abéliens sans torsion de groupes de Lie.

par :

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Contents

Summary

The first part of this thesis focuses on the theme of group actions on smooth manifolds and cohomology. Our contribution was to introduce a new cohomology attached to group actions on smooth manifolds called *"Cohomology of co-invariant differential forms"*. More precisely, let M be a manifold endowed with an action of a group Γ, we are interersted in studying the graded vector space spanned by $\omega - \gamma^* \omega$ where ω is a differential form with compact support and $\gamma \in \Gamma$. The results of this part are the subject of [1].

In the second part of this thesis, we introduce and study two new invariants associated to any connected Lie group. More precisely, given a connected Lie group G we define $\mathrm{p}(G)$ as *the maximal integer p such that $\mathbb{Z}^p$ is isomorphic to a discrete subgroup of G*, likewise we define $\mathrm{q}(G)$ to be the *maximal integer q for which $\mathbb{R}^q$ is isomorphic to a closed subgroup of G*. Our study concerns connected nilpotent Lie groups and the relations of these two invariants with a well-known algebraic invariants, namely the rank of the fundamental group $\pi_1(G)$ and $\mathscr{M}(\mathfrak{g})$, i.e the maximum among the dimensions of abelian subalgebras of the Lie algebra $\mathfrak{g}$ of G. The full extent of these results are presented in [3].

Chapter 1

General Introduction

In this chapter

This chapter provides the main guidelines by which we proceed throughout this document. The first two sections contain the technical layout and statements of the central results, these will be discussed with further details in Chapters 2 and 3. The final part of this chapter is dedicated to the discussion of open questions and problems that were raised during the period of preparation of this thesis and which may take part in future research plans.

1.1 Part 1: Cohomology of Co-invariant Differential Forms

The first part of the thesis, which is the topic of Chapter 2, is for the most part an exposition of the article entitled *"Cohomology of Co-invariant Differential Forms"* that was published in *"Journal of Lie Theory"*. In this work, we introduce a new complex $\Omega_c(M)_\Gamma$ that is attached to an action by diffeomorphisms of a group Γ on a smooth manifold M and which consists of *co-invariant differential forms* i.e compactly supported forms ω that can be expressed as a finite sum $\omega = \sum_i \omega_i - \gamma_i^* \omega_i$ with $\omega_i \in \Omega_c(M)$ and $\gamma_i \in \Gamma$. The reason for this specific labeling is that in many situations, co-invariant differential forms share a complementary relationship with invariant differential forms i.e differential forms ω on M satisfying $\omega = \gamma^* \omega$ for any $\gamma \in \Gamma$, for instance when M is a compact manifold and Γ is a finite group, then it is shown that $\Omega(M) = \Omega(M)_\Gamma \oplus \Omega(M)^\Gamma$ where $\Omega(M)^\Gamma$ denotes the complex of invariant forms on M. The main goal of the article was to prove that in more general settings, the same behavior is still preserved at the cohomology level. We start with the case of an isometric action on a compact Riemannian manifold as a natural generalization of finite group actions, here is a statement of the main results of this part:

Theorem 1.1.0.1. *Let Γ be a group acting by isometries on a compact oriented Riemannian manifold M. Then the map $\Phi : \mathrm{H}^p(\Omega(M)_\Gamma) \oplus \mathrm{H}^p(\Omega(M)^\Gamma) \longrightarrow \mathrm{H}^p(M)$ given by the expression $\Phi([\omega]_\Gamma \oplus [\eta]^\Gamma) = [\omega + \eta]$ is an isomorphism satisfying:*

$$\Phi(\mathrm{H}^p(\Omega(M)_\Gamma)) = \mathrm{H}^p(M)_\Gamma \quad \textit{and} \quad \Phi(\mathrm{H}^p(\Omega(M)^\Gamma)) = \mathrm{H}^p(M)^\Gamma. \tag{1.1}$$

In particular if $\rho(\Gamma) \subset \mathrm{Isom}(M)^0$ then $\mathrm{H}^p(M) \simeq \mathrm{H}^p(\Omega(M)^\Gamma)$.

The proof of this Theorem relies on the theory of harmonic forms on compact manifolds presented in Appendix B. As a consequence, we get another proof of the following classic result:

Corollary 1.1.0.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{H}(\mathfrak{g})$ denote the Lie algebra cohomology of $\mathfrak{g}$, then $\mathrm{H}^p(G) \simeq \mathrm{H}^p(\mathfrak{g})$.*

The next step in the article was to discuss the relationship between the cohomologies of invariant and co-invariant differential forms in the context of properly discontinuous actions, the motivation for this choice is that this type of group actions represents the natural extension of finite group actions into the non-compact framework (in fact, in the compact case properly discontinuous actions are exactly finite group actions). Let Γ be a group acting properly discontinuous by diffeomorphisms on a smooth manifold M, as it is the case with finite group actions, one is able in this setting to define the *average* $m(\omega)$ of a compactly supported differential form ω by the expression $m(\omega) = \sum_{\gamma \in \Gamma} \gamma^* \omega$, note that this does make sense since the previous sum is finite when ω is restricted to any relatively compact subset of M and therefore results in a smooth Γ-invariant form on M, moreover it can be shown that $\operatorname{supp} m(\omega)/\Gamma$ is compact i.e $m(\omega)$ has Γ-compact support. In summary we get a linear map $m : \Omega_c(M) \longrightarrow \Omega(M)^\Gamma_{\Gamma c}$ where $\Omega(M)^\Gamma_{\Gamma_c}$ is the complex of Γ-invariant differential forms on M with Γ-compact support.

Proposition 1.1.0.1. *Let Γ be a group acting properly discontinuously on a smooth* *manifold M. The average map $m : \Omega_c(M) \longrightarrow \Omega(M)^{\Gamma}_{\Gamma c}$ is surjective and* $\ker(m) = \Omega_c(M)_{\Gamma}$.

In order to prove this Proposition, we made use of the following technical Lemma which gives the existence of a special type of functions:

LEMMA 1.1.0.1 (CUTOFF FUNCTIONS)**.** *Let Γ be a discrete group acting properly discontinously on a smooth manifold M. There exists a positive function $\phi : M \longrightarrow \mathbb{R}$ which is $\mathscr{C}^{\infty}$ such that for any compact $B \subset M/\Gamma$,* $\mathrm{supp}(\phi) \cap \pi^{-1}(B)$ *is compact. Furthermore we have:*

$$\sum_{\gamma \in \Gamma} \phi \circ \gamma = 1. \tag{1.2}$$

The function $\phi \in \mathscr{C}^{\infty}(M)$ is called a Γ-cutoff function.

Here is the statement of the main Theorem of this part:

THEOREM 1.1.0.2. *Let Γ be a group that acts properly discontinuously on a manifold M. Then we have the following short exact sequence:*

$$0 \to \Omega_c(M)_{\Gamma} \xrightarrow{\iota} \Omega_c(M) \xrightarrow{m} \Omega(M)^{\Gamma}_{\Gamma c} \to 0, \tag{1.3}$$

which in turn gives rise to a long exact cohomology sequence:

$$\cdots \to \mathrm{H}^p(\Omega_c(M)_{\Gamma}) \xrightarrow{\iota} \mathrm{H}^p_c(M) \xrightarrow{m} \mathrm{H}^p(\Omega(M)^{\Gamma}_{\Gamma c}) \xrightarrow{\delta} \mathrm{H}^{p+1}(\Omega_c(M)_{\Gamma}) \to \dots. \tag{1.4}$$

Moreover, the connecting homomorphism $\delta : \mathrm{H}^p(\Omega(M)^{\Gamma}_{\Gamma c}) \longrightarrow \mathrm{H}^{p+1}(\Omega_c(M)_{\Gamma})$ is given by the expression:

$$\delta([\omega]^{\Gamma}) = [\mathrm{d}\phi \wedge \omega]_{\Gamma},$$

for any cutoff function $\phi \in \mathscr{C}^{\infty}(M)$.

To end the paragraph, it is worth to mention that the Cohomology of co-invariant differential forms, has been developed as a kind of generalization of the *Cohomology of divergence forms* introduced in [4] by Pr. A. Abouqateb in order to resolve some problems of group actions on manifolds where the Lie group structure is absent i.e actions of topological groups or when the Lie algebra does not provide much information about the action of the Lie group, this happens for instance when one considers the action of a discrete Lie group on a manifold, in which case the Lie algebra of the group is trivial.

1.2 Part 2: Maximal Abelian Torsion-free Subgroups of Lie Groups

In this part, we give a brief review of the paper *"On a type of Maximal Abelian Torsion-free subgroups of Connected Lie groups"* that were recently published in the scientific journal *" Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg"*. The complete context of the presented statements as well as their full proofs can be consulted with more details in Chapter 3. In this article, we introduce two integers $\mathrm{p}(G)$ and $\mathrm{q}(G)$ that are naturally associated to a connected Lie group G and which measure to some extent the maximal size of abelian torsion-free Lie subgroups of G, namely:

$$\mathrm{p}(G) = \max\{p \in \mathbb{N},\ \mathbb{Z}^p \text{ is isomorphic to a discrete subgroup of } G\}.$$

$$\mathrm{q}(G) = \max\{q \in \mathbb{N},\ \mathbb{R}^q \text{ is isomorphic to a closed subgroup of } G\}.$$

Before proceeding to the main study, the first step was to make sure that these integers are non-trivial (nonzero for a large class of Lie groups) and well-defined (are indeed integers and not infinite), this is provided by the following result:

PROPOSITION 1.2.0.1. *Let G be a noncompact connected Lie group. Then:*

$$1 \leq q(G) \leq p(G) \leq \dim(G/K),$$

where K is a maximal compact subgroup of G. In particular $\mathrm{p}(G)$ *is finite.*

The next Proposition summarizes some general properties of the integers $\mathrm{p}(G)$ and $\mathrm{q}(G)$:

PROPOSITION 1.2.0.2. *Let G be a connected Lie group. We have the following properties:*

1. *Assume that $G = G_1 \times G_2$ for some connected Lie groups G_1 and G_2, then:*

$$\mathrm{p}(G) = \mathrm{p}(G_1) + \mathrm{p}(G_2) \ \textit{and} \ \mathrm{q}(G) = \mathrm{q}(G_1) + \mathrm{q}(G_2).$$

2. *Let $\tilde{G} \xrightarrow{\pi} G$ be a finite cover of Lie groups, then $\mathrm{p}(\tilde{G}) = \mathrm{p}(G)$ and $\mathrm{q}(\tilde{G}) = \mathrm{q}(G)$.*

It should be noted that 2 does not hold for an infinite cover of Lie groups (for more details, see Examples 3.3.0.2 and 3.3.0.3). The remaining part shifts from the general case into more specific situations and illustrates how the integers $\mathrm{p}(G)$ and $\mathrm{q}(G)$ may differ significantly depending on the nature of the Lie group G. It also points to the relationship that these invariants may share with the abelian dimension $\mathscr{M}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G i.e the maximal dimension of an abelian subalgebra of the Lie algebra $\mathfrak{g}$. Here is an example when this occurs:

PROPOSITION 1.2.0.3. *Let G be an exponential Lie group. Then:*

$$\mathrm{p}(G) = \mathrm{q}(G) = \mathscr{M}(\mathfrak{g}).$$

Simply connected nilpotent Lie groups represents an important class of exponential Lie groups and one may naturally ask whether a general nilpotent Lie group G exhibits the same behavior from the perspective of the integer $\mathrm{p}(G)$ and $\mathrm{q}(G)$. The first results in this direction are stated as follows:

PROPOSITION 1.2.0.4. *Let G be a linear connected nilpotent Lie group, then:*

$$\mathrm{p}(G) = \mathrm{q}(G) = \mathscr{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(G)).$$

THEOREM 1.2.0.1. *Let G be a connected nilpotent Lie group, then:*

$$\mathrm{q}(G) = \mathscr{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(\mathrm{G})).$$

To this end it may seem that the same result may hold for any connected nilpotent Lie group, but this turns out to be false and the we provided a counter-example to this situation by considering a connected nilpotent Lie group G such that $\mathrm{p}(G) \neq \mathrm{q}(G)$ (for more details, see Example 3.5.0.1). In fact, it is shown by the analogue Theorem for $\mathrm{p}(G)$ in the case of a general nilpotent Lie group is not as easily stated:

THEOREM 1.2.0.2. *Let G be a connected nilpotent Lie group and denote* $\mathfrak{g}$ *its Lie algebra. Then:*

$$\mathrm{p}(G) = \dim(\mathfrak{n}_0) - \mathrm{rank}(\pi_1(\mathrm{G})),$$

where $\mathfrak{n}_0$ *is of maximal dimension among all* 2*-step nilpotent Lie subalgebras* $\mathfrak{n}$ *of* $\mathfrak{g}$ *such that the Lie group* $(\mathfrak{n}, *)$ *admits a lattice* Γ *satisfying* $[\Gamma, \Gamma] \subset \ker(\exp_G) \subset \Gamma$.

1.3 Perspectives

The results obtained during the course of this thesis raise many open questions and a handful of situations remain to be explored in the future. We state here a set of problems, some of which are already part of the literature, that we think are approachable using our methods and might be central topic of discussion in upcoming works.

1.3.1 Isometric Actions on Pseudo-Riemannian Manifolds

Let (M,g) be a compact manifold endowed with a Pseudo-Riemannian metric g and let Γ be a group acting on M by Pseudo-Riemannian isometries.
PROBLEM 1. *Can we formulate an analogue of Theorem 1.1.0.1 in this situation ? What happens in the Lorentzian case ?*

In constrast to the Riemannian case, Pseudo-Riemannian structures do not always exist in general and relies on a number of topological requirements that the manifold must satisfy, for instance a Lorentzian metric exists on a compact manifold M if and only if M admits a nowhere vanishing vector field i.e its Euler class must vanish. These constraints alone make the question more challenging and looking for either answer is a topic that deserves to be addressed.

1.3.2 Existence of compact Clifford-Klein Forms

Let G be a Lie group and H a connected Lie subgoup. If Γ is any discrete subgroup of G acting properly discontinuously and freely on the homogeneous space G/H, the double quotient $\Gamma\backslash G/H$ is a smooth manifold locally modeled on G/H called a Clifford-Klein form of G/H. We say that $\Gamma \subset G \supset H$ is a *Clifford-Klein triplet*, if furthermore the double quotient $\Gamma\backslash G/H$ is compact we say that the triplet $\Gamma \subset G \supset H$ is compact. Clifford-Klein forms has been studied by many authors (see [29, 39, 40, 41] for instance).

PROBLEM 2. *Can we use the results of Theorem 1.1.0.2 in the context of homegeneous space in order to study the existence problem of compact Clifford-Klein forms. Will this approach lead to some topological obstructions ?*

1.3.3 Links Between the Cohomologies of Divergence forms and Co-invariant forms

Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on smooth manifold M by diffeomorphisms, and let Γ be a subgroup of G. In [4], Pr. A. Abouqateb introduced the complex of divergence forms $C_{\mathfrak{g}}(M)$ which consists of differential forms $\omega \in \Omega_c(M)$ that can be written as a finite sum $\omega = \sum_i L_{X_i}\eta_i$ with $X_i \in \mathfrak{g}$ and $\eta_i \in \Omega_c(M)$. When the Lie group G is connected, it is straightforward to check that $\Omega_c(M)_\Gamma \subset C_{\mathfrak{g}}(M)$, and this inclusion in turn induces a homomorphism $\iota : H^p(\Omega_c(M)_\Gamma) \longrightarrow H^p_{\mathfrak{g}}(M)$ between the cohomologies of the complexes involved.

PROBLEM 3. *Study the homomorphism* $\iota : H^p(\Omega_c(M)_\Gamma) \longrightarrow H^p_{\mathfrak{g}}(M)$ *in general and in the case when* Γ *is a lattice of the Lie group* G.

1.3.4 The invariants p(G) and q(G) in more general situations

The computation of the integers p(G) and q(G) for a general Lie group G is a difficult problem, but restricting the nature of the Lie group G could make it more approachable as more results concerning the structure of the Lie group become available, this is for instance what has been shown in Theorems 1.2.0.1, 1.2.0.2 and Propositions 1.2.0.3, 1.2.0.4:

PROBLEM 5. *Can we prove a similar set of results or generalizations of the previous results for* p(G) *and* q(G) *when* G *is a connected solvable or semisimple Lie group? What can be said about transformation groups (for instance the isometry group of a manifold)?*

Chapter 2

Cohomology of co-invariant Differential forms

In this chapter

2.1 Introduction

The main goal of this chapter is to introduce a cohomological invariant attached to a group action on a manifold, and which can be used to look for obstructions of the existence of certain types of group actions. This new invariant is in a broad sense a generalization of the cohomology of divergence forms (introduced in [4]) when the acting group fails to be a Lie group (or is discrete in which case its Lie algebra is trivial). This usage of cohomology to study group actions has become quite frequent, examples of this practice may be found in [31], [39], [40] and [41] for instance.

We start by fixing some notations. Let V be a vector space and $\varphi : \Gamma \times V \longrightarrow V$ an action of group Γ on V by linear isomorphisms, put $\varphi(\gamma, v) := \gamma \cdot v$. Denote by V^Γ the vector subspace of V consisting of Γ-invariant vectors and V_Γ the vector subspace of Γ-co-invariant vectors, i.e. the vector subspace given by:

$$V_\Gamma = \mathrm{span}\{v - \gamma.v, v \in V, \gamma \in \Gamma\}.$$

Let M be a smooth n-dimensional manifold. We denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M. Let $\rho : \Gamma \to \mathrm{Diff}(M)$ be an action of a group Γ on M by diffeomorphisms. For an r-form ω on M and an element $\gamma \in \Gamma$, we denote $\gamma^*\omega$ the pull-back of ω by the diffeomorphism $\rho(\gamma)$. This gives rise to a linear action $(\gamma, \omega) \longrightarrow (\gamma^{-1})^*\omega$ of the group Γ on both $\Omega^r(M)$ and $\Omega^r_c(M)$ to which corresponds a space of Γ-*invariant* r-forms $\Omega^r(M)^\Gamma$ and a space of Γ-*co-invariant* r-forms $\Omega^r_c(M)_\Gamma$. It is easy to check that the graded vector spaces $\Omega(M)^\Gamma := \oplus_r \Omega^r(M)^\Gamma$ and $\Omega_c(M)_\Gamma := \oplus_r \Omega^r_c(M)_\Gamma$ are stable under the usual de Rham differential and hence define two cohomologies $\mathrm{H}(\Omega(M)^\Gamma)$ and $\mathrm{H}(\Omega_c(M)_\Gamma)$. To avoid confusion, the cohomology class of a closed differential form ω on M will be written $[\omega]_\Gamma$ in $\mathrm{H}^p(\Omega_c(M)_\Gamma)$, $[\omega]^\Gamma$ in $\mathrm{H}^p(\Omega(M)^\Gamma)$ and simply $[\omega]$ in $\mathrm{H}^p(M)$. The cohomology $\mathrm{H}(\Omega(M)^\Gamma)$ also known as the *cohomology of invariant forms* has been studied by many authors (for instance see [22, 35, 46]), however the cohomology $\mathrm{H}(\Omega_c(M)_\Gamma)$, which we call *cohomology of* Γ-*coinvariant forms*, is to our knowledge new and constitutes our main object of study. In order to study a smooth action of a group Γ on a manifold M, we will exhibit relationships relating the differential complexes $\Omega_c(M)_\Gamma$, $\Omega(M)^\Gamma$ and $\Omega_c(M)$ in various situations, this allows to illustrate the interplay between their respective cohomologies by means of direct sum decompositions or exact sequences, depending on the case of study which maybe viewed as obstructions for the existence of certain types of group actions (Isometric actions or properly discontinuous actions for instance). We note that this method of proceeding is similar to the one that can be found in the article of Morita [39] in which he used a cohomology to find an obstruction for the existence of Clifford-Klein forms.

This chapter is organized in the following manner: We start by some general results of linear group actions on vector spaces which will be useful for the development that will follow. We then address the main problem of the chapter in three parts. The first part is concerned with the action of a finite group Γ on a compact manifold M, in this context we show that $\Omega(M) = \Omega(M)_\Gamma \oplus \Omega(M)^\Gamma$, and as a consequence, we get that:

$$\mathrm{H}^p(M) \simeq \mathrm{H}^p(\Omega(M)_\Gamma) \oplus \mathrm{H}^p(\Omega(M)^\Gamma). \tag{2.1}$$

The action of the group Γ on $\Omega(M)$ given by the pull-back operation induces a linear action of Γ on the cohomology of M, hence it gives rise to the vector subspaces $\mathrm{H}(M)^\Gamma$ and $\mathrm{H}(M)_\Gamma$ of $\mathrm{H}(M)$. We prove that:

$$\mathrm{H}^p(M)^\Gamma \simeq \mathrm{H}^p(\Omega(M)^\Gamma) \text{ and } \mathrm{H}^p(M)_\Gamma \simeq \mathrm{H}^p(\Omega(M)_\Gamma). \tag{2.2}$$

The second part of this chapter is an extension of the first in which we widen our context in order to include isometric actions of an arbitrary group Γ on a compact orientable Riemannian manifold M, the goal is to show that the relations (2.1) and (2.2) still holds in this new setting, this can be interpreted as an obstruction for the existence of Riemannian isometric actions. This part makes use of Hodge theory of harmonic forms (Appendix B). We conclude by applying our results to get a new proof of a well-known theorem on Lie algebra cohomology (see [19], Theorem III, p.163). Finally the third section studies the case of a properly discontinuous action of a group Γ on a smooth manifold M that is not necessarily compact, this again is another generalization of a finite action already seen in the first section. The main result of this part is a long exact sequences relating the cohomologies $\mathrm{H}(\Omega_c(M)_\Gamma)$, $\mathrm{H}_c(M)$ and $\mathrm{H}(\Omega(M)^\Gamma)$ generalizing (2.1) to this situation, the involved operators are defined and studied at the beginning of the paragraph. We then derive some consequences of this result. It is interesting to compare this exact sequence to the one found in [4, Theorem 5.3].

2.2 Group Actions on Vector Spaces

Let V be a vector space and Γ a group acting on V by linear isomorphisms.

PROPOSITION 2.2.0.1. *Suppose that V is finite dimensional and that Γ acts orthogonally on V with respect to a scalar product, then we have the following decomposition:*

$$V = V_\Gamma \overset{\perp}{\oplus} V^\Gamma. \tag{2.3}$$

Proof. For all $v \in V^\Gamma$, $w \in V$ and $\gamma \in \Gamma$, we have:

$$\langle v, w - \gamma w\rangle = \langle v, w\rangle - \langle \gamma^{-1} v, w\rangle = \langle v, w\rangle - \langle v, w\rangle = 0.$$

Thus V^Γ and V_Γ are orthogonal subspaces of V. Now fix $v \in (V_\Gamma \oplus V^\Gamma)^\perp$, since $v \in (V_\Gamma)^\perp$ then for every $\gamma \in \Gamma$, $w \in V$ we have:

$$\langle v - \gamma v, w\rangle = \langle v, w - \gamma^{-1} w\rangle = 0.$$

This gives that $v = \gamma v$ for all $\gamma \in \Gamma$ i.e $v \in V^\Gamma$, but $v \in (V^\Gamma)^\perp$ as well hence $v = 0$. We conclude that $(V_\Gamma \oplus V^\Gamma)^\perp = 0$, this leads to the desired result. □

As a consequence of this result, we get the following:

Corollary 2.2.0.1. *Let Γ be a finite group acting linearly on a finite dimensional vector space V, then:*

$$V = V_\Gamma \oplus V^\Gamma. \tag{2.4}$$

Proof. From an arbitrary scalar product $\langle\, ,\, \rangle_0$ on V, one can define a Γ-invariant scalar product $\langle\, ,\, \rangle$ by setting for all $v, w \in V$:

$$\langle v, w\rangle = \sum_{\gamma\in\Gamma} \langle \gamma v, \gamma w\rangle_0.$$

Now formula (2.4) is a consequence of Proposition 2.2.0.1. □

The preceding result remains true for any vector space V, not necessarily finite dimensional, provided that the group Γ is finite. In order to prove this fact, we consider the linear map $\mathrm{m} : V \longrightarrow V$ given by:

$$\mathrm{m}(v) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \gamma v \tag{2.5}$$

where $|\Gamma|$ is the cardinal of Γ. Then m has the following properties:

Proposition 2.2.0.2. *Let Γ be a finite group acting linearly on a vector space V. The map $\mathrm{m} : V \longrightarrow V$ defined in (2.5) is a linear projection with $\mathrm{Im}(\mathrm{m}) = V^\Gamma$ and $\ker(\mathrm{m}) = V_\Gamma$.*

Proof. First, observe that for all $v \in V$ and $\alpha \in \Gamma$, $\alpha\mathrm{m}(v) = \mathrm{m}(v)$. Indeed:

$$\alpha\mathrm{m}(v) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} (\alpha\gamma) v = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \gamma v = \mathrm{m}(v).$$

It follows that $\mathrm{m}(v) \in V^\Gamma$ for all $v \in V$ and thus $\mathrm{Im}(\mathrm{m}) \subset V^\Gamma$. Conversely if $w \in V^\Gamma$, it is clear that $w = \mathrm{m}(w)$, hence $V^\Gamma \subset \mathrm{Im}(\mathrm{m})$. Next notice that since $\mathrm{m}(v) \in V^\Gamma$ then $\mathrm{m}^2(v) = \mathrm{m}(v)$ i.e m is a linear projection. It is only left to prove that $\ker(\mathrm{m}) = V_\Gamma$, for all $v \in V$ and $\gamma \in \Gamma$, formula (2.5) gives that $\mathrm{m}(v - \gamma v) = 0$, thus $V_\Gamma \subset \ker(\mathrm{m})$. Conversely, fix $v \in \ker(\mathrm{m})$ and define for every $\gamma \in \Gamma$, $v_\gamma = \frac{1}{|\Gamma|}(v - \gamma v) \in V_\Gamma$. We get that:

$$\sum_{\gamma \in \Gamma} v_\gamma = v - \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \gamma v = v - \mathrm{m}(v) = v.$$

Thus $v \in V_\Gamma$, which means that $\ker(\mathrm{m}) \subset V_\Gamma$ hence the equality. □

In view of the decomposition $V = \ker(\mathrm{m}) \oplus \mathrm{Im}(\mathrm{m})$, we obtain:

Corollary 2.2.0.2. *Let Γ be a finite group acting linearly on a vector space V. Then:*

$$V = V_\Gamma \oplus V^\Gamma. \tag{2.6}$$

2.3 Action of a Finite Group on a Compact Manifold

In this section, M is a compact manifold and Γ a finite group. The goal is to express the relationship between the invariant and co-invariant forms relative to the action of Γ on M.

Proposition 2.3.0.1. *Let Γ be a finite group acting by diffeomorphisms on a compact manifold M. Then we have the following properties:*

1. $\Omega(M) = \Omega(M)_\Gamma \oplus \Omega(M)^\Gamma$.
2. *The map $\Phi : \mathrm{H}^p(\Omega(M)^\Gamma) \oplus \mathrm{H}^p(\Omega(M)_\Gamma) \longrightarrow \mathrm{H}^p(M)$, $\Phi([\omega]^\Gamma \oplus [\eta]_\Gamma) = [\omega + \eta]$ is an isomorphism. Moreover:*

$$\Phi(\mathrm{H}^p(\Omega(M)^\Gamma) = \mathrm{H}^p(M)^\Gamma \;\; \textit{and} \;\; \Phi(\mathrm{H}^p(\Omega(M)_\Gamma) = \mathrm{H}^p(M)_\Gamma.$$

Proof. In view of Corollary 2.2.0.2, we obtain 1. by taking $V = \Omega(M)$ and the linear action given by the pull-back operation. If instead we take $V = \mathrm{H}^p(M)$, we obtain the vector space decomposition $\mathrm{H}^p(M) = \mathrm{H}^p(M)_\Gamma \oplus \mathrm{H}^p(M)^\Gamma$. It follows from 1. that the map Φ is indeed an isomorphism, moreover since the cohomology class of a closed Γ-invariant form on M is also Γ-invariant we get that $\Phi(\mathrm{H}^p(\Omega(M)^\Gamma)) \subset \mathrm{H}^p(M)^\Gamma$. Conversely let $m : \Omega(M) \longrightarrow \Omega(M)$ be the average map given by (2.5) and choose $[\omega] \in \mathrm{H}^p(M)^\Gamma$ i.e $[\omega] = [\gamma^*\omega]$ for any $\gamma \in \Gamma$, then clearly $m(\omega)$ is a closed Γ-invariant form on M satisfying $[\omega] = [m(\omega)] = \Phi[m(\omega)]^\Gamma$, it follows that $\Phi(\mathrm{H}^p(\Omega(M)^\Gamma)) = \mathrm{H}^p(M)^\Gamma$. On the other hand it is clear that $\mathrm{H}^p(M)_\Gamma \subset \Phi(\mathrm{H}^p(\Omega(M)_\Gamma))$.

Since M is compact it has finite dimensional cohomology and since:

$$\dim \mathrm{H}^p(M) = \dim \mathrm{H}^p(M)^{\Gamma} + \dim \mathrm{H}^p(M)_{\Gamma} = \dim \mathrm{H}^p(\Omega(M)^{\Gamma}) + \dim \mathrm{H}^p(\Omega(M)_{\Gamma}),$$

we get that $\dim \mathrm{H}^p(M)_{\Gamma} = \dim \mathrm{H}^p(\Omega(M)_{\Gamma})$ thus $\Phi(\mathrm{H}^p(\Omega(M)_{\Gamma})) = \mathrm{H}^p(M)_{\Gamma}$. □

In particular if Γ is a finite group that acts on a compact manifold M by diffeomorphisms, we have $\dim \mathrm{H}^p(\Omega(M)_{\Gamma}) \leq \dim \mathrm{H}^p(M)$ and $\dim \mathrm{H}^p(\Omega(M)^{\Gamma}) \leq \dim \mathrm{H}^p(M)$ hence all the involved cohomologies are finite dimensional.

