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Abstract

This thesis falls under the theme of pseudo-Riemannian geometry in the setting of Lie groups. Its purpose it to present a number of results on left-invariant Einstein Lorentzian metrics on nilpotent Lie groups as an extension of the well-known classical Riemannian case. The content of this thesis fits nicely in the subject literature since most of its results complete previous works that were initiated by many authors, some of which are even generalizations of earlier studies into broader contexts. The general outline can be divided into two major parts:
The first part is concerned with the preliminaries of our study of Lorentzian left-invariant Einstein metrics on nilpotent Lie groups. The main theorem states that if the center of such a Lie group is degenerate then it must be Ricci-flat and its Lie algebra can be obtained by the double extension process from an abelian Euclidean Lie algebra. We also show that all nilpotent Lie groups up to dimension 5 endowed with a Lorentzian Einstein left-invariant metric have degenerate center and we use this fact to give a complete classification of these metrics. We show that if $\mathfrak{g}$ is the Lie algebra of a nilpotent Lie group endowed with a Lorentzian left-invariant Einstein metric with non-zero scalar curvature then the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is non-degenerate Euclidean, the derived ideal $[\mathfrak{g},\mathfrak{g}]$ is non-degenerate Lorentzian and $Z(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$. We give the first examples of Ricci-flat Lorentzian nilpotent Lie algebra with non-degenerate center. The results in this part have been published in [21]. The second part can be seen as a starting point for the study of Einstein Lorentzian nilpotent Lie groups with non-degenerate center as it carries over the machinery previously developed in order to treat the case of 3-step nilpotent Lie groups. The principal theorem of this part is the classification of all Einstein Lorentzian 3-step nilpotent Lie groups with 1-dimensional non-degenerate center, its proof, while long and somewhat difficult, gives insight into many different properties and aspects that were not apparent before, and the techniques used for the proof seem promising for a future inspection. The material laid out in this part was published in [16].

Keywords: Lorentzian manifolds, Nilpotent Lie groups, Nilpotent Lie algebras, Flat manifolds, Ricci curvature, Einstein metrics...

Résumé

Cette thèse se situe dans le cadre des groupes de Lie pseudo-Riemanniens. Son objectif est de présenter un nombre de résultats autour des métriques Lorentziennes d'Einstein sur les groupes de Lie nilpotents comme une extension du cas Riemannien classique. Le contenu de cette thèse est bien placé dans la litérature mathématique puisque la majorité des résultats mis en évidence donnent un contexte plus large à des travaux déjà initiés par plusieurs auteurs. Cette thèse comporte deux parties majeures :
La première partie concerne les préliminaires de notre étude des métriques Lorentziennes d'Einstein invariantes à gauche sur les groupes de Lie nilpotents. Le théorème central affirme que si le centre d'un tel groupe de Lie est dégénéré alors il est forcément Ricci-plat et son algèbre de Lie peut être obtenue par le procédé de double extension à partir d'une algèbre de Lie Abélienne Euclidienne. On montre aussi que tous les groupes de Lie dimension inférieure ou égale à 5 munis d'une métrique Lorentzienne d'Einstein invariante à gauche possèdent un centre dégénéré, nous utilisons ce fait pour donner une classification complète de ces métriques. On montre que si $\mathfrak{g}$ est l'algèbre de Lie d'un groupe de Lie nilpotent qui est munit d'une métrique Lorentzienne d'Einstein invariante à gauche de courbure scalaire non nulle, alors le centre $Z(\mathfrak{g})$ de $\mathfrak{g}$ est non dégénéré Euclidien, son idéal dérivé $[\mathfrak{g},\mathfrak{g}]$ est non dégénéré Lorentzien et $Z(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$. Les résultats de cette partie ont été publiés dans [21]. La deuxième partie peut être vue comme le début de l'étude des groupes de Lie nilpotents Lorentziens d'Einstein dont le centre est non dégénéré, nous utilisons ici la même machine précedemment développée afin de traiter le cas des groupes de Lie 3-step nilpotents. Le théorème principal de cette partie est la classification de tous les groupes de Lie 3-step nilpotents Lorentziens d'Einstein de centre unidimensionel non dégénéré. La preuve de ce thèorème permet d'éclaircir de nouveaux aspects de l'étude globale et les techniques utilisés permettent de s'ouvrire sur de nouvelles perspectives. Le contenu de cette partie a fait l'objet de [16].

Mots-clé : Variétés lorentziennes, Groupes de Lie nilpotents, Algèbres de Lie nilpotentes, Variétés plates, Courbure de Ricci, Métriques d'Einstein...

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1

Introduction

1.1 Historical Notes

The study of left-invariant, pseudo-Riemannian Lie groups, and more generally left-invariant structures on Lie groups, is a central topic of differential geometry that has attracted the interest of many mathematicians over the past few decades, primarily because it allows to bring difficult geometric problems into a more approachable setting allowing to deal with a number of issues from an algebraic standpoint.

A smooth structure on a differentiable manifold is in many instances described by a tensor field, and so questions concerning the existence of certain structures with specific properties can be formulated through partial differential equations that must be satisfied by the coefficients of the corresponding tensor field relatively to some local coordinate system, however deciding whether a given partial differential equation admits a solution is not an easy task in general and therefore not much can be said about the situation at hand. In contrast, when the underlying manifold is a Lie group and the structure is left-invariant, the problem can be entirely expressed in terms of a system of linear or quadratic equations at the Lie algebra level. This is especially true for questions concerning the existence of pseudo-riemannian metrics with certain curvature requirements. This approach turned out to be very efficient when looking for examples or counter-examples, and in many situations, it even provided key insights that served to build arguments for the general case.

1.2 Research Highlights

The purpose of this thesis was to investigate left-invariant Einstein Lorentzian metrics on nilpotent Lie groups, this was done with the collaboration and supervision of Professor Boucetta Mohamed and the intent was to develop a set of results that could potentially lead to the classification of these structures. A classical result in this subject is due to Milnor and deals with the Riemannian case, it is stated as follows:

Theorem 1.2.1 ([19], Theorem 2.4)**.** *Any left-invariant Riemannian metric on a nilpotent non-abelian connected Lie group has a direction of strictly positive Ricci curvature and a direction of strictly negative Ricci curvature.*

An obvious consequence of this result is that *the only nilpotent Lie groups that can be equipped with a left-invariant, Einstein Riemannian metric are abelian groups*. The indefinite case however, is highly non-trivial with only few known examples (see for instance [6]), and it is mainly for this reason that we set out to improve on the current state of the art. The first stage of the inspection was based on papers due to M. Boucetta (see [5]) and M. Guediri & M. Bin Asfour (see [18]) which settled the case for 2-step nilpotent Lie groups, the main results in these papers are stated as follows:

Theorem 1.2.2 ([18], Lemma 14)**.** *Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be a Ricci-flat Lorentzian non abelian 2-step nilpotent Lie algebra. Then $Z(\mathfrak{g})$ is degenerate.*

Theorem 1.2.3 ([5], Proposition 3.4)**.** *Any 2-step nilpotent, pseudo-Euclidean Einstein Lie algebra must be Ricci-flat.*

Theorem 1.2.4 ([18], Theorem 15)**.** *Let $\mathfrak{g}$ be any 2-step nilpotent, non-abelian Lie algebra. Then $\mathfrak{g}$ admits a Ricci-flat Lorentzian metric if and only if $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{n}$ i.e a direct sum of an abelian Lie algebra and a nilpotent Lie algebra $\mathfrak{n}$ such that the Lie brackets of $\mathfrak{n}$ are expressed relatively to a basis $\mathscr{B} = \{e, z_1, \ldots, z_p, \bar{e}, e_1, \ldots, e_q\}$ as follows:*

$$[\bar{e}, e_i] = \alpha_i e + \sum_{k=1}^{p} c_{ik} z_k, \quad [e_i, e_j] = a_{ij} e, \quad 1 \leq i, j \leq q, \tag{1.1}$$

with $\sum_{i,j=1}^{q} a_{ij}^2 = 2\sum_{i=1}^{q}\sum_{k=1}^{p} c_{ik}^2$. Moreover the basis $\mathscr{B}$ can be chosen Lorentzian, in particular the restriction of the metric to $[\mathfrak{g},\mathfrak{g}]$ is degenerate.

It was observed in these works that there was a certain interplay between the Einstein aspect of the metric and the degeneracy of the center of the Lie group. Following these steps, the goal of our first paper was to look further into this relationship and its implications in the case of general nilpotent Lie groups, this has led us to give a detailed description of the structure of Einstein Lorentzian nilpotent Lie groups with degenerate center, generalizing therefore a central result in [18], and ultimately classifying all Einstein Lorentzian nilpotent Lie groups of dimension less than 5 (see [21] for details).
The second part of our research focused on Einstein Lorentzian nilpotent Lie groups with non-degenerate center, particularly the 3-step nilpotent case, we gave the first known example of such Lie groups in [21], disproving in the process a long-standing conjecture due to M. Boucetta which stated that *every Einstein Lorentzian nilpotent Lie group has a degenerate center*. The central result of this part was the classification of all such Lie groups when the center is 1-dimensional (for details, see [16]).