2.4 Action by Isometries on a compact manifold

Let M be a compact manifold and Γ a group. Recall that when Γ is a finite group acting on M, then M admits a Γ-invariant Riemannian metric. A natural generalization of the preceding situation is then to consider Γ acting by isometries on the manifold M. Fix a Riemannian metric $\langle\,,\,\rangle$ on M and let $G := \mathrm{Isom}(M, \langle\,,\,\rangle)$ be the isometry group of $(M, \langle\,,\,\rangle)$. Let $\rho : \Gamma \longrightarrow \mathrm{Diff}(M)$ be an action of the group Γ on M by Riemannian isometries, this means that $\rho(\Gamma) \subset \mathrm{Isom}(M, \langle\,,\,\rangle)$. We then denote G^0 the identity component of the isometry group G. The goal of this paragraph is to give an extension of the results presented in (2.1) and (2.2) to this new setting, and on the other hand, use those results to find topological obstructions for the existence of isometric actions. We start by showing that the induced action of Γ on the cohomology of M is in fact equivalent to a finite group action, to do this we need this key lemma which can be found in [20]:

Lemma 2.4.0.1. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold. Any isometry $\varphi \in G^0$ in the identity component is homotopic to* Id_M.

Denote $\Gamma_0 = \rho(\Gamma)/\rho(\Gamma) \cap G^0$, this is clearly a group since $\rho(\Gamma) \cap G^0 \lhd \rho(\Gamma)$, moreover Γ_0 is finite, this is due to the compactness of the isometry group G. Denote $[\gamma]$ the equivalence class of an element γ in $\rho(\Gamma)$.

Proposition 2.4.0.1. *The action $\Gamma_0 \times \mathrm{H}^p(M) \longrightarrow \mathrm{H}^p(M)$ given by $([\gamma], [\omega]) \mapsto [\gamma^*\omega]$ is well-defined, furthermore:*

$$\mathrm{H}^p(M)_{\Gamma} = \mathrm{H}^p(M)_{\Gamma_0} \;\; \textit{and} \;\; \mathrm{H}^p(M)^{\Gamma} = \mathrm{H}^p(M)^{\Gamma_0}. \tag{2.7}$$

Proof. Take two elements γ_1 and γ_2 in Γ satisfying $[\gamma_1] = [\gamma_2]$. This meanss $\gamma_2 = \gamma_0\gamma_1$ for some element $\gamma_0 \in \rho(\Gamma) \cap G^0$. Hence for all $[\omega] \in \mathrm{H}^p(M)$, the preceding lemma gives that:

$$[\gamma_2^*\omega] = [\gamma_1^*(\gamma_0^*\omega)] = \gamma_1^*[\gamma_0^*\omega] = \gamma_1^*[\omega] = [\gamma_1^*\omega].$$

This shows that the action depends only on the equivalence class in question, it is thus well-defined. Now (2.7) follows from $[\gamma^*\omega] = [\omega]$ for any $\gamma \in \rho(\Gamma) \cap G^0$. □

In view of (2.2.0.2) and the last proposition, we conclude that:

Corollary 2.4.0.1. *Let M be a compact Riemannian manifold and Γ a group that acts on M by Riemannian isometries. Then:*

$$\mathrm{H}^p(M) = \mathrm{H}^p(M)_\Gamma \oplus \mathrm{H}^p(M)^\Gamma.$$

Furthermore if $\rho(\Gamma) \subset \mathrm{Isom}(M)^0$ *then* $\mathrm{H}^p(M)_\Gamma = 0$.

We dedicate the rest of the paragraph to show that the results (2.1) and (2.2) are still available in a more general setting. The claims and proofs presented here relies on the theory of harmonic forms on compact oriented Riemannian manifolds, the notations and main results can be consulted in Appendix B. In what follows $(M, \langle\, ,\, \rangle)$ denotes a compact oriented Riemannian manifold with Riemannian volume element $\mathrm{d}V$.

Lemma 2.4.0.2. *For any $\gamma \in \mathrm{Isom}(M)$, we have $*\gamma^* = \deg(\gamma)\gamma^* *$ where $\deg(\gamma)$ is the degree of γ (equal to 1 if γ is orientation preserving and -1 otherwise).*

Proof. Fix an orthonormal frame $\mathscr{R} = \{E_1, \dots, E_n\}$ on an open subset $U \subset M$ with dual co-frame $\{\varepsilon^1, \dots, \varepsilon^n\}$. We claim that $\{\gamma^*\varepsilon^1, \dots, \gamma^*\varepsilon^n\}$ is the dual co-frame of the orthonormal frame $\mathscr{R}_\gamma = \{\gamma_* E_1, \dots, \gamma_* E_n\}$ defined on $\gamma^{-1}U$, indeed this is a result of the following computation:

$$(\gamma^*\varepsilon^i)(\gamma_* E_j)(x) = \varepsilon^i_{\gamma x}(T_x\gamma((\gamma_* E_j)_x)) = \varepsilon^i_{\gamma x}(E_j(\gamma x)) = \delta_{ij}.$$

It follows that $(\gamma^*\varepsilon_I, \gamma^*\varepsilon_J) = \delta_{IJ}$ where $I = \{i_1 < \cdots < i_p\}$ and $\varepsilon_I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_p}$. Next consider two p-forms ω and η, and write $\omega = \sum_I f_I \varepsilon_I$ and $\eta = \sum_J g_J \varepsilon_J$ on U. Then:

$$(\gamma^*\omega, \gamma^*\eta) = \gamma^*\left(\sum_I f_I g_I\right) = \gamma^*(\omega, \eta).$$

Denote $\sigma = \gamma^{-1}$, since $\gamma^*(\mathrm{d}V) = \deg(\gamma)\mathrm{d}V$, the preceding relation gives that:

$$\omega \wedge (\deg(\gamma)\gamma^*(*\eta)) = \deg(\gamma)\gamma^*(\sigma^*\omega \wedge *\eta) = \deg(\gamma)\gamma^*(\sigma^*\omega, \eta)\gamma^*(\mathrm{d}V) = (\omega, \gamma^*\eta)\mathrm{d}V.$$

The result then follows from the definition of the Hodge-$*$ operator. □

COROLLARY 2.4.0.2. *Any $\gamma \in \mathrm{Isom}(M)$ induces an isometry γ^* on $(\Omega(M), \langle\ ,\ \rangle)$, more precisely if ω and η are differential forms on M then:*

$$\langle \gamma^*\omega, \gamma^*\eta\rangle = \langle \omega, \eta\rangle.$$

Proof. In view of the preceding Lemma, we have:

$$\int_M \gamma^*\omega \wedge *\gamma^*\eta = \deg(\gamma)\int_M \gamma^*\omega \wedge \gamma^*(*\eta) = \deg(\gamma)\int_M \gamma^*(\omega \wedge *\eta) = \deg(\gamma)^2\int_M \omega \wedge *\eta.$$

Since $\deg(\gamma)^2 = 1$, we obtain that $\langle \gamma^*\omega, \gamma^*\eta\rangle = \langle \omega, \eta\rangle$. □

COROLLARY 2.4.0.3. *For all $\gamma \in \Gamma$, we have $\gamma^*\delta = \delta\gamma^*$ and $\gamma^*\Delta = \Delta\gamma^*$. In particular, for any harmonic form ω on M, $\gamma^*\omega$ is also harmonic.*

Proof. Let $\eta \in \Omega^p(M)$ et $\gamma \in \Gamma$, we have by the preceding lemma:

$$\delta(\gamma^*\eta) = (-1)^{n(p+1)+1} * d(*\gamma^*\eta) = (-1)^{n(p+1)+1}\gamma^*(*d*\eta) = \gamma^*(\delta\eta).$$

This shows that $\gamma^*\delta = \delta\gamma^*$. Now $\gamma^*\Delta = \Delta\gamma^*$ follows from $\Delta = d\delta + \delta d$. □

A consequence of Corollary 2.4.0.3 is that the induced linear action of Γ on $\Omega(M)$ restricts to the space $\mathcal{H}(M)$ of harmonic forms on M , let $\mathcal{H}(M)_\Gamma$ (resp. $\mathcal{H}(M)^\Gamma$) denote the corresponding graded vector space of Γ-co-invariant (resp. Γ-invariant) harmonic forms.

THEOREM 2.4.0.1 (HODGE DECOMPOSITION FOR $\Omega(M)_\Gamma$ AND $\Omega(M)^\Gamma$).
Let $\rho : \Gamma \longrightarrow \mathrm{Isom}(M)$ be an action by isometries of a group Γ on a compact oriented Riemannian manifold M. Then:

1. $\Omega^p(M)_\Gamma = \mathrm{d}(\Omega^{p-1}(M)_\Gamma) \overset{\perp}{\oplus} \delta(\Omega^{p+1}(M)_\Gamma) \overset{\perp}{\oplus} \mathcal{H}^p(M)_\Gamma$.
2. $\Omega^p(M)^\Gamma = \mathrm{d}(\Omega^{p-1}(M)^\Gamma) \overset{\perp}{\oplus} \delta(\Omega^{p+1}(M)^\Gamma) \overset{\perp}{\oplus} \mathcal{H}^p(M)^\Gamma$.
3. $\mathcal{H}^p(M) = \mathcal{H}^p(M)_\Gamma \overset{\perp}{\oplus} \mathcal{H}^p(M)^\Gamma$.

Proof. By Corollary 2.4.0.2 the group Γ acts by isometries on $\mathcal{H}(M)$ and since $\mathcal{H}^p(M)$ is a finite dimensional vector space, then 3. follows from Proposition 2.2.0.1. The decomposition in 1. is straightforward, indeed choose $\omega \in \Omega^p(M)$ and use B.3.0.1 to write $\omega = \mathrm{d}\alpha + \delta\beta + \eta$

where $\alpha \in \Omega^{p-1}(M)$, $\beta \in \Omega^{p+1}(M)$ and $\eta \in \mathcal{H}^p(M)$. Then:

$$\omega - \gamma^*\omega = \mathrm{d}(\alpha - \gamma^*\alpha) + \delta(\beta - \gamma^*\beta) + (\eta - \gamma^*\eta).$$

This proves the claim. For 2. let $\omega \in \Omega^p(M)^\Gamma$, again by B.3.0.1 we can write $\omega = \mathrm{d}\alpha + \delta + \eta$ with $\alpha \in \Omega^{p-1}(M)$, $\beta \in \Omega^{p+1}(M)$ and $\eta \in \mathcal{H}^p(M)$. For any $\gamma \in \Gamma$ we have that:

$$0 = \omega - \gamma^*\omega = \mathrm{d}(\alpha - \gamma^*\alpha) + \delta(\beta - \gamma^*\beta) + (\eta - \gamma^*\eta).$$

Uniqueness of the Hodge decomposition gives that $\mathrm{d}\alpha = \gamma^*\mathrm{d}\alpha$, $\delta\beta = \gamma^*\delta\beta$ and $\eta = \gamma^*\eta$ therefore $\eta \in \mathcal{H}^p(M)^\Gamma$. Next we prove that α and β can be chosen Γ-invariant, more precisely we show that we can write $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$ with $\alpha_1 \in \Omega^{p-1}(M)^\Gamma$, $\beta_1 \in \Omega^{p+1}(M)^\Gamma$ and such that $\mathrm{d}\alpha_2 = 0$, $\delta\beta_2 = 0$. In order to do so let $\alpha = \mathrm{d}\mu + \delta\nu + \lambda$ and $\beta = \mathrm{d}\hat{\mu} + \delta\hat{\nu} + \hat{\lambda}$ be the Hodge decompositions of α and β respectively, then for any $\gamma \in \Gamma$:

$$\mathrm{d}(\alpha - \gamma^*\alpha) = \mathrm{d}\delta(\nu - \gamma^*\nu) \quad \text{and} \quad \delta(\beta - \gamma^*\beta) = \delta\mathrm{d}(\hat{\mu} - \gamma^*\hat{\mu}).$$

This implies that $\mathrm{d}\delta(\nu - \gamma^*\nu) = 0$ and $\delta\mathrm{d}(\hat{\mu} - \gamma^*\hat{\mu}) = 0$. Therefore:

$$\langle \delta(\nu - \gamma^*\nu), \delta(\nu - \gamma^*\nu)\rangle = \langle \nu - \gamma^*\nu, \mathrm{d}\delta(\nu - \gamma^*\nu)\rangle = 0.$$

Thus $\delta\nu - \gamma^*\delta\nu = 0$ for any $\gamma \in \Gamma$ and hence $\delta\nu \in \Omega^{p-1}(M)^\Gamma$. In a similar way we can show that $\mathrm{d}\hat{\mu} \in \Omega^{p+1}(M)^\Gamma$. Put $\alpha_1 = \delta\nu$, $\alpha_2 = \mathrm{d}\mu + \lambda$, $\beta_1 = \mathrm{d}\hat{\mu}$ and $\beta_2 = \delta\hat{\nu} + \hat{\lambda}$. We get that:

$$\omega = \mathrm{d}(\alpha_1 + \alpha_2) + \delta(\beta_1 + \beta_2) + \eta = \mathrm{d}\alpha_1 + \delta\beta_1 + \eta,$$

such that $\alpha_1 \in \Omega^{p-1}(M)^\Gamma$, $\beta_1 \in \Omega^{p+1}(M)^\Gamma$ and $\eta \in \mathcal{H}^p(M)^\Gamma$. This completes the proof. □

We can now state and prove the main theorem of this paragraph:

THEOREM 2.4.0.2. *Let Γ be a group acting by isometries on a compact oriented Riemannian manifold M. Then the map $\Phi : \mathrm{H}^p(\Omega(M)_\Gamma) \oplus \mathrm{H}^p(\Omega(M)^\Gamma) \longrightarrow \mathrm{H}^p(M)$ given by the expression $\Phi([\omega]_\Gamma \oplus [\eta]^\Gamma) = [\omega + \eta]$ is an isomorphism satisfying:*

$$\Phi(\mathrm{H}^p(\Omega(M)_\Gamma)) = \mathrm{H}^p(M)_\Gamma \quad \textit{and} \quad \Phi(\mathrm{H}^p(\Omega(M)^\Gamma)) = \mathrm{H}^p(M)^\Gamma. \tag{2.8}$$

In particular if $\rho(\Gamma) \subset \mathrm{Isom}(M)^0$ then $\mathrm{H}^p(M) \simeq \mathrm{H}^p(\Omega(M)^\Gamma)$.

Proof. From the first two decompositions of Theorem 2.4.0.1 we get that there exists an isomorphism:

$$\Phi_1 : \mathrm{H}^p(\Omega(M)_\Gamma) \oplus \mathrm{H}^p(\Omega(M)^\Gamma) \longrightarrow \mathscr{H}^p(M)_\Gamma \oplus \mathscr{H}^p(M)^\Gamma \overset{(3)}{=} \mathscr{H}^p(M),$$

and $\Phi = \Phi_1 \circ j$ where $j : \mathscr{H}^p(M) \longrightarrow \mathrm{H}^p(M)$ is the isomorphism in Corollary B.3.0.1. It is clear that $j(\mathscr{H}^p(M)_\Gamma) = \mathrm{H}^p(M)_\Gamma$ and $j(\mathscr{H}^p(M)^\Gamma) = \mathrm{H}^p(M)^\Gamma$ therefore we get (2.8). Finally if $\rho(\Gamma)$ is contained in the identity component of $\mathrm{Isom}(M)$ then from Corollary 2.4.0.1 we get that $\mathrm{H}^p(M)_\Gamma = 0$ and thus $\mathrm{H}^p(\Omega(M)_\Gamma) = 0$ as desired. □

The first point of Proposition 2.3.0.1 however, does not generalize to this case, this is best illustrated in the following example:

Example 2.4.0.1. *Let $\mathbb{T}^n$ be the n-dimensional torus which we view as the quotient $\mathbb{R}^n/\mathbb{Z}^n$ with natural projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{T}^n$. Choose a generator $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ of $\mathbb{T}^n$ this means that $\mathbb{T}^n = \overline{\langle \pi(a) \rangle}$ or equivalently that the real numbers $1, a_1, \ldots, a_n$ are $\mathbb{Q}$-linearly independant. Denote $\gamma \in \mathrm{Diff}(\mathbb{T}^n)$ the group multiplication by $\pi(a)$ and $\rho : \mathbb{Z} \longrightarrow \mathrm{Diff}(\mathbb{T}^n)$ the corresponding $\mathbb{Z}$-action, i.e $\rho(n) = \gamma^n$. Clearly $\rho(\mathbb{Z}) \subset \mathrm{Isom}(\mathbb{T}^n, \langle\, ,\, \rangle)$ for any (bi-)invariant Riemannian metric $\langle\, ,\, \rangle$ on $\mathbb{T}^n$.*
Since γ has dense orbits in $\mathbb{T}^n$ then $\mathscr{C}^\infty(\mathbb{T}^n)^\mathbb{Z} = \mathbb{R}$. Furthermore since γ is orientation preserving any function $g \in \mathscr{C}^\infty(\mathbb{T}^n)_\mathbb{Z}$ must satisfy $I(g) = 0$ where $I : \mathscr{C}^\infty(\mathbb{T}^n) \longrightarrow \mathbb{R}$ is the operator given by the expression $I(g) = \int_{\mathbb{T}^n} g(x)dx$. It is easy to check that:

$$\mathscr{C}^\infty(\mathbb{T}^n) = \ker(I) \oplus \mathbb{R}.$$

We say that $a \in \mathbb{R}^n$ is a Liouville vector if there exists $A > 0$ such that for any $\tau > 0$ there exists $m_\tau \in \mathbb{Z}^n$ satisfying:

$$|1 - e^{2\pi i \langle m_\tau, a \rangle}| \geq \frac{A}{|m_\tau|^\tau}.$$

*When $a \in \mathbb{R}^n$ is a Liouville vector, A. El Kacimi and H. Hmili in [**?**, Theorem 1.4] used Fourier series to construct an infinite family $\{g_k,\, k \in \mathbb{N}^*\}$ of smooth functions on $\mathbb{T}^n$ satisfying $I(g_k) = 0$ and such that the equation $f - \gamma^* f = g_k$ doesn't have a solution. In particular we obtain that $\mathscr{C}^\infty(\mathbb{T}^n)_\mathbb{Z} \subsetneq \ker(I)$ and therefore $\mathscr{C}^\infty(\mathbb{T}^n) \neq \mathscr{C}^\infty(\mathbb{T}^n)_\mathbb{Z} \oplus \mathscr{C}^\infty(\mathbb{T}^n)^\mathbb{Z}$ in this case.*

As an application, we give new proofs of the following known results:

Proposition 2.4.0.2. *Let G be a compact connected Lie group acting smoothly on a compact orientable manifold M. Then:*

$$\mathrm{H}^p(M) \simeq \mathrm{H}^p(\Omega(M)^G).$$

Proof. Let $\langle\,,\,\rangle$ be any Riemannian metric on M and denote $\rho : G \longrightarrow \mathrm{Diff}(M)$ the action of the group G on M. Let $x \in M$ and X, Y two smooth vector fields on M. The map $G \longrightarrow \mathbb{R}$ given by $g \mapsto \langle \mathrm{T}_x\rho(g)(X_x), \mathrm{T}_x\rho(g)(Y_x)\rangle_{gx}$ is smooth, we put:

$$\langle\langle X_x, Y_x\rangle\rangle_x := \int_G \langle \mathrm{T}_x\rho(g)(X_x), \mathrm{T}_x\rho(g)(Y_x)\rangle^0_{gx}\mathrm{d}g.$$

This defines a G-invariant Riemannian metric $\langle\langle\,,\,\rangle\rangle$ on M i.e $\rho(G) \subset \mathrm{Isom}(M, \langle\langle\,,\,\rangle\rangle)$. Since G is connected and ρ is continuous, $\rho(G)$ is contained in the identity component of $\mathrm{Isom}(M, \langle\langle\,,\,\rangle\rangle)$, hence $\mathrm{H}^p(\Omega(M)_G) = 0$ and thus $\mathrm{H}^p(M) \simeq \mathrm{H}^p(\Omega(M)^G)$. □

COROLLARY 2.4.0.4. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{H}(\mathfrak{g})$ denote the Lie algebra cohomology of $\mathfrak{g}$, then $\mathrm{H}^p(G) \simeq \mathrm{H}^p(\mathfrak{g})$.*

Proof. The homomorphism $\rho : G \longrightarrow \mathrm{Diff}(G)$, $g \mapsto \ell_g$ where ℓ_g denotes the left multiplication on G defines a smooth isometric action on G with respect to any left invariant Riemannian metric $\langle\,,\,\rangle$ on G. Thus by Proposition 2.4.0.2, $\mathrm{H}^p(G) \simeq \mathrm{H}^p(G)^G \simeq \mathrm{H}^p(\mathfrak{g})$. □

2.5 Group Actions on Noncompact Manifolds

We turn now to the study of the action of a group on a manifold which is not necessarily compact in contrast to the preceding situations. In what follows, we set a group Γ acting properly discontinously on a smooth manifold M. When the manifold M is compact, the group Γ must be finite, therefore properly discontinuous actions are a natural extension of finite actions on compact manifolds. A main goal in this paragraph is therefore to give an analogue to the decomposition (2.1) in the current situation. As a matter of fact, the process of averaging described in the compact case still makes sense in the case of properly discontinuous actions. Indeed, the *average* map $m : \Omega_c(M) \longrightarrow \Omega(M)^\Gamma$ given by the expression $m(\omega) = \sum_{\gamma\in\Gamma}\gamma^*\omega$ is well-defined and linear. Note that this map is not a projection contrary to the compact case since $\Omega(M)^\Gamma$ may contain differential forms which are not compactly supported. In fact, m is not even surjective, to see this it is sufficient to consider the case when Γ is a finite group acting on a noncompact manifold M, we obtain that for any $\omega \in \Omega_c(M)$, the average form $m(\omega)$ is compactly supported. However, up to reducing its range we can show that m can be made surjective. The key to proceed lies in the following important lemma, which was originally proved in [**?**] in the case of a properly discontinuous and free action:

LEMMA 2.5.0.1 (CUTOFF FUNCTIONS). *Let Γ be a discrete group acting properly discontinously on a smooth manifold M. There exists a positive function $\phi : M \longrightarrow \mathbb{R}$ which is $\mathscr{C}^\infty$ such that for any compact $B \subset M/\Gamma$, $\operatorname{supp}(\phi) \cap \pi^{-1}(B)$ is compact. Furthermore we have:*

$$\sum_{\gamma \in \Gamma} \phi \circ \gamma = 1. \tag{2.9}$$

The function $\phi \in \mathscr{C}^\infty(M)$ is called a Γ-cutoff function.

Proof. Set a locally finite cover $\mathscr{V} := \{V_n,\, n \in \mathbb{N}\}$ of M/Γ by relatively compact open subsets. There exists a locally finite open cover $\mathscr{W} := \{W_n,\, n \in \mathbb{N}\}$ of M/Γ such that W_n is a relatively compact subset and $\bar{V}_n \subset W_n$. To see this, denote $J_1 = \{i \in \mathbb{N},\ \bar{V}_1 \cap V_i \neq \emptyset\}$. Since $\bar{V}_1$ is compact and $\mathscr{V}$ is a locally finite cover of M/Γ, we obtain that J_1 is finite and moreover:

$$\bar{V}_1 \subset \bigcup_{j \in J_1} V_j \ \text{ et } \ \bar{V}_1 \cap \bigcup_{j \notin J_1} V_j = \emptyset.$$

Thus we can find a relatively compact open set W_1 containing $\bar{V}_1$ such that:

$$W_1 \subset \bigcup_{j \in J_1} V_j \ \text{ et } \ W_1 \cap \bigcup_{j \notin J_1} V_j = \emptyset.$$

Hence the family $\mathscr{W}_1 = \{W_1\} \cup \{V_j,\, j \geq 2\}$ is a locally finite cover of M/Γ by relatively compact open sets. Repeating this process on the family $\mathscr{W}_1$, we obtain by induction the desired open cover. Next we show that there exists relatively compact open subsets U_n and O_n of M satisfying $\pi(U_n) = V_n$, $\pi(O_n) = W_n$ and $\bar{U}_n \subset O_n$. Indeed, we start with any pair of relatively compact open subsets $\hat{U}_n$ and $\hat{O}_n$ of M such that $\pi(\hat{U}_n) = V_n$ and $\pi(\hat{O}_n) = W_n$. Since $V_n \subset W_n$ then $\hat{U}_n \subset \bigcup_{\gamma \in \Gamma} \gamma \cdot \hat{O}_n$. We then put:

$$U_n = \hat{O}_n \cap \bigcup_{\gamma \in \Gamma} \gamma^{-1} \cdot \hat{U}_n.$$

Thus U_n is a relatively compact open subset of M and it is clear that $\pi(U_n) \subset V_n$. Conversely let $x \in V_n$, we can write $x = \pi(a)$ for some $a \in \hat{U}_n$. Since $\hat{U}_n \subset \bigcup_{\gamma \in \Gamma} \gamma \cdot \hat{O}_n$ we can find $\gamma \in \Gamma$ such that $\gamma^{-1} a \in \gamma^{-1} \hat{U}_n \cap \hat{O}_n$ and from $\pi(\gamma^{-1} a) = x$ we get that $x \in \pi(U_n)$. Next for any $n \in \mathbb{N}$ define $S_n = \{\gamma \in \Gamma,\ \bar{U}_n \cap \gamma \cdot \hat{O}_n \neq \emptyset\}$ which is a finite set since Γ acts properly discontinuously on M. Since $\bar{V}_n \subset W_n$, then $\bar{U}_n \subset \bigcup_{\gamma \in \Gamma} \gamma \cdot \hat{O}_n$ and we deduce that:

$$U_n \subset \bigcup_{\gamma \in S_n} \gamma \cdot \hat{O}_n := O_n.$$

After this tedious construction, we can begin the proof of the lemma. Let $g_n \in \mathscr{C}_c^\infty(M)$ such that $0 \le g_n \le 1$, $\operatorname{supp}(g_n) \subset O_n$ and $g_n = 1$ on U_n, next put $g = \sum_{n\in\mathbb{N}} g_n$. We claim that the function g is well-defined, positive and smooth. Indeed, choose any relatively compact open subset $U \subset M$ and let $J = \{n \in \mathbb{N}, \bar{U} \cap O_n \neq \emptyset\}$. Since $\mathscr{W}$ is a locally finite cover of M/Γ then $(O_n)_{n\in\mathbb{N}}$ is a locally finite cover of M and thus the set J is finite. Hence:

$$g_{|U} = \sum_{j\in J} g_{j_{|U}} \in \mathscr{C}^\infty(U).$$

This shows that g is a well-defined smooth function on M. Let $B \subset M/\Gamma$ be a compact subset and put $I = \{n \in \mathbb{N}, \pi^{-1}(B) \cap O_n \neq \emptyset\}$. Since $\pi(O_n) = W_n$ then:

$$I \subset \{n \in \mathbb{N}, \pi^{-1}(B) \cap \pi^{-1}(W_n) \neq \emptyset\} \subset \{n \in \mathbb{N}, B \cap W_n \neq \emptyset\} \text{ which is finite.}$$

Thus $\pi^{-1}(B) \cap \operatorname{supp}(g) \subset \pi^{-1}(B) \cap \bigcup_{n\in\mathbb{N}} O_n = \pi^{-1}(B) \cap \bigcup_{i\in I} O_i$. It is clear that $\bigcup_{i\in I} O_i$ is relatively compact, and therefore $\pi^{-1}(B) \cap \operatorname{supp}(g)$ is compact. Next let $x \in M$, then $\pi(x) \in V_n$ for some $n \in \mathbb{N}$. This means that there exists $\gamma \in \Gamma$ such that $\gamma x \in U_n$ and thus $g(\gamma x) > 0$. Hence $\sum_{\gamma\in\Gamma} g(\gamma x) > 0$. Finally put:

$$\phi = \frac{g}{\sum_{\gamma\in\Gamma} g \circ \gamma}.$$

This gives that $\operatorname{supp}(\phi) = \operatorname{supp}(g)$ and $\sum_{\gamma\in\Gamma} \phi \circ \gamma = 1$. □

Remark 2.5.0.1. *When M/Γ is compact, writing $\operatorname{supp}(\phi) = \operatorname{supp}(\phi) \cap \pi^{-1}(M/\Gamma)$ shows that any cutoff function ϕ on M must be compactly supported.*

We say that a Γ-invariant form ω on M has Γ-compact support if $(\operatorname{supp} \omega)/\Gamma$ is compact. We denote $\Omega(M)^\Gamma_{\Gamma c}$ the space of Γ-invariant forms on M with Γ-compact support, it is a differential subalgebra of $\Omega(M)$.

Remark 2.5.0.2. *For any $\omega \in \Omega_c(M)$ the average form $m(\omega)$ has Γ-compact support. Indeed, it is clear that $\operatorname{supp} m(\omega) \subset \bigcup_{\gamma\in\Gamma} \gamma \cdot \operatorname{supp}(\omega)$, hence $\pi(\operatorname{supp} m(\omega)) \subset \pi(\operatorname{supp} \omega)$ which is a compact subset of M/Γ. In other words $m(\omega) \in \Omega(M)^\Gamma_{\Gamma c}$.*

Proposition 2.5.0.1. *Let Γ be a group acting properly discontinuously on a smooth manifold M. The average map $m : \Omega_c(M) \longrightarrow \Omega(M)^\Gamma_{\Gamma c}$ is surjective and $\ker(m) = \Omega_c(M)_\Gamma$.*

Proof. Let $\eta \in \Omega(M)^\Gamma_{\Gamma c}$ and choose a Γ-cutoff function $\phi \in \mathscr{C}^\infty(M)$. Denote $K := \pi(\operatorname{supp} \eta)$, since $\operatorname{supp}(\eta)$ is Γ-invariant we obtain that $\operatorname{supp}(\eta) = \pi^{-1}(K)$, furthermore K is a compact set. It follows that the form $\omega = \phi\eta$ has compact support in M, thus:

$$m(\omega) = \sum_{\gamma\in\Gamma} \gamma^*\omega = \sum_{\gamma\in\Gamma} \gamma^*(\phi\eta) = \left(\sum_{\gamma\in\Gamma} \gamma^*\phi\right)\eta = \eta.$$

This shows that m is surjective. On the other hand, given $\alpha \in \Gamma$ and $\omega \in \Omega_c(M)$, we have:

$$m(\omega - \alpha^*\omega) = \sum_{\gamma\in\Gamma} \gamma^*\omega - \sum_{\gamma\in\Gamma} \gamma^*\alpha^*\omega = \sum_{\gamma\in\Gamma} \gamma^*\omega - \sum_{\gamma\in\Gamma} \gamma^*\omega = 0.$$

Hence $\Omega_c(M)_\Gamma \subset \ker(m)$. Conversely let $\omega \in \ker(m)$, for any $\gamma \in \Gamma$ define $\omega_\gamma := \phi_\gamma\omega - \gamma^*(\phi_\gamma\omega)$ with $\phi_\gamma := \phi \circ \gamma^{-1}$. It is clear that $\omega_\gamma \in \Omega_c(M)_\Gamma$, we claim that $\omega_\gamma = 0$ except for a finite family $A \subset \Gamma$. To see this put $K = \pi(\operatorname{supp}\omega)$, then K is compact subset of M/Γ and thus by Lemma 2.5.0.1, the subset $\operatorname{supp}(\phi) \cap \pi^{-1}(K) = \bigcup_{\gamma\in\Gamma}(\operatorname{supp}(\phi) \cap \gamma\cdot\operatorname{supp}\omega)$ is compact. Next put $A := \{\gamma \in \Gamma,\ (\operatorname{supp}\phi \cap \pi^{-1}(K)) \cap \gamma\cdot\operatorname{supp}\omega \neq \emptyset\}$, since Γ acts properly discontinuous on the manifold M we obtain that A is a finite subset of Γ, Moreover it is straightforward to check that $A = \{\gamma \in \Gamma,\ \operatorname{supp}\phi \cap \gamma\cdot\operatorname{supp}\omega \neq \emptyset\}$. Hence for $\alpha \in \Gamma \setminus A$, it follows that:

$$\operatorname{supp}(\phi_\alpha\omega) \subset (\alpha^{-1}\operatorname{supp}\phi) \cap \operatorname{supp}\omega = \alpha^{-1}(\operatorname{supp}\phi \cap \alpha\operatorname{supp}\omega) = \emptyset.$$

Consequently $\phi_\alpha\omega = 0$ and thus $\omega_\alpha = 0$ for all $\alpha \in \Gamma \setminus A$. As a result:

$$\sum_{\gamma\in A} \omega_\gamma = \sum_{\gamma\in\Gamma} \omega_\gamma = \left(\sum_{\gamma\in\Gamma} \phi_\gamma\right)\omega - \phi\left(\sum_{\gamma\in\Gamma} \gamma^*\omega\right) = \omega - \phi m(\omega) = \omega.$$

In summary $\omega = \sum_{\gamma\in A} \omega_\gamma \in \Omega_c(M)_\Gamma$. Thus $\ker(m) \subset \Omega_c(M)_\Gamma$. □

This leads to the main Theorem of this paragraph:

THEOREM 2.5.0.1. *Let Γ be a group that acts properly discontinuously on a manifold M. Then we have the following short exact sequence:*

$$0 \to \Omega_c(M)_\Gamma \xrightarrow{\iota} \Omega_c(M) \xrightarrow{m} \Omega(M)^\Gamma_{\Gamma c} \to 0, \tag{2.10}$$

which in turn gives rise to a long exact cohomology sequence:

$$\cdots \to \mathrm{H}^p(\Omega_c(M)_\Gamma) \xrightarrow{\iota} \mathrm{H}^p_c(M) \xrightarrow{m} \mathrm{H}^p(\Omega(M)^\Gamma_{\Gamma c}) \xrightarrow{\delta} \mathrm{H}^{p+1}(\Omega_c(M)_\Gamma) \to \dots. \tag{2.11}$$

Moreover, the connecting homomorphism $\delta : \mathrm{H}^p(\Omega(M)^\Gamma_{\Gamma c}) \longrightarrow \mathrm{H}^{p+1}(\Omega_c(M)_\Gamma)$ is given by the *expression:*

$$\delta([\omega]^\Gamma) = [\mathrm{d}\phi \wedge \omega]_\Gamma,$$

for any cutoff function $\phi \in \mathscr{C}^\infty(M)$ (satisfying (2.9)).

Proof. The exactness of the sequence (2.10) easily follows from Propostion 2.5.0.1 in which case the cohomology sequence (2.11) is straightforward. In order to obtain the expression of the operator δ, fix a closed form $\omega \in \Omega^p(M)^\Gamma_{\Gamma c}$. If we write $\delta([\omega]^\Gamma) = [\beta]_\Gamma$ then $\beta = \mathrm{d}\alpha$ for some $\alpha \in \Omega^p_c(M)$ satisfying $m(\alpha) = \omega$. For $\alpha = \phi\omega$, we conclude that $\beta = \mathrm{d}\phi \wedge \omega$. □

REMARK 2.5.0.3. *Consider a cutoff function $\phi \in \mathscr{C}^\infty(M)$ (satisfying (2.9)). Then*

$$\mathrm{d}\left(\sum_{\gamma\in\Gamma} \gamma^*\phi\right) = \sum_{\gamma\in\Gamma} \gamma^* \mathrm{d}\phi = 0.$$

It follows that $\mathrm{d}\phi \in \Omega(M)_\Gamma$. When M/Γ is compact, we obtain that $\mathrm{d}\phi \in \Omega_c(M)_\Gamma$ in this case we denote $[\theta]_\Gamma \in \mathrm{H}^1(\Omega_c(M)_\Gamma)$ its cohomology class.

REMARK 2.5.0.4. *In the exact sequence (2.11) we have $\ker(\iota) = \mathrm{Im}(\delta)$, hence the co-invariant classes of the form $[\mathrm{d}\phi \wedge \omega]_\Gamma$ are exactly the classes of $\mathrm{H}(\Omega_c(M)_\Gamma)$ which are exact in $\mathrm{H}_c(M)$.*

We give some consequences of the preceding exact sequence:

COROLLARY 2.5.0.1. *Let Γ be a group acting properly discontinuously on a smooth manifold M. Then $\mathrm{H}^0(\Omega_c(M)_\Gamma) = 0$ and if M is compact, the mapping $\iota : \mathrm{H}^p(\Omega_c(M)_\Gamma) \longrightarrow \mathrm{H}^p_c(M)$ is injective and the map $m : \mathrm{H}^p_c(M) \longrightarrow \mathrm{H}^p(\Omega(M)^\Gamma)$ is surjective.*

Proof. Let $\phi \in \mathscr{C}^\infty(M)$ be a cutoff function satisfying (2.9). Let $[f]_\Gamma \in \mathrm{H}^0(\Omega_c(M)_\Gamma)$ i.e a constant function satisfying $\sum_{\gamma\in\Gamma} \gamma^* f = 0$. If M is noncompact, then $f = 0$ since it is constant and compactly supported. If M is compact, then Γ must be finite since it acts properly discontinuously on M thus $\sum_{\gamma\in\Gamma} \gamma^* f = |\Gamma| f = 0$, hence $f = 0$. If M is compact then Γ is finite we can choose $\phi = 1/|\Gamma|$ as a cutoff function. Let $\delta : \mathrm{H}^p(\Omega(M)^\Gamma_{\Gamma c}) \longrightarrow \mathrm{H}^{p+1}(\Omega_c(M)_\Gamma)$ be the connecting homomorphism in the cohomology sequence (2.11), from the previous remark we get that $\delta([\omega]^\Gamma) = [\mathrm{d}\phi \wedge \omega]_\Gamma = 0$ for every closed form $\omega \in \Omega(M)^\Gamma$. It follows from the exactness of (2.11) that $\ker(\iota) = \mathrm{Im}(\delta) = 0$ and $\mathrm{Im}(m) = \ker(\delta) = \Omega(M)^\Gamma_c$. We conclude that ι is injective et m is surjective. □

PROPOSITION 2.5.0.2. *Let Γ be a group acting properly discontinuously on a noncompact connected manifold M. Then M/Γ is compact if and only if $\iota : \mathrm{H}^1(\Omega_c(M)_\Gamma) \longrightarrow \mathrm{H}^1_c(M)$ is not injective.*

Proof. Let $\phi \in \mathscr{C}^\infty(M)$ be a cutoff function. Assume M/Γ is compact, then $\phi \in \mathscr{C}^\infty_c(M)$. Suppose that $[\mathrm{d}\phi]_\Gamma = 0$, then we can find $\psi \in \Omega^0_c(M)_\Gamma$ such that $\mathrm{d}\phi = \mathrm{d}\psi$ and since M is connected the difference $\phi - \psi$ is constant with compact support, which implies that $\phi = \psi$ but this is impossible as $\phi \notin \Omega^0_c(M)_\Gamma$. We obtain that $[d\phi]_\Gamma \neq 0$, and since $\iota[\mathrm{d}\phi]_\Gamma = [\mathrm{d}\phi] = 0$ we conclude that ι is not injective.
Conversely if M/Γ is not compact, let $\omega \in \Omega_c(M)_\Gamma$ be a closed form such that $[\omega] = 0$. We can write $\omega = df$ for some $f \in \mathscr{C}^\infty_c(M)$. Since $\mathrm{d}m(f) = m(\omega) = 0$ we obtain that $m(f)$ is a constant function with Γ-compact support. If $m(f) \neq 0$ then we would have $\operatorname{supp} m(f) = M$ and consequently $\operatorname{supp} m(f)/\Gamma = M/\Gamma$ which is noncompact, this leads to a contradiction. Thus $m(f) = 0$ i.e $f \in \mathscr{C}^\infty_c(M)_\Gamma$ and therefore $[\omega]_\Gamma = 0$. We conclude that ι is injective. □

Corollary 2.5.0.2. *Let M be a contractible smooth manifold, and Γ a group acting properly discontinuously on M. Then M/Γ is compact if and only if* $\mathrm{H}^1(\Omega_c(M)_\Gamma) \neq 0$.

Corollary 2.5.0.3. *Let M be a contractible manifold and $\rho : \Gamma \to \mathrm{Diff}(M)$ be a properly discontinuous action with compact orbit space M/Γ. Then for any $1 \leq p \leq n$, we have*

$$\mathrm{H}^p(\Omega_c(M)_\Gamma) \simeq \mathrm{H}^{p-1}(\Omega(M)^\Gamma).$$

In particular, $\mathrm{H}^1(\Omega_c(M)_\Gamma) = \mathrm{Span}\{[\theta]_\Gamma\}$.

Let $\psi \in \mathscr{C}^\infty(M)$ be a cutoff function with respect to the action of Γ on M. Let U, V be open subsets of M and choose a partition of unity $\{\phi_U, \phi_V\}$ subordinate to $\{U, V\}$ then put:

$$\hat{\phi}_U = \sum_{\gamma \in \Gamma} \gamma^*(\psi\phi_U), \quad \hat{\phi}_V = \sum_{\gamma \in \Gamma} \gamma^*(\psi\phi_V).$$

It is then straightforward to check that $\{\hat{\phi}_U, \hat{\phi}_V\}$ is a Γ-invariant partition of unity subordinate to $\{U, V\}$. As a consequence of Proposition 2.6.2.4 we get the following result:

Corollary 2.5.0.4. *Let Γ be a group that acts properly discontinuously on a smooth manifold M by diffeomorphisms. For any Γ-invariant open subsets U, V of M, the long exact cohomology sequences* (2.14) *and* (2.16) *hold.*

Example 2.5.0.1 (Clifford-Klein forms). *Let G be a Lie group and $H \subset G$ a connected Lie subgoup. If Γ is any discrete subgroup of G acting properly discontinuously and freely on the homogeneous space G/H, the double quotient $\Gamma\backslash G/H$ is a smooth manifold locally modeled on G/H called a Clifford-Klein form of G/H. We say that $\Gamma \subset G \supset H$ is a Clifford-Klein triplet, if furthermore $\Gamma\backslash G/H$ is compact we say that the triplet $\Gamma \subset G \supset H$ is compact. Clifford-Klein forms has been studied by many authors (see [29] for instance).*

For every compact Clifford-Klein form $\Gamma\backslash G/H$, by virtue of Theorem 2.5.0.1, we have a long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^p(\Omega_c(G/H)_\Gamma) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}^p_c(G/H) \xrightarrow{\mathrm{H}(m)} \mathrm{H}^p(\Omega(G/H)^\Gamma) \xrightarrow{\delta} \mathrm{H}^{p+1}(\Omega_c(G/H)_\Gamma) \to \dots$$

In particular, when H is a maximal compact subgroup of a connected Lie group G, the manifold G/H is then a contractible, which leads to:

$$\mathrm{H}^p(\Omega_c(G/H)_\Gamma) \simeq \mathrm{H}^{p-1}(\Omega(G/H)^\Gamma)$$

and when the action of Γ on G/H is free, we have $\mathrm{H}^p(\Omega_c(G/H)_\Gamma) \simeq \mathrm{H}^{p-1}(\Gamma\backslash G/H)$.

Example 2.5.0.2. *Let Σ_g be a connected compact Riemann surface of genus $g \geq 2$. The fundamental group of Σ_g can be identified with a discrete subgroup Γ_g of the projective group:*

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{I}\},$$

so that Σ_g is identified with the orbit space $\mathbb{H}/\Gamma_g$ of the action of Γ_g on the Poincaré half-plane $\mathbb{H}$. This action is given by:

$$(A,z) \longmapsto \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for every $z \in \mathbb{H}$ and any matrix $A \in \mathrm{SL}(2,\mathbb{R})$. The 2-cohomology space of Γ_g-co-invariant forms on $\mathbb{H}$ is then isomorphic to the 1-cohomology space of Σ_g, hence $\dim(\mathrm{H}^2(\Omega_c(\mathbb{H})_{\Gamma_g}) = 2g$.

Example 2.5.0.3 (Nilmanifolds). *A compact nilmanifold is the quotient of a simply connected nilpotent Lie group G by a discrete subgroup $\Gamma \subset G$ such that G/Γ is compact (one can see [45] for details). The simplest example being the n-dimensional torus viewed as the natural quotient of $\mathbb{R}^n$ by $\mathbb{Z}^n$.*

The cohomology of Γ-invariant forms on G can naturally be identified with the cohomology of G/Γ and a famous theorem of Nomizu (see [42]) asserts that the cohomology $\mathrm{H}(G/\Gamma)$ of the nilmanifold is isomorphic to $\mathrm{H}(\mathfrak{g})$ the cohomology of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. It follows from Corollary 2.5.0.3 that for every $p = 1,\dots,n$,

$$\mathrm{H}^p(\Omega(G)^\Gamma) \simeq \mathrm{H}^p(\mathfrak{g}) \ \ \text{and} \ \ \mathrm{H}^p(\Omega_c(G)_\Gamma) \simeq \mathrm{H}^{p-1}(\mathfrak{g}).$$

In particular, for the usual action by translations of $\mathbb{Z}^n$ on $\mathbb{R}^n$, we obtain that:

$$\dim \mathrm{H}^p(\Omega_c(\mathbb{R}^n)_{\mathbb{Z}^n}) = C_n^{p-1}.$$

2.6 Properties of Co-invariant Cohomology

In this paragraph we give some basic properties of co-invariant cohomology.

2.6.1 Homotopy Invariance

Let M and N be smooth manifolds and Γ a group acting on both M and N by diffeomorphisms. Let $f_i : M \longrightarrow N$ be a smooth Γ-equivariant map with $i = 0, 1$, then $f_i^* \circ \gamma^* = \gamma^* \circ f_i^*$ for any $\gamma \in \Gamma$ and it follows that $f_i^*(\Omega(N)_\Gamma) \subset \Omega(M)_\Gamma$. If furthermore the map f_i is proper then $f_i^*(\Omega_c(N)_\Gamma) \subset \Omega_c(M)_\Gamma$. Let $F : M \times \mathbb{R} \longrightarrow N$ be a smooth homotopy between f_0 and f_1 and $K : \Omega(N) \longrightarrow \Omega(M)$ the corresponding chain homotopy given by (B.3) recall that:

$$K \circ d + d \circ K = f_1^* - f_0^*.$$

For any $\gamma \in \Gamma$ and $(x,t) \in M \times \mathbb{R}$ put $\gamma \cdot (x,t) := (\gamma x, t)$, this defines an action by diffeomorphisms of the group Γ on $M \times \mathbb{R}$. In the case where F is Γ-equivariant, then it is easy to see that $K \circ \gamma^* = \gamma^* \circ K$ and therefore $K(\Omega^p(N)_\Gamma) \subset \Omega^{p-1}(M)_\Gamma$, if moreover F is a proper homotopy then $K(\Omega_c^p(N)_\Gamma) \subset \Omega_c^{p-1}(M)_\Gamma$. As a result, we get the following proposition:

PROPOSITION 2.6.1.1. *Let Γ be a group acting on smooth manifolds M and N by diffeomorphisms. Let $f, g : M \longrightarrow N$ be Γ-equivariant smooth maps and suppose $F : M \times \mathbb{R} \longrightarrow N$ is a Γ-equivariant homotopy sending f to g. Then the cohomology maps:*

$$f^*, g^* : \mathrm{H}^p(\Omega(N)_\Gamma) \longrightarrow \mathrm{H}^p(\Omega(M)_\Gamma),$$

induced by the pullback operation are equal. If F is a proper homotopy, the maps f and g are proper as well and $f^*, g^* : \mathrm{H}^p(\Omega_c(N)_\Gamma) \longrightarrow \mathrm{H}^p(\Omega_c(M)_\Gamma)$ are also equal.

2.6.2 Mayer-Vietoris sequence

Let Γ be a group acting on a smooth manifold M by diffeomorphisms. Let $U, V \subset M$ be open subsets and consider the Mayer-Vietoris exact sequence:

$$0 \longrightarrow \Omega(U \cup V) \xrightarrow{r} \Omega(U) \oplus \Omega(V) \xrightarrow{J} \Omega(U \cap V) \longrightarrow 0,$$

where $r(\omega) = (\omega_{|U}, \omega_{|V})$ and $\jmath(\omega,\eta) = \eta_{|U\cap V} - \omega_{|U\cap V}$. Assume now that U and V are Γ-invariant, it is easy to check that for any $\gamma \in \Gamma$, $r\circ\gamma^* = \gamma^*\circ r$ and $\jmath\circ\gamma^* = \gamma^*\circ\jmath$, consequently:

$$r(\Omega(U\cup V)_\Gamma) \subset \Omega(U)_\Gamma \oplus \Omega(V)_\Gamma, \quad \jmath(\Omega(U)_\Gamma \oplus \Omega(V)_\Gamma) \subset \Omega(U\cap V)_\Gamma.$$

This allows to obtain the following sequence:

$$0 \longrightarrow \Omega(U\cup V)_\Gamma \xrightarrow{r_\Gamma} \Omega(U)_\Gamma \oplus \Omega(V)_\Gamma \xrightarrow{\jmath_\Gamma} \Omega(U\cap V)_\Gamma \longrightarrow 0, \tag{2.12}$$

where r_Γ and $\jmath_\Gamma$ are the restriction corresponding to r and $\jmath$ respectively.

PROPOSITION 2.6.2.1. *Let Γ be a group that acts by diffeomorphisms on a smooth manifold M and let $\{U,V\}$ be Γ-invariant open subsets of M. Assume that there exists a Γ-invariant partition of unity $\{\phi_U,\phi_V\}$ subordinate to $\{U,V\}$. Then (2.12) is an exact sequence.*

Proof. It is clear that r_Γ is injective, let $\alpha \in \Omega(U\cap V)_\Gamma$ which can be expressed as a finite sum $\alpha = \sum_i \alpha_i - \gamma_i^*\alpha_i$ with $\alpha_i \in \Omega(U\cap V)$ and $\gamma_i \in \Gamma$. Since $\jmath$ is a surjective map then we can write $\alpha_i = \omega_i - \eta_i$ such that $\omega_i \in \Omega(U)$ and $\eta_i \in \Omega(V)$. It follows that $\alpha = \jmath_\Gamma(\omega,\eta)$ with:

$$\omega = \sum_i \omega_i - \gamma_i^*\omega_i \quad \text{and} \quad \eta = \sum_i \eta_i - \gamma_i^*\eta_i. \tag{2.13}$$

Thus $\jmath_\Gamma$ is surjective. On the other hand we clearly have $\mathrm{Im}(r_\Gamma) \subset \ker(\jmath_\Gamma)$, conversely choose $\omega \in \Omega(U)_\Gamma$ and $\eta \in \Omega(V)_\Gamma$ such that $\omega = \eta$ on $U\cap V$ i.e $(\omega,\eta) \in \ker(\jmath_\Gamma)$ and write:

$$\omega = \sum_i \omega_i - \gamma_i^*\omega_i \quad \text{and} \quad \eta = \sum_j \eta_j - \gamma_j^*\eta_j.$$

Since $\ker(\jmath) = \mathrm{Im}(r)$, then $\omega = \alpha_{|U}$ and $\eta = \alpha_{|V}$ for some $\alpha \in \Omega(U\cup V)$. It is clear that $\phi_U\omega_i$ and $\phi_V\eta_j$ define global forms on $U\cup V$. Thus by Γ-invariance of ϕ_U and ϕ_V we get that:

$$\alpha = \phi_U\alpha + \phi_V\alpha = \sum_i (\phi_U\omega_i) - \gamma_i^*(\phi_U\omega_i) + \sum_j (\phi_V\eta_j) - \sigma_j^*(\phi_V\eta_j) \in \Omega(U\cup V)_\Gamma.$$

It follows that $(\omega,\eta) = r_\Gamma(\alpha)$. This ends the proof. □

PROPOSITION 2.6.2.2. *Let M be a smooth manifold, Γ a group acting on M by diffeomorphisms and $U,V \subset M$ arbitrary open subsets. Assume that there exists a Γ-invariant partition of unity $\{\phi_U,\phi_V\}$ subordinate to $\{U,V\}$. Then the short exact sequence (2.12) in-*

duce the following long exact cohomology sequence:

$$\cdots \to \mathrm{H}^p(\Omega(U\cup V)_\Gamma) \xrightarrow{r_\Gamma} \mathrm{H}^p(\Omega(U)_\Gamma)\oplus \mathrm{H}^p(\Omega(V)_\Gamma) \xrightarrow{\jmath_\Gamma} \mathrm{H}^p(\Omega(U\cap V)_\Gamma) \xrightarrow{\partial} \mathrm{H}^{p+1}(\Omega(U\cup V)_\Gamma) \xrightarrow{r_\Gamma} \dots \tag{2.14}$$

where $\partial : \mathrm{H}^p(\Omega(U\cap V)_\Gamma) \longrightarrow \mathrm{H}^{p+1}(\Omega(U\cup V)_\Gamma)$ *denote the connecting homomorphism.*

For any open subset $W \subset M$ we can view $\Omega_c(W)$ as the subcomplex of forms $\omega \in \Omega_c(M)$ such that $\mathrm{supp}(\omega) \subset W$. Recall that the short Mayer-Vietoris sequence for compact supports is given by:

$$0 \longrightarrow \Omega_c(U\cap V) \xrightarrow{\iota} \Omega_c(U)\oplus\Omega_c(V) \xrightarrow{\rho} \Omega_c(U\cup V) \longrightarrow 0,$$

where ι and ρ are given by $\iota(\omega) = (\omega, -\omega)$ and $\rho(\omega,\eta) = \omega+\eta$. It is easy to see that $\iota\circ\gamma^* = \gamma^*\circ\iota$ and $\rho\circ\gamma^* = \gamma^*\circ\rho$. Thus we get the following sequence:

$$0 \longrightarrow \Omega_c(U\cap V)_\Gamma \xrightarrow{\iota_\Gamma} \Omega_c(U)_\Gamma\oplus\Omega_c(V)_\Gamma \xrightarrow{\rho_\Gamma} \Omega_c(U\cup V)_\Gamma \longrightarrow 0, \tag{2.15}$$

where ι_Γ and ρ_Γ are the restriction maps corresponding to ι and ρ respectively.

Proposition 2.6.2.3. *Assume that there exists a* Γ*-invariant partition of unity* $\{\phi_U, \phi_V\}$ *subordinate to* $\{U, V\}$*. Then* (2.15) *is an exact sequence.*

Proof. Following the proof of Proposition 2.6.2.1 we can show in a similar way that ι_Γ is injective, ρ_Γ is surjective and $\mathrm{Im}(\iota_\Gamma) \subset \ker(\rho_\Gamma)$. Conversely let $\omega \in \Omega_c(U)_\Gamma$ and $\eta \in \Omega_c(V)_\Gamma$ such that $\omega+\eta = 0$ on $U\cup V$. Since $\ker(\rho) = \mathrm{Im}(\iota)$ we can find $\alpha \in \Omega_c(U\cap V)$ such that $\omega = \alpha$ and $\eta = -\alpha$. Now ϕ_U and ϕ_V are Γ-invariant, $\mathrm{supp}(\phi_U) \subset U$ and $\mathrm{supp}(\phi_V) \subset V$, therefore we get that $\phi_V\omega, \phi_U\eta \in \Omega_c(U\cap V)_\Gamma$ and it follows that:

$$\alpha = \phi_U\alpha + \phi_V\alpha = \phi_V\omega - \phi_U\eta \in \Omega_c(U\cap V)_\Gamma.$$

Thus $(\omega,\eta) = \iota_\Gamma(\alpha)$, this ends the proof. □

Proposition 2.6.2.4. *Let* M *be a smooth manifold,* Γ *a group acting on* M *by diffeomorphisms and* $U, V \subset M$ *arbitrary open subsets. Assume that there exists a* Γ*-invariant partition of unity* $\{\phi_U, \phi_V\}$ *subordinate to* $\{U, V\}$*. Then the short exact sequence* (2.15) *induce the following long exact cohomology sequence:*

$$\cdots \to \mathrm{H}^p(\Omega_c(U\cap V)_\Gamma) \xrightarrow{\iota_\Gamma} \mathrm{H}^p(\Omega_c(U)_\Gamma)\oplus \mathrm{H}^p(\Omega_c(V)_\Gamma) \xrightarrow{\rho_\Gamma} \mathrm{H}^p(\Omega_c(U\cup V)_\Gamma) \xrightarrow{\partial} \mathrm{H}^{p+1}(\Omega_c(U\cap V)_\Gamma) \xrightarrow{\iota_\Gamma} \dots \tag{2.16}$$

where $\partial : \mathrm{H}^p(\Omega_c(U\cup V)_\Gamma) \longrightarrow \mathrm{H}^{p+1}(\Omega_c(U\cap V)_\Gamma)$ *denote the connecting homomorphism.*

2.7 Examples with infinite dimensional co-invariant cohomology

The goal of this part is to show that without additional hypothesis on the nature of a group action on a smooth manifold (for instance being properly discontinuous or an isometric action), the co-invariant cohomology is not well-behaved in general. To illustrate this we give examples in which the co-invariant cohomology is actually infinite-dimensional on a compact smooth manifold contrary to the de Rham cohomology.

2.7.1 The Hyperbolic Torus

Let $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $\mathrm{tr}(A) > 2$. It is easy to check that $A = PDP^{-1}$ with $P \in \mathrm{GL}(2,\mathbb{R})$ and $D = \mathrm{diag}(\lambda, \lambda^{-1})$. Clearly $\lambda > 0$ and $\lambda \neq 1$. Hence it makes sense to set $D^t = \mathrm{diag}(\lambda^t, \lambda^{-t})$ and define $A^t = PD^tP^{-1}$ for any $t \in \mathbb{R}$. Next we define the Lie group homomorphism:

$$\phi : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{R}^2), \;\; t \mapsto A^t$$

The hyperbolic torus $\mathbb{T}^3_A$ is the smooth manifold defined as $(\mathbb{R}^2 \rtimes_\phi \mathbb{R})/(\mathbb{Z}^2 \rtimes_\phi \mathbb{Z})$. The natural projection $\mathbb{R}^2 \rtimes_\phi \mathbb{R} \xrightarrow{p} \mathbb{R}$ induce a fiber bundle structure $\mathbb{T}^3_A \xrightarrow{p} \mathbb{S}^1$ with fiber type $\mathbb{T}^2$ such that $p[x,y,t] = [t]$. If $(1,a)$ and $(1,b)$ are the eigenvectors of A respectively associated to the eigenvalues λ and λ^{-1} then:

$$v = (1,a,0), \;\; w = (1,b,0) \;\; \text{and} \;\; e = (0,0,-\log(\lambda)^{-1}),$$

forms a basis of $\mathfrak{g} = \mathrm{Lie}(\mathbb{R}^2 \rtimes_\phi \mathbb{R})$, and we can check that:

$$[v,w]_{\mathfrak{g}_3} = 0, \;\; [e,v]_{\mathfrak{g}_3} = -v, \;\; \text{and} \;\; [e,w]_{\mathfrak{g}_3} = w. \tag{2.17}$$

Denote X, Y and Z the left invariant vector fields on $\mathbb{R}^2 \rtimes_\phi \mathbb{R}$ associated to v, w and e respectively, then $\{X,Y,Z\}$ defines a parallelism on $\mathbb{T}^A_3$, a direct calculation leads to:

$$X = \lambda^t \left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right), \;\; Y = \lambda^{-t}\left(\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \;\; \text{and} \;\; Z = -\log(\lambda)^{-1}\frac{\partial}{\partial t}. \tag{2.18}$$

Now denote α, β and θ the dual forms associated to X, Y and Z respectively. It is clear that the vector fields X and Y of $\mathbb{T}^3_A$ are tangent to the fibers of the fiber bundle $\mathbb{T}^3_A \xrightarrow{p} \mathbb{S}^1$, and that $\theta = -(\log\lambda)p^*(\sigma)$ where σ is the invariant volume form on $\mathbb{S}^1$ satisfying $\int_{\mathbb{S}^1} \sigma = 1$.