1.3 Papers Outline

Let $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle)$ be a *pseudo-Euclidean Lie algebra*, i.e, a Lie algebra endowed with a pseudo-Euclidean product. The *Levi-Civita product* of $\mathfrak{h}$ is the bilinear map $\mathrm{L} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ given by Koszul's formula

$$2\langle \mathrm{L}_u v, w\rangle = \langle [u,v], w\rangle + \langle [w,u], v\rangle + \langle [w,v], u\rangle. \tag{1.2}$$

For any $u, v \in \mathfrak{h}$, $\mathrm{L}_u : \mathfrak{h} \longrightarrow \mathfrak{h}$ is skew-symmetric and $[u,v] = \mathrm{L}_u v - \mathrm{L}_v u$. The curvature of $\mathfrak{h}$ is given by

$$K(u,v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v].$$

The Ricci curvature $\mathrm{ric} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{R}$ and its Ricci operator $\mathrm{Ric} : \mathfrak{h} \longrightarrow \mathfrak{h}$ are defined by

$$\langle \mathrm{Ric}(u), v\rangle = \mathrm{ric}(u,v) = \mathrm{tr}\,(w \longrightarrow K(u,w)v).$$

A pseudo-Euclidean Lie algebra is called *flat* (resp. *Ricci-flat*) if $K = 0$ (resp. $\mathrm{ric} = 0$). It is called λ-Einstein if there exists a constant $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{h}}$. In the case of nilpotent Lie algebras, the Ricci curvature is given by:

$$\mathrm{ric}(u,v) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(J_u \circ J_v), \tag{1.3}$$

where J_u is the skew-symmetric endomorphism given by $J_u(v) = \mathrm{ad}_v^* u$. Moreover, if $\mathcal{J}_1$ and $\mathcal{J}_2$ denote the symmetric endomorphisms given by

$$\langle \mathcal{J}_1 u, v\rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*),\ \langle \mathcal{J}_2 u, v\rangle = -\mathrm{tr}(J_u \circ J_v) = \mathrm{tr}(J_u \circ J_v^*). \tag{1.4}$$

then the Ricci operator has the following expression

$$\mathrm{Ric} = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2, \tag{1.5}$$

On Einstein Lorentzian nilpotent Lie groups [21]. The goal of this work was to give a description of Einstein Lorentzian nilpotent Lie groups with degenerate center, following the lines of a study that was initiated by M. Boucetta in [5]. As it is the case for left-invariant structures in general, the problem can be entirely treated at the Lie algebra level without any reference to the group in question. The main theorem of this paper states that any *Einstein Lorentzian nilpotent Lie algebra with degenerate center is Ricci-flat and can be obtained by a double extension from a Euclidean vector space with prescribed parameters*, a rigorous account of the double extension process can be found in [3] and its adaptation to our situation was discussed in details in [21]. The official statement of the Theorem is as follows:

Theorem 1.3.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein nilpotent non abelian Lorentzian Lie algebra and suppose that there exists $e \in Z(\mathfrak{g})$ a central isotropic vector and denote $\mathcal{I} = \mathbb{R}e$. Then:*

1. *$Z(\mathfrak{g})$ is degenerate and $\lambda = 0$.*

2. *$\mathcal{I}^\perp$ is an ideal and $\mathfrak{g}_0 = \mathcal{I}^\perp/\mathcal{I}$ is a Euclidean abelian Lie algebra.*

3. *$\mathfrak{g}$ is obtained from $\mathfrak{g}_0$ by the double extension process with admissible data $(K, D, 0, b)$ and D is nilpotent.*

This theorem generalizes a result of M. Guediri & M. Bin Asfour that deals with the 2-step nilpotent case ([18], Theorem 15). An important stream of thoughts that was dominant throughout the paper was to bring the influence of the Ricci curvature over the metric nature of the center and the derived ideal (either degenerate or non-degenerate), first found in [5], into a broader setting, this set of results can be summarized as follows:

Theorem 1.3.2. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent non abelian Lie algebra.*

1. *If $[\mathfrak{g}, \mathfrak{g}]$ is non degenerate then it is Lorentzian.*

2. *If $Z(\mathfrak{g})$ is nondegenerate then it is Euclidean.*

3. *$[\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp \subset Z(\mathfrak{g})$ and if $[\mathfrak{g}, \mathfrak{g}]$ is degenerate then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is Ricci flat.*

The following result, first proved in [18], is obtained as a corollary of the previous theorem:

Corollary 1.3.1. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be an Einstein Lorentzian non abelian 2-step nilpotent Lie algebra. Then $Z(\mathfrak{g})$ is degenerate.*

In the same spirit, we were able to prove the following result that can be perceived as a slight improvement on Corollary 1.3.1, but which turned out to be essential for the upcoming developement:

Proposition 1.3.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a Ricci flat Lorentzian nilpotent non abelian Lie algebra such that* $\dim[\mathfrak{g}, \mathfrak{g}] = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) + 1$. *Then $Z(\mathfrak{g})$ is degenerate.*

By combining Theorems 1.3.2 and 1.3.1 and Proposition 1.3.1, we were rewarded with a complete classification of Einstein Lorentzian nilpotent Lie algebras of dimension less 5 (see Theorems 2.5.2 and 2.5.3 in Chapter 2 for the exact statement).

Classification of Einstein Lorentzian 3-nilpotent Lie groups with 1-dimensional non-degenerate center [16]. This work can be considered as a continuation of [21] and is concerned with the study of Einstein Lorentzian nilpotent Lie groups with non-degenerate center. This class of Lie groups is very large and contains in particular Einstein Lorentzian nilpotent Lie groups with nonvanishing scalar curvature (Theorem 1.3.1), the minimal dimension required for this phenomenon to occur is 6 and the first known example of such Lie groups was given in [21] (see Example 1 in Chapter 2). As a first step towards a general study, we focus on the 3-step nilpotent setting, the first main theorem of the paper can be stated as follows:

Theorem 1.3.3. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian 3-step nilpotent Lie algebra with nondegenerate center. Then $\lambda \geq 0$.*

By restricting to the case where the center is 1-dimensional, we were able to give a full classification of these Lie groups, which surprisingly enough, are shown to only exist in dimensions 6 and 7, the precise statement is as follows:

Theorem 1.3.4. *Let $\mathfrak{h}$ be a 3-step nilpotent Lie algebra with $\dim \mathrm{Z}(\mathfrak{h}) = 1$. Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathfrak{h}$ such that $\mathrm{Z}(\mathfrak{h})$ is non-degenerate, then $\langle\ ,\ \rangle$ is Einstein if and only if it is Ricci-flat and $(\mathfrak{h}, \langle\ ,\ \rangle)$ has one of the following forms :*

(i) $\dim \mathfrak{h} = 6$ *and $\mathfrak{h}$ is isomorphic to $\mathrm{L}_{6,19}(-1)$, i.e., $\mathfrak{h}$ has a basis $(f_i)_{i=1}^6$ such that the non vanishing Lie brackets are*

$$[f_1, f_2] = f_4, [f_1, f_3] = f_5, [f_2, f_4] = f_6, [f_3, f_5] = -f_6$$

and the metric is given by :

$$\langle\ ,\ \rangle := f_1^* \otimes f_1^* + 2f_2^* \otimes f_2^* + 2f_3^* \otimes f_3^* + 4\alpha^4 f_6^* \otimes f_6^* - 2\alpha^2 f_4^* \odot f_5^*, \quad \alpha \neq 0. \tag{1.6}$$

(ii) $\dim \mathfrak{h} = 7$ *and $\mathfrak{h}$ is isomorphic to the nilpotent Lie algebras $147E$ found in the classification given in [8](p. 57). In precise terms, there exists a basis $\{f_i\}_{i=1}^7$ of $\mathfrak{h}$ where the non vanishing Lie bracket are given by :*

$$[f_1, f_2] = f_5,\ [f_1, f_3] = f_6,\ [f_2, f_3] = f_4,\ [f_6, f_2] = (1-r)f_7,\ [f_5, f_3] = -rf_7,\ [f_4, f_1] = f_7, \tag{1.7}$$

with $0 < r < 1$, and the metric has the form:

$$\langle\ ,\ \rangle = f_1^* \otimes f_1^* + f_2^* \otimes f_2^* + f_3^* \otimes f_3^* - a f_4^* \otimes f_4^* + ar f_5^* \otimes f_5^* + a(1-r) f_6^* \otimes f_6^* + a^2 f_7^* \otimes f_7^*, \quad a > 0. \tag{1.8}$$

1.4 Future research

While the results obtained in the course of this thesis may set the ground for any future inquiry on the subject, there is still room for more elaborate arguments and methods, and as it is usually the case with research, one ends up with more questions than answers. We name here a few that we think are relevant for any further developement on the matter and might even be at the heart of some paper down the line:

Question 1: *Is there a complete classification of Einstein, Lorentzian 3-step nilpotent Lie algebras with non-degenerate center, similar to Theorem 3.1.1 ?*

We think that this is a legitimate question and a natural sequel to the work present in [16], one reason is because the machinery to proceed has been partially developed so that one only needs to adapt the methods to this more general setting by dropping the hypothesis about the dimension of the center.

Question 2: *Is there a method that allows the construction of Einstein Lorentzian nilpotent Lie groups with nonvanishing scalar curvature ?*

So far the only known Example in this category of Lie groups is due do D. Conti & F. Rossi (see [6] and also [16], Example 1), and naturally one needs further examples in order to make reasonable statements.

Question 3: *Is there a possible classification when one restricts to a specific class of Einstein Lorentzian nilpotent Lie algebras ?*

Although a complete classification is far reaching, certain classes of nilpotent Lie algebras, for instance filiform or characteristically nilpotent Lie algebras, enjoy special properties that could make a classification within the realm of possibilities. In any case, these situations need a careful study.

Question 4: *Can we prove a similar set of results for Einstein nilpotent Lie groups of arbitrary signature ?*

The Lorentzian case is only an instance of the more general non-degenerate setting, which lacks its presence in the mathematical litterature.

2

On Einstein Lorentzian nilpotent Lie groups

2.1 Introduction

A pseudo-Riemannian manifold (M, g) is called Einstein if its Ricci tensor $\mathrm{Ric} : \mathrm{TM} \longrightarrow \mathrm{TM}$ satisfies $\mathrm{Ric} = \lambda \mathrm{Id}_{TM}$ for some constant $\lambda \in \mathbb{R}$. When $\lambda = 0$, (M, g) is called Ricci-flat. Pseudo-Riemannian Einstein manifolds present a central topic of differential geometry and an active area of research. The subclass of Lorentzian Einstein manifolds has attracted a particular interest due to its importance in the physics of general relativity (see [4]). Homogeneous Riemannian manifolds were intensively studied and the Alekseevskii's conjecture (see [4]) has driven a profound exploration of Einstein left invariant Riemannian metrics on Lie groups leading to some outstanding results (see [10, 13]). However, the study of left invariant Einstein pseudo-Riemannian metrics on Lie groups is at beginning. In [1, 14], flat Lorentzian left invariant metrics on Lie groups has been studied, in [15] flat left invariant metrics of signature $(2, n-2)$ on nilpotent Lie groups has been characterized, Ricci-flat Lorentzian left invariant metrics on 2-step nilpotent Lie groups has been investigated in [5, 18] and in [7, 2], all four dimensional Lie algebras of Einstein Lorentzian Lie groups were given. The study of pseudo-Riemannian Einstein left invariant metric with non vanishing scalar curvature has been initiated in [6].