Suppose in what follows that the eigenvalue λ of A is irrational, then from the relation:

$$A\begin{pmatrix}1\\a\end{pmatrix}=\lambda\begin{pmatrix}1\\a\end{pmatrix}$$

we deduce that $a\in\mathbb{R}\setminus\mathbb{Q}$. This remark leads to:

PROPOSITION 2.7.1.1. *The orbits of the vector field X defined in* (2.18) *are dense in the fibers of the fiber bundle $\mathbb{T}^3_A \xrightarrow{p} \mathbb{S}^1$. In particular for any $f\in\mathscr{C}^\infty(\mathbb{T}^3_A)$, $X(f)=0$ is equivalent to $f=p^*(\phi)$ for some $\phi\in\mathscr{C}^\infty(\mathbb{S}^1)$.*

Proof. We shall identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. Fix $[t]\in\mathbb{S}^1$ and consider the diffeomorphism:

$$\Phi_t:\mathbb{T}^2\longrightarrow p^{-1}[t],\;\; [x,y]\mapsto[x,y,t].$$

Then define the vector field $\hat{X}$ on $\mathbb{T}^2$ given by:

$$\hat{X}_{[x,y]}=T_{[x,y,t]}\phi_t^{-1}(X_{[x,y,t]})=\lambda^t\left(\frac{\partial}{\partial x}+a\frac{\partial}{\partial y}\right).$$

Since a is irrational we get that the family $\{1,a\}$ is $\mathbb{Q}$-linearly independent and thus the orbits of $\hat{X}$ are dense in $\mathbb{T}^2$, consequently the orbits of $\hat{X}$ are dense in $p^{-1}[t]$, this proves the assertion since t is arbitrary. □

Put $M=\mathbb{T}^3_A$. It is straightforward to check that:

$$d\alpha=-\alpha\wedge\theta,\;\; d\beta=\beta\wedge\theta,\;\; d\theta=0$$

and that $L_X\alpha=-\theta$ and $L_X\beta=L_X\theta=0$ thus $L_X(\alpha\wedge\beta\wedge\theta)=0$. The vector field X induce a discrete action $\rho:\mathbb{Z}\longrightarrow \mathrm{Diff}(M)$ given by $\rho(n)(x)=\phi^X_n(x)$. For convenience, denote $\gamma:=\rho(1)$.

CALCULATING $\mathrm{H}^0(\Omega(M)_\rho)$: Choose $f\in\Omega^0(M)_\rho$ such that $df=0$, then f is a constant function equal to $g-\gamma^*g$ for some $g\in\mathscr{C}^\infty(M)$. Consequently we obtain that:

$$\int_M f\alpha\wedge\beta\wedge\theta=\int_M(g-\gamma^*g)\alpha\wedge\beta\wedge\theta=\int_M g\alpha\wedge\beta\wedge\theta-\int_M\gamma^*(g\alpha\wedge\beta\wedge\theta)=0.$$

Thus $f=0$ and we conclude that $\mathrm{H}^0(\Omega(M)_\rho)=0$.

CALCULATING $\mathrm{H}^1(\Omega(M)_\rho)$: We prove that $\mathrm{H}^1(\Omega(M)_\rho)$ is infinite dimensional. In order to do so, we prove that the map $p^*:\Omega^1(\mathbb{S}^1)\longrightarrow \mathrm{H}^1(\Omega(M)_\rho)$ is well-defined and injective or equivalently we show that $p^*(\Omega^1(\mathbb{S}^1))\subset \mathrm{Z}^1(\Omega(M)_\rho)$ and $p^*(\Omega^1(\mathbb{S}^1))\cap \mathrm{B}^1(\Omega(M)_\rho)=0$. An element $\eta\in p^*(\Omega^1(\mathbb{S}^1))$ can always be written as $\eta=p^*(\phi)\theta$ where $\phi\in\mathscr{C}^\infty(\mathbb{S}^1)$. Since $L_X\theta=0$ and $X(p^*\phi)=0$ then $L_X\eta=0$ and thus $\eta=(\phi_t^X)^*\eta$. Therefore we get that:

$$\eta=\int_0^1(\phi_t^X)^*\eta dt=-p^*(\phi)\int_0^1(\phi_t^X)^*(L_X\alpha)dt=p^*(\phi)\alpha-\gamma^*(p^*(\phi)\alpha).$$

Moreover observe that $d\eta=0$, hence we deduce that $p^*(\Omega^1(\mathbb{S}^1))\subset \mathrm{Z}^1(\Omega(M)_\rho)$. Now assume that $\eta=d(g-\gamma^*g)$ then clearly $X(g-\gamma^*g)=0$ and $Z(g-\gamma^*g)=p^*(\phi)$, thus according to Proposition 2.7.1.1, there exists $\psi\in\mathscr{C}^\infty(\mathbb{S}^1)$ such that $g-\gamma^*g=p^*\psi$. By induction we can show that for any $n\in\mathbb{N}$, $g=\rho(n)^*g+np^*\psi$ which then leads to:

$$|p^*\psi|\leq\frac{1}{n}|g-\rho(n)^*g|\leq\frac{2}{n}\|g\|_\infty\underset{n\to+\infty}{\longrightarrow}0.$$

Hence $p^*\psi=g-\gamma^*g=0$ and so $\eta=0$. Thus $p^*(\Omega^1(\mathbb{S}^1))\cap \mathrm{B}^1(\Omega(M)_\rho)=0$.

Calculating $\mathrm{H}^2(\Omega(M)_\rho)$: We will show that $p^*(\Omega^1(\mathbb{S}^1))\wedge\beta\subset \mathrm{H}^2(\Omega(M)_\rho)$. To do this, we fix a 2-form $\eta=p^*(\phi)\theta\wedge\beta$ with $\phi\in\mathscr{C}^\infty(\mathbb{S}^1)$. We can easily check that $d\eta=0$, moreover from the previous calculations and the fact that $L_X\beta=0$ we get that $\beta=\gamma^*\beta$, therefore:

$$p^*(\phi)\theta\wedge\beta=(p^*\phi\alpha\wedge\beta)-\gamma^*(p^*(\phi)\alpha\wedge\beta).$$

Hence $p^*(\Omega^1(\mathbb{S}^1))\wedge\beta\subset \mathrm{Z}^2(\Omega(M)_\rho)$. Now assume that $\eta=d(\omega-\gamma^*\omega)$, in order to proceed we need the following lemma:

Lemma 2.7.1.1. *Let $f\in\mathscr{C}^\infty(\mathrm{T}_A^3)$ then for every $s\in\mathbb{R}$ we have the following formula:*

$$Z\big((\phi_s^X)^*(f)\big)=-s(\phi_s^X)^*\big(X(f)\big)+(\phi_s^X)^*\big(Z(f)\big).\tag{2.19}$$

In particular $Z(\gamma^*f)=-X(\gamma^*f)+\gamma^*(Z(f))$ and $i_Z\circ\gamma^*=-\gamma^*\circ i_X+\gamma^*\circ i_Z$.

Proof. For any $(x,y,t)\in\mathbb{R}^3$, a straightforward computation gives that:

$$\begin{aligned}
Z\big((\phi_s^X)^*(f)\big)(x,y,t) &= -\frac{1}{\log\lambda}d(f\circ\phi_s^X)_{(x,y,t)}(0,0,1)\\
&= -\frac{1}{\log\lambda}\frac{d}{du}_{|u=0}(f\circ\phi_s^X)(x,y,t+u)\\
&= -\frac{1}{\log\lambda}\frac{d}{du}_{|u=0}f(s\lambda^{t+u}+x, as\lambda^{t+u}+y, t+u)\\
&= -\frac{1}{\log\lambda}(df)_{\phi_s^X(x,y,t)}(s\log(\lambda)\lambda^t, as\log(\lambda)\lambda^t, 1)\\
&= -s(df)_{\phi_s^X(x,y,t)}(\lambda^t, a\lambda^t, 0)-\frac{1}{\log\lambda}(df)_{\phi_s^X(x,y,t)}(0,0,1)\\
&= -s(X(f)\circ\phi_s^X)(x,y,t)+(Z(f)\circ\phi_s^X)(x,y,t).
\end{aligned}$$

Which achieves the proof. □

Corollary 2.7.1.1. *Let $f\in\mathscr{C}^\infty(M)$ and assume that $f=\gamma^*f$. Then $X(f)=0$ and consequently $f=p^*\psi$ with $\psi\in\mathscr{C}^\infty(\mathbb{S}^1)$.*

Proof. Since $f=\gamma^*f$ we get that for every $n\in\mathbb{Z}$, $f=(\gamma^n)^*(f)$ thus the preceding lemma gives that:

$$Z(f)=-nX(f)+(\gamma^n)^*(Z(f)).$$

Consequently we obtain that for every $n\in\mathbb{Z}$:

$$|X(f)|\leq\frac{1}{n}\big(\|Z(f)\|_\infty+\|(\gamma^n)^*(Z(f))\|_\infty\big)\leq\frac{2}{n}\|Z(f)\|_\infty,$$

which leads to $X(f)=0$ and achieves the proof. □

From $\eta=d(\omega-\gamma^*\omega)$ and $[X,Y]=0$ we get that $i_Xi_Y(d\omega)=\gamma^*(i_Xi_Yd\omega)$ and therefore according to Corollary 2.7.1.1 we can write $i_Xi_Yd\omega=p^*\psi$ for some $\psi\in\mathscr{C}^\infty(\mathbb{S}^1)$. On the other hand we get from Lemma 2.7.1.1 that:

$$p^*\phi-i_Xi_Y(d\omega)=\gamma^*(i_Zi_Yd\omega)-i_Zi_Yd\omega.$$

It follows that $p^*(\phi-\psi)=\gamma^*(i_Zi_Yd\omega)-i_Zi_Yd\omega$, as before we can prove that $p^*(\phi-\psi)=0$, so we deduce that $p^*\phi=p^*\psi=i_Xi_Y(d\omega)$. Now if we write $\omega=f\alpha+g\beta+h\theta$, then we get that $p^*\phi = X(g) - Y(f)$. From $X(p^*\phi) = Y(p^*\phi) = 0$ we get that for every $s \in \mathbb{R}$:

$$\begin{aligned}
s^2 p^*\phi &= \int_0^s \int_0^s (\phi_t^X)^*(\phi_u^Y)^*(p^*\phi)dudt \\
&= \int_0^s \int_0^s (\phi_t^X)^*(\phi_u^Y)^*(X(g))dudt - \int_0^s \int_0^s (\phi_t^X)^*(\phi_u^Y)^*(Y(f))dudt \\
&= \int_0^s (\phi_t^X)^* X\left(\int_0^s (\phi_u^Y)^*(g)du\right)dt - \int_0^s (\phi_u^Y)^* Y\left(\int_0^s (\phi_t^X)^*(f)dt\right)du \\
&= (\phi_s^X)^*\left(\int_0^s (\phi_u^Y)^*(g)du\right) - \int_0^s (\phi_u^Y)^*(g)du - (\phi_s^Y)^*\left(\int_0^s (\phi_t^X)^*(f)dt\right) + \int_0^s (\phi_t^X)^*(f)dt.
\end{aligned}$$

It follows that:

$$s^2|p^*\phi| \leq 2\left\|\int_0^s (\phi_u^Y)^*(g)du\right\|_\infty + 2\left\|\int_0^s (\phi_t^X)^*(f)dt\right\|_\infty \leq 2|s|(\|g\|_\infty + \|f\|_\infty)$$

Hence $|p^*\phi| \leq \frac{1}{s}(\|g\|_\infty + \|f\|_\infty) \underset{s\to+\infty}{\longrightarrow} 0$. Thus $\eta = 0$ and $p^*(\Omega^1(\mathbb{S}^1)) \wedge \beta \cap \mathrm{B}^2(\Omega(M)_\rho) = 0$, in particular this proves that $\mathrm{H}^2(\Omega(M)_\rho)$ is infinite dimensional.

Calculating $\mathrm{H}^3(\Omega(M)_\rho)$: The elements of $\Omega^3(M)_\rho$ are of the form $(f - \gamma^* f)\alpha \wedge \beta \wedge \theta$ for some $f \in \mathscr{C}^\infty(M)$. Put:

$$c = \frac{\int_M f\alpha \wedge \beta \wedge \theta}{\alpha \wedge \beta \wedge \theta}, \text{ then } \int_M (f - c)\alpha \wedge \beta \wedge \theta = 0.$$

Thus $(f - c)\alpha \wedge \beta \wedge \theta = d\omega$ and since $L_X(\alpha \wedge \beta \wedge \theta) = 0$ it follows that $(\gamma^* f - c)\alpha \wedge \beta \wedge \theta = d(\gamma^*\omega)$ and therefore:

$$(f - \gamma^* f)\alpha \wedge \beta \wedge \theta = d(\omega - \gamma^*\omega),$$

i.e $\mathrm{H}^3(\Omega(M)_\rho) = 0$.

Chapter 3

On a type of maximal abelian torsion free subgroups of connected Lie groups

In this chapter

3.1 Introduction and statement of main results

For an arbitrary Lie group G we can define the two integers:

$$\mathrm{p}(G) = \max\{p \in \mathbb{N},\ \mathbb{Z}^p \text{ is isomorphic to a discrete subgroup of } G\}.$$

$$\mathrm{q}(G) = \max\{q \in \mathbb{N},\ \mathbb{R}^q \text{ is isomorphic to a closed subgroup of } G\}.$$

The main purpose of this chapter is to find situations where it is possible to give explicit calculations of these invariants and relationships with other Lie algebra invariants. In [11] the authors were interested by the maximum among the dimensions of abelian subalgebras of a given Lie algebra $\mathfrak{g}$, so they introduced an integer $\mathscr{M}(\mathfrak{g})$ which they called the *maximal abelian dimension* of $\mathfrak{g}$ and they had given explicit calculations of $\mathscr{M}(\mathfrak{g})$ for some nilpotent Lie algebras. We shall see how $\mathscr{M}(\mathfrak{g})$ relates to $\mathrm{p}(G)$, $\mathrm{q}(G)$ in the case of an exponential Lie group G with Lie algebra $\mathfrak{g}$ and in the case of a nilpotent Lie group G.

Another problem closely related to our work is treated in [21] where the author introduced the notion of an envelope of an abelian subgroup of a Lie group, in mathematical terms given a Lie group G and an abelian subgroup A of G, an envelope of A in G is any connected abelian subgroup of G containing A. The author has shown that in general an envelope of an abelian subgroup might not exist and had given a counter-example in the nilpotent case, however he proved [21, Proposition 2 p. 142] that an abelian subgroup of a connected nilpotent linear Lie group always has an envelope. We inspect nilpotent Lie groups and give in Theorem 3.5.0.2 a result in the same spirit, showing where lies the difference between $\mathrm{p}(G)$ and $\mathrm{q}(G)$. Discrete cyclic subgroups of a Lie group G have also been studied from a theoretical measure point of view one can see [50] for details.

The chapter is organised as follows: in the first paragraph we give general results concerning $\mathrm{p}(G)$ and $\mathrm{q}(G)$ and we show that $\mathrm{p}(G)$ is finite for connected Lie groups. In the next section, we treat the effect of a finite cover on $\mathrm{p}(G)$, the following section specializes in describing the invariants $\mathrm{p}(G)$ and $\mathrm{q}(G)$ when G is an exponential Lie group, a class that includes simply connected nilpotent Lie groups. We then drop the simple connectedness condition and focalize on describing $\mathrm{p}(G)$ and $\mathrm{q}(G)$ for connected nilpotent Lie groups, this gives a way of comparing $\mathrm{p}(G)$ and $\mathrm{q}(G)$ in this case.

Acknowledgement

This work started with the natural question of whether a noncompact Lie group contains an infinite cyclic discrete subgroup. The authors wish to thank Aziz El Kacimi Alaoui for giving them a first proof of the Proposition 3.2.0.1.

3.2 Generalities on $\mathrm{p}(G)$ and $\mathrm{q}(G)$

Let G be a Lie group. It is then clear that $\mathrm{p}(G) = \mathrm{q}(G) = 0$ when G is compact, so we will be interested throughout the whole paper by the non-compact case. We should note that $\mathbb{Z}^p$ is isomorphic to a discrete subgroup of G means that we can find elements $\gamma_1, \cdots, \gamma_p$ of G satisfying $\gamma_i\gamma_j = \gamma_j\gamma_i$ for all i, j such that the following map

$$\mathbb{Z}^p \to G, \ \ (m_1, \cdots, m_p) \mapsto \gamma_1^{m_1} \cdots \gamma_p^{m_p}$$

is injective and the generated group $\langle\gamma_1,\cdots,\gamma_p\rangle$ is a discrete subgroup of G. Likewise, the abelian group $\mathbb{R}^p$ is isomorphic to a closed subgroup of G means that there exists commuting one parameter subgroups $\exp(t_1X_1),\cdots,\exp(t_pX_p)$ such that the following group homomorphism

$$\mathbb{R}^p \to G,\ \ (t_1,\cdots,t_p)\mapsto \exp(t_1X_1)\cdots\exp(t_pX_p)$$

is a topological isomorphism onto a closed subgroup of G. When $H\subset G$ is a closed subgroup we have evidently $\mathrm{p}(H)\le \mathrm{p}(G)$ and $\mathrm{q}(H)\le \mathrm{q}(G)$.

PROPOSITION 3.2.0.1. *Let G be a noncompact connected Lie group then:*

$$1\le \mathrm{q}(G)\le \mathrm{p}(G).$$

Proof. From [9, Theorem 6] or [6], there exists a compact subgroup $K\subset G$ and 1-parameter subgroups $H_1,\ldots,H_p$ of G such that the map:

$$\phi: K\times H_1\times\cdots\times H_p \longrightarrow G,\ \ (k,h_1,\ldots,h_p)\mapsto kh_1\ldots h_p$$

is a diffeomorphism. Thus $G=K\bar{H}_1\ldots\bar{H}_p$ where $\bar{H}_i$ is the closure of H_i in G, and since the group G is noncompact there must exist some noncompact $\bar{H}_i$. Combining this with the fact that H_i is a 1-parameter subgroup we get that H_i must be closed in G and hence isomorphic to $\mathbb{R}$, which proves that $\mathrm{q}(G)\ge 1$. For the right hand inequality, we observe that any Lie group imbedding $\psi:\mathbb{R}^q\longrightarrow G$ can be restricted to an injective homomorphism $\varphi:\mathbb{Z}^q\longrightarrow G$ with discrete image in G by setting $\varphi:=\psi_{|\mathbb{Z}^q}$, thus $q\le \mathrm{p}(G)$ and since q was arbitrary we get that $\mathrm{q}(G)\le\mathrm{p}(G)$. □

The equality $\mathrm{p}(G)=1$ does occur, this is the case when G is isomorphic to $\mathbb{R}$, a non trivial situation when this occurs is illustrated in the following explicit calculation:

EXAMPLE 3.2.0.1 (Explicit calculation for the Affine group of $\mathbb{R}$).
Denote $G:=\mathrm{Aff}(\mathbb{R})^\circ$ the identity component of the affine group of $\mathbb{R}$, which can be defined as the matrix group:

$$\mathrm{Aff}(\mathbb{R})^\circ=\left\{\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix} /\ a>0,\ b\in\mathbb{R}\right\}$$

It is known that $\mathrm{Aff}(\mathbb{R})^\circ\simeq\mathbb{R}_+^\times\ltimes_\rho\mathbb{R}$ with $\rho:\mathbb{R}_+^\times\to\mathrm{Aut}(\mathbb{R})$, $\rho(a)(b)=a\cdot b$. For any injective homomorphism $\varphi:\mathbb{Z}^2\longrightarrow\mathrm{Aff}(\mathbb{R})^\circ$ we will show that $\varphi(\mathbb{Z}^2)$ cannot be discrete in $\mathrm{Aff}(\mathbb{R})^\circ$. Denote $\gamma_1:=\varphi(0,1)=(a_1,b_1)$ and $\gamma_2:=\varphi(1,0)=(a_2,b_2)$. From the equality $\gamma_1\gamma_2=\gamma_2\gamma_1$, we get that:

$$b_1(a_2-1)=b_2(a_1-1).$$

By the injectivity of φ, it follows that $b_1 = 0$ if and only if $b_2 = 0$ (i.e $\gamma_1, \gamma_2 \in \mathbb{R}_+^\times$), similarly we have that $a_1 = 1$ if and only if $a_2 = 1$ (i.e $\gamma_1, \gamma_2 \in \mathbb{R}$). In either case, $\varphi(\mathbb{Z}^2)$ cannot be discrete in $\mathrm{Aff}(\mathbb{R})^\circ$. Therefore we can suppose that γ_1 and γ_2 are neither in $\mathbb{R}_+^\times$ nor $\mathbb{R}$ i.e $a_1 \neq 1$, $a_2 \neq 1$ and $b_1 b_2 \neq 0$. We can easily check that for all $n, m \in \mathbb{Z}$:

$$\varphi(n,m) = \gamma_1^n \gamma_2^m = \left(a_1^n a_2^m, \frac{b_1}{1-a_1}(1 - a_1^n a_2^m) \right).$$

Since φ is injective we get that for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$, $a_1^n a_2^m \neq 1$ or in other terms that $\ln(a_1)/\ln(a_2) \neq n/m$ hence $\alpha := \ln(a_1)/\ln(a_2) \in \mathbb{R} \setminus \mathbb{Q}$. Now $\mathbb{Z} + \alpha\mathbb{Z}$ is dense in $\mathbb{R}$, we can therefore choose non-stationary sequences $(p_n)_n$ and $(q_n)_n$ in $\mathbb{Z}$ such that:

$$p_n + \alpha q_n = \frac{\ln(a_1^{p_n} a_2^{q_n})}{\ln(a_1)} \underset{n \to +\infty}{\longrightarrow} 0, \ \ \textit{thus} \ \ \lim_{n \to +\infty} a_1^{p_n} a_2^{q_n} = 1.$$

If we put $A_n = \gamma_1^{p_n} \gamma_2^{q_n}$, we get that $(A_n)_n$ is a non-stationary sequence of $\varphi(\mathbb{Z}^2)$ which converges to I_2, therefore $\varphi(\mathbb{Z}^2)$ cannot be discrete in $\mathrm{Aff}(\mathbb{R})^\circ$. We conclude that $\mathrm{p}(\mathrm{Aff}(\mathbb{R})^\circ) = 1$ and by Proposition 3.2.0.1, $\mathrm{q}(\mathrm{Aff}(\mathbb{R})^\circ) = 1$.

It is clear that the quantity $\mathrm{q}(G)$ is finite and in fact $\mathrm{q}(G) \leq \dim(G)$, however the finiteness of $\mathrm{p}(G)$ is not as clear so we address it in what follows.

PROPOSITION 3.2.0.2. *Let G be a connected Lie group. Then:*

$$1 \leq q(G) \leq p(G) \leq \dim(G/K),$$

where K is a maximal compact subgroup of G. In particular $\mathrm{p}(G)$ is finite.

Proof. Let K be a maximal compact subgroup G, it is well-known that K exists and is unique up to conjugation since G has finitely many connected components, moreover the quotient space G/K is a contractible manifold. Let Γ be a discrete subgroup of G which is isomorphic to $\mathbb{Z}^p$, the natural action of Γ on G/K given by:

$$\Gamma \times G/K \longrightarrow G/K, \ \ (\gamma, [g]) \mapsto [\gamma g],$$

is properly discontinuous since K is compact. This action is also free, indeed since the action is properly discontinous, the isotropy groups Γ_x, $x \in G/K$ are finite hence trivial which follows from the fact that Γ is isomorphic to $\mathbb{Z}^p$ which has no nontrivial finite subgroups. This gives that the natural projection $G/K \xrightarrow{\pi} \Gamma\backslash G/K$ is a covering of smooth manifolds, and since G/K is contractible we get that $\Gamma\backslash G/K$ is a classifying space for Γ (cf. [38]). Now since the group Γ is isomorphic to $\mathbb{Z}^p$ and the p-dimensional torus $\mathbb{T}^p$ is a classfying space for $\mathbb{Z}^p$, we obtain that $\Gamma\backslash G/K$ must be homotopy equivalent to $\mathbb{T}^p$ and in particular they have the same de-Rham cohomology:

$$\mathrm{H}^k(\mathbb{T}^p) \cong \mathrm{H}^k(\Gamma\backslash G/K).$$

Now suppose by contradiction that we can choose $p > \dim(G/K)$, the preceding remark would give that $\mathrm{H}^p(\Gamma\backslash G/K) \cong \mathrm{H}^p(\mathbb{T}^p) \cong \mathbb{R}$, but this is clearly impossible since the double quotient $\Gamma\backslash G/K$ is a smooth manifold with $\dim(\Gamma\backslash G/K) = \dim(G/K)$ hence $\mathrm{H}^k(\Gamma\backslash G/K)$ must be trivial for any $k > \dim(G/K)$. Since p was arbitrary we get that $\mathrm{p}(G) \le \dim(G/K)$. □

Corollary 3.2.0.1. *Let G be a connected Lie group and C any compact subgroup of G. Then:*

$$\mathrm{p}(G) \le \dim(G) - \dim(C).$$

Example 3.2.0.2. *Choose a compact connected Lie group K and put $G = \mathbb{R}^p \times K$. Then it is clear that $p \le \mathrm{p}(G) \le \dim(G/K)$, and so $\mathrm{p}(G) = \dim(G/K)$ in this case.*

Example 3.2.0.3. *Let $G = \mathrm{Aff}(\mathbb{R})^\circ$ the identity component of the affine group of $\mathbb{R}$. It is clear that any compact subgroup of G is trivial thus $K = \{e_G\}$ is a maximal compact subgroup of the group G which means that $\dim(G/K) = \dim(G) = 2$, however $\mathrm{p}(G) = 1$ according to Example 3.2.0.1. So $\mathrm{p}(G) < \dim(G/K)$ in this case.*

Proposition 3.2.0.3. *Let G_1 and G_2 be connected Lie groups and $G = G_1 \times G_2$, then:*

$$\mathrm{q}(G) = \mathrm{q}(G_1) + \mathrm{q}(G_2).$$

Proof. It is clear that if A_i is a closed abelian subgroup of G_i isomorphic to $\mathbb{R}^{q_i}$ for $i = 1,2$ then the subgroup $A_1 \times A_2$ is closed in G and isomorphic to $\mathbb{R}^{q_1+q_2}$ thus $q_1 + q_2 \le \mathrm{q}(G)$ and therefore $\mathrm{q}(G_1) + \mathrm{q}(G_2) \le \mathrm{q}(G)$. Conversely, let A be a closed abelian subgroup of G isomorphic to $\mathbb{R}^r$, let $\mathrm{pr}_i : G \longrightarrow G_i$ be the projection on the i-th component for $i = 1,2$. Next, denote $A_i := \mathrm{pr}_i(A)$ then its closure $\overline{A}_i$ in G_i is a closed connected abelian subgroup of G_i thus isomorphic to some $\mathbb{R}^{q_i} \times \mathbb{T}^{k_i}$, $i = 1,2$. Clearly $A \subset \overline{A}_1 \times \overline{A}_2$ and therefore:

$$r = \dim(A) \le \mathrm{q}(\overline{A}_1 \times \overline{A}_2) = q_1 + q_2 \le \mathrm{q}(G_1) + \mathrm{q}(G_2).$$

We conclude that $\mathrm{q}(G) \le \mathrm{q}(G_1) + \mathrm{q}(G_2)$. □

An analogous claim for the discrete setting is more subtle, so before addressing it we need a lemma:

LEMMA 3.2.0.1. *Let K be a compact Lie group and Γ be a discrete subgroup of $\mathbb{R}^p \times K$ isomorphic to $\mathbb{Z}^r$ then $r \leq p$.*

Proof. Let $\varphi : \mathbb{Z}^r \longrightarrow \mathbb{R}^p \times K$ be an injective group homomorphism such that $\varphi(\mathbb{Z}^r) = \Gamma$ and denote $\mathrm{pr}_1 : \mathbb{R}^p \times K \longrightarrow \mathbb{R}^p$ the first projection. Put $\psi := \mathrm{pr}_1 \circ \phi$, then:

$$\psi^{-1}(0) = \varphi^{-1}(\mathrm{pr}_1^{-1}(0)) = \varphi^{-1}(\Gamma \cap \{0\} \times K),$$

since $\Gamma \cap (\{0\} \times K)$ is finite and φ is injective we get that $\psi^{-1}(0)$ is a finite subgroup of $\mathbb{Z}^r$ thus trivial, it follows that $\psi : \mathbb{Z}^r \longrightarrow \mathbb{R}^p$ is injective. Next choose a sequence $(\gamma_n)_n$ in $\mathbb{R}^p$ converging to 0 , then $(\gamma_n)_n$ is contained in some compact neighborhood $\bar{U}$ of 0 in $\mathbb{R}^p$. For any $n \in \mathbb{N}$, put $\gamma_n = \psi(g_n)$, we can write $\varphi(g_n) = (\gamma_n, k_n)$ for some $k_n \in K$, then (γ_n, k_n) belongs to $(\bar{U} \times K) \cap \Gamma$ which is finite since Γ is discrete in $\mathbb{R}^p \times K$, therefore $(\gamma_n, k_n)_n$ can only take finitely many values and it follows that the sequence $(\gamma_n)_n$ is stationary. We conclude that $\psi(\mathbb{Z}^r)$ is a discrete subgroup of $\mathbb{R}^p$ isomorphic to $\mathbb{Z}^r$ thus $r \leq p$. □

PROPOSITION 3.2.0.4. *Let G_1 and G_2 be connected Lie groups and $G = G_1 \times G_2$, then:*

$$\mathrm{p}(G) = \mathrm{p}(G_1) + \mathrm{p}(G_2).$$

Proof. Let $\varphi_i : \mathbb{Z}^{p_i} \longrightarrow G_i$ $i = 1,2$ be an injective homomorphism with discrete image then the map $\varphi : \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2} \longrightarrow G$ given by $\varphi(u_1, u_2) = (\varphi_1(u_1), \varphi_2(u_2))$ is an injective homomorphism with discrete image therefore $p_1 + p_2 \leq \mathrm{p}(G)$ and since the p_1, p_2 were arbitrary we get that $\mathrm{p}(G_1) + \mathrm{p}(G_2) \leq \mathrm{p}(G)$. Conversely, let $\varphi : \mathbb{Z}^r \longrightarrow G$ be an injective homomorphism with discrete image and put $\Gamma := \varphi(\mathbb{Z}^r)$. For $i = 1,2$, define $H_i := \overline{\mathrm{pr}_i(\Gamma)}$ where $\mathrm{pr}_i : G \longrightarrow G_i$ is the projection on the i-th component and put $H := H_1 \times H_2$. It is clear that H_i is an abelian closed subgroup of G_i thus H is an abelian closed subgroup of G containing Γ. According to [51] each H_i is isomorphic to $\mathbb{R}^{q_i} \times \mathbb{Z}^{p_i} \times \mathbb{T}^{k_i} \times F_i$ where F_i is a finite abelian group, hence H is isomorphic to $\mathbb{R}^q \times \mathbb{Z}^p \times \mathbb{T}^k \times F$ with $q = q_1 + q_2$, $p = p_1 + p_2$, $k = k_1 + k_2$ and $F = F_1 \times F_2$. Consequently H can be realized as a closed subgroup of $\mathbb{R}^{p+q} \times \mathbb{T}^k \times F$ so by Lemma 3.2.0.1:

$$r \leq p + q \leq p_1 + p_2 + q_1 + q_2 \leq \mathrm{p}(G_1) + \mathrm{p}(G_2).$$

Since the integer r was arbitrary we get that $\mathrm{p}(G) \leq \mathrm{p}(G_1) + \mathrm{p}(G_2)$. □

3.3 Finite cover of Lie groups

In this paragraph we discuss the behavior of $\mathrm{p}(G)$ and $\mathrm{q}(G)$ with respect to some finite cover of Lie groups $\tilde{G} \xrightarrow{\pi} G$.