In this chapter, we study Einstein Lorentzian left invariant metrics on nilpotent Lie groups. As in any study involving left invariant structures on Lie groups, we can consider the problem at the Lie algebra level. Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a nilpotent Lorentzian Lie algebra with Ricci operator $\mathrm{Ric} : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{g}}$. Our main results can be stated as follows :

1. If the center $\mathrm{Z}(\mathfrak{g})$ of $\mathfrak{g}$ is nondegenerate then it is Euclidean and if the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ is nondegenerate then it is Lorentzian.

2. If $[\mathfrak{g}, \mathfrak{g}]$ is degenerate then $Z(\mathfrak{g})$ is degenerate and the metric is Ricci-flat.

3. If the scalar curvature of $\mathfrak{g}$ is non zero then $Z(\mathfrak{g})$ is nondegenerate Euclidean, $[\mathfrak{g},\mathfrak{g}]$ is nondegenerate Lorentzian and $Z(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$.

4. If $Z(\mathfrak{g})$ is degenerate then $\mathfrak{g}$ is Ricci-flat and $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is obtained by the process of double extension from an abelian Euclidean Lie algebra. The process of double extension has been introduced by Medina-Revoy [3] in the context of bi-invariant pseudo-Riemannian metrics on Lie groups and turned out to be efficient in many other situations. We adapt this process to our case and, in addition to our main result, we use it to construct a large class of Einstein Lorentzian Lie algebras (not necessarily nilpotent). We also recover the description of 2-step nilpotent Lorentzian Lie algebras obtained in [18].

5. If $\mathfrak{g}$ is Ricci-flat non-abelian, and $\dim[\mathfrak{g},\mathfrak{g}] = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]) + 1$ then $Z(\mathfrak{g})$ is degenerate.

6. If $\dim\mathfrak{g} \leq 5$ then the center of $\mathfrak{g}$ is degenerate. In this case we give a complete classification of all such Lie algebras.

7. We give the first examples Ricci-flat Lorentzian nilpotent Lie algebras with nondegenerate center. It is worth to mention that this differs from the flat case. Indeed, it has been shown (see [14]) that if a nilpotent Lie group G is endowed with a flat left-invariant metric which is either Lorentzian or of signature $(2, n-2)$ then its center must be degenerate.

8. We give another proof of the main result in [6] by using a formula known in the Euclidean context (see Propositions 2.3.6-2.3.7)

The chapter is organized as follows. In Section 2.2, we establish two lemmas and we give a useful expression of the Ricci operator involving our main tool : a family of skew-symmetric endomorphisms which we call structure endomorphisms. In Section 2.3, we prove some general results on Einstein Lorentzian nilpotent Lie algebras. In Section 2.4, we describe the process of double extension which permits to construct a large class of Einstein Lorentzian Lie algebras and we prove our main result (see Theorem 2.4.1), then we show that Lorentzian Einstein nilpotent Lie algebras up to dimension 5 satisfy the hypothesis of this theorem and we give the list of such algebras. Finally, we give the first examples of Ricci-flat Lorentzian nilpotent Lie algebras with nondegenerate center. This widely opens the door for a future study of this particular class.

2.2 Ricci curvature of pseudo-Euclidean Lie algebras

The purpose of this paragraph is to fix the notations that shall be used throughout the chapter, this is done by defining several operators related to the metric structure on a pseudo-Euclidean Lie algebra, particularly its curvature, we then proceed to give many properties of these operators as well as their basis expression. This step is crucial for the upcoming development, since the proof of many central results relies on the computational material introduced in this paragraph. We also introduce some facts concerning pseudo-Euclidean vector spaces in the form of Lemmas at the end of the paragraph for later use. A detailed account on pseudo-Euclidean vector spaces is given in Appendix B, the preliminary notions on pseudo-Euclidean Lie algebras such as the Levi-Civita product and the various flavors of curvature are the subject of Appendix C.

Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra and denote $\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$, $(u,v)\mapsto u\cdot v$ its *Levi-Civita product*, we recall the Koszul formula (*cf.* (C.1))

$$2\langle u\cdot v,w\rangle \;=\; \langle [u,v],w\rangle+\langle [w,u],v\rangle+\langle [w,v],u\rangle. \tag{2.1}$$

For any $u\in\mathfrak{g}$, we shall denote $\mathrm{L}_u,\mathrm{R}_u:\mathfrak{g}\longrightarrow\mathfrak{g}$ be the corresponding left and right multiplications i.e $\mathrm{R}_v(u)=\mathrm{L}_u v=u\cdot v$. By Koszul formula (2.1), we get that $\mathrm{L}_u:\mathfrak{g}\longrightarrow\mathfrak{g}$ is skew-symmetric and $\mathrm{ad}_u=\mathrm{L}_u-\mathrm{R}_u$. The curvature $K:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathrm{End}(\mathfrak{g})$ of $\mathfrak{g}$ can then be expressed in these terms as:

$$\begin{aligned} K(u,v)w &= \mathrm{L}_{[u,v]}w-[\mathrm{L}_u,\mathrm{L}_v]w \\ &= [\mathrm{R}_w,\mathrm{L}_u](v)-\mathrm{R}_w\circ\mathrm{R}_u(v)+\mathrm{R}_{uw}(v). \end{aligned}$$

From the last relation, we deduce that the Ricci curvature $\mathrm{ric}:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathbb{R}$ of $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is given by:

$$\mathrm{ric}(u,v) \;=\; -\mathrm{tr}(\mathrm{R}_u\circ\mathrm{R}_v)+\mathrm{tr}(\mathrm{R}_{uv}). \tag{2.2}$$

In order to make more use of the Ricci curvature, we introduce $H\in\mathfrak{g}$ and $J:\mathfrak{g}\longrightarrow\mathrm{so}(\mathfrak{g},\langle\ ,\ \rangle)$ such that for any $u,v\in\mathfrak{g}$,

$$\langle H,u\rangle=\mathrm{tr}(\mathrm{ad}_u)\quad\text{and}\quad J_u(v)=\mathrm{ad}_v^*(u). \tag{2.3}$$

Note that $H\in[\mathfrak{g},\mathfrak{g}]^\perp$ and $H=0$ if and only if $\mathfrak{g}$ is unimodular. In these notations (2.1) can be rewritten as:

$$2\langle R_v u,w\rangle=-\langle \mathrm{ad}_v(u),w\rangle-\langle \mathrm{ad}_v^*(u),w\rangle-\langle J_v(u),w\rangle. \tag{2.4}$$

Proposition 2.2.1. *Let $\mathfrak{g}$ be a pseudo-Euclidean Lie algebra. We have:*

$$\begin{aligned} \mathrm{ric}(u,v) &= -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u\circ\mathrm{ad}_v)-\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u\circ\mathrm{ad}_v^*)-\frac{1}{4}\mathrm{tr}(J_u\circ J_v) \\ &\quad -\frac{1}{2}\langle \mathrm{ad}_H u,v\rangle-\frac{1}{2}\langle \mathrm{ad}_H v,u\rangle. \end{aligned}$$

Proof. It is a consequence of (2.2), the following formula which can be deduced from (2.4)

$$\mathrm{R}_u=-\frac{1}{2}(\mathrm{ad}_u+\mathrm{ad}_u^*)-\frac{1}{2}J_u,$$

and the following computation. For any orthonormal basis $(e_1,\dots,e_n)$ of $\mathfrak{g}$, with $\epsilon_i = \langle e_i, e_i\rangle$:

$$\begin{aligned}
\mathrm{tr}(\mathrm{R}_{uv}) &= \sum_{i=1}^{n} \epsilon_i \langle \mathrm{L}_{e_i}(u\cdot v), e_i\rangle \\
&\overset{(2.1)}{=} -\sum_{i=1}^{n} \epsilon_i \langle [u\cdot v, e_i], e_i\rangle \\
&= -\mathrm{tr}(\mathrm{ad}_{uv}) \\
&= -\langle H, u\cdot v\rangle \\
&= -\frac{1}{2}\langle \mathrm{ad}_H u, v\rangle - \frac{1}{2}\langle \mathrm{ad}_H v, u\rangle.
\end{aligned}$$

The result is then a matter of simple calculation. □

Recall that any nilpotent Lie algebra is unimodular and has vanishing Killing form, this leads to the following observation:

Proposition 2.2.2. *If $\mathfrak{g}$ is a nilpotent pseudo-Euclidean Lie algebra, then:*

$$\mathrm{ric}(u,v) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(J_u \circ J_v).$$

In particular, its Ricci operator $\mathrm{Ric} : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is given by*

$$\mathrm{Ric} = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2, \tag{2.5}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are the auto-adjoint endomorphisms given by

$$\langle \mathcal{J}_1(u), v\rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) \quad \textit{and} \quad \langle \mathcal{J}_2(u), v\rangle = -\mathrm{tr}(J_u \circ J_v). \tag{2.6}$$

Remark 1. *The endomorphisms J_u are skew-symmetric and $J_u = 0$ if and only if $u \in [\mathfrak{g},\mathfrak{g}]^\perp$. As a result, if $\langle\ ,\ \rangle$ is Euclidean then we get for any $u \in \mathfrak{g}$, $\langle \mathcal{J}_i(u), u\rangle \geq 0$ $(i = 1, 2)$,* $\ker \mathcal{J}_1 = Z(\mathfrak{g})$ *and* $\ker \mathcal{J}_2 = [\mathfrak{g},\mathfrak{g}]^\perp$.

The operators $\mathcal{J}_1$ and $\mathcal{J}_2$ will play a crucial role in our study so we are going to express them in a useful way. This is based on the notion of structure endomorphisms we now introduce. Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra and $(e_1,\dots,e_p)$ a basis of $\mathfrak{g}$. For any $u, v \in \mathfrak{g}$, the Lie bracket can be written as:

$$[u,v] = \sum_{i=1}^{p} \langle S_i u, v\rangle e_i, \tag{2.7}$$

where $S_i : \mathfrak{g} \longrightarrow \mathfrak{g}$ are skew-symmetric endomorphisms with respect to $\langle\ ,\ \rangle$. The family of operators $(S_1,\dots,S_p)$ will be called *structure endomorphisms* of $\mathfrak{g}$ associated to $(e_1,\dots,e_p)$. Note that $Z(\mathfrak{g}) = \cap_{i=1}^{p} \ker S_i$, furthermore one can see easily from (2.7) and the definition of J in (2.3) that for any $u \in \mathfrak{g}$,

$$J_u = \sum_{i=1}^{p} \langle u, e_i\rangle S_i. \tag{2.8}$$

The following Proposition is of central importance and will be used in many instances.

Proposition 2.2.3. *Let $\mathfrak{g}$ be a pseudo-Euclidean Lie algebra and let $(S_1,\ldots,S_p)$ be structure endomorphisms corresponding to a basis $(e_1,\ldots,e_p)$ of $\mathfrak{g}$. Then:*

$$\mathcal{J}_1 = -\sum_{i,j=1}^{p}\langle e_i,e_j\rangle S_i\circ S_j \quad \textit{and} \quad \mathcal{J}_2 u = -\sum_{i,j=1}^{p}\langle e_i,u\rangle \mathrm{tr}(S_i\circ S_j)e_j. \tag{2.9}$$

In particular, $\mathrm{tr}\mathcal{J}_1 = \mathrm{tr}\mathcal{J}_2$.

Proof. The expression of $\mathcal{J}_2$ is an immediate consequence of (2.6) and (2.8). As for $\mathcal{J}_1$ we have that :

$$\begin{aligned}(\mathrm{ad}_u\circ \mathrm{ad}_u^*)(v) &= (\mathrm{ad}_u\circ J_v)(u)\\ &\overset{(2.7)}{=} \sum_{i=1}^{p}\langle S_iu,J_vu\rangle e_i\\ &\overset{(2.8)}{=} \sum_{i,j}\langle S_iu,S_ju\rangle\langle v,e_j\rangle e_i\\ &= -\sum_{i,j}\langle (S_j\circ S_i)(u),u\rangle K_{i,j}v,\end{aligned}$$

where $K_{i,j}v=\langle v,e_j\rangle e_i$. Clearly $\mathrm{tr}(K_{i,j})=\langle e_i,e_j\rangle$, thus $\mathrm{tr}(\mathrm{ad}_u\circ\mathrm{ad}_v^*) = -\sum_{i,j}\langle (S_j\circ S_i)(u),v\rangle\langle e_i,e_j\rangle$. This gives the desired formula of $\mathcal{J}_1$. □

We close the paragraph by the following two Lemmas on skew-symmetric operators of Lorentzian vector spaces:

Lemma 2.2.1. *Let $(V,\langle\ ,\ \rangle)$ be a Lorentzian vector space, e an isotropic vector and A a skew-symmetric endomorphism. Then $\langle Ae,Ae\rangle \geq 0$. Moreover, $\langle Ae,Ae\rangle = 0$ if and only if $Ae=\alpha e$ with $\alpha\in\mathbb{R}$.*

Proof. We choose an isotropic vector $\bar{e}$ of V such that $\langle e,\bar{e}\rangle = 1$ and we fix an orthonormal basis $(f_1,\ldots,f_r)$ of $\{e,\bar{e}\}^\perp$. Since A is skew-symmetric, we have

$$Ae=\alpha e+\sum_{i=1}^{r}a_if_i \quad \text{and} \quad \langle Ae,Ae\rangle = \sum_{i=1}^{r}a_i^2,$$

and the result follows. □

Lemma 2.2.2. *Let $(V,\langle\ ,\ \rangle)$ be a Lorentzian vector space, e an isotropic vector and A a skew-symmetric endomorphism such that $A(e)=0$. Then:*

1. $\mathrm{tr}(A^2)\leq 0$,

2. $\mathrm{tr}(A^2)=0$ *if and only if for any $x\in e^\perp$, $A(x)=\lambda(x)e$ and in this case* $\mathrm{tr}(A\circ B)=0$ *for any skew-symmetric endomorphism satisfying* $B(e)=0$.

Proof. We choose a Lorentzian basis $\mathbb{B} = (e, \bar{e}, f_1, \ldots, f_n)$ of V such that $(e, f_1, \ldots, f_n)$ is a basis of $\{e\}^\perp$, $\bar{e}$ is isotropic, $\langle e, \bar{e} \rangle = 1$ and $(f_1, \ldots, f_n)$ is an orthonormal basis of $\{e, \bar{e}\}^\perp$. First observe that the restriction of $\langle\ ,\ \rangle$ to $\{e\}^\perp$ is nonnegative and for any $x \in \{e\}^\perp$, $\langle x, x \rangle = 0$ if and only if $x = \alpha e$. Now $Af_i \in \{e\}^\perp$ for any $i = 1, \ldots, n$ and so:

$$\mathrm{tr}(A^2) = \langle A^2(e), \bar{e} \rangle + \langle A^2(\bar{e}), e \rangle - \sum_{i=1}^{n} \langle Af_i, Af_i \rangle = -\sum_{i=1}^{n} \langle Af_i, Af_i \rangle = -\sum_{i,j=1}^{n} \langle Af_i, f_j \rangle^2.$$

This shows that $\mathrm{tr}(A^2) \leq 0$ and $\mathrm{tr}(A^2) = 0$ if and only if $Af_i = \alpha_i e$ for all $i = 1, \ldots, n$. In this case, if B is skew-symmetric and $B(e) = 0$ then $B(f_i) = \beta_i e$, therefore:

$$\mathrm{tr}(A \circ B) = -\langle B(e), A(\bar{e}) \rangle - \langle A(\bar{e}), B(e) \rangle - \sum_{i=1}^{n} \langle Af_i, Bf_i \rangle = 0.$$

This proves the claim. □

2.3 Some results on Einstein Lorentzian nilpotent Lie algebras

The principal goal of this section is to prove a set of results typical to the Lorentzian case, and which characterizes the signature of the center $Z(\mathfrak{g})$ and the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ of an Einstein Lorentzian nilpotent Lie algebra of $\mathfrak{g}$. We draw a number of consequences from these results, for instance we obtain as corollaries some known facts on the Ricci curvature of Einstein Lorentzian 2-step nilpotent Lie algebras, we also give a slight generalization of these results that includes some Lorentzian Ricci-flat 3-step nilpotent Lie algebras. Finally, we use our approach to recover some results first proved in [6].

Before going further, let us first give the following remark which we will use frequently: Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra. From the definition of J in (2.3), one can easily deduce that $\ker J = [\mathfrak{g}, \mathfrak{g}]^\perp$ and hence:

$$Z(\mathfrak{g}) \subset \ker \mathcal{J}_1 := M \quad \text{and} \quad [\mathfrak{g}, \mathfrak{g}]^\perp \subset \ker \mathcal{J}_2 := N. \tag{2.10}$$

Since $\mathcal{J}_1$ and $\mathcal{J}_2$ are symmetric with respect to $\langle\ ,\ \rangle$,

$$\mathrm{Im}\mathcal{J}_1 = M^\perp \subset Z(\mathfrak{g})^\perp \quad \text{and} \quad \mathrm{Im}\mathcal{J}_2 = N^\perp \subset [\mathfrak{g}, \mathfrak{g}]. \tag{2.11}$$

Proposition 2.3.1. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent non abelian Lie algebra. If $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate then it is Lorentzian.*

Proof. We reason by contradiction and suppose that $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate Euclidean, choose an orthonormal basis $(e_1, \ldots, e_d)$ of $[\mathfrak{g}, \mathfrak{g}]$ and denote by $(S_1, \ldots, S_d)$ the associated structure endomorphisms. According to (2.5) and (2.9), we have

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}, \quad \mathcal{J}_1 = -\sum_{i=1}^{d} S_i^2 \quad \text{and} \quad \mathcal{J}_2 u = -\sum_{i,j=1}^{d} \langle u, e_i \rangle \mathrm{tr}(S_i \circ S_j) e_j.$$

Since $\mathfrak{g}$ is nilpotent then $\dim[\mathfrak{g},\mathfrak{g}]^\perp \geq 2$ and we can choose a couple $(e,\bar{e})$ of isotropic vectors in $[\mathfrak{g},\mathfrak{g}]^\perp$ such that $\langle e,\bar{e}\rangle = 1$. By replacing in the relations above and using (2.10), we get:

$$\frac{1}{2}\mathcal{J}_1 e = -\lambda e,\ \frac{1}{2}\mathcal{J}_1 \bar{e} = -\lambda \bar{e} \quad \text{and} \quad \sum_{i=1}^{d}\langle S_i e, S_i e\rangle = \sum_{i=1}^{d}\langle S_i \bar{e}, S_i \bar{e}\rangle = 0.$$

By using Lemma 2.2.1, we deduce that for any $i \in \{1,\ldots,d\}$, $S_i e = \alpha_i e$ and $S_i \bar{e} = -\alpha_i \bar{e}$ and hence:

$$\lambda = \frac{1}{2}\sum_{i=1}^{d}\alpha_i^2 \geq 0.$$

For $i = 1,\ldots,d$, S_i is skew-symmetric and leaves invariant $\mathrm{span}\{e,\bar{e}\}$ so it leaves invariant its orthogonal. We denote by K_i the restriction of S_i to the Euclidean vector space $\{e,\bar{e}\}^\perp$. We have $\mathrm{tr}(S_i^2) = 2\alpha_i^2 + \mathrm{tr}(K_i^2)$ and $\mathrm{tr}(K_i^2) \leq 0$. Now, since $\mathrm{tr}(\mathcal{J}_1) = \mathrm{tr}(\mathcal{J}_2)$, we get:

$$(\dim\mathfrak{g})\lambda = -\frac{1}{4}\mathrm{tr}(\mathcal{J}_1) = \frac{1}{4}\sum_{i=1}^{d}(2\alpha_i^2 + \mathrm{tr}(K_i^2)) = \lambda + \frac{1}{4}\sum_{i=1}^{d}\mathrm{tr}(K_i^2).$$