PROPOSITION 3.3.0.1. *Let $\tilde{G} \xrightarrow{\pi} G$ be a finite cover of connected Lie groups. Then:*

$$\mathrm{p}(G) = \mathrm{p}(\tilde{G}).$$

Proof. Write $F := \ker(\pi)$, then F is a finite central subgroup of $\tilde{G}$, i.e $F \subset \mathrm{Z}(\tilde{G})$. We start by showing that $\mathrm{p}(\tilde{G}) \le \mathrm{p}(G)$. Let $\varphi : \mathbb{Z}^p \longrightarrow \tilde{G}$ be an injective homomorphism with discrete image, define $\psi := \pi \circ \varphi$, we claim that ψ is injective. Indeed $\ker(\psi) = \varphi^{-1}(F)$ and since φ is injective and F is finite, $\varphi^{-1}(F)$ is a finite subgroup of $\mathbb{Z}^p$ hence trivial and $\ker\psi = \{0\}$. To show that $\psi(\mathbb{Z}^p)$ is discrete in G it suffices to show that $V \cap \psi(\mathbb{Z}^p) = \{e_G\}$ for some neighborhood V of e_G in G. Since $\varphi(\mathbb{Z}^p)$ is discrete in $\tilde{G}$ and F is finite then $\varphi(\mathbb{Z}^p)\cdot F$ is discrete in $\tilde{G}$, thus $U \cap (\varphi(\mathbb{Z}^p)\cdot F) = \{e_{\tilde{G}}\}$ for some open neighborhood U of $e_{\tilde{G}}$, put $V = \pi(U)$. Let $z \in V \cap \psi(\mathbb{Z}^p)$, we can find $u \in U$ and $y \in \varphi(\mathbb{Z}^p)$ such that $\pi(u) = \pi(y) = z$, then $u = y \cdot f$ and therefore $u = e_{\tilde{G}}$ and $y = f^{-1}$, consequently we obtain that $z = \pi(f^{-1}) = e_G$. We conclude that $\psi(\mathbb{Z}^p)$ is a discrete subgroup of G, choosing $p = \mathrm{p}(\tilde{G})$ we get that $\mathrm{p}(\tilde{G}) \le \mathrm{p}(G)$.
Conversely, we show that $\mathrm{p}(G) \le \mathrm{p}(\tilde{G})$. Let $\varphi : \mathbb{Z}^p \longrightarrow G$ be an injective homomorphism such that $\varphi(\mathbb{Z}^p)$ is discrete in G. If $\{e_1,\dots,e_p\}$ is the canonical basis of $\mathbb{R}^p$ and write $\gamma_i := \varphi(e_i)$, then $\varphi(\mathbb{Z}^p) = \langle \gamma_1,\dots,\gamma_p \rangle$. Now choose $\tilde{\gamma}_1,\dots,\tilde{\gamma}_p \in \tilde{G}$ such that $\pi(\tilde{\gamma}_i) = \gamma_i$, since $\gamma_i\gamma_j = \gamma_j\gamma_i$ and $\pi : \tilde{G} \longrightarrow G$ is a group homomorphism with $\ker(\pi) = F$ we obtain that

$$\tilde{\gamma}_i\tilde{\gamma}_i = (\tilde{\gamma}_j\tilde{\gamma}_i)\cdot f_{ij} \text{ for some } f_{ij} \in F.$$

Since F is finite we can find $k \in \mathbb{N}^*$ such that $f^k = e_{\tilde{G}}$ for every $f \in F$. We can check that for every $p \in \mathbb{N}$, $\tilde{\gamma}_i^p\tilde{\gamma}_j = \tilde{\gamma}_j\tilde{\gamma}_i^p f_{ij}^p$ thus $\tilde{\gamma}_i^k\tilde{\gamma}_j = \tilde{\gamma}_j\tilde{\gamma}_i^k$ and consequently $\tilde{\gamma}_i^k\tilde{\gamma}_j^k = \tilde{\gamma}_j^k\tilde{\gamma}_i^k$.
Therefore we get an homomorphism $\tilde{\varphi} : \mathbb{Z}^p \longrightarrow G$ by setting $\tilde{\varphi}(e_i) := \tilde{\gamma}_i^k$, moreover if we define $\psi : \mathbb{Z}^p \longrightarrow G$ by $\psi(u) = \varphi(u)^k$ we get that ψ is an injective homomorphism, also ψ has a discrete image in G, this follows from $\psi(\mathbb{Z}^p) = \varphi(k\mathbb{Z} \times \cdots \times k\mathbb{Z})$, furthermore $\psi = \pi \circ \tilde{\varphi}$, in particular $\tilde{\varphi}$ is also injective. To show that $\tilde{\varphi}(\mathbb{Z}^p)$ is a discrete subgroup of $\tilde{G}$, let V be an evenly covered open neighborhood of e_G in G such that $V \cap \psi(\mathbb{Z}^p) = \{e_G\}$ and let U be an open neighborhood of $e_{\tilde{G}}$ in $\tilde{G}$ such that the restriction $\pi_{|U} : U \longrightarrow V$ is a diffeomorphism. Let $y \in U \cap \tilde{\varphi}(\mathbb{Z}^p)$ then it is clear that $\pi(y) \in V \cap \psi(\mathbb{Z}^p)$ and thus $\pi(y) = e_G$ or in other terms that $y \in F$. Consequently, $y^k = e_{\tilde{G}}$ but since y is also in $\tilde{\varphi}(\mathbb{Z}^p)$ we get that $y = e_{\tilde{G}}$. We conclude that $\tilde{\varphi}(\mathbb{Z}^p)$ is discrete in $\tilde{G}$, if we set $p = \mathrm{p}(G)$ then we get $\mathrm{p}(G) \le \mathrm{p}(\tilde{G})$. □

EXAMPLE 3.3.0.1. *Let* $\mathrm{PSL}(2,\mathbb{R})$ *be the projective special linear group, which is defined as:*

$$\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{Id}\};$$

this group cannot admit a discrete subgroup isomorphic to $\mathbb{Z}^2$ *[27, Corollary* 2.3.7*]. This is equivalent to saying that* $\mathrm{p}(\mathrm{PSL}(2,\mathbb{R})) = 1$*. Hence, from Proposition 3.3.0.1 we obtain:*

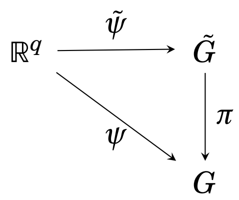
$$\mathrm{p}(\mathrm{SL}(2,\mathbb{R})) = 1,$$

although we knew that as a manifold $\mathrm{SL}(2,\mathbb{R})$ *is diffeomorphic to* $S^1 \times \mathbb{R}^2$.

We now show that the same result holds for the invariant $\mathrm{q}(G)$ as well.

PROPOSITION 3.3.0.2. *Let* $\tilde{G} \xrightarrow{\pi} G$ *be a finite cover of Lie groups. Then* $\mathrm{q}(\tilde{G}) = \mathrm{q}(G)$.

Proof. Denote $F := \ker \pi \subset \mathrm{Z}(\tilde{G})$ and consider the commutative diagram:



Then $\tilde{\psi}$ is an imbedding of Lie groups if and only if ψ is an imbedding of Lie groups. Indeed we have that $\ker(\psi) = \tilde{\psi}^{-1}(F)$, assume that $\psi : \mathbb{R}^q \longrightarrow \psi(\mathbb{R}^q)$ is an isomorphism of Lie groups, then $\tilde{\psi} : \mathbb{R}^q \longrightarrow \tilde{G}$ is injective. Let $(v_n)_n$ be a sequence in $\mathbb{R}^q$ such that $(\tilde{\psi}(v_n))_n$ converges to $e_{\tilde{G}}$ then $(\psi(v_n))_n$ converges to e_G and since ψ is an isomorphism onto its image, it follows that $(v_n)_n$ converges to 0, thus $\tilde{\psi}$ is an isomorphism onto its image. Conversely, suppose that $\tilde{\psi} : \mathbb{R}^q \longrightarrow \tilde{\psi}(\mathbb{R}^q)$ is an isomorphism, then $\ker(\psi)$ is a finite subgroup of $\mathbb{R}^q$ which is torsion-free, therefore $\ker(\psi) = \{0\}$. Let $(v_n)_n$ be a sequence in $\mathbb{R}^q$ such that $(\psi(v_n))_n$ converges to e_G in G, since $\tilde{\psi} = \pi \circ \psi$ we can find $f_n \in F$ such that $(f_n \tilde{\psi}(v_n))_n$ converges to $e_{\tilde{G}}$. Thus $(f_n \tilde{\psi}(v_n))^k = \tilde{\psi}(k v_n)$ when $k = |F|$ and since $(\tilde{\psi}(v_n)^k)_n$ converges to $e_{\tilde{G}}$ and $\tilde{\psi}$ is an isomorphism onto its image, it follows that $(k v_n)_n$ converges to 0 in $\mathbb{R}^q$ and therefore the sequence $(v_n)_n$ converges to 0 in $\mathbb{R}^q$ and ψ is an isomorphism onto its image.

In particular this implies that $\mathrm{q}(\tilde{G}) \leq \mathrm{q}(G)$, on the other hand if $\psi : \mathbb{R}^q \longrightarrow G$ is an imbedding of Lie groups put $A = \psi(\mathbb{R}^q)$ and $\mathfrak{a} = \mathrm{Lie}(A)$ then the restriction $\exp_{G|\mathfrak{a}} : (\mathfrak{a}, +) \longrightarrow A$ is an isomorphism. Now define $\tilde{\psi} : \mathbb{R}^q \longrightarrow \tilde{G}$ by the formula:

$$\tilde{\psi}(v) = \exp_{\tilde{G}} \circ (\pi')^{-1} \circ (\exp_{G|\mathfrak{a}})^{-1} \circ \psi(v).$$

It is straightforward to check that $\tilde{\psi}$ is a Lie group homomorphism satisfying $\pi \circ \tilde{\psi} = \psi$ thus by the preceding remarks $\tilde{\psi}$ is a Lie group imbedding and we conclude that $\mathrm{q}(G) \leq \mathrm{q}(\tilde{G})$. □

The following examples show that Propositions 3.3.0.1 and 3.3.0.2 are not true for an infinite covering of Lie groups $\tilde{G} \xrightarrow{\pi} G$:

EXAMPLE 3.3.0.2. *An obvious counter-example is the case of $\tilde{G} = \mathbb{R}$ and $G = \mathbb{S}^1$, it is clear that $\mathrm{p}(\tilde{G}) = \mathrm{q}(\tilde{G}) = 1$ and $\mathrm{p}(G) = \mathrm{q}(G) = 0$.*

The next example illustrates a situation where the claim of Proposition 3.3.0.2 does hold and Proposition 3.3.0.1 does not:

EXAMPLE 3.3.0.3. *Take $\tilde{G} := \widetilde{\mathrm{SL}}(2,\mathbb{R})$ the universal cover of $G := \mathrm{SL}(2,\mathbb{R})$. It is well known that $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \mathbb{Z}$ which can be identified with a central subgroup of $\tilde{G}$, namely $\ker(\pi)$. Let $\Gamma := \langle \gamma \rangle$ be any discrete subgroup of G isomorphic to $\mathbb{Z}$ and put $\tilde{\Gamma} := \pi^{-1}(\Gamma)$. Clearly $\tilde{\Gamma}$ is a discrete subgroup of $\tilde{G}$ (it is the inverse image of a discrete subgroup by the covering map). Moreover if we choose $\gamma_1 \in \tilde{G}$ such that $\pi(\gamma_1) = \gamma$, then:*

$$\tilde{\Gamma} = \langle \gamma_1 \rangle \cdot \ker(\pi) = \{\gamma_1^n \gamma_2^m,\ n, m \in \mathbb{Z}\},$$

where γ_2 is any generator of $\ker(\pi) \simeq \mathbb{Z}$, in particular $\tilde{\Gamma}$ is abelian. To conclude we show that $\tilde{\Gamma}$ is torsion-free, indeed if $\gamma_1^n \gamma_2^m = e$ then $e = \pi(\gamma_1^n \gamma_2^m) = \gamma^n$ and since Γ is torsion-free we get that $n = 0$ and hence $\gamma_2^m = e$, again since $\ker(\pi)$ is torsion-free $m = 0$. In summary we conclude that $\tilde{G}$ admits a discrete subgroup isomorphic to $\mathbb{Z}^2$, this means that $\mathrm{p}(\tilde{G}) \geq 2$ while according to Example 3.3.0.1, $\mathrm{p}(G) = 1 = \mathrm{q}(G)$.
In contrast, the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ only admits 1-dimensional abelian subalgebras and therefore the dimension of closed connected abelian subgroups of both G and $\tilde{G}$ cannot exceed 1, in particular using Proposition 3.2.0.2, $\mathrm{q}(G) = \mathrm{q}(\tilde{G}) = 1$.

3.4 The case of an exponential Lie group

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We say that G is exponential when the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism. In particular, G is a simply connected and contractible Lie group.

LEMMA 3.4.0.1. *Let G be an exponential Lie group and let $\alpha, \beta : \mathbb{R} \longrightarrow G$ be 1-parameter subgroups of G such that $\alpha(1) = \beta(1)$. Then $\alpha = \beta$.*

Proof. If we write $\alpha(t) = \exp(tX)$ and $\beta(t) = \exp(tY)$, then the condition $\alpha(1) = \beta(1)$ is equivalent to $\exp(X) = \exp(Y)$ which gives that $X = Y$ since the group G is exponential, and so we conclude that $\alpha = \beta$. □

Lemma 3.4.0.2. *Let G be an exponential Lie group and let $X, Y \in \mathfrak{g}$. Then $[X,Y] = 0$ if and only if $\exp(X)\exp(Y) = \exp(Y)\exp(X)$.*

Proof. Define the 1-parameter subgroups α and β on G given by:

$$\alpha(t) = \exp(Y)\exp(tX)\exp(-Y) \quad \text{and} \quad \beta(t) = \exp(tX).$$

Then $\exp(X)\exp(Y) = \exp(Y)\exp(X)$ is equivalent to $\alpha(1) = \beta(1)$ which by the preceding lemma gives that $\alpha = \beta$, i.e for every $t \in \mathbb{R}$:

$$\exp(Y)\exp(tX)\exp(-Y) = \exp(tX),$$

hence $\exp(Y)\exp(tX) = \exp(tX)\exp(Y)$. Now applying this reasoning a second time we get that for every $t, s \in \mathbb{R}$:

$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX),$$

which is equivalent to $[X,Y] = 0$. The converse is straightforward. □

Since the exponential map is a diffeomorphism, any abelian subgroup A of G is of the form $A = \exp(\mathfrak{a})$ where $\mathfrak{a} = \mathrm{Lie}(A)$, thus $\mathrm{q}(G)$ is equal to the maximal possible dimension of abelian Lie subalgebras in $\mathfrak{g}$, which is denoted $\mathscr{M}(\mathfrak{g})$ and called the *maximal abelian dimension* of $\mathfrak{g}$ (see [11]). The same holds for $\mathrm{p}(G)$ by the following proposition:

Proposition 3.4.0.1. *Let G be an exponential Lie group. Then:*

$$\mathrm{p}(G) = \mathrm{q}(G) = \mathscr{M}(\mathfrak{g}).$$

Proof. Let $\varphi : \mathbb{Z}^p \longrightarrow G$ be an injective homomorphism with discrete image. For $i = 1, \dots, p$, denote $\varphi(e_i) = \exp(X_i)$ and $\mathfrak{a} = \mathrm{span}\{X_1, \dots, X_p\}$. Since:

$$\exp(X_i)\exp(X_j) = \exp(X_j)\exp(X_i),$$

we obtain from the preceding lemma that $[X_i, X_j] = 0$, hence $\mathfrak{a}$ is an abelian Lie subalgebra of $\mathfrak{g}$ of dimension $\leq p$. Furthermore $A = \exp(\mathfrak{a})$ is a closed abelian subgroup of G, and since $\varphi(\mathbb{Z}^p)$ is a discrete subgroup of A we get necessarily that $\dim A \geq p$, hence $\dim \mathfrak{a} = p$. Conversely, given any p-dimensional abelian Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, choose a basis $v_1, \dots, v_p$

of $\mathfrak{a}$ and consider the map:

$$\varphi : \mathbb{Z}^p \longrightarrow G, \ (n_1, \dots, n_p) \mapsto \exp(n_1 v_1 + \cdots + n_p v_p).$$

Then $\varphi : \mathbb{Z}^p \longrightarrow G$ is clearly an injective homomorphism with discrete image. In particular, we obtain that $\mathrm{p}(G)$ is equal to the maximal possible dimension of an abelian Lie subalgebra of $\mathfrak{g}$. □

3.5 The case of a nilpotent Lie group

A simply connected nilpotent Lie group is a particular case of an exponential Lie group, so Propostion 3.4.0.1 applies to this case. More generally we have the following result:

PROPOSITION 3.5.0.1. *Let G be a linear connected nilpotent Lie group, then:*

$$\mathrm{p}(G) = \mathrm{q}(G) = \mathcal{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(G)). \tag{3.1}$$

Proof. Recall from ([24], Theorem 15.2.7) that a connected linear Lie group is isomorphic to the semi-direct product $B \rtimes H$ where B is a simply connected solvable Lie group and H is linearly real reductive, moreover the radical R of G is isomorphic to the semi-direct product $B \rtimes T$ where T is a maximal torus in H. In the case where the group G is solvable we get that $G = R \simeq B \rtimes T$, in particular when G is nilpotent then B is a nilpotent simply connected Lie group and since T is compact it must be central in G, therefore $G \simeq B \times T$. It follows from Propositions 3.2.0.3, 3.2.0.4 and 3.4.0.1 that:

$$\mathrm{p}(G) = \mathrm{p}(B) = \mathrm{q}(B) = \mathrm{q}(G). \tag{3.2}$$

Next denote $\mathfrak{b} = \mathrm{Lie}(B)$ and $\mathfrak{t} = \mathrm{Lie}(T)$, it is clear since $\mathfrak{g}$ is isomorphic to the Lie algebra product $\mathfrak{b} \times \mathfrak{t}$ that $\mathcal{M}(\mathfrak{b}) + \dim(\mathfrak{t}) \le \mathcal{M}(\mathfrak{g})$. Conversely, it is possible to assume $\mathfrak{g} = \mathfrak{b} \times \mathfrak{t}$ so that we have $\mathfrak{b} \simeq \mathfrak{g}/\mathfrak{t}$ as a Lie algebra, hence if $\mathfrak{a} \subset \mathfrak{g}$ is an abelian Lie subalgebra of maximal dimension we get that $\mathfrak{a}/\mathfrak{t}$ is an abelian subalgebra of $\mathfrak{b}$ and thus $\mathcal{M}(\mathfrak{g}) - \dim(\mathfrak{t}) \le \mathcal{M}(\mathfrak{b})$. In summary using formula (3.2) and Proposition 3.4.0.1 we get that:

$$\mathrm{p}(G) = \mathrm{q}(G) = \mathcal{M}(\mathfrak{b}) = \mathcal{M}(\mathfrak{g}) - \dim(\mathfrak{t}),$$

we conclude by noticing that $\dim(\mathfrak{t}) = \mathrm{rank}(\pi_1(T)) = \mathrm{rank}(\pi_1(G))$. □

Proposition 3.5.0.1 gives a glimpse of how things might turn out in a general setting. In fact, we next show that equation (3.1) still holds for $\mathrm{q}(G)$ for any connected nilpotent Lie group G. Henceforth, we denote $\mathscr{L}_1(G) = \mathrm{span}\{\ker(\exp_G)\}$, then $\mathscr{L}_1(G) \subset \mathrm{Z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. Next put $K_1(G) := \exp_G(\mathscr{L}_1(G))$, this is a torus i.e. a compact connected abelian Lie subgroup of G (see [24] p. 439) whose dimension is equal to $\mathrm{rank}(\pi_1(\mathrm{G}))$ (see [45] p.32 for a more general situation).

Now we are in a position to state our first main result concerning $\mathrm{q}(G)$.

THEOREM 3.5.0.1. *Let G be a connected nilpotent Lie group, then:*

$$\mathrm{q}(G) = \mathscr{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(\mathrm{G})).$$

Proof. Choose a closed subgroup V of G isomorphic to $\mathbb{R}^{\mathrm{q}(G)}$ and denote its Lie algebra $\mathfrak{v}$, then $\exp_V : (\mathfrak{v}, +) \longrightarrow V$ is an isomorphism of Lie groups, in particular $\mathfrak{v} \cap \ker(\exp_G) = \{0\}$. We claim that $\mathscr{L}_1(G) \cap \mathfrak{v} = \{0\}$, indeed suppose by contradiction that there exists some nontrivial element $Y \in \mathscr{L}_1(G) \cap \mathfrak{v}$, then we get $\exp_G(\mathbb{R}Y) \subset V \cap K_1(G)$. Since $Y \in \mathfrak{v}$ we get that the group $\exp_G(\mathbb{R}Y)$ is closed in G and isomorphic to $\mathbb{R}$, on the other hand $\exp_G(\mathbb{R}Y)$ is contained in $K_1(G)$ so it must be compact, which is impossible. Now choose a maximal abelian Lie subalgebra $\mathfrak{b}$ with $\mathfrak{v} \subset \mathfrak{b} \subset \mathfrak{g}$ and a maximal dimensional Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Using the maximality of both $\mathfrak{a}$ and $\mathfrak{b}$, we get that $\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{a} \cap \mathfrak{b}$ hence $\mathscr{L}_1(G) \subset \mathfrak{a} \cap \mathfrak{b}$ and therefore $\mathfrak{v} \oplus \mathscr{L}_1(G) \subset \mathfrak{b}$. Now:

$$\mathscr{M}(\mathfrak{g}) = \dim(\mathfrak{a}) \geq \dim(\mathfrak{b}) \geq \dim(\mathscr{L}_1(G)) + \dim(\mathfrak{v}).$$

Since $\dim(\mathfrak{v}) = \mathrm{q}(G)$ and $\dim \mathscr{L}_1(G) = \mathrm{rank}(\pi_1(\mathrm{G}))$ we conclude that:

$$\mathscr{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(\mathrm{G})) \geq \mathrm{q}(\mathrm{G}).$$

Conversely let $\mathfrak{a} \subset \mathfrak{g}$ be an abelian Lie subalgebra of maximal dimension and define the subroup $A = \exp_G(\mathfrak{a})$, we claim that A is a closed in G. Indeed, denote by $\bar{A}$ the closure of A which is a Lie subgroup of G by the classical closed subgroup theorem, and put $\bar{\mathfrak{a}} = \mathrm{Lie}(\bar{A})$. Since A is abelian and connected, then the same is true for $\bar{A}$ and thus $\bar{A} = \exp_G(\bar{\mathfrak{a}})$. Now from $A \subset \bar{A}$ we get that $\mathfrak{a} \subset \bar{\mathfrak{a}}$ and by maximality of $\mathfrak{a}$ we get that $\bar{\mathfrak{a}} = \mathfrak{a}$ therefore $\bar{A} = A$. On the other hand, $\mathscr{L}_1(G) \subset \mathrm{Z}(\mathfrak{g}) \subset \mathfrak{a}$ so we can write $\mathfrak{a} = \mathfrak{w} \oplus \mathscr{L}_1(G)$. Put $W = \exp_G(\mathfrak{w})$, it is straightforward to check that the map:

$$\phi : \mathfrak{w} \times K_1(G) \longrightarrow A, \ \ (w, h) \mapsto \exp_G(w)h,$$

is an isomorphism of Lie groups, in particular $W = \phi(\mathfrak{w})$ is closed in A hence closed in G. Thus we obtain:

$$\mathrm{q}(G) \geq \dim(W) = \dim(\mathfrak{a}) - \dim(\mathscr{L}_1(G)) = \mathscr{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(\mathrm{G})).$$

which ends the proof. □

The following example shows that in contrast to the simply connected case, it can occur that $\mathrm{p}(G) \neq \mathrm{q}(G)$ for a connected nilpotent Lie group G:

Example 3.5.0.1. *Let $\mathfrak{g}$ be the 3-dimensional Heisenberg Lie algebra with its usual Lie bracket structure given by $[X_1, X_2] = X_3$, it is clear that $\mathrm{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}X_3$. Next consider the nilpotent Lie group $G := (g, *)/\mathbb{Z}X_3$,[1] then clearly $\pi_1(G) \cong \mathbb{Z}X_3$ and thus $\mathrm{rank}(\pi_1(\mathrm{G})) = 1$. From [11] we get that $\mathscr{M}(\mathfrak{g}) = 2$ and therefore Theorem 3.5.0.1 gives that:*

$$\mathrm{q}(G) = \mathscr{M}(\mathfrak{g}) - \mathrm{rank}(\pi_1(\mathrm{G})) = 1.$$

We will show that $\mathrm{p}(G) \geq 2$. Denote $\bar{X}$ the equivalence class of $X \in \mathfrak{g}$, from the relation:

$$X_1 * X_2 = X_1 + X_2 + \frac{1}{2}[X_1, X_2] = X_2 * X_1 + X_3, \tag{3.3}$$

we deduce that $\bar{X}_1 * \bar{X}_2 = \bar{X}_2 * \bar{X}_1$, and therefore the homomorphism $\varphi : \mathbb{Z}^2 \longrightarrow G$ given by the expression $\varphi(e_i) = \bar{X}_i$ for $i = 1, 2$ is well-defined. We claim that φ is injective and has discrete image in G. Indeed let $n, m \in \mathbb{Z}$ such that $\varphi(n, m) = e_G$, thus $(n\bar{X}_1) * (m\bar{X}_2) = e_G$. Using formula (3.3), we can find $p \in \mathbb{Z}$ for which:

$$2nX_1 + 2mX_2 + nmX_3 = pX_3,$$

hence $n, m = 0$. On the other hand if $g_n = (p_n\bar{X}_1) * (q_n\bar{X}_2)$ is a sequence in $\varphi(\mathbb{Z}^2)$ converging to e_G, then $((p_nX_1) * (q_nX_2) * (r_nX_3))_n$ converges in $(\mathfrak{g}, *)$ to the identity element for some integer sequence $(r_n)_n$. Using relation (3.3) we obtain that:

$$(p_nX_1) * (q_nX_2) * (r_nX_3) = p_nX_1 + q_nX_2 + ((p_nq_n)/2 + r_n)X_3,$$

thus the sequences $(p_n)_n$ and $(q_n)_n$ converge to 0 in $\mathbb{Z}$ and are therefore stationary, consequently $(g_n)_n$ is stationary. In summary, we get that $\mathrm{p}(G) \geq 2 > \mathrm{q}(G)$.

Now we are going to give a similar description for the invariant $\mathrm{p}(G)$ for an arbitrary

[1] This is not a linear group as it is shown in [24] p. 336

connected nilpotent Lie group G. In order to prove the main result of this paragraph (Theorem 3.5.0.2), we need some preparation. For this, we use some results concerning lattices in nilpotent Lie groups, the necessary details can be found in Appendix (C). Our theorem is stated as follows:

THEOREM 3.5.0.2. *Let G be a connected nilpotent Lie group and denote $\mathfrak{g}$ its Lie algebra. Then:*

$$\mathrm{p}(G) = \dim(\mathfrak{n}_0) - \mathrm{rank}(\pi_1(\mathrm{G})),$$

where $\mathfrak{n}_0$ is of maximal dimension among all 2-step nilpotent Lie subalgebras $\mathfrak{n}$ of $\mathfrak{g}$ such that the Lie group $(\mathfrak{n}, *)$ admits a lattice Γ satisfying $[\Gamma,\Gamma] \subset \ker(\exp_G) \subset \Gamma$.