This shows that $\lambda \leq 0$. By combining the results obtained so far, we deduce that $\lambda = 0$ and for all $i = 1,\ldots,d$, $\mathrm{tr}(K_i^2) = 0$ and $\alpha_i^2 = 0$ which implies that $S_i = 0$. Thus $\mathfrak{g}$ is abelian which is a contradiction, this proves our claim. □

Proposition 2.3.2. *Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be an Einstein pseudo-Euclidean non-abelian nilpotent Lie algebra. If $Z(\mathfrak{g})$ is non-degenerate then $Z(\mathfrak{g})^\perp$ is not Euclidean.*

Proof. Denote by $(p,q) = (-,\ldots-,+,\ldots,+)$ the signature of $\langle\ ,\ \rangle$. We reason by contradiction and assume that $Z(\mathfrak{g})$ is non-degenerate and $Z(\mathfrak{g})^\perp$ is Euclidean. This implies in particular that $\dim \mathrm{Z}(\mathfrak{g})^\perp \leq q$ and therefore $\dim \mathrm{Z}(\mathfrak{g}) \geq p$. Consequently, we can choose an orthogonal family $(e_1,\ldots,e_p)$ in $Z(\mathfrak{g})$ such that $\langle e_i,e_i\rangle = -1$ for $i = 1,\ldots,p$. Write $\mathfrak{g} = \mathrm{span}\{e_1,\ldots,e_p\}\oplus\mathfrak{g}_0$, where $\mathfrak{g}_0 = \{e_1,\ldots,e_p\}^\perp$. For any $u,v \in \mathfrak{g}_0$, put:

$$[u,v] = \sum_{i=1}^{p}\langle K_i u, v\rangle e_i + [u,v]_0, \tag{2.12}$$

where $K_i : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$ are skew-symmetric endomorphisms and $[u,v]_0 \in \mathfrak{g}_0$. Let $\langle\ ,\ \rangle_0$ denote the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{g}_0$. It is obvious that $(\mathfrak{g}_0,[\ ,\ ]_0,\langle\ ,\ \rangle_0)$ is a Euclidean nilpotent Lie algebra. We claim that if $(\mathfrak{g},\langle\ ,\ \rangle)$ is Einstein i.e $\mathrm{Ric} = \lambda\mathrm{Id}_{\mathfrak{g}}$ then we have $\lambda = \frac{1}{4}\mathrm{tr}(K_i^2) \leq 0$ for all $i = 1,\ldots,p$. Moreover if $\mathrm{Ric}_{\langle\ ,\ \rangle_0}$ is the Ricci operator of $(\mathfrak{g}_0,[\ ,\ ]_0,\langle\ ,\ \rangle_0)$ then:

$$\mathrm{Ric}_{\langle\ ,\ \rangle_0} = \lambda\mathrm{Id}_{\mathfrak{g}_0} + \frac{1}{2}\sum_{i=1}^{p}K_i^2. \tag{$*$}$$

This implies that the Ricci curvature of $(\mathfrak{g}_0,\langle\ ,\ \rangle_0)$ is nonpositive. However a non-abelian nilpotent Euclidean Lie algebra has always a Ricci negative direction and a Ricci positive direction (see [19, Theorem 2.4]). So the only possibility is $K_i = 0$ for $i = 1,\ldots,p$ and $\mathfrak{g}_0$ is abelian. We get a contradiction in view of (2.12), which completes the proof.

Let us prove our claim. We choose an orthonormal basis $\mathbb{B}_1 = (f_1,\dots,f_q)$ of $\mathfrak{g}_0$. Then clearly $\mathbb{B} = (e_1,\dots,e_p,f_1,\dots,f_q)$ is an orthonormal basis of $\mathfrak{g}$. Denote by $(S_1,\dots,S_p,T_1,\dots,T_q)$ the structure endomorphisms of $(\mathfrak{g},\langle\ ,\ \rangle)$ with respect to $\mathbb{B}$ and $(M_1,\dots,M_q)$ the structure endomorphisms of $(\mathfrak{g}_0,\langle\ ,\ \rangle_0)$ with respect to $\mathbb{B}_1$. Every S_i and T_i vanish on $Z(\mathfrak{g})$ and hence leave $\mathfrak{g}_0$ invariant. By using (2.12), one can easily see that, for $i = 1,\dots,p$ and $j = 1,\dots,q$:

$$(S_i)_{|\mathfrak{g}_0} = K_i \quad \text{and} \quad (T_j)_{|\mathfrak{g}_0} = M_j.$$

If $\mathfrak{g}$ is Einstein then according to (2.9), we have:

$$-\frac{1}{2}\sum_{i=1}^{p} S_i^2 + \frac{1}{2}\sum_{i=1}^{q} T_i^2 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}, \qquad (**)$$

where

$$\mathcal{J}_2 = -\sum_{i,j}\langle e_i,\bullet\rangle \mathrm{tr}(K_i\circ K_j)e_j - \sum_{i,j}\langle f_i,\bullet\rangle \mathrm{tr}(M_i\circ M_j)f_j - \sum_{i,j}\langle f_i,\bullet\rangle \mathrm{tr}(M_i\circ K_j)e_j - \sum_{i,j}\langle e_i,\bullet\rangle \mathrm{tr}(K_i\circ M_j)f_j.$$

If we evaluate the relation $(**)$ at e_i, we get

$$\frac{1}{4}\mathrm{tr}(K_i^2)e_i + \frac{1}{4}\sum_{j=1}^{q}\mathrm{tr}(K_i\circ M_j)f_j = \lambda e_i.$$

This is equivalent to $\lambda = \frac{1}{4}\mathrm{tr}(K_i^2)$ and $\mathrm{tr}(K_i\circ M_j) = 0$ for $i\in\{1,\dots,p\}$ and $j\in\{1,\dots,q\}$. This implies that if we restrict $(**)$ to $\mathfrak{g}_0$ we get the desired relation $(*)$. □

Corollary 2.3.1. *Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be an Einstein Lorentzian non-abelian nilpotent Lie algebra. If $Z(\mathfrak{g})$ is non-degenerate then it is Euclidean.*

The following result was first found by Guediri in [18] and served as a key ingredient in the classification of Einstein Lorentzian 2-step nilpotent Lie algebras. It can also be deduced from the preceding results.

Corollary 2.3.2. *Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be an Einstein Lorentzian non abelian 2-step nilpotent Lie algebra. Then $Z(\mathfrak{g})$ is degenerate.*

Proof. Suppose that $Z(\mathfrak{g})$ is non-degenerate. According to Corollary 2.3.1, $Z(\mathfrak{g})$ is non-degenerate Euclidean. But $\mathfrak{g}$ is 2-step nilpotent and hence $[\mathfrak{g},\mathfrak{g}]\subset Z(\mathfrak{g})$. Thus $[\mathfrak{g},\mathfrak{g}]$ is non-degenerate Euclidean which contradicts Proposition 2.3.1. □

Proposition 2.3.3. *Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent Lie algebra such that $[\mathfrak{g},\mathfrak{g}]$ is degenerate then $[\mathfrak{g},\mathfrak{g}]\cap[\mathfrak{g},\mathfrak{g}]^\perp\subset Z(\mathfrak{g})$ and $(\mathfrak{g},\langle\ ,\ \rangle)$ is Ricci flat.*

Proof. Let e be a generator of $[\mathfrak{g},\mathfrak{g}]\cap[\mathfrak{g},\mathfrak{g}]^\perp$. Then there exists a basis $(e,\bar{e},f_1,\dots,f_d,g_1,\dots,g_s)$ of $\mathfrak{g}$ such that $(e,f_1,\dots,f_d)$ is a basis of $[\mathfrak{g},\mathfrak{g}]$, $(e,g_1,\dots,g_s)$ is basis of $[\mathfrak{g},\mathfrak{g}]^\perp$, $(e,\bar{e})$ are co-isotropic i.e $\langle e,\bar{e}\rangle = 1$ and $(f_1,\dots,f_d,g_1,\dots,g_s)$ is an orthonormal basis of $\{e,\bar{e}\}^\perp$. Next denote by $(A,S_1,\dots,S_d)$ the associated structure endomorphisms, i.e. for any $u,v\in\mathfrak{g}$,

$$[u,v] = \langle Au,v\rangle e + \sum_{i=1}^{d}\langle S_iu,v\rangle f_i.$$

According to (2.5) and (2.9), we have:

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}} \quad \text{and} \quad \mathcal{J}_1 = -\sum_{i=1}^{d} S_i^2.$$

Since $e \in [\mathfrak{g},\mathfrak{g}]^{\perp}$ and it is isotropic, we have $\mathcal{J}_2 e = 0$, $-\frac{1}{2}\mathcal{J}_1 e = \lambda e$, and so $\sum_{j=1}^{d}\langle S_j e, S_j e\rangle = 0$. Using Lemma 2.2.1, we get $S_j e = a_j e$ for any $j = 1,\ldots,d$ and hence $\lambda = \frac{1}{2}\sum_{i=1}^{d} a_i^2 \geq 0$. On the other hand, since $\mathrm{tr}(\mathcal{J}_1) = \mathrm{tr}(\mathcal{J}_2)$, it follows that:

$$(\dim \mathfrak{g})\lambda = -\frac{1}{4}\mathrm{tr}(\mathcal{J}_1) = \frac{1}{4}\sum_{j=1}^{d}\mathrm{tr}(S_j^2).$$

Furthermore we have:

$$\begin{aligned}\mathrm{tr}(S_j^2) &= \langle S_j^2 e, \bar{e}\rangle + \langle S_j^2 \bar{e}, e\rangle + \sum_l \langle S_j^2 f_l, f_l\rangle + \sum_l \langle S_j^2 g_l, g_l\rangle \\ &= 2a_j^2 - \sum_l \langle S_j f_l, S_j f_l\rangle - \sum_l \langle S_j g_l, S_j g_l\rangle.\end{aligned}$$

Since S_j leaves invariant e, it leaves invariant its orthogonal $\mathrm{span}\{e, f_l, g_k\}$. But the restriction of $\langle\ ,\ \rangle$ to $\mathrm{span}\{e, f_l, g_k\}$ is nonnegative. So $\langle S_j f_l, S_j f_l\rangle \geq 0$ and $\langle S_j g_l, S_j g_l\rangle \geq 0$. Thus:

$$(\dim \mathfrak{g} - 1)\lambda = -\sum_{l,j}\langle S_j f_l, S_j f_l\rangle - \sum_{l,j}\langle S_j g_l, S_j g_l\rangle \leq 0.$$

But we have already shown that $\lambda \geq 0$. We conclude that $\lambda = 0$ and $S_j(e) = 0$ for $j = 1,\ldots,p$. This implies that for any $u \in \mathfrak{g}$, $[e,u] = \langle A(e), u\rangle e$. But ad_u is nilpotent and hence $[e,u] = 0$ which completes the proof. □

Corollary 2.3.3. *Let $\mathfrak{g}$ be a nilpotent Lorentzian Einstein Lie algebra. Suppose that $[\mathfrak{g},\mathfrak{g}]$ is degenerate, then $\mathrm{Z}(\mathfrak{g})$ is also degenerate.*

Proposition 2.3.4. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a Ricci-flat Lorentzian nilpotent non-abelian Lie algebra such that $\dim[\mathfrak{g},\mathfrak{g}] = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]) + 1$. Then $Z(\mathfrak{g})$ is degenerate.*

Proof. Suppose that $Z(\mathfrak{g})$ is non-degenerate. According to Propositions 2.3.1 and 2.3.3 and Corollary 2.3.1, $Z(\mathfrak{g})$ is Euclidean and $[\mathfrak{g},\mathfrak{g}]$ is Lorentzian and hence there exists an orthonormal basis $(e_1,\ldots,e_r)$ of $[\mathfrak{g},\mathfrak{g}]$ such that $e_i \in Z(\mathfrak{g})$ for $i = 1,\ldots,r-1$ and $\langle e_r, e_r\rangle = -1$. We denote by $(S_1,\ldots,S_r)$ the structure endomorphisms associated to $(e_1,\ldots,e_r)$. We have:

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = 0, \quad \mathcal{J}_1 = S_r^2 - \sum_{j=1}^{r-1} S_i^2 \quad \text{and} \quad \mathcal{J}_2(u) = -\sum_{i,j}\langle e_i, u\rangle \mathrm{tr}(S_i \circ S_j) e_j.$$

Since $Z(\mathfrak{g}) \subset \ker \mathcal{J}_1$, we get $\mathcal{J}_2(e_i) = 0$ for $i = 1,\ldots,r-1$. This is equivalent to $\mathrm{tr}(S_i \circ S_j) = 0$ for $i = 1,\ldots,r$ and $j = 1\ldots,r-1$ and hence:

$$\mathcal{J}_2(u) = \langle e_r, u\rangle \mathrm{tr}(S_r^2) e_r.$$

But $\mathrm{tr}(\mathcal{J}_1) = \mathrm{tr}(\mathcal{J}_2) = 0$ so $\mathcal{J}_1 = \mathcal{J}_2 = 0$. This implies, by virtue of (2.6), that $\mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y^*) = 0$ for any $x, y \in \mathfrak{g}$. For $x \in \mathfrak{g}$, put

$$\mathrm{ad}_x(e_r) = \alpha_1 e_1 + \ldots + \alpha_r e_r.$$

So for any $k \in \mathbb{N}$, $\mathrm{ad}_x^k(e_r) = \alpha_r^k e_r + u_k$ with $u_k \in \mathrm{Z}(\mathfrak{g})$ but since $\mathfrak{g}$ is nilpotent and $e_r \notin \mathrm{Z}(\mathfrak{g})$ then $\alpha_r = 0$ and hence $\mathrm{ad}_x(e_r) \in Z(\mathfrak{g})$. If $(f_1, \ldots, f_q)$ is an orthonormal basis of $[\mathfrak{g}, \mathfrak{g}]^\perp$, then:

$$\begin{aligned} 0 &= \mathrm{tr}(\mathrm{ad}_{e_r} \circ \mathrm{ad}_{e_r}^*) \\ &= \sum_{i=1}^{r-1} \langle \mathrm{ad}_{e_r}(e_i), \mathrm{ad}_{e_r}(e_i) \rangle + \sum_{i=1}^{q} \langle \mathrm{ad}_{e_r}(f_i), \mathrm{ad}_{e_r}(f_i) \rangle \\ &= \sum_{i=1}^{q} \langle \mathrm{ad}_{f_i}(e_r), \mathrm{ad}_{f_i}(e_r) \rangle. \end{aligned}$$

But $\mathrm{ad}_{f_i}(e_r) \in Z(\mathfrak{g})$ and $Z(\mathfrak{g})$ is Euclidean thus $\mathrm{ad}_{f_i}(e_r) = 0$ for $i = 1, \ldots, q$ an hence $e_r \in Z(\mathfrak{g})$ which is a contradiction. This completes the proof. □

Using our approach, we recover some results obtained in [6].

Proposition 2.3.5. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a nilpotent pseudo-Euclidean Lie algebra.*

1. *If $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is Einstein with $\lambda \neq 0$ then $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$.*
2. *If $\dim Z(\mathfrak{g}) \geq \dim[\mathfrak{g}, \mathfrak{g}]$ then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is Einstein if and only if it is Ricci flat.*

In particular, if $\mathfrak{g}$ is 2-nilpotent then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is Einstein if and only if it is Ricci flat.

Proof. Suppose that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is nilpotent and Einstein with $\lambda \neq 0$, i.e.

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}.$$

Put $M := \ker(\mathcal{J}_1)$ and $N := \ker(\mathcal{J}_2)$. By virtue of (2.10) and (2.11), this implies that:

$$Z(\mathfrak{g}) \subset \mathrm{Im}\mathcal{J}_2 \subset [\mathfrak{g}, \mathfrak{g}].$$

It also implies that $M \cap N = \{0\}$. But, if $\dim Z(\mathfrak{g}) \geq \dim[\mathfrak{g}, \mathfrak{g}]$ then

$$\dim M + \dim N \geq \dim Z(\mathfrak{g}) + \dim[\mathfrak{g}, \mathfrak{g}]^\perp \geq \dim \mathfrak{g}$$

and hence $\mathfrak{g} = M \oplus N$. This contradicts $\mathrm{tr}(\mathcal{J}_1) = \mathrm{tr}(\mathcal{J}_2)$. □

One of the main results in [6] is that if a pseudo-Euclidean Einstein nilpotent Lie algebra has a derivation with a non vanishing trace then it is Ricci flat. We give another proof of this fact based on (2.13). This formula was established in the Euclidean context in [12] by using the Ricci tensor as a moment map. We prove this formula in the general case by a direct computation.

Proposition 2.3.6. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra and let Q denote the symmetric endomorphism $Q = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2$. Then for any orthonormal basis $(e_1, \ldots, e_p)$ of $\mathfrak{g}$ and any endomorphism E of $\mathfrak{g}$, we have*

$$\mathrm{tr}(QE) = \frac{1}{4}\sum_{i,j} \epsilon_i\epsilon_j \langle E([e_i, e_j]) - [E(e_i), e_j] - [e_i, E(e_j)], [e_i, e_j]\rangle, \tag{2.13}$$

where $\langle e_i, e_i\rangle = \epsilon_i$.

Proof. We denote by $(S_1, \ldots, S_p)$ the structures endomorphisms associated to $(e_1, \ldots, e_p)$. From (2.8), we get $S_i = \epsilon_i J_{e_i}$ and by using (2.9) we get:

$$QE(u) = \frac{1}{2}\sum_{i=1}^{p} \epsilon_i J_{e_i}^2 E(u) - \frac{1}{4}\sum_{i,j=1}^{p} \epsilon_i\epsilon_j \langle e_i, E(u)\rangle \mathrm{tr}(J_{e_i} \circ J_{e_j}) e_j.$$