We start by proving a series of important lemmas:

LEMMA 3.5.0.1. *Let $\tilde{N}$ be a simply connected nilpotent Lie group and Γ be a lattice in $\tilde{N}$.*

1. *If Γ is abelian then $\tilde{N}$ is abelian.*
2. *Let N_1 be a connected closed subgroup of $\tilde{N}$ and Γ_1 a lattice in N_1. Assume that $\Gamma_1 \lhd \Gamma$, then $N_1 \lhd \tilde{N}$ and $\Gamma/\Gamma \cap N_1$ is a lattice in $\tilde{N}/N_1$.*

Proof.
1. Since Γ is a lattice in $\tilde{N}$ then Γ must be finitely generated and torsion-free according to Theorem C.3.0.6, hence isomorphic to a certain $\mathbb{Z}^k$, denote $\sigma : \mathbb{Z}^k \longrightarrow \Gamma$ this isomorphism. Since $\mathbb{Z}^k$ is a lattice of $\mathbb{R}^k$, it follows by Corollary C.3.0.6 that σ can be extended to an isomorphism $\mathbb{R}^k \longrightarrow \tilde{N}$ so $\tilde{N}$ is abelian.

2. First note that N_1 is also simply connected, this follows from $N_1 = \exp_{\tilde{N}}(\mathfrak{n}_1)$ and the fact that $\exp_{\tilde{N}}$ is a diffeomorphism, where $\mathfrak{n}_1$ is the Lie algebra of N_1. Since $\Gamma_1 \lhd \Gamma$, then for any $\gamma \in \Gamma$:

$$\Gamma_1 = \gamma\Gamma_1\gamma^{-1} \subset \gamma N_1 \gamma^{-1},$$

thus $\Gamma_1 \subset N_1 \cap (\gamma N_1 \gamma^{-1}) \subset N_1$, but Γ_1 is a lattice in N_1 hence $N_1 \subset \gamma N_1 \gamma^{-1}$. It follows that the subgroup N_1 is normalized by Γ and since Γ is a lattice in $\tilde{N}$ we get that N_1 is normal in $\tilde{N}$ according to Corollary C.3.0.1. Let $\pi : \tilde{N} \longrightarrow \tilde{N}/N_1$ the natural projection and choose a compact subset $C \subset N_1$ such that $N_1 = \Gamma_1 C$. Then:

$$\pi^{-1}(\pi(\Gamma)) = \Gamma \cdot N_1 = \Gamma \cdot \Gamma_1 \cdot C = \Gamma \cdot C,$$

thus $\pi(\Gamma)$ is a closed subgroup of $\tilde{N}/N_1$ and since it is countable at most then it must be discrete in $\tilde{N}/N_1$. To prove that $\pi(\Gamma)$ is a lattice in $\tilde{N}/N_1$, let $H \subset \tilde{N}/N_1$ be a connected subgoup containing $\pi(\Gamma)$, then $\Gamma \subset \pi^{-1}(H) \subset \tilde{N}$. Now since $H \simeq \pi^{-1}(H)/N_1$ and N_1 are connected we get that $\pi^{-1}(H)$ is connected as well, therefore $\pi^{-1}(H) = \tilde{N}$ and $H = \tilde{N}/N_1$. The result then follows from Theorem C.3.0.1. □

LEMMA 3.5.0.2. *Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and denote by Γ a discrete subgroup of $(\mathfrak{g}, *)$ satisfying $\ker(\exp_G) \subset \Gamma$. Then the subgroup $\Gamma_0 := \exp_G(\Gamma)$ is discrete in G.*

Proof. Choose a sequence $(X_n)_n$ in Γ such that $(\exp_G(X_n))_n$ converges to e_G in G. Next fix an open neighborhood U of e_G in G and a neighborhood V of 0 in $\mathfrak{g}$ such that the restriction $\exp_{G|V} : V \longrightarrow U$ is a diffeomorphism. Since $\exp_G(X_n) \in U$ for $n \geq N$, we can find $Y_n \in V$ such that $\exp_G(Y_n) = \exp_G(X_n)$. Then $(Y_n)_n$ converges to 0 in $\mathfrak{g}$ moreover there exists $Z_n \in \ker(\exp_G)$ such that $Y_n = X_n * Z_n$ and since $\ker(\exp_G) \subset \Gamma$ we get that $Y_n \in \Gamma$. It follows from the discreteness of Γ in $\mathfrak{g}$ that $(Y_n)_n$ must be stationary hence $(\exp_G(X_n))_n$ is stationary which shows that Γ_0 is discrete in G. □

LEMMA 3.5.0.3. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ and Γ a discrete subgroup of $(\mathfrak{g}, *)$. Denote $\mathfrak{h} := \mathrm{span}(\ker(\exp_G))$ and $\Gamma_0 := \exp_G(\Gamma)$. Suppose that Γ satisfies:*

$$[\Gamma, \Gamma] \subset \ker(\exp_G) \subset \Gamma.$$

Then $\exp_G(\Gamma \cap \mathfrak{h})$ is a finite central subgroup of G and $\Gamma_0/\exp_G(\Gamma \cap \mathfrak{h}) \subset G/\exp_G(\Gamma \cap \mathfrak{h})$ is a discrete abelian, finitely generated, torsion-free subgroup isomorphic as an abstract group to $\Gamma/\Gamma \cap \mathfrak{h}$.

Proof. It is clear that $\Gamma \cap \mathfrak{h}$ is a discrete abelian subgroup of $(\mathfrak{h}, +)$ containing $\ker(\exp_G)$. If we let $\pi : \mathfrak{h} \longrightarrow \mathfrak{h}/\ker(\exp_G)$ be the natural projection then:

$$\pi^{-1}(\pi(\Gamma \cap \mathfrak{h})) = \Gamma \cap \mathfrak{h} + \ker(\exp_G) = \Gamma \cap \mathfrak{h}.$$

It follows that $\pi(\Gamma \cap \mathfrak{h})$ is a closed countable subgroup of $\mathfrak{h}/\ker(\exp_G)$ so it must be discrete, and since $\mathfrak{h}/\ker(\exp_G)$ is compact then $\pi(\Gamma \cap \mathfrak{h}) \simeq \Gamma \cap \mathfrak{h}/\ker(\exp_G)$ is finite. Now the exponential map $\exp_G$ induce a bijection $\Gamma \cap \mathfrak{h}/\ker(\exp_G) \mapsto \exp_G(\Gamma \cap \mathfrak{h})$ hence $\exp_G(\Gamma \cap \mathfrak{h})$ is finite. On the other hand $\Gamma \cap \mathfrak{h} \subset \mathrm{Z}(\mathfrak{g})$ gives that $\exp_G(\Gamma \cap \mathfrak{h})$ is a central subgroup in G. Using Lemma 3.5.0.2, we obtain that Γ_0 is discrete in G hence the quotient $\Gamma_0/\exp_G(\Gamma \cap \mathfrak{h})$ is a discrete subgroup of $G/\exp_G(\Gamma \cap \mathfrak{h})$.

Now from $[\Gamma,\Gamma]\subset\Gamma\cap\mathfrak{h}$ we obtain that $\Gamma\cap\mathfrak{h}$ is a normal subgroup of Γ and $\Gamma/\Gamma\cap\mathfrak{h}$ is abelian, moreover since Γ and $\Gamma\cap\mathfrak{h}$ are finitely generated, it follows that $\Gamma/\Gamma\cap\mathfrak{h}$ is finitely generated as well. Let $X\in\Gamma$ such that $nX\in\Gamma\cap\mathfrak{h}$ for some $n\in\mathbb{Z}$ then $X\in\mathfrak{h}$ thus $X\in\Gamma\cap\mathfrak{h}$ and it follows that $\Gamma/\Gamma\cap\mathfrak{h}$ is torsion-free. Consider now the homomorphism:

$$\psi:\Gamma/\Gamma\cap\mathfrak{h}\longrightarrow\Gamma_0/\exp_G(\Gamma\cap\mathfrak{h}),\ \ \bar{X}\mapsto[\exp_G(X)].$$

It is straightforward to check that ψ is a well-defined isomorphism of abstract groups, which achieves the proof. □

Proof of Theorem 3.5.0.2. Let $\mathfrak{n}$ be a 2-step nilpotent subalgebra of $\mathfrak{g}$ such that $(\mathfrak{n},*)$ admits a lattice Γ with the property $[\Gamma,\Gamma]\subset\ker(\exp_G)\subset\Gamma$. Denote $N:=\exp_G(\mathfrak{n})$ and $\Gamma_0:=\exp_G(\Gamma)$. According to Lemma 3.5.0.2 Γ_0 is a discrete subgroup of G. Choose a compact subset $C\subset\mathfrak{n}$ such that $\mathfrak{n}=\Gamma*C$ then:

$$N=\exp_G(\mathfrak{n})=\exp_G(\Gamma)\exp_G(C)=\Gamma_0\exp_G(C).$$

thus N is closed in G. On the other hand, $\exp_G(\Gamma\cap\mathscr{L}_1(G))$ is a finite central subgorup of G by Lemma 3.5.0.3. Let $N_0:=N/\exp_G(\Gamma\cap\mathscr{L}_1(G))$ and denote by $\pi:N\longrightarrow N_0$ the canonical projection, again Lemma 3.5.0.3 gives that $\Gamma_0/\exp_G(\Gamma\cap\mathscr{L}_1(G))$ is a discrete abelian, finitely generated, torsion-free subgroup of N_0 isomorphic as an abstract group to $\Gamma/\Gamma\cap\mathscr{L}_1(G)$ thus:

$$\mathrm{p}(N_0)\geq\mathrm{rg}(\Gamma/\Gamma\cap\mathscr{L}_1(G))=\mathrm{rg}(\Gamma)-\mathrm{rg}(\Gamma\cap\mathscr{L}_1(G))=\dim\mathfrak{n}-\dim\mathscr{L}_1(G).$$

Since $N\overset{\pi}{\longrightarrow}N_0$ is a finite cover, Proposition 3.3.0.1 gives that $\mathrm{p}(N)=\mathrm{p}(N_0)$, moreover the closedness of N in G results in $\mathrm{p}(G)\geq\mathrm{p}(N)$. Finally, using $\dim\mathscr{L}_1(G)=\mathrm{rg}(\pi_1(G))$ we conclude that:

$$\mathrm{p}(G)\geq\dim(\mathfrak{n})-\mathrm{rg}(\pi_1(G)).$$

Conversely, put $\mathrm{p}(G):=p$ and let $\varphi:\mathbb{Z}^p\longrightarrow G$ be an injective homomorphism with discrete image. Denote $\tilde{G}\overset{\pi}{\longrightarrow}G$ the universal cover of G and put $\Gamma=\pi^{-1}(\varphi(\mathbb{Z}^p))$, then $\ker(\pi)\subset\Gamma$ and since π is a covering map we get that Γ is discrete in $\tilde{G}$. Moreover for any $\gamma_1,\gamma_2\in\Gamma$ we have that:

$$\pi(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})=\pi(\gamma_1)\pi(\gamma_2)\pi(\gamma_1)^{-1}\pi(\gamma_2)^{-1}=e_G.$$

Thus $\pi[\Gamma,\Gamma]=\{e_G\}$ which means that $[\Gamma,\Gamma]\subset\ker(\pi)\subset Z(\tilde{G})$ so Γ is 2-step nilpotent. Let $\tilde{N}$

be the Zariski closure of Γ in $\tilde{G}$ then denote $\mathfrak{n} := \mathrm{Lie}(\tilde{N})$ and $N := \pi(\tilde{N})$. Choose a compact subset $C \subset \tilde{N}$ such that $\tilde{N} = \Gamma \cdot C$, then $N = \varphi(\mathbb{Z}^p)\exp_G(C)$ and since $\varphi(\mathbb{Z}^p) \subset G$ is discrete we get that N is closed in G, in particular $\mathrm{p}(G) \leq \mathrm{p}(N)$. On the other hand $K_1(G)$ is a compact subgroup of G and is contained in N, therefore by Corollary 3.2.0.1 we get that:

$$\mathrm{p}(G) \leq \dim(N) - \dim(K_1(G)) = \dim(\mathfrak{n}) - \mathrm{rg}(\pi_1(G)).$$

We now prove that $\mathfrak{n}$ is 2-step nilpotent. Indeed let N_1 the Zariski closure of $[\Gamma,\Gamma]$ in $\tilde{N}$, and put $\mathfrak{n}_1 = \mathrm{Lie}(N_1)$. It is clear that since Γ is 2-step nilpotent then $[\Gamma,\Gamma]$ is an abelian normal subgroup of Γ hence by Lemma 3.5.0.1, N_1 is an abelian normal subgroup of $\tilde{N}$ and $\Gamma/\Gamma \cap N_1$ is a lattice in $\tilde{N}/N_1$. Now since $[\Gamma,\Gamma] \subset \Gamma \cap N_1$ it follows that $\Gamma/\Gamma \cap N_1$ is an abelian group, thus the quotient $\tilde{N}/N_1$ must be abelian according to Lemma 3.5.0.1. This means that $[\mathfrak{n},\mathfrak{n}] \subset \mathfrak{n}_1$. From $[\Gamma,[\Gamma,\Gamma]] = \{e_G\}$ we obtain that $\gamma = \tau\gamma\tau^{-1}$ for any $\gamma \in \Gamma$ and for any $\tau \in [\Gamma,\Gamma]$, so $\sigma_{\tau|\Gamma} = \mathrm{Id}_\Gamma$ where $\sigma_\tau : \tilde{N} \longrightarrow \tilde{N}$ is given by $\sigma_\tau(g) = \tau g \tau^{-1}$. Since σ_τ and $\mathrm{Id}_{\tilde{N}}$ coincide on Γ, they must coincide on $\tilde{N}$ in view of Theorem C.3.0.4 which means that $\tau g \tau^{-1} = g$ for any $g \in \tilde{N}$ which is the same as $[\Gamma,\Gamma] \subset Z(\tilde{N})$ thus $N_1 \subset Z(\tilde{N})$ and consequently we deduce that $[\mathfrak{n},\mathfrak{n}] \subset \mathfrak{n}_1 \subset \mathrm{Z}(\mathfrak{n})$. Hence $\mathfrak{n}$ is 2-step nilpotent Lie subalgebra of $\mathfrak{g}$. Finally observe that in the case where $\tilde{G} = (\mathfrak{g},*)$ and $\pi = \exp_G$, we get that $\tilde{N} = (\mathfrak{n},*)$ and Γ is a lattice of $(\mathfrak{n},*)$ satisfying:

$$[\Gamma,\Gamma] \subset \ker(\exp_G) \subset \Gamma,$$

which achieves the proof. □

REMARK 3.5.0.1. *Notice that in the proof of the preceding result, we can actually deduce that any discrete subgroup of a connected nilpotent Lie group G isomorphic to $\mathbb{Z}^p$ is contained in a connected closed 2-step nilpotent subgroup of G.*

COROLLARY 3.5.0.1. *Let G be a connected nilpotent Lie group and T a maximal compact subgroup of G. Denote $A := T \cap \overline{[G,G]}$, then G/A is a linear Lie group and $\mathrm{p}(G) \leq \mathrm{p}(G/A)$.*

Proof. First observe that since T is a compact subgroup of the nilpotent Lie group G, then T is central and therefore A is a compact normal subgroup of G. Moreover T/A is a maximal compact subgroup of G/A. Since G/A is a nilpotent Lie group, according to [24, Theorem 15.2.9] G/A is linear if and only if:

$$\mathfrak{t}/\mathfrak{a} \cap [\mathfrak{g}/\mathfrak{a}, \mathfrak{g}/\mathfrak{a}] = \{0\},$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{a} = \mathrm{Lie}(A)$. Let $\pi : G \longrightarrow G/A$ be the natural projection then from $\pi^{-1}([G/A,G/A]) = [G,G]\cdot A \subset \overline{[G,G]}$ we get that:

$$\pi^{-1}(T/A \cap [G/A,G/A]) = \pi^{-1}(T/A) \cap \pi^{-1}([G/A,G/A]) \subset T \cap \overline{[G,G]}.$$

It follows that $T/A \cap [G/A,G/A]$ is trivial and therefore its Lie algebra $\mathfrak{t}/\mathfrak{a} \cap [\mathfrak{g}/\mathfrak{a},\mathfrak{g}/\mathfrak{a}]$ must also be trivial, thus G/A is a linear nilpotent Lie group. For the second claim recall that according to Theorem 3.5.0.2:

$$\mathrm{p}(G) = \dim(\mathfrak{n}_0) - \mathrm{rg}(\pi_1(G)),$$

where $\mathfrak{n}_0$ is of maximal dimension among all 2-step nilpotent Lie subalgebras $\mathfrak{n} \subset \mathfrak{g}$ such that $(\mathfrak{n},*)$ admits a lattice Γ satisfying the inclusion $[\Gamma,\Gamma] \subset \ker(\exp_G) \subset \Gamma$. Now $[\Gamma,\Gamma]$ is a lattice in $[\mathfrak{n}_0,\mathfrak{n}_0]$[2] and $\mathfrak{t} := \mathrm{span}\{\ker(\exp_G)\}$[3] therefore we get that $[\mathfrak{n}_0,\mathfrak{n}_0] \subset \mathfrak{t} \subset \mathfrak{n}_0$ and since $[\mathfrak{n}_0,\mathfrak{n}_0] \subset [\mathfrak{g},\mathfrak{g}]$ it follows that $[\mathfrak{n}_0,\mathfrak{n}_0] \subset \mathfrak{a} \subset \mathfrak{n}_0$. On the other hand since G/A is a linear nilpotent Lie group, Proposition 3.5.0.1 gives that:

$$\mathrm{p}(G/A) = \mathscr{M}(\mathfrak{g}/\mathfrak{a}) - \mathrm{rg}(\pi_1(G/A)).$$

Let $\mathfrak{b} \subset \mathfrak{g}/\mathfrak{a}$ be an abelian subalgebra of maximal dimension then denote $\pi_0 : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$ the natural projection and $\hat{\mathfrak{n}} := \pi_0^{-1}(\mathfrak{b})$ where . From $[\mathfrak{b},\mathfrak{b}] = 0$ we get that $[\hat{\mathfrak{n}},\hat{\mathfrak{n}}] \subset \mathfrak{a} \subset \mathrm{Z}(\mathfrak{g})$ and we deduce that $\hat{\mathfrak{n}}$ is a 2-step nilpotent Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, in fact from the maximality of $\mathfrak{b}$ we obtain that $\hat{\mathfrak{n}}$ is of maximal dimension among all 2-step nilpotent Lie algebras $\mathfrak{n} \subset \mathfrak{g}$ satisfying $[\mathfrak{n},\mathfrak{n}] \subset \mathfrak{a} \subset \mathfrak{n}$, in particular from the previous observation we get that $\dim \mathfrak{n}_0 \leq \dim \hat{\mathfrak{n}}$. In summary:

$$\begin{aligned}
\mathrm{p}(G) \leq \dim(\hat{\mathfrak{n}}) - \mathrm{rg}(\pi_1(G)) &\leq \dim(\mathfrak{b}) + \dim(\mathfrak{a}) - \mathrm{rg}(\pi_1(G)) \\
&\leq \mathscr{M}(\mathfrak{g}/\mathfrak{a}) + \dim(\mathfrak{a}) - \mathrm{rg}(\pi_1(G)) \\
&\leq \mathrm{p}(G/A) + \dim(\mathfrak{a}) + \mathrm{rg}(\pi_1(G/A)) - \mathrm{rg}(\pi_1(G)).
\end{aligned}$$

Since T and T/A are maximal compact subgroups of G and G/A respectively, it follows that $\pi_1(G) \simeq \pi_1(T)$ and $\pi_1(G/A) \simeq \pi_1(T/A)$, moreover since both T and T/A are tori we deduce that $\mathrm{rg}(\pi_1(G/A)) = \dim(T) - \dim(A) = \mathrm{rg}(\pi_1(G)) - \dim(A)$. Thus from the previous

[2] To see this let N_1 be the Zariski closure of $[\Gamma,\Gamma]$ in $N := (\mathfrak{n}_0,*)$ then $[\Gamma,\Gamma] \subset N_1 \subset [N,N]$, it follows from Lemma 3.5.0.1 that $\Gamma/\Gamma \cap N_1$ is a lattice in N/N_1, but since $\Gamma/\Gamma \cap N_1$ is abelian, Lemma 3.5.0.1 gives that N/N_1 is abelian and so $[N,N] \subset N_1$. Therefore $[\Gamma,\Gamma]$ is a lattice in $[N,N]$.

[3] The maximal compact subgroup $T \subset G$ is exactly $K_1(G)$.

computation we conclude that $\mathrm{p}(G) \leq \mathrm{p}(G/A)$. □

COROLLARY 3.5.0.2. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that $(\mathfrak{g},*)$ admits a lattice Γ and put $G = (\mathfrak{g},*)/\Gamma \cap \mathrm{Z}(\mathfrak{g})$. Then:*

$$\mathrm{p}(G) = \dim(\mathfrak{g}) - \dim \mathrm{Z}(\mathfrak{g}).$$

Proof. Clearly $\ker(\exp_G) = \Gamma \cap \mathrm{Z}(\mathfrak{g})$. Since $\mathfrak{g}$ is 2-step nilpotent then $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Z}(\mathfrak{g})$ and it follows that $[\Gamma,\Gamma] \subset \ker(\exp_G) \subset \Gamma$. From Theorem 3.5.0.2 we have $\mathrm{p}(G) = \dim \mathfrak{g} - \mathrm{rank}(\pi_1(\mathrm{G}))$. Now $\Gamma \cap \mathrm{Z}(\mathfrak{g})$ is a lattice in $\mathrm{Z}(\mathfrak{g})$ by Corollary C.3.0.7, and since $\pi_1(G) \simeq \Gamma \cap \mathrm{Z}(G)$ we conclude that $\mathrm{rank}(\pi_1(\mathrm{G})) = \dim \mathrm{Z}(\mathfrak{g})$. □

EXAMPLE 3.5.0.2. *Let $\mathfrak{g} = \mathfrak{h}_k$ be the $(2k+1)$-dimensional Heisenberg Lie algebra, i.e the Lie algebra with basis $\mathscr{B} = \{e_1, \ldots, e_{2k+1}\}$ and Lie bracket given by:*

$$[e_{2i}, e_{2i+1}] = e_1, \quad i = 1, \ldots, k.$$

The Lie algebra $\mathfrak{g}$ is 2-step nilpotent with $\mathrm{Z}(\mathfrak{g}) = \mathrm{span}\{e_1\}$, put $G = (\mathfrak{g},*)/\mathbb{Z}e_1$. As before the Lie algebra structure on $\mathfrak{g}$ is rational with respect to $\mathscr{B}$ hence the group $\Gamma := \langle \mathrm{span}_{\mathbb{Z}}(\mathscr{B}) \rangle$ is a lattice in $(\mathfrak{g},*)$ and it is straightforward to check that $\Gamma \cap \mathrm{Z}(\mathfrak{g}) = \mathbb{Z}e_1$. By Corollary 3.5.0.2 we get that:

$$\mathrm{p}(G) = 2k.$$

On the other hand, $\mathscr{M}(\mathfrak{g}) = k+1$ by [11, Theorem 5.4]. Thus from Proposition 3.4.0.1 and Theorem 3.5.0.1 we get that:

$$\mathrm{q}(G) = k \ \ \text{and} \ \ \mathrm{p}(\mathfrak{g},*) = \mathrm{q}(\mathfrak{g},*) = k+1.$$

Appendix A

Group Actions, Fiber Bundles and Classifying Bundles

In this chapter

In what follows, unless otherwise stated a space will always mean a Hausdorff, paracompact topological space and a map will always mean a continuous map.

A.1 Group Actions

Definition A.1.0.1. *Let G be a topological group. We call a* continuous action *of the group G on a space B any map $\varphi : G \times B \longrightarrow B$, satisfying the following properties:*

1. *For any $x \in B$, $\varphi(e,x) = x$.*

2. *For any $x \in B$ and any $g,h \in G$, $\varphi(gh,x) = \varphi(g,\varphi(h,x))$.*

When there is no ambiguity, we write $\varphi(g,x) := g \cdot x$. This definition says that a continuous action $\varphi : G \times B \longrightarrow B$ is just a group homomorphism $\rho : G \longrightarrow \mathrm{Homeo}(B)$ that is continuous when $\mathrm{Homeo}(B)$ is endowed with the compact-open topology, the corresponding action of G on B is then obtained by setting $\varphi(g,x) := \rho(g)(x)$. The *orbit* of an element $x \in B$ under the action of the group G is the subset $G\cdot x := \{g\cdot x,\ g\in G\}$. The *isotropy subgroup* at an element $x\in B$ is the subgroup $G_x = \{g\in G,\ g\cdot x = x\}$ of G.

DEFINITION A.1.0.2. *A continuous group action of G on B is called:*

1. ***(Effective)**: If the associated group homomorphism $\rho : G \longrightarrow \mathrm{Homeo}(B)$ is injective.*
2. ***(Transitive)**: If for some (hence all) $x\in B$, $B = G\cdot x$.*
3. ***(Free)**: If for every $x\in B$ the isotropy subgroup G_x is trivial.*
4. ***(Proper)**: If for any compact subset $C\subset B$, the set:*

$$G_C := \{g\in G,\ g\cdot C\cap C\neq\emptyset\},$$

is a compact subset of G. When the topology of G is discrete we say that the action is properly discontinuous, in which case G_C is finite.

A continuous group action $\varphi : G\times M \longrightarrow M$ is called *smooth* if G is a Lie group, M is a smooth manifold and φ is a smooth map. In view of the preceding remark, this induces a continuous homomorphism $\rho : G \longrightarrow \mathrm{Diff}(M)$. Denote $\mathfrak{g}$ the Lie algebra of G and $\mathcal{X}(M)$ the Lie algebra of vector fields on the manifold M, the action of the Lie group G on the manifold M can naturally give rise to a Lie algebra homomorphism $\rho' : \mathfrak{g} \longrightarrow \mathcal{X}(M)$ by the formula:

$$\rho'(X)_x := \frac{d}{dt}_{|t=0}(\exp_G(tX)\cdot x).$$

This is called the *infinitesimal action* of G on M associated to φ. More generally, let M be a smooth manifold and let $\mathfrak{g}$ be a finite dimensional Lie algebra, any Lie algebra homomorphism $\rho' : \mathfrak{g} \longrightarrow \mathcal{X}(M)$ is called a *Lie algebra action* and the vector fields $\rho'(X)$ are called *fundamental vector fields*, in this case we say that M is a $\mathfrak{g}$-manifold. Every such action defines a singular foliation on the manifold M as follows:

Given $x\in M$, the leaf through x (also the $\mathfrak{g}$-orbit of x) is the immersed submanifold $\mathfrak{g}(x)\subset M$ consisting of points of the form $y = \phi^{X_1}_{t_1}\circ\cdots\circ\phi^{X_r}_{t_r}$ for some $X_1,\ldots,X_r\in\mathfrak{g}$, $t_1,\ldots,t_r\in\mathbb{R}$ and where ϕ^X the flow of the fundamental vector field $\rho'(X)$. The topology of $\mathfrak{g}(x)$ is the finest topology for which the maps:

$$(t_1,\ldots,t_k)\mapsto \phi^{X_1}_{t_1}\circ\cdots\circ\phi^{X_k}_{t_k}(x),$$

are continuous (when these maps are well-defined) for every $X_1,\ldots,X_k\in\mathfrak{g}$. Let $\mathfrak{g}_x$ be the kernel of the linear map $\rho'_x : \mathfrak{g} \longrightarrow \mathrm{T}_xM$, $X\mapsto\rho'(X)_x$, then $\dim\mathfrak{g}(x) = \dim\mathfrak{g} - \dim\mathfrak{g}_x$. Given a smooth map $f : M \longrightarrow N$ between two $\mathfrak{g}$-manifolds, we say that it is $\mathfrak{g}$-equivariant if it satisfies $f(\mathfrak{g}(x)) \subset \mathfrak{g}(f(x))$. Finally for any $y,z \in M$, declare $y \sim z$ if and only if $\mathfrak{g}(y) = \mathfrak{g}(z)$, the set of equivalence classes with respect to this relation will be denoted $M/\mathfrak{g}$, it is called the orbit space of the $\mathfrak{g}$-action and is endowed with the quotient topology.

PROPOSITION A.1.0.1. *Let $\varphi : G \times M \longrightarrow M$ be a smooth action of a Lie group G on a smooth manifold M. Denote $\rho' : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ the corresponding infinitesimal action. Then for any $x \in M$ we have $\mathfrak{g}(x) = G \cdot x$ and therefore $M/\mathfrak{g} = M/G$. Moreover, let N be another G-manifold, a smooth map $f : M \longrightarrow N$ is $\mathfrak{g}$-equivariant if and only if it is G-equivariant, i.e $f(g \cdot x) = g \cdot f(x)$.*

A Lie algebra action $\rho' : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ is called *complete* if $\rho'(X)$ is a complete vector field for every $X \in \mathfrak{g}$, the action is called *weakly complete* if there exists a subset $S \subset \mathfrak{g}$ that generates $\mathfrak{g}$ such that $\rho'(X)$ is complete for every $X \in S$.

REMARK A.1.0.1. *Let $\rho' : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ be the infinitesimal action corresponding to a Lie group action $\varphi : G \times M \longrightarrow M$. By definition, it is clear that the integral curve of $\rho'(X)$ through a point $x \in M$ is exactly the curve:*

$$\gamma : \mathbb{R} \longrightarrow M, \quad \gamma(t) := \exp(tX) \cdot x.$$

In this sense infinitesimal actions are always complete.

THEOREM A.1.0.1 (PALAIS). *Every weakly-complete Lie algebra action can be seen as the infinitesimal description of a global action of a connected simply connected Lie group. In particular, weakly complete Lie algebra actions are complete.*

A.2 Fiber bundles, Vector bundles and Principal bundles

DEFINITION A.2.0.1. *A fiber bundle over a space B is a triple (E,π,F) of topological spaces E and F and a surjective map $\pi : E \longrightarrow M$ such that E is locally trivial over B: There exists an open cover $(U_\alpha)_\alpha$ of M and a family of homeomorphisms $\phi_\alpha : U_\alpha \times F \longrightarrow \pi^{-1}(U_\alpha)$ called local trivializations such that the following diagram is commutative:*

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow{\phi_\alpha} & \pi^{-1}(U_\alpha) \\ & \searrow^{\mathrm{pr}_1} & \downarrow \pi \\ & & U_\alpha \end{array}$$

In the preceding definition, B is called the *base space* and E the *total space* of the fiber bundle (E,π,F). It is clear that $\pi^{-1}(x) \simeq F$ for any $x \in B$ therefore F is called the *fiber type* of the fiber bundle. Fiber bundles will be denoted $F \hookrightarrow E \xrightarrow{\pi} B$ or simply $E \xrightarrow{\pi} B$ when there is no ambiguity. A natural example of a fiber bundle is the *trivial bundle* $B \times F \xrightarrow{\mathrm{pr}_1} B$. Finally, a fiber bundle is said to be *smooth* if the spaces B, E and F are smooth manifolds and all the maps involved are smooth.

DEFINITION A.2.0.2. *Let $E \xrightarrow{\pi} B$ be a fiber bundle over B with fiber type F and let $\mathscr{F}$ be a family of local trivializations $\phi_\alpha : U_\alpha \times F \longrightarrow \pi^{-1}(U_\alpha)$ covering B. We call transition functions of $E \xrightarrow{\pi} B$ (relative to $\mathscr{F}$) the family of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F)$ given by:*

$$(\phi_\beta^{-1} \circ \phi_\alpha)(x,p) = (x, g_{\alpha\beta}(x)(p)),$$

for any $x \in U_\alpha \cap U_\beta$ and $p \in F$.

In these notations, if G is a subgroup of $\mathrm{Homeo}(F)$ with $g_{\alpha\beta}(U_\alpha \cap U_\beta) \subset G$ then G is called the *structure group* of $E \xrightarrow{\pi} B$. It can be shown that transition maps are continuous whenever $\mathrm{Homeo}(F)$ is given the compact-open topology.

DEFINITION A.2.0.3. *A (continuous) section of a fiber bundle $E \xrightarrow{\pi} B$ is a map $\sigma : B \longrightarrow E$ satisfying $\pi \circ \sigma = \mathrm{Id}_B$.*

DEFINITION A.2.0.4. *Given fiber bundles $E_i \xrightarrow{\pi_i} B_i$, $i = 1,2$, a fiber bundle homomorphism is a pair of maps $f : B_1 \longrightarrow B_2$ and $g : E_1 \longrightarrow E_2$ such that the following diagram is commutative:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\;g\;} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow[\;f\;]{} & B_2 \end{array}$$

The pair (f,g) is called an isomorphism if $g : E_1 \longrightarrow E_2$ is a homeomorphism (in which case the map $f : B_1 \longrightarrow B_2$ is automatically a homeomorphism).

DEFINITION A.2.0.5. *If $E_i \xrightarrow{\pi_i} B$, $i = 1,2$ are fiber bundles over the same base space B, then a bundle morphism over B is just a map $g : E_1 \longrightarrow E_2$ satisfying $\pi_2 \circ g = \pi_1$ (in the sense of the preceding definition, this is just the pair (Id_B, g)). It is called a bundle isomorphism over M if the map $g : E_1 \longrightarrow E_2$ is an homeomorphism.*

When a fiber bundle $E \xrightarrow{\pi} B$ is isomorphic to the trivial bundle $B \times F \xrightarrow{\mathrm{pr}_1} B$, it is called *trivial*. Every fiber bundle is by definition locally isomorphic to the trivial bundle.

DEFINITION A.2.0.6. *We call a $\mathbb{K}$-vector bundle over a space B any fiber bundle $E \xrightarrow{\pi} B$*

whose fiber type F is a $\mathbb{K}$-vector space and whose structure group is $\mathrm{GL}(F,\mathbb{K})$*,* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$*.*

It follows that the fibers of a $\mathbb{K}$-vector bundle $E \xrightarrow{\pi} B$ are naturally endowed with a $\mathbb{K}$-vector space structure.

Definition A.2.0.7. *An homomorphism of $\mathbb{K}$-vector bundles $E_i \xrightarrow{\pi_i} B_i$, $i=1,2$ is a bundle homomorphism $(f,g):(B_1,E_1)\longrightarrow(B_2,E_2)$ such that g is $\mathbb{K}$-linear when restricted to the fibers of $E_1 \xrightarrow{\pi_1} B_1$.*

A *local frame* of a vector bundle $E \xrightarrow{\pi} B$ is any family $\sigma_1,\dots,\sigma_r : U \longrightarrow E$ of local sections defined on an open subset $U\subset B$ such that $\{\sigma_1(x),\dots,\sigma_r(x)\}$ forms a basis of the fiber $\pi^{-1}(x)$ for all $x\in U$.

Proposition A.2.0.1. *A vector bundle $E \xrightarrow{\pi} M$ is trivial if and only if it admits a global frame.*

Definition A.2.0.8. *Let G be a topological group. A principal G-bundle over a space B is a fiber bundle $P \xrightarrow{\pi} B$ whose total space is endowed with a fiber preserving continuous group action $P\times G\longrightarrow P$, $(p,g)\mapsto p\cdot g$ such that there exists a family $\phi_\alpha : U_\alpha\times G\longrightarrow \pi^{-1}(U_\alpha)$ of G-equivariant local trivializations covering B (the action of G on $U_\alpha\times G$ is naturally given by $(x,g)\cdot h := (x,g\cdot h)$).*

Let $P \xrightarrow{\pi} B$ be a principal G-bundle and let $\phi_\alpha : U_\alpha\times G\longrightarrow \pi^{-1}(U_\alpha)$ be a family of G-equivariant local trivializations covering B. We can check that the domain of the transition maps $g_{\alpha\beta} : U_\alpha\cap U_\beta \longrightarrow \mathrm{Homeo}(G)$ is always contained in $\mathrm{R}(G)$ the group of right translations of G and which can naturally be identified to G, thus we can say that G is the structure group of the principal G-bundle.

Definition A.2.0.9. *A morphism of principal G-bundles $P_i \xrightarrow{\pi_i} B_i$, $i=1,2$ is a morphism of fiber bundles $(f,g):(M_1,B_1)\longrightarrow(M_2,B_2)$ such that g is G-equivariant.*

Proposition A.2.0.2. *A principal G-bundle $P \xrightarrow{\pi} B$ is trivial if and only if it admits a global continuous section.*

When a principal G-bundle $P \xrightarrow{\pi} B$ is smooth as a fiber bundle and G is a Lie group that acts by diffeomorphisms on P, we say that it is a smooth principal G-bundle. For a principal G-bundle the group G acts properly and freely on the total space P and the base space B is homeomorphic to P/G. A partial converse was proven by R. S. Palais in [44], it can be stated as follows:

Theorem A.2.0.1. *Let G be a Lie group acting smoothly, properly and freely on a smooth manifold M, then M/G admits a unique structure of smooth manifold such that the natural projection $M \xrightarrow{\pi} M/G$ is a smooth principal G-bundle.*

A.3 Operations on Fiber bundles

PULL-BACK: Fix a map $f : X \longrightarrow B$ and let $E \xrightarrow{\pi} M$ be a fiber bundle over B with fiber type F. Set:

$$f^{-1}E := \{(x,v) \in X \times E,\ f(x) = \pi(v)\}.$$

It is straightforward to check that $f^{-1}E \xrightarrow{\mathrm{pr}_1} X$ is a fiber bundle over X with the same fiber type F. It is called the *pull-back* of $E \xrightarrow{\pi} B$ via f and it satisfies that the following diagram is commutative:

$$\begin{array}{ccc} f^{-1}E & \xrightarrow{\mathrm{pr}_2} & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & B \end{array}$$

If $E \xrightarrow{\pi} B$ is a vector bundle (resp. principal G-bundle) , the pull-back $f^{-1}E \xrightarrow{\mathrm{pr}_1} X$ is also a vector bundle (resp. a principal G-bundle).

THEOREM A.3.0.1. *Let $E \xrightarrow{\pi} B$ be a fiber bundle and suppose that the maps $f_0 : X \longrightarrow B$ and $f_1 : X \longrightarrow B$ are homotopic. Then the respective pull-back bundles via f_0 and f_1 are isomorphic, i.e $f_0^*(E) \simeq f_1^*(E)$.*

When $f_0, f_1 : X \longrightarrow B$ are homotopic maps and $E \xrightarrow{\pi} B$ is a principal G-bundle then the pullbacks $f_0^*(E)$ and $f_1^*(E)$ are isomorphic *as principal G-bundles*. Similarly, if E is a vector bundle we get that the pullbacks $f_0^*(E)$ and $f_1^*(E)$ are isomorphic *as vector bundles*.

COROLLARY A.3.0.1. *Let $E \xrightarrow{\pi} B$ be a fiber bundle and suppose that B is a contractible space. Then E is trivial.*

FIBERED PRODUCT: Let $E_i \xrightarrow{\pi_i} B$, $i = 1,2$ be two fiber bundles over B with respective fiber types F_i. Define:

$$E_1 \times_B E_2 = \{(u,v) \in E_1 \times E_2,\ \pi_1(u) = \pi_2(v)\}.$$

The map $\pi : E_1 \times_B E_2 \xrightarrow{\mathrm{pr}_i} E_i \xrightarrow{\pi_i} B$ is surjective and $E_1 \times_B E_2 \xrightarrow{\pi} B$ is a fiber bundle with fiber type $F_1 \times F_2$, it is called the *fibered product* of E_1 and E_2 over B. If $E_i \xrightarrow{\pi_i} B$ are vector bundles then the resulting fibered product is a vector bundle, it is called the *Whitney sum* of E_1 and E_2 and its total space is usually denoted $E_1 \oplus E_2$.

ASSOCIATED BUNDLES: Let $P \xrightarrow{\pi} B$ be a principal G-bundle, F a topological space and fix a continuous action $\rho : G \longrightarrow \mathrm{Homeo}(F)$. Consider the continuous action of G on $P \times F$

given by the expression:

$$(p,z)\cdot g := (p\cdot g, \rho(g^{-1})(z)),$$

and denote the corresponding orbit space $P\times_\rho F := (P\times F)/G$. The map $\pi_0 : P\times_\rho F \longrightarrow B$ given by $\pi_0([p,z]) = \pi(p)$ is well-defined and $P\times_\rho F \xrightarrow{\pi_0} B$ is a fiber bunder over B with fiber type F *associated to* the principal G-bundle $P \xrightarrow{\pi} M$. When F is a vector space and and the action of G on F is linear i.e $\rho(G)\subset \mathrm{GL}(F)$, the resulting associated bundle is a vector bundle.

A.4 Classifying bundles

Let $P \xrightarrow{\pi} B$ is a principal G-bundle and denote $[X,B]$ the space of homotopic classes of maps $X \longrightarrow B$. Let $\mathrm{Prin}_G(X)$ be the space of isomorphism classes of principal G-bundles over X. According to the remark following Theorem A.3.0.1, the pullback operation induces a correspondence:

$$[X,B] \longrightarrow \mathrm{Prin}_G(X), \;\; [f]\mapsto [f^*P].$$

DEFINITION A.4.0.1. *A principal G-bundle* $\mathrm{EG} \xrightarrow{\pi} \mathrm{BG}$ *is called universal if the total space* EG *is contractible.*

J. Milnor have shown in [38] that universal principal G-bundles always exists for any topological group G.

THEOREM A.4.0.1. *Suppose that* $\mathrm{EG} \xrightarrow{\pi} \mathrm{BG}$ *is a universal principal G-bundle and X is a CW-complex. Then the correspondance* $[X,\mathrm{BG}] \longrightarrow \mathrm{Prin}_G(X)$ *given by* $[f]\mapsto[f^*(\mathrm{EG})]$ *is bijective.*

The space BG will be called a *classifying space* for the group G, and if $P \xrightarrow{\pi} X$ is a principal G-bundle, any map $f : X \longrightarrow \mathrm{BG}$ such that $P \simeq f^*(\mathrm{EG})$ will be called a *classifying map* for P.

Concerning uniqueness properties of universal principal G-bundles, we state the following results:

COROLLARY A.4.0.1. *The classifying space* BG *can be taken to have the homotopy type of a CW-complex.*

In what follows a classifying space BG will always be assumed a CW-complex.

THEOREM A.4.0.2. *Let* $\mathrm{EG}\longrightarrow\mathrm{BG}$ *and* $\mathrm{E'G}\longrightarrow\mathrm{B'G}$ *be two universal principal G-bundles. There exists a homotopy equivalence* $\mathrm{B'G}\longrightarrow\mathrm{BG}$ *that is covered by a G-equivariant homo-*

topy equivalence $\mathrm{E'G} \longrightarrow \mathrm{EG}$*. In this sense, universal principal* G*-bundles are unique up to homotopy equivalence.*

We thus have a well-defined correspondence $G \mapsto \mathrm{BG}$ from the category of topological groups to the category of homotopy classes of CW-complexes, which is functorial according to the following theorem:

THEOREM A.4.0.3. *To each homomorphism* $\phi : G \longrightarrow H$ *of topological groups is associated a natural homotopy class* $\mathrm{B}\phi \in [\mathrm{BG}, \mathrm{BH}]$ *such that if* $\phi \in \mathrm{Hom}(G, H)$ *and* $\psi \in \mathrm{Hom}(H, K)$ *then* $[\mathrm{B}(\phi \circ \psi)] = [\mathrm{B}\phi \circ \mathrm{B}\psi]$ *and* $\mathrm{BId} = \mathrm{Id}$.

We achieve this section with a result on the universal bundle associated to a subgroup of a given topological group:

PROPOSITION A.4.0.1. *Let* $H \overset{\iota}{\hookrightarrow} G$ *be an inclusion of topological groups such that the canonical projection* $G \longrightarrow G/H$ *is a principal* H*-bundle. Then we can take* $\mathrm{EH} = \mathrm{EG}$ *for a total space and* $\mathrm{BH} = \mathrm{EG} \times_G (G/H)$ *for a classifying space.*

If H is a closed subgroup of a Lie group G, then the canonical projection $G \longrightarrow G/H$ is always a principal H-bundle and thus the preceding result holds in this case.

Appendix B

De Rham Cohomology

In this chapter

B.1 Differential Complexes and Cohomology

Let $V = \oplus_p V^p$ be a graded vector space. a linear map $\mathrm{d}: V \longrightarrow V$ is called a *differential* on V if it satisfies $\mathrm{d}(V^p) \subset V^{p+1}$ and $\mathrm{d}^2 = 0$. The couple (V, d) consisting of a graded vector space and a differential is called a *differential complex*, in this case we call a d-*cochain* any element $v \in V$. We say that $v \in V$ is a d-*cocycle* if $\mathrm{d}v = 0$, and we say that v is a d-*coboundary* if $w = \mathrm{d}u$ for some $u \in V$, we denote by $\mathrm{Z_d}(V)$ the set of d-cocycles and $\mathrm{B_d}(V)$ the set of d-coboundaries. It is clear that $\mathrm{Z_d}(V) := \oplus_p \mathrm{Z}^p_\mathrm{d}(V)$ and $\mathrm{B_d}(V) := \oplus_p \mathrm{B}^p_\mathrm{d}(V)$ with:

$$\mathrm{Z}^p_\mathrm{d}(V) = \ker(\mathrm{d}: V^p \longrightarrow V^{p+1}), \quad \mathrm{B}^p_\mathrm{d}(V) = \mathrm{Im}(\mathrm{d}: V^{p-1} \longrightarrow V^p),$$

hence we can see that $\mathrm{Z_d}(V)$ and $\mathrm{B_d}(V)$ are graded vector subspaces of V. Moreover the relation $\mathrm{d}^2 = 0$ implies that $\mathrm{B}^p_\mathrm{d}(V) \subset \mathrm{Z}^p_\mathrm{d}(V)$. The *p-th cohomology group* of the differential complex (V, d) is defined as the vector space:

$$\mathrm{H}^p_\mathrm{d}(V) := \mathrm{Z}^p_\mathrm{d}(V)/\mathrm{B}^p_\mathrm{d}(V) = \frac{\ker(\mathrm{d}: V^p \longrightarrow V^{p+1})}{\mathrm{Im}(\mathrm{d}: V^{p-1} \longrightarrow V^p)}.$$

Finally we call the *cohomology* of (V,d) the graded vector space $\mathrm{H}_{\mathrm{d}}(V) = \oplus_p \mathrm{H}^p_{\mathrm{d}}(V)$, the cohomology class of a d-cocycle $v \in V$ will be denoted $[v]_{\mathrm{d}}$. We call a *subcomplex* of a differential complex (V,d) is any graded vector subspace $W := \oplus_p W^p$ such that $W^p \subset V^p$ and which is stable under the differential operator d, it is clear that (W,d) is a differential complex in its own right. Consider the following sequence of sets and arrows:

$$\dots \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \xrightarrow{f_{i+1}} V_{i+2} \xrightarrow{f_{i+1}} \dots \tag{B.1}$$

We say that (B.1) is an *exact sequence* of vector spaces if the sets V_i are vector spaces, the maps $f_i : V_i \longrightarrow V_{i+1}$ are linear and $\ker(f_{i+1}) = \mathrm{Im}(f_i)$, in this case the sequence (B.1) is called *short* if it is of the form $0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \to 0$, otherwise we say that (B.1) is a *long exact sequence*. In this sense, the cohomology of a differential complex (V,d) is an artifact measuring the exactness of the sequence of vector spaces $\dots \xrightarrow{\mathrm{d}} V^p \xrightarrow{\mathrm{d}} V^{p+1} \xrightarrow{\mathrm{d}} V^{p+2} \xrightarrow{\mathrm{d}} \dots$

Given two graded vector spaces $V_1 = \oplus_p V_1^p$ and $V_2 = \oplus_p V_2^p$, an *homomorphism of graded vector space* is any linear map $f : V_1 \longrightarrow V_2$ which respects the graduation i.e $f(V_1^p) \subset V_2^p$, in the case where V_i is a differential complex with differential operator d_i, then $f : V_1 \longrightarrow V_2$ is called a *complex homomorphism* it it further satisfies the relation $f \circ \mathrm{d}_1 = \mathrm{d}_2 \circ f$. It is straightforward to see that *any complex homomorphism* $f : (V_1,\mathrm{d}_1) \longrightarrow (V_2,\mathrm{d}_2)$ *induce maps* $f : \mathrm{H}^p_{\mathrm{d}_1}(V_1) \longrightarrow \mathrm{H}^p_{\mathrm{d}_2}(V_2)$ *at the cohomology level*. Finally, an exact sequence of differential complexes is any exact sequence of vector spaces provided that the involved spaces are differential complexes and the involved maps are complex homomorphisms.

THEOREM B.1.0.1 (FIVE LEMMA). *Let (V_i,d_i) be a differential complex. Any short exact sequence $0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \to 0$ of differential complexes induce the following long exact sequence in cohomology:*

$$\dots \longrightarrow \mathrm{H}^p_{\mathrm{d}_1}(V_1) \xrightarrow{f_1} \mathrm{H}^p_{\mathrm{d}_2}(V_2) \xrightarrow{f_2} \mathrm{H}^p_{\mathrm{d}_3}(V_3) \xrightarrow{\partial} \mathrm{H}^{p+1}_{\mathrm{d}_1}(V_1) \xrightarrow{f_1} \dots.$$

The boundary operator $\partial : \mathrm{H}^p_{\mathrm{d}_3}(V_3) \longrightarrow \mathrm{H}^{p+1}_{\mathrm{d}_1}(V_1)$ is given by $\partial[u]_{\mathrm{d}_3} = f_1^{-1}(\mathrm{d}_2(f_2^{-1}[u]_{\mathrm{d}_3}))$ i.e for any class $[u]_{\mathrm{d}_3} \in \mathrm{H}^p_{\mathrm{d}_3}(V_3)$, $\partial[u]_{\mathrm{d}_3} = [w]_{\mathrm{d}_1}$ such that $f_1(w) = \mathrm{d}_2 v$ and $f_2(v) = u$, the definition of $\partial[u]_{\mathrm{d}_3}$ is independant of the choice of v and w. ∂ is called the connecting homomorphism.

B.2 Generalities on De Rham Cohomology

Let M be a smooth manifold, denote $\Omega^p(M)$ the vector space of smooth p-forms on the manifold M and $\Omega^p_c(M)$ the vector space of smooth p-forms on M with compact support.

Denote $\Omega(M) = \oplus_p \Omega^p(M)$ and $\Omega_c(M) = \oplus_p \Omega_c^p(M)$, it is clear that $\Omega_c(M) \subset \Omega(M)$ with equality when M is compact. Define the linear map $\mathrm{d} : \Omega(M) \longrightarrow \Omega(M)$ by the expression:

$$\mathrm{d}\omega(X_0,..,X_p) = \sum_{i=0}^{p}(-1)^i X_i \omega(X_0,..,\hat{X}_i,..,X_p) + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,..,\hat{X}_i,..,\hat{X}_j,..,X_p).$$

Clearly $\mathrm{d}(\Omega^p(M)) \subset \Omega^{p+1}(M)$ and $\mathrm{d}^2 = 0$. The operator d is called the *de Rham differential* of the manifold M, it follows that $(\Omega(M),\mathrm{d})$ is a differential complex, i.e the *de Rham complex*, moreover $\mathrm{d}(\Omega_c(M)) \subset \Omega_c(M)$ making $\Omega_c(M)$ a subcomplex of $(\Omega(M),\mathrm{d})$. In these notations, a d-cocycle is called a *closed form* and a d-coboundary is called an *exact form*.

DEFINITION B.2.0.1. *Let M be a smooth manifold. The p-th de Rham cohomology group of the manifold M, denoted $\mathrm{H}^p_{dR}(M)$ is defined as the p-th cohomology group of the differential complex $(\Omega(M),\mathrm{d})$, the graded vector space $\mathrm{H}_{dR}(M) := \oplus_p \mathrm{H}^p_{dR}(V)$ is called the de Rham cohomology of M. Finally, the cohomology of the complex $(\Omega_c(M),\mathrm{d})$, denoted $\mathrm{H}_c(M)$ is called the cohomology with compact support on M.*

Recall that $\Omega^p(M) = 0$ for $p > n$ when M is an n-dimensional manifold, as a consequence we get that $\mathrm{H}^p_{dR}(M) = 0$ and $\mathrm{H}^p_c(M) = 0$ for $p > n$. On the other hand if M is connected any function $f \in \mathscr{C}^\infty(M)$ satisfying $\mathrm{d}f = 0$ is constant moreover if f has compact support and M is non-compact then $f = 0$, in terms of cohomology groups of M this simply states that $\mathrm{H}^0_{dR}(M) = \mathbb{R}$ and $\mathrm{H}^0_c(M) = 0$ when M is non-compact. As a consequence if M consists of a single point then $\mathrm{H}^p(M) = 0$ for $p > 0$. Now suppose that $M = \amalg_\alpha M_\alpha$ is the union of its connected components. A differential form on M is smooth if and only if it is smooth on each connected component, it follows that $\Omega^p(M) = \oplus_\alpha \Omega^p(M_\alpha)$ and $\mathrm{H}^p_{dR}(M) = \oplus_\alpha \mathrm{H}^p_{dR}(M_\alpha)$.

DEFINITION B.2.0.2. *Let M and N be smooth manifolds and let $f,g : M \longrightarrow N$ be smooth maps. Then f and g are said to be homotopic if there exists a smooth map $F : \mathbb{R} \times M \longrightarrow N$ such that for any $x \in M$:*

$$F(t,x) = f(x),\ t \leq 0 \ \textit{and}\ F(t,x) = g(x),\ t \geq 1.$$

We say that F is a proper homotopy if for any compact $K \subset N$ the subset $F^{-1}(K) \cap [0,1]$ is compact, that is $F_{|[0,1]\times M}$ is a proper map.

Any smooth map $f : M \longrightarrow N$ defines a *pull-back* operation on differential forms, i.e a linear map $f^* : \Omega(N) \longrightarrow \Omega(M)$ given by:

$$(f^*\omega)_x(v_1,\dots,v_p) := \omega_{(f(x))}(\mathrm{T}_x f(v_1),\dots,\mathrm{T}_x f(v_p)).$$

It is straightforward to check that $f^* \circ \mathrm{d} = \mathrm{d} \circ f^*$ therefore we get a well-defined induced map $f^* : \mathrm{H}_{dR}(N) \longrightarrow \mathrm{H}_{dR}(M)$. Any vector field X on M can be naturally extended over the manifold $M \times \mathbb{R}$ by setting $\hat{X} := (X, 0)$. Let $i_X : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ the *contraction* corresponding to the vector field $X \in \mathcal{X}(M)$, to any form $\eta \in \Omega^p(M \times \mathbb{R})$, one can associate the form $\xi \in \Omega^{p-1}(M)$ given by:

$$\xi_x(X_{1|x}, \dots, X_{p-1|x}) := \int_0^1 (i_{\frac{\partial}{\partial t}} \eta)_{(x,s)}(\hat{X}_{1|(x,s)}, \dots, \hat{X}_{p-1|(x,s)}) ds, \tag{B.2}$$

where $\frac{\partial}{\partial t}$ is the vector field generating $\mathrm{T}\mathbb{R}$. Let $f, g : M \longrightarrow N$ be homotopic smooth maps and denote $F : \mathbb{R} \times M \longrightarrow N$. Composing F^* with the map $\Omega^p(M \times \mathbb{R}) \mapsto \Omega^{p-1}(M)$ given by formula (B.2), one gets a linear operator $K : \Omega^p(N) \longrightarrow \Omega^{p-1}(M)$ explicitely given by:

$$(K\omega)_x(X_{1|x}, \dots, X_{p-1|x}) := \int_0^1 (i_{\frac{\partial}{\partial t}} F^* \omega)_{(x,s)}(\hat{X}_{1|(x,s)}, \dots, \hat{X}_{p-1|(x,s)}) ds. \tag{B.3}$$

We can check by a direct computation that $K(\mathrm{d}\omega) + \mathrm{d}(K\omega) = f^*\omega - g^*\omega$, more generally:

Definition B.2.0.3. *Let $f, g : (V, \mathrm{d}) \longrightarrow (W, \mathrm{d})$ be two complex homomorphisms. Call f and g chain homotopic if there exists a linear map $K : V \longrightarrow W$ such that:*

$$K \circ \mathrm{d} - \mathrm{d} \circ K = f - g.$$

Proposition B.2.0.1. *Assume $f, g : (V, \mathrm{d}) \longrightarrow (W, \mathrm{d})$ are chain homotopic complex homomorphisms, then the induced maps $f, g : \mathrm{H}_{\mathrm{d}}(V) \longrightarrow \mathrm{H}_{\mathrm{d}}(W)$ are the equal, i.e $f([u]_{\mathrm{d}}) = g([u]_{\mathrm{d}})$ for any $[u]_{\mathrm{d}} \in \mathrm{H}_{\mathrm{d}}(V)$.*

In the case of smooth manifolds M and N we have seen that homotopic maps induce a chain homotopy between the pull-backs f^* and g^* given by formula (B.3). As a consequence of proposition (B.2.0.1) we get the following result:

Theorem B.2.0.1. *Let $f, g : M \longrightarrow N$ be smooth homotopic maps between smooth manifolds M and N. Then the induced maps $f^*, g^* : \mathrm{H}_{dR}(N) \longrightarrow \mathrm{H}_{dR}(M)$ are equal.*

Recall that two manifolds M and N are said to be *homotopy equivalent* if we can find smooth maps $f : M \longrightarrow N$ and $g : N \longrightarrow M$ such that $f \circ g$ is homotopic to Id_N and $g \circ f$ is homotopic to Id_M. If $M \subset N$ we call a *retraction* any a smooth map $r : N \longrightarrow M$ such that $r \circ \iota = \mathrm{Id}_M$, if we further have that $\iota \circ r$ is homotopic to Id_N we say that M is a *retract by deformation* of N this is a particular case of a homotopy equivalence. Finally if the manifold M is a point we say that N is contractible.

Corollary B.2.0.1. *Let M and N be smooth manifolds and $f : M \longrightarrow N$ an homotopy* *equivalence, then* $f^* : \mathrm{H}_{dR}(N) \longrightarrow \mathrm{H}_{dR}(M)$ *is an isomorphism. If* M *is a retract by deformation of* N *the inclusion* $\iota : M \longrightarrow N$ *induces an isomorphism in cohomology, in particular when* M *is a point we get* $\mathrm{H}^p_{dR}(N) = 0$ *for any* $p > 0$.

Let M be a smooth n-dimensional manifold. We say that M is orientable if we can find a nowhere vanishing n-form on M.

THEOREM B.2.0.2 (POINCARÉ DUALITY). *Let* M *be a orientable smooth manifold of dimension* n. *Then the bilinear map:*

$$B : \mathrm{H}^p(M) \times \mathrm{H}^{n-p}_c(M) \longrightarrow \mathbb{R}, \quad ([\omega],[\eta]_c) \mapsto \int_M \omega \wedge \eta,$$

is non-degenerate, hence it defines an isomorphism $B^\# : \mathrm{H}^p(M) \xrightarrow{\simeq} \mathrm{H}^{n-p}_c(M)$.