Let us compute:

$$\begin{aligned}
\mathrm{tr}(QE) &= \sum_{j=1}^{p} \epsilon_j \langle QE(e_j), e_j\rangle \\
&= -\frac{1}{2}\sum_{i,j=1}^{p} \epsilon_i\epsilon_j \langle J_{e_i}E(e_j), J_{e_i}(e_j)\rangle - \frac{1}{4}\sum_{i,j=1}^{p} \epsilon_i\epsilon_j \langle e_i, E(e_j)\rangle \mathrm{tr}(J_{e_i} \circ J_{e_j}) \\
&= -\frac{1}{2}\sum_{i,j=1}^{p} \epsilon_j\epsilon_i \langle e_i, [E(e_j), J_{e_i}(e_j)]\rangle + \frac{1}{4}\sum_{i,j,l=1}^{p} \epsilon_i\epsilon_l\epsilon_j \langle e_i, E(e_j)\rangle \langle J_{e_j}e_l, J_{e_i}e_l\rangle \\
&= -\frac{1}{2}\sum_{i,j,l=1}^{p} \epsilon_j\epsilon_i\epsilon_l \langle J_{e_i}(e_j), e_l\rangle \langle e_i, [E(e_j), e_l]\rangle + \frac{1}{4}\sum_{j,l=1}^{p} \epsilon_l\epsilon_j \langle J_{e_j}e_l, J_{E(e_j)}e_l\rangle \\
&= -\frac{1}{2}\sum_{i,j,l=1}^{p} \epsilon_j\epsilon_i\epsilon_l \langle e_i, [e_j, e_l]\rangle \langle e_i, [E(e_j), e_l]\rangle + \frac{1}{4}\sum_{i,j,l=1}^{p} \epsilon_l\epsilon_j\epsilon_i \langle J_{e_j}e_l, e_i\rangle \langle e_i, J_{E(e_j)}e_l\rangle \\
&= -\frac{1}{2}\sum_{j,l=1}^{p} \epsilon_j\epsilon_l \langle [e_j, e_l], [E(e_j), e_l]\rangle + \frac{1}{4}\sum_{i,j,l=1}^{p} \epsilon_l\epsilon_j\epsilon_i \langle e_j, [e_l, e_i]\rangle \langle [e_l, e_i], E(e_j)\rangle \\
&= -\frac{1}{2}\sum_{j,l=1}^{p} \epsilon_j\epsilon_l \langle [e_j, e_l], [E(e_j), e_l]\rangle + \frac{1}{4}\sum_{i,l=1}^{p} \epsilon_l\epsilon_i \langle [e_l, e_i], E([e_l, e_i])\rangle \\
&= -\frac{1}{4}\sum_{j,l=1}^{p} \epsilon_j\epsilon_l \langle [e_j, e_l], [E(e_j), e_l]\rangle - \frac{1}{4}\sum_{j,l=1}^{p} \epsilon_j\epsilon_l \langle [e_j, e_l], [e_j, E(e_l)]\rangle + \frac{1}{4}\sum_{i,l=1}^{p} \epsilon_l\epsilon_i \langle [e_l, e_i], E([e_l, e_i])\rangle,
\end{aligned}$$

and the formula follows. □

From Proposition 2.3.6 we get the following important result:

Proposition 2.3.7. *([6, Theorem 4.1]) Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean nilpotent Lie algebra having a derivation with non-zero trace. Then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is Einstein if and only if it is Ricci flat.*

Proof. Let $D \in \mathrm{Der}(\mathfrak{g})$ such that $\mathrm{tr}(D) \neq 0$. Write $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{g}}$, using formula (2.5) and (2.13) we get that $\lambda \mathrm{tr}(D) = 0$ and therefore $\lambda = 0$. □

Remark 2. *The derivations of nilpotent Lie algebras have been widely studied and computed (see [17]). It turns out that nilpotent Lie algebras having a derivation with non null trace are the most common. For instance, any nilpotent Lie algebra up to dimension 6 has this property and most of the nilpotent Lie algebras of dimension 7 have this property (see [6]).*

2.4 Einstein Lorentzian nilpotent Lie algebras with degenerate center

In this section, we give a complete description of Einstein Lorentzian nilpotent Lie algebras with degenerate center. We will show that these Lie algebras are obtained by a double extension process of an abelian Euclidean Lie algebra. The double extension process was introduced by Medina-Revoy in [3] in the context of quadratic Lie algebras. It turned out to be useful in many other situations. We give here a version of this process adapted to our study.
Consider a Euclidean vector space $(V,\langle\ ,\ \rangle_0)$, $b\in V$, $K,D:V\longrightarrow V$ two endomorphisms of V such that K is skew-symmetric. We endow the vector space $\mathfrak{g}=\mathbb{R}e\oplus V\oplus\mathbb{R}\bar{e}$ with the inner product $\langle\ ,\ \rangle$ which extends $\langle\ ,\ \rangle_0$ so that span$\{e,\bar{e}\}$ and V are orthogonal, e and $\bar{e}$ are isotropic and satisfy $\langle e,\bar{e}\rangle=1$. We also define on $\mathfrak{g}$ the bracket:

$$[\bar{e},e]=\mu e,\quad [\bar{e},u]=D(u)+\langle b,u\rangle_0 e\quad\text{and}\quad [u,v]=\langle K(u),v\rangle_0 e,\quad u,v\in V. \tag{2.14}$$

Proposition 2.4.1. *Suppose that $(\mathfrak{g},\langle\ ,\ \rangle,[\ ,\ ])$ is obtained by a double extension process from a Euclidean vector space $(V,\langle\ ,\ \rangle_0)$ with parameters (K,D,μ,b) (i.e as in (2.14)), then:*

(*i*) *$(\mathfrak{g},[\ ,\ ])$ is a Lie algebra if and only if :*

$$KD+D^*K=\mu K.$$

In this case $(\mathfrak{g},[\ ,\ ])$ is nilpotent if and only if $\mu=0$ and D is nilpotent.

(*ii*) *$(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is an Einstein Lorentzian Lie algebra if and only if*

$$KD+D^*K=\mu K\quad\textit{and}\quad 4\mu\mathrm{tr}(D)=\mathrm{tr}(K^2)+2\mathrm{tr}(D^2)+2\mathrm{tr}(DD^*).$$

In this case, it is Ricci flat.

Proof. The bracket $[\ ,\ ]$ is a Lie bracket if and only if for any $v,w\in V$,

$$[\bar{e},[v,w]]+[w,[\bar{e},v]]+[v,[w,\bar{e}]]=\langle(\mu K-K\circ D-D^*\circ K)(v),w\rangle_0 e=0.$$

Therefore, $(\mathfrak{g},[\ ,\ ])$ is a Lie algebra if and only if $\mu K=K\circ D+D^*\circ K$ and it is easy to see that $(\mathfrak{g},[\ ,\ ])$ is nilpotent if and only if $\mu=0$ and D is a nilpotent endomorphism.
We will now compute the Ricci curvature of $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ by using the formula

$$\mathrm{ric}(u,v)=-\frac{1}{2}B(u,v)-\frac{1}{2}\langle\mathcal{J}_1(u),v\rangle+\frac{1}{4}\langle\mathcal{J}_2(u),v\rangle-\frac{1}{2}\langle\mathrm{ad}_H u,v\rangle-\frac{1}{2}\langle\mathrm{ad}_H v,u\rangle,$$

where B is the Killing form and H is the vector defined in (2.3).
We choose an orthonormal basis $(f_1,\dots,f_n)$ of V and we denote by $(K_0,\bar{K},S_1,\dots,S_n)$ the structure endomorphisms of $(e,\bar{e},f_1,\dots,f_n)$. By a direct computation, we get that B and H are given by

$$H=(\mu+\mathrm{tr}(D))e,\ \mathbb{R}e\oplus V\subset\ker B\quad\text{and}\quad B(\bar{e},\bar{e})=\mu^2+\mathrm{tr}(D^2).$$

On the other hand, $\bar{K}=0$ and for any $u,v\in\mathfrak{g}$

$$\langle K_0(u),v\rangle=\langle[u,v],\bar{e}\rangle\quad\text{and}\quad\langle S_i(u),v\rangle=\langle[u,v],f_i\rangle,\ u,v\in\mathfrak{g},i=1,\dots,n.$$

This gives that

$$\begin{cases}K_0(e)=-\mu e, K_0(\bar{e})=\mu\bar{e}+b,\ K_0(f_i)=K(f_i),\\ S_i(e)=0, S_i(f_j)=-\langle D^*(f_i),f_j\rangle e\quad\text{and}\quad S_i(\bar{e})=D^*(f_i).\end{cases}$$

From these relations, one can easily deduce that $\mathrm{tr}(K_0\circ S_i)=\mathrm{tr}(S_i\circ S_j)=0$ for $i,j=1,\dots,n$ and hence

$$\mathcal{J}_1=-\sum_{i=1}^n S_i^2\quad\text{and}\quad\mathcal{J}_2=-\langle e,\bullet\rangle\mathrm{tr}(K_0^2)e.$$

Using these expressions, a careful computation gives

$$\mathbb{R}e\oplus V\subset\ker\mathrm{ric}\quad\text{and}\quad\mathrm{ric}(\bar{e},\bar{e})=-\frac{1}{2}\mathrm{tr}(D^2)-\frac{1}{2}\mathrm{tr}(DD^*)-\frac{1}{4}\mathrm{tr}(K^2)+\mu\mathrm{tr}(D).$$

This completes the proof. □

Any data (K,D,μ,b) satisfying the conditions in Proposition 2.4.1 is called *admissible*. We can now state the main theorem of this section, which gives the structure of Einstein Lorentzian nilpotent Lie algebras with degenerate center.

Theorem 2.4.1. *Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be an Einstein nilpotent non abelian Lorentzian Lie algebra and suppose that there exists $e\in Z(\mathfrak{g})$ a central isotropic vector. Then:*

1. *$Z(\mathfrak{g})$ is degenerate and $\mathfrak{g}$ is Ricci-flat.*
2. *$\mathfrak{g}$ is obtained by a double extension process with admissible data $(K,D,0,b)$ and D is nilpotent from a Euclidean vector space V.*

Proof. Let $\mathcal{I}:=\mathbb{R}e$ and choose an orthonormal basis $\mathbb{B}=(e,\bar{e},f_1,\dots,f_n)$ of $\mathfrak{g}$ such that $\bar{e}$ is isotropic with $\langle e,\bar{e}\rangle=1$, $(e,f_1,\dots,f_n)$ is a basis of $\mathcal{I}^\perp$ and $(f_1,\dots,f_n)$ is an orthonormal basis of $\{e,\bar{e}\}^\perp$. Denote $(K,\bar{K},S_1,\dots,S_n)$ the structure endomorphisms of $\mathbb{B}$, i.e. for any $u,v\in\mathfrak{g}$:

$$[u,v]=\langle Ku,v\rangle e+\langle\bar{K}u,v\rangle\bar{e}+\sum_{i=1}^n\langle S_iu,v\rangle f_i.$$