Let M be a smooth manifold and let U, V be open subset of M. Consider the following sequence of differential complex homomorphisms:

$$0 \to \Omega(U \cup V) \xrightarrow{j} \Omega(U) \oplus \Omega(V) \xrightarrow{k} \Omega(U \cap V) \to 0, \tag{B.4}$$

where $j : \Omega(U \cup V) \longrightarrow \Omega(U) \oplus \Omega(V)$ and $k : \Omega(U) \oplus \Omega(V) \longrightarrow \Omega(U \cap V)$ are given by:

$$j(\omega) = \omega_{|U} \oplus \omega_{|V}, \quad k(\omega \oplus \eta) = \omega_{|U\cap V} - \eta_{|U \cap V}.$$

We can check that (B.4) is an exact sequence of differential complexes, hence the Five-lemma B.1.0.1 gives the following consequence:

THEOREM B.2.0.3 (MAYER-VIETORIS SEQUENCE). *Let* M *be a smooth manifold and choose open subsets* $U, V \subset M$, *then* (B.4) *gives rise to the following exact cohomology sequence:*

$$\ldots \mathrm{H}^p(U \cup V) \xrightarrow{j} \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \xrightarrow{k} \mathrm{H}^p(U \cap V) \xrightarrow{\partial} \mathrm{H}^{p+1}(U \cup V) \to \ldots.$$

where $\partial : \mathrm{H}^p(U \cap V) \longrightarrow \mathrm{H}^{p+1}(U \cup V)$ *is the connecting homomorphism. This sequence is called the Mayer-Vietoris sequence for de Rham cohomology.*

Let $U \subset M$ be an open subset and $\omega \in \Omega^p_c(U)$. One can naturally extend ω into a smooth p-form with compact support on any open subset $W \subset M$ containing U by setting $\omega = 0$ on $W \setminus U$. Therefore for open subsets $U, V \subset M$, this remark allows one to define complex homomorphisms $r : \Omega_c(U \cap V) \longrightarrow \Omega_c(U) \oplus \Omega_c(V)$ and $\rho : \Omega_c(U) \oplus \Omega_c(V) \longrightarrow \Omega_c(U \cup V)$ given by $r(\omega) = (\omega, -\omega)$ and $\rho(\omega, \eta) = \omega + \eta$. We check that the following sequence is exact:

$$0 \to \Omega_c(U \cap V) \xrightarrow{r} \Omega_c(U) \oplus \Omega_c(V) \xrightarrow{\rho} \Omega_c(U \cap V) \to 0. \tag{B.5}$$

Again by the Five-lemma B.1.0.1 we obtain:

THEOREM B.2.0.4 (MAYER-VIETORIS FOR COMPACT SUPPORTS). *Let M be a smooth manifold and let U,V be open subsets of M, then* (B.5) *gives the following exact cohomology sequence:*

$$\dots \mathrm{H}_c^p(U\cap V) \xrightarrow{r} \mathrm{H}_c^p(U)\oplus \mathrm{H}_c^p(V) \xrightarrow{\rho} \mathrm{H}_c^p(U\cup V) \xrightarrow{\partial} \mathrm{H}_c^{p+1}(U\cap V) \to \dots.$$

where $\partial : \mathrm{H}_c^p(U\cup V) \longrightarrow \mathrm{H}_c^{p+1}(U\cap V)$ is the connecting homomorphism. This sequence is called the Mayer-Vietoris sequence for de Rham cohomology.

B.3 Harmonic forms on compact manifolds

We devote this section to review the main notions and results of Hodge theory on compact manifolds. In this paragraph, $(M,\langle\ ,\ \rangle)$ will always denote a compact, orientable Riemannian manifold with volume element $\mathrm{d}V$. The Riemannian structure on M allows us to define a Euclidean structure on the vector bundle $\Lambda^p T^*M$ for all $p\in\mathbb{N}$ as follows:
Fix an open subset $U\subset M$ and choose a local orthonormal frame $\{E_1,\dots,E_n\}$ on U then denote $\{\varepsilon^1,\dots,\varepsilon^n\}$ its dual frame. Define on $\Lambda^p T^*M$ the Euclidean structure $(\ ,\ )$ given by:

$$(\varepsilon_I,\varepsilon_J)=\delta_{IJ}, \tag{B.6}$$

with $I=\{i_1\le\cdots\le i_p\}$ and $\varepsilon_I=\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_p}$.

PROPOSITION B.3.0.1. *Let $(M,\langle\ ,\ \rangle)$ be a Riemannian manifold. The local Euclidean structure $(\ ,\ )$ given by* (B.6) *does not depend on the choice of the local orthonormal frame, as a result it defines a global Euclidean structure on the vector bundle $\Lambda^p T^*M$ for all $p\in\mathbb{N}$.*

PROPOSITION B.3.0.2. *Let $(M,\langle\ ,\ \rangle)$ be a compact oriented Riemannian manifold with Riemannian volume element* $\mathrm{d}V$.

1. *For all $\eta\in\Lambda^pT^*M$, there exists a unique element $*\eta\in\Lambda^{n-p}T^*M$ defined by the relation $\omega\wedge *\eta=(\omega,\eta)\mathrm{d}V$, for all $\omega\in\Lambda^pT^*M$.*

2. *The map $*:\Lambda^pT^*M\longrightarrow\Lambda^{n-p}T^*M$ given by $\eta\longrightarrow *\eta$ is an isomorphism of vector bundles satisfying $**\eta=(-1)^{p(n-p)}\eta$. It is called the Hodge $*$-operator.*

3. *Let $\{\varepsilon^1,\dots,\varepsilon^n\}$ an orthonormal co-frame on M defined on some open subset U, then*

$$*(\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_p})=sgn(\sigma)\varepsilon^{j_1}\wedge\cdots\wedge\varepsilon^{j_{n-p}}.$$

with $sgn(\sigma)$ being the signature of the permutation $\sigma := (i_1,\dots,i_p,j_1,\dots,j_{n-p})$ of the set $\{1,\dots,n\}$.

DEFINITION B.3.0.1. *Let M be a compact oriented Riemannian manifold. The Laplace-Beltrami operator $\Delta : \Omega(M) \longrightarrow \Omega(M)$ is the linear map given by:*

$$\Delta = \delta \mathrm{d} + \mathrm{d}\delta.$$

where $\delta : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ called the co-differential is given by $\delta\omega = (-1)^{n(p+1)+1} * d * \omega$.

PROPOSITION B.3.0.3. *Let M be a compact oriented Riemannian manifold. For all $p \in \mathbb{N}$ the mapping $\Omega^p(M) \times \Omega^p(M) \longrightarrow \mathbb{R}$ given by:*

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta$$

defines a scalar product on $\Omega^p(M)$. It can be further extended to a scalar product on $\Omega(M)$ by declaring $\langle \Omega^p(M), \Omega^q(M) \rangle = 0$ for $p \neq q$. We denote $\|\alpha\| = \langle \alpha, \alpha \rangle^{\frac{1}{2}}$.

PROPOSITION B.3.0.4. *Let M be a compact oriented Riemannian manifold. We have the following properties:*

1. *The Laplace-Beltrami operator Δ commutes with the Hodge $*$-operator, i.e: $*\Delta = \Delta *$.*
2. *The co-differential δ and the differential d are conjugates in $(\Omega(M), \langle\ ,\ \rangle)$, more precisely:*
$$\langle \mathrm{d}\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$
3. *The Laplace-Beltrami operator Δ is self-adjoint in $(\Omega(M), \langle\ ,\ \rangle)$.*
4. *For all $\alpha \in \Omega(M)$, $\Delta\alpha = 0$ if and only if $d\alpha = 0$ et $\delta\alpha = 0$.*

DEFINITION B.3.0.2. *Let M be a compact oriented Riemannian manifold. Define the vector space $\mathcal{H}^p(M) = \{\omega \in \Omega^p(M),\ \Delta\omega = 0\}$. The elements of $\mathcal{H}^p(M)$ are called the harmonic p-forms on M.*

THEOREM B.3.0.1 (HODGE DECOMPOSITION)**.** *Let M be a compact oriented Riemannian manifold. The vector space $\mathcal{H}^p(M)$ is finite dimensional for all $0 \le p \le n$ and we have the following decomposition of $\Omega^p(M)$:*

$$\begin{aligned}
\Omega^p(M) &= \Delta(\Omega^p(M)) \overset{\perp}{\oplus} \mathcal{H}^p(M) \\
&= d\delta(\Omega^p(M)) \overset{\perp}{\oplus} \delta d(\Omega^p(M)) \overset{\perp}{\oplus} \mathcal{H}^p(M) \\
&= d(\Omega^{p-1}(M)) \overset{\perp}{\oplus} \delta(\Omega^{p+1}(M)) \overset{\perp}{\oplus} \mathcal{H}^p(M)
\end{aligned}$$

As a consequence of this theorem, we have the following results:

COROLLARY B.3.0.1. *Any cohomology class in* $\mathrm{H}^p_{dR}(M)$ *possesses a unique harmonic representative, more precisely the mapping* $\jmath : \mathscr{H}^p(M) \longrightarrow \mathrm{H}^p_{dR}(M)$ *given by* $\jmath(\omega) = [\omega]$ *is an isomorphism of vector spaces.*

COROLLARY B.3.0.2. *Let* M *be a compact orientable manifold, then its de Rham cohomology* $\mathrm{H}_{dR}(M)$ *is finite dimensional.*

Appendix C

Lattices of Nilpotent Lie groups

In this chapter

C.1 Generalities on Lattices of Topological Groups

Let G be a locally compact topological group. A subgroup H of G is said to be a *lattice* if H is discrete in G and the quotient G/H carries a finite G-invariant measure. We say that H is *uniform* in G if the quotient G/H is compact. The goal of this paragraph is to present in a unified manner the general results concerning lattices of topological groups. Assume in what follows that G is a locally compact topological group and $H \subset G$ is a closed subgroup. It is known (see [17, Th. 2.10 and 2.20]) that G carries a left *Haar measure* μ_G which is unique up to scaling by a positive factor, this corresponds to a continuous positive linear functional:

$$\mathscr{C}_c^0(G) \longrightarrow \mathbb{R}, \quad f \mapsto \int_G f(g)dg, \tag{C.1}$$

i.e a *Radon measure*. For any $x \in G$, denote $\ell_x : G \longrightarrow G$ the corresponding left group multiplication, in view of the correspondence (C.1) the left invariance of μ_G translates to the relation:

$$\int_G f\circ \ell_x(g)dg := \int_G f(x\cdot g)dg = \int_G f(g)dg,$$

for any $f\in \mathscr{C}_c^0(G)$. Denote $r_x : G \longrightarrow G$ the right multiplication by $x\in G$, then the positive linear functional:

$$\mathscr{C}_c^0(G)\longrightarrow \mathbb{R},\ \ f\mapsto \int_G f\circ r_x(g)dg = \int_G f(g\cdot x)dg,$$

defines a Borel measure on G which is left-invariant since $\ell_y\circ r_x = r_x\circ \ell_y$ hence a Haar measure on G and so by uniqueness of μ_G there exists $\Delta_G(x)\in\mathbb{R}_+$ such that:

$$\int_G f(g\cdot x)dg = \Delta_G(x)\int_G f(g)dg.$$

Continuity under integral symbol implies that $\Delta_G : G\longrightarrow \mathbb{R}_+$ is a continuous function called the *modular function* of G, it is clearly independant of the choice of the Haar measure μ_G. Moreover if we choose $f\in C_c^0(G)$ such that $\int_G f(g)dg \neq 0$, then using the relations $r_{e_G} = \mathrm{Id}_G$ and $r_{xy^{-1}} = r_x\circ r_y^{-1}$ we obtain that:

$$\Delta_G(e_G) = 1 \ \text{ and } \ \Delta_G(xy^{-1}) = \Delta_G(x)\Delta_G(y)^{-1}.$$

Hence $\Delta_G : G\longrightarrow \mathbb{R}_+$ is a group homomorphism into the multiplicative group $(\mathbb{R}_+,\times)$ i.e *a character*. Likewise, H being a locally compact topological group in its own right, it does admit a left Haar measure μ_H and a modular function Δ_H. The group G is said to be *unimodular* if $\Delta_G = 1$. Let μ be a Borel measure on the homogeneous space G/H and choose a continuous homomorphism $\chi : G\longrightarrow \mathbb{R}_+$. Then μ is said to be *semi G-invariant with character* χ if for every $g\in G$ and any measurable subset $E\subset G/H$ we have:

$$\mu(g\cdot E) = \chi(g)\mu(E).$$

THEOREM C.1.0.1. *Let G be a locally compact topological group and H a closed subgroup of G. The homogeneous space G/H admits a semi G-invariant measure if and only if the homomorphism* $\Delta_G\Delta_H^{-1} : H\longrightarrow \mathbb{R}^+$ *can be extended to a continuous homomorphism on all of G. Moreover, given any homomorphism* $u : G\longrightarrow \mathbb{R}_+$ *such that* $u_{|H} = \Delta_G\Delta_H^{-1}$*, then G/H admits a semi G-invariant Borel measure with character u and this measure is unique up to a scalar multiple.*

A semi G-invariant measure μ on G/H is said to be G-invariant if it corresponds to the trivial character 1. Now assume that G/H admits a finite semi G-invariant measure μ with character χ, then choosing a function $f \in \mathscr{C}_c^0(G/H)$ with $\int_{G/H} f(x)d\mu(x) \neq 0$ one obtains for any $g \in G$:

$$\chi(g)\left|\int_{G/H} f(x)d\mu(x)\right| = \left|\int_{G/H} f(g\cdot x)d\mu(x)\right| \leq \|f\|_\infty \left|\int_{G/H} d\mu(x)\right|.$$

It follows that χ is a bounded group homomorphism, which is only possible if χ is trivial. On the other hand when H is discrete in G, the Borel measure on H defined by the linear functional:

$$\mathscr{C}_c^0(H) \longrightarrow \mathbb{R},\ \ f \mapsto \sum_{h\in H} f(h),$$

is a Haar measure on H which is also right invariant, hence $\Delta_H = 1$ in this case. It follows that if H is a lattice in G then G/H admits a finite semi G-invariant measure with obvious character Δ_G, and by the finiteness of the measure the preceding remark leads to $\Delta_G = 1$. So in summary:

COROLLARY C.1.0.1. *Let G be a locally compact group. If G admits a lattice then it is unimodular*

In particular all compact groups are unimodular. For a Lie group G, one can speak of smooth functions and since $\mathscr{C}_c^\infty(G)$ is dense in $\mathscr{C}_c^0(G)$ any Haar measure $\mathscr{C}_c^0(G) \longrightarrow \mathbb{R}$ is completely determined by its behavior on $\mathscr{C}_c^\infty(G)$. Therefore in order to construct a Haar measur on a Lie group G, we fix an orientation on G then consider a left-invariant volume form ω on G corresponding to this orientation (which is entirely determined by a generator of $\bigwedge^n(\mathfrak{g}^*)$ with $\mathfrak{g} = \mathrm{Lie}(G)$ and $n = \dim(G)$). The Haar measure on G is then given by the usual integration operator:

$$\mathscr{C}_c^\infty(G) \longrightarrow \mathbb{R},\ \ f \mapsto \int_G f\omega.$$

Since the modular function Δ_G is a continuous group homomorphism, it is automatically smooth and in fact for any $x \in G$:

$$\int_G (f\circ r_x)\omega = \int_G r_x^*(f r_{x^{-1}}^*\omega) = \int_G r_x^*(f c_{x^{-1}}^*\omega) = \det(\mathrm{Ad}_{x^{-1}})\int_G f\omega.$$

Thus $\Delta_G(x) = \det(\mathrm{Ad}_{x^{-1}})$ for any $x \in G$, and the relation $\det\exp(A) = e^{\mathrm{tr}(A)}$ leads to the following result:

COROLLARY C.1.0.2. *Let G be a connected Lie group with Lie algebra* $\mathfrak{g}$. *If G admits a lattice then* $\mathrm{tr}(\mathrm{ad}_X) = 0$ *for any* $X \in \mathfrak{g}$.

C.2 Generalities on Nilpotent Lie Groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g,h \in G$ set $[g,h] := ghg^{-1}h^{-1}$. Let A,B be subsets of G and denote $[A,B]$ the subgroup of G generated by $[g,h]$ where $g \in A$, $h \in B$. The *derived series* of G (resp. of $\mathfrak{g}$) is the sequence of normal subgroups of G (resp. ideals of $\mathfrak{g}$) defined by:

$$\mathscr{D}^0(G) := G,\ \mathscr{D}^k(G) := [\mathscr{D}^{k-1}(G),\mathscr{D}^{k-1}(G)]\ (\text{resp. } \mathscr{D}^0(\mathfrak{g}) := \mathfrak{g},\ \mathscr{D}^k(\mathfrak{g}) := [\mathscr{D}^{k-1}(\mathfrak{g}),\mathscr{D}^{k-1}(\mathfrak{g})]).$$

Similarly, the sequence of normal subgroups of G (resp. of ideals of $\mathfrak{g}$):

$$\mathscr{C}^0(G) := G,\ \mathscr{C}^k(G) := [\mathscr{C}^{k-1}(G),G]\ (\text{resp. } \mathscr{C}^0(\mathfrak{g}) := \mathfrak{g},\ \mathscr{C}^k(\mathfrak{g}) := [\mathscr{C}^{k-1}(\mathfrak{g}),\mathfrak{g}]),$$

is the *descending central series* of G (resp. of $\mathfrak{g}$). Clearly $\mathscr{D}^k(G) \subset \mathscr{C}^k(G)$ and $\mathscr{D}^k(\mathfrak{g}) \subset \mathscr{C}^k(\mathfrak{g})$.

DEFINITION C.2.0.1. *A Lie group G (resp. Lie algebra $\mathfrak{g}$) is called nilpotent if its descending central series is trivial.*

PROPOSITION C.2.0.1. *A connected Lie group is nilpotent if and only if its Lie algebra is nilpotent.*

Let G be a connected nilpotent Lie group and denote $\mathfrak{g}$ its Lie algebra. Then the polynomial product $* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by Campbell-Dynkin-Baker-Hausdorff formula:

$$X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots$$

defines a simply connected nilpotent Lie group structure $(\mathfrak{g},*)$ with exponential map given by $\exp_{\mathfrak{g}} = \mathrm{Id}_{\mathfrak{g}}$ and Lie algebra $\mathrm{Lie}(\mathfrak{g},*) = \mathfrak{g}$, moreover the center $\mathrm{Z}(\mathfrak{g},*)$ of the group $(\mathfrak{g},*)$ coincides with the center $\mathrm{Z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The exponential map $\exp_G : \mathfrak{g} \longrightarrow G$ is the universal covering morphism of G and in particular it is surjective and therefore the fundamental group $\pi_1(G)$ of G can be identified with the subgroup $\ker(\exp_G) \subset \mathrm{Z}(\mathfrak{g},*)$. For any $x, y \in \ker(\exp_G)$ we can check that $x + y = x * y$, so $\ker(\exp_G)$ can also be seen as a discrete subgroup of the additive group $(\mathfrak{g},+)$. Hence there exists an integer r, that will be called the rank of $\pi_1(G)$ and denoted by $\mathrm{rank}(\pi_1(\mathrm{G}))$, such that $\pi_1(G) \cong \mathbb{Z}^r$.

PROPOSITION C.2.0.2. *Any nilpotent Lie group is unimodular.*

C.3 Lattices of Nilpotent Lie Groups

In order to state results concerning lattices of nilpotent Lie groups, we need to introduce some preliminary notions from algebraic geometry:

Given a polynomial ideal $\mathrm{I} \subset \mathbb{C}[X_1,\ldots,X_n]$ we define $\mathcal{Z}(\mathrm{I}) \subset \mathbb{C}^n$ to be the set of zeroes of polynomial elements in I. A subset $A \subset \mathbb{C}^n$ is said to be *algebraic* in $\mathbb{C}^n$ if $A = \mathcal{Z}(\mathrm{I})$ for some polynomial ideal I. We can check that the family of all algebraic subsets of $\mathbb{C}^n$ forms the closed (*Zariski-closed*) sets of a topology on $\mathbb{C}^n$ called the *Zariski topology*. Let $A \subset \mathbb{C}^n$ be any subset and denote $\mathrm{I}(A)$ the ideal consisting of polynomials P on $\mathbb{C}^n$ such that $P_{|A} = 0$. The *Zariski closure* of A in $\mathbb{C}^n$ is exactly the subset $B = \mathcal{Z}(\mathrm{I}(A))$. Finally, any algebraic subset of $\mathbb{C}^n$ with the induced Zariski topology is referred to as an *affine algebraic variety*.

Let us give $\mathrm{GL}(n,\mathbb{C})$ a structure of affine algebraic variety using the following remark: Let $g := (g_{ij})_{i,j}$ be any element of $\mathrm{GL}(n,\mathbb{C})$, then the condition $\det(g) \neq 0$ is equivalent to stating that $y\det(g) - 1 = 0$ for some $y \in \mathbb{C}$ which we may rewrite as the polynomial equation $P(y,g) = 0$ with $P \in \mathbb{C}[Y,(X_{ij})_{i,j}]$. Now consider the imbedding:

$$\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathbb{C}^{n^2+1}, \quad g \mapsto (g, \det(g)^{-1}).$$

Use this imbedding to identify $\mathrm{GL}(n,\mathbb{C})$ with its image, it is clear by the preceding remark that $\mathrm{GL}(n,\mathbb{C}) = \mathcal{Z}(P)$ thus $\mathrm{GL}(n,\mathbb{C})$ can be given the structure of an affine algebraic variety. A *linear algebraic group* is any Zariski-closed subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$. Since linear algebraic groups are zeroes of polynomials, they are therefore Lie groups for the usual topology of $\mathrm{GL}(n,\mathbb{C})$.
A matrix $A \in \mathrm{GL}(n,\mathbb{C})$ is said to be *unipotent* if $(A - I_n)^n = 0$. Let $\mathrm{U}_n(\mathbb{C})$ be the set of all unipotent matrices, then $\mathrm{U}_n(\mathbb{C})$ is an algebraic subgroup of $\mathrm{GL}(n,\mathbb{C})$, its Lie algebra $\mathfrak{u}$ is the set of all nilpotent elements of $\mathfrak{gl}(n,\mathbb{C})$ and the exponential map:

$$\exp : \mathfrak{u} \longrightarrow \mathrm{U}_n(\mathbb{C}),$$

is a diffeomorphism and in fact a polynomial map. Any subgroup of $\mathrm{U}_n(\mathbb{C})$ will be called a *unipotent* group. The following theorem determines when a closed subgroup of a simply connected nilpotent Lie group is a lattice. The proof of this theorem and all the results in this paragraph can be found in [45].

THEOREM C.3.0.1. *Let N be a simply connected nilpotent Lie group and $\Gamma \subset N$ a closed subgroup. The following assertions are equivalent:*

1. *There exists a faithful unipotent representation $\rho : N \longrightarrow \mathrm{GL}(n,\mathbb{R})$ such that $\rho(N)$ and $\rho(\Gamma)$ have the same Zariski closure in $\mathrm{GL}(n,\mathbb{C})$.*
2. *The quotient group N/Γ is compact.*
3. *The quotient group N/Γ carries a finite invariant measure.*
4. *There are no proper connected closed subgroups of N containing Γ.*
5. *For any faithful unipotent representation $\rho : N \longrightarrow \mathrm{GL}(n,\mathbb{R})$, $\rho(N)$ and $\rho(\Gamma)$ have the same Zariski closure.*

Next we state some straightforward consequences of this Theorem:

COROLLARY C.3.0.1. *Let H be a uniform subgroup of a simply connected nilpotent Lie group N then $H \cap \mathscr{D}^k(N)$ is a uniform subgroup of $\mathscr{D}^k(N)$ and $H \cap \mathscr{C}^k(N)$ is a uniform subgroup of $\mathscr{C}^k(N)$.*

PROPOSITION C.3.0.1. *Let N be a simply connected nilpotent Lie group and $H \subset N$ a closed uniform subgroup. A connected Lie group U of N is normal in N if and only if it is normalized by H.*

COROLLARY C.3.0.2. *Let H be a closed uniform subgroup of a nilpotent Lie group N and let H^0 be the identity component of H. Then H^0 is a normal subgroup of N*

The next proposition expresses what the Zariski closure of a subgroup H in a simply connected Lie group N means in terms of the Lie group topology without requiring an external algebraic group.

PROPOSITION C.3.0.2. *Let N be a simply connected nilpotent Lie group and H any subgroup. Then H is contained in a unique minimal connected closed subgroup $\tilde{H}$ of N. If furthermore H is closed in N then $\tilde{H}/H$ is compact.*

In what follows, the Zariski closure of a closed subgroup H in a simply connected nilpotent Lie group N is defined to be the smallest closed connected subgroup $\tilde{H} \subset N$ containing H. We now state some results about the structure of lattices of nilpotent Lie groups:

THEOREM C.3.0.2. *Any subgroup of a finitely generated nilpotent group is finitely generated.*

Let Γ be a finitely generated nilpotent group, we call a *filtration* of Γ any sequence of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$ such that $\Gamma_i \lhd \Gamma_{i-1}$ and Γ_{i-1}/Γ_i is abelian. Such a filtration always exists and we define the *rank* of Γ to be the integer:

$$\mathrm{rg}(\Gamma) := \sum_{i=1}^{k} \mathrm{rg}(\Gamma_{i-1}/\Gamma_i).$$

This integer is independant of the choice of the filtration.

THEOREM C.3.0.3. *Let N be a simply connected nilpotent Lie group and Γ a discrete subgroup of N. Denote $\tilde{\Gamma}$ the Zariski closure of Γ in N, then Γ is finitely generated and:*

$$\mathrm{rg}(\Gamma) = \dim(\tilde{\Gamma}).$$

COROLLARY C.3.0.3. *Let N be a nilpotent simply connected Lie group and H a closed subgroup. Let H^0 be the identity component of H and $\tilde{H}$ the Zariski closure of H in N. Then:*

$$\dim \tilde{H} = \dim H + \mathrm{rg}(H/H^0).$$

COROLLARY C.3.0.4. *Let N be a connected nilpotent Lie group (not necessarily simply connected) and let $\Gamma \subset N$ be any discrete subgroup. Then Γ is finitely generated.*

The next set of results shows how a homomorphism of simply connected nilpotent Lie groups is completely determined by its behavior on a lattice. As a consequence, two simply connected nilpotent Lie groups giving rise to the same nilmanifolds (quotient by a lattice) are the same.

THEOREM C.3.0.4. *Let N and V be two nilpotent simply connected groups and let H be a uniform subgroup of N. Then any continuous homomorphism $\rho : H \longrightarrow \mathrm{V}$ can be extended in a unique manner to a continuous (hence smooth) homomorphism $\tilde{\rho} : N \longrightarrow \mathrm{V}$.*

COROLLARY C.3.0.5. *Let N be a simply connected nilpotent Lie group and $H \subset N$ a closed uniform subgroup. Then any automorphism of H extends to a unique automorphism of N.*

COROLLARY C.3.0.6. *Let N_1 and N_2 be simply connected nilpotent Lie groups and H_1, H_2 uniform closed subgroups of N_1 and N_2 respectively. Any isomorphism of H_1 on H_2 extends to an isomorphism of N_1 on N_2. In particular, N_1/H_1 and N_2/H_2 are homeomorphic. Conversely if H_1 and H_2 are uniform lattices such that N_1/H_1 and N_2/H_2 are homeomorphic then N_1 and N_2 are isomorphic.*

We end this paragraph with the famous Malcev criterion which allows to decide whether a simply connected nilpotent Lie group admits a lattice just by looking at its Lie algebra structure.

THEOREM C.3.0.5 (MALCEV)**.** *Let N be a simply connected Lie group and denote $\mathfrak{n}$ its Lie algebra. Then N admits a lattice if and only if $\mathfrak{n}$ admits a basis with respect to which the constants of structure are rational.*

A more precise statement of the preceding theorem is the following: Let $\mathfrak{n}$ be a nilpotent Lie algebra with respect to which the constants of structure are rational. Let $\mathfrak{n}_0$ be the $\mathbb{Q}$-vector space spanned by this basis, if $\mathscr{L}$ is a lattice of maximal rank in $(\mathfrak{n},+)$ contained in $\mathfrak{n}_0$ then the group generated by $\exp\mathscr{L}$ is a lattice in N. Conversely if Γ is a lattice in N, then the $\mathbb{Z}$-span of $\exp^{-1}\Gamma$ is a lattice of maximal rank in $(\mathfrak{n},+)$ such that the constants of structure of $\mathfrak{n}$ with respect to any basis in $\mathscr{L} := \mathrm{span}_{\mathbb{Z}}(\exp^{-1}\Gamma)$ are rational. We conclude with two important consequences of Malcev's Theorem, but first some terminology:
The *ascending central series* of a Lie group G is the family of ideals $(\mathscr{C}'_k(G))_k$ defined inductively by $\mathscr{C}'_0(G) = \{e_G\}$ and $\mathscr{C}'_k(G) = \pi^{-1}(\mathrm{Z}(G/\mathscr{C}'_{k-1}(G)))$ where $\pi : G \longrightarrow G/\mathscr{C}'_{k-1}(G)$ is the natural projection. Explicitly, we get that:

$$\mathscr{C}'_k(G) = \{g \in G,\ ghg^{-1}h^{-1} \in \mathscr{C}'_{k-1}(G) \ \text{ for any } h \in G\}.$$

In particular, $\mathscr{C}'_1(G) = \mathrm{Z}(G)$ and it can be shown that the Lie group G is nilpotent if and only if the sequence $(\mathscr{C}'_k(G))_k$ reaches G i.e $\mathscr{C}'_k(G) = G$ for k large enough.

COROLLARY C.3.0.7. *Let N be a simply connected nilpotent Lie group and $\Gamma \subset N$ a lattice. Then $\mathscr{C}'_k(N) \cap \Gamma$ is a lattice in $\mathscr{C}'_k(N)$, in particular the intersection of Γ with the center of N is a lattice of the center.*

THEOREM C.3.0.6. *A group Γ is isomorphic to a lattice in a simply connected nilpotent Lie group if and only if Γ is nilpotent, torsion-free and finitely generated.*

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