According to (2.5) and (2.9), we have that:

$$\begin{cases} -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}},\ \mathcal{J}_1 = -\bar{K}\circ K - K\circ\bar{K} - \sum_{j=1}^{n} S_j^2, \\ \mathcal{J}_2 = -\left[\langle e,\bullet\rangle \mathrm{tr}(K^2) + \sum_{i=1}^{n}\mathrm{tr}(K\circ S_i)\langle f_i,\bullet\rangle\right]e - \left[\langle \bar{e},\bullet\rangle \mathrm{tr}(\bar{K}^2) + \sum_{i=1}^{n}\mathrm{tr}(\bar{K}\circ S_i)\langle f_i,\bullet\rangle\right]\bar{e} - \langle e,\bullet\rangle\mathrm{tr}(K\circ\bar{K})\bar{e} \\ -\langle \bar{e},\bullet\rangle \mathrm{tr}(K\circ\bar{K})e - \sum_{i=1}^{n}\langle e,\bullet\rangle\mathrm{tr}(K\circ S_i)f_i - \sum_{i=1}^{n}\langle\bar{e},\bullet\rangle\mathrm{tr}(\bar{K}\circ S_i)f_i - \sum_{i,j=1}^{n}\langle f_i,\bullet\rangle\mathrm{tr}(S_i\circ S_j)f_j. \end{cases} \tag{2.15}$$

Since $e\in Z(\mathfrak{g})$ then $K(e)=\bar{K}(e)=S_i(e)=0$ for all $i=1,\dots,n$ and thus $\mathcal{J}_1(e)=0$. This implies that $\frac{1}{4}\mathcal{J}_2(e)=\lambda e$, which is equivalent to:

$$\frac{1}{4}\mathrm{tr}(K\circ\bar{K}) = -\lambda \quad \text{and} \quad \mathrm{tr}(\bar{K}^2) = \mathrm{tr}(\bar{K}\circ S_i) = 0 \text{ for } i=1,\dots,n.$$

According to Lemma 2.2.2, we get that for any $x\in\mathcal{I}^\perp$, $\bar{K}(x)=\alpha(x)e$ and $-4\lambda=\mathrm{tr}(\bar{K}\circ K)=0$. On the other hand, since $\mathrm{tr}(\mathcal{J}_1)=\mathrm{tr}(\mathcal{J}_2)$ then from the first relation in system (2.15) we deduce that $\mathrm{tr}(\mathcal{J}_1) = -\sum_{i=1}^n \mathrm{tr}(S_i^2)=0$. Again, Lemma 2.2.2 along with $\mathrm{tr}(S_i^2)=0$ gives that for any $x\in\mathcal{I}^\perp$, $S_i(x)=s_i(x)e$ and $\mathrm{tr}(K\circ S_i)=\mathrm{tr}(S_i\circ S_j)=0$ for $i,j\in\{1,\dots,n\}$. By skew-symmetry, we deduce that, for $j=1,\dots,n$

$$\bar{K}(\bar{e}) = -\sum_{i=1}^{n}\alpha(f_i)f_i \quad \text{and} \quad S_j(\bar{e}) = -\sum_{i=1}^{n}s_j(f_i)f_i.$$

On the other hand, for any $u\in\mathcal{I}^\perp$,

$$[\bar{e},u] = \langle K(\bar{e}),u\rangle e - \alpha(u)\bar{e} - \sum_{i=1}^{n}s_i(u)f_i.$$

But since ad_u is nilpotent then we must have $\alpha(u)=0$ for any $u\in\mathcal{I}^\perp$ and thus $\bar{K}=0$. To sum up, if we put $V=\mathrm{span}\{f_1,\dots,f_n\}$ and define $D:V\longrightarrow V$ by $D(u)=\sum_{i=1}^n\langle S_i(\bar{e}),u\rangle f_i$, then:

$$\begin{cases} [u,v] = \langle Ku,v\rangle e,\ u,v\in V, \\ [\bar{e},u] = \langle K(\bar{e}),u\rangle e + D(u),\ u\in V, \\ \mathcal{J}_2 = \langle e,\bullet\rangle \mathrm{tr}(K^2)e, \\ -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = 0,\ \mathcal{J}_1 = -\sum_{j=1}^n S_j^2. \end{cases}$$

This completes the proof. □

As an application of Theorem 1.3.1 we recover the following results due to Guediri [18, Lemma 14 and Theorem 15] :

Corollary 2.4.1. *Let $\mathfrak{g}$ be an Einstein Lorentzian 2-step nilpotent Lie algebra. Then $Z(\mathfrak{g})$ is degenerate and $\mathfrak{g}$ is Ricci-flat.*

Theorem 2.4.2. *Let $\mathfrak{g}$ be a 2-step nilpotent, non-abelian Lie algebra. Then $\mathfrak{g}$ admits a Ricci-flat Lorentzian metric if and only if $\mathfrak{g}=\mathbb{R}^n\oplus\mathfrak{n}$ (a direct sum of Lie algebras) such that $\mathfrak{n}$ is a*

Lie algebra for which the Lie brackets are expressed in a basis $\mathscr{B} = \{e, z_1, \ldots, z_p, \bar{e}, e_1, \ldots, e_q\}$ as follows :

$$[\bar{e}, e_i] = \alpha_i e + \sum_{k=1}^{p} c_{ik} z_k, \quad [e_i, e_j] = a_{ij} e, \quad 1 \le i, j \le q, \tag{2.16}$$

with $\sum_{i,j=1}^{q} a_{ij}^2 = 2 \sum_{i=1}^{q} \sum_{k=1}^{p} c_{ik}^2$. Moreover the basis $\mathscr{B}$ can be chosen Lorentzian, in particular the restriction of the metric to $[\mathfrak{g}, \mathfrak{g}]$ is degenerate.

Proof. Suppose that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a 2-step nilpotent Lorentzian, Ricci-flat Lie algebra. By virtue of Corollary 2.4.1, $\mathrm{Z}(\mathfrak{g})$ is degenerate and Theorem 1.3.1 implies that $\mathfrak{g}$ is given by a process of double extension from a Euclidean vector space V_0 with parameters $(K, D, 0, b)$, i.e $\mathfrak{g} = \mathbb{R}e \oplus V_0 \oplus \mathbb{R}\bar{e}$ where $e, \bar{e}$ are isotropic vectors satisfying $\langle \bar{e}, e \rangle = 1$ and, for any $u, v \in V_0$,

$$[\bar{e}, u] = D(u) + \langle b, u \rangle e, \quad [u, v] = \langle K(u), v \rangle e. \tag{2.17}$$

Moreover, Proposition 2.4.1 implies that $D^2 = 0$ and

$$K \circ D + D^* \circ K = 0, \qquad 2\mathrm{tr}(DD^*) = -\mathrm{tr}(K^2). \tag{2.18}$$

First, we observe that $\mathrm{Im}(D) \subset \mathrm{Z}(\mathfrak{g}) \cap V_0$. Indeed, given $w \in \mathfrak{g}$ and $u \in V_0$ we have that :

$$[w, Du] = [w, Du + \langle b, u \rangle e] = [w, [\bar{e}, u]] = 0.$$

Write $V_0 = (V_0 \cap \mathrm{Z}(\mathfrak{g})) \overset{\perp}{\oplus} W_0$ and $V_0 \cap \mathrm{Z}(\mathfrak{g}) = \mathrm{Im}(D) \overset{\perp}{\oplus} S$, then S is an abelian Lie subalgebra of $\mathfrak{g}$ since it is contained in $\mathrm{Z}(\mathfrak{g})$ and we have that $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{n}$ with $\mathfrak{n} = \mathbb{R}e \oplus \mathbb{R}\bar{e} \oplus \mathrm{Im}(D) \oplus W_0$, moreover using (2.17) we can check that $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$. Next, let $\{z_1, \ldots, z_p\}$ be a Euclidean basis of $\mathrm{Im}(D)$ and let $\{e_1, \ldots, e_q\}$ be a Euclidean basis of W_0. Write :

$$D(e_i) = \sum_{k=1}^{p} c_{ik} z_k, \quad \langle b, e_i \rangle = \alpha_i, \quad \langle K(e_i), e_j \rangle = a_{ij}.$$

Then it follows that :

$$[\bar{e}, e_i] = \alpha_i e + \sum_{k=1}^{p} c_{ik} z_k, \quad [e_i, e_j] = a_{ij} e, \quad 1 \le i, j \le q.$$

Now

$$\mathrm{tr}(K^2) = -\sum_{i,j=1}^{q} a_{ij}^2 = 2\mathrm{tr}(DD^*) = \sum_{i=1}^{q} \sum_{k=1}^{p} c_{ik}^2$$

and we conclude by (2.18). Conversely, for any 2-step nilpotent Lie algebra $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{n}$ satisfying (2.16), the equation $\mathrm{Ric}_{\mathfrak{g}} = 0$ follows from a straightforward calculation. □

2.5 Classification of Einstein Lorentzian nilpotent Lie algebras of dimension ≤ 5

In this section, we give a complete description of the Lorentzian Lie algebras associated to all Einstein Lorentzian nilpotent Lie groups of dimension ≤ 5. This classification is based on Theorem 1.3.1 and the following result.

Theorem 2.5.1. *Let* $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ *be an Einstein Lorentzian nilpotent Lie algebra of dimension less than* 5*. Then the center of* $\mathfrak{g}$ *is degenerate.*

Proof. We use the classification of nilpotent Lie algebras up to dimension 6 given by [9]. We will also use Corollary 2.4.1 and Proposition 2.3.4.
There is a unique nilpotent Lie algebra in dimension 3 which is $L_{3,2}$ and it is 2-step nilpotent hence we can apply Corollary 2.3.1. In dimension 4, there is two nilpotent Lie algebras namely $L_{3,2}\oplus\mathbb{R}$ whose center is degenerate by Corollary 2.4.1 and $L_{4,3}$ whose Lie bracket is given by

$$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4.$$

It is clear that $\mathrm{L}_{4,3}$ satisfies the hypothesis of Proposition 2.3.4. Five dimensional nilpotent Lie algebras can be listed as in Table 2.2.

We can see that apart from $L_{5,6}$ and $L_{5,7}$ all the other Lie algebras are either 2-step nilpotent or satisfy the hypothesis of Proposition 2.3.4. Let us now study $L_{5,6}$ and $L_{5,7}$.
If we denote by $\mathfrak{g}$ either $L_{5,6}$ or $L_{5,7}$, one can see that

$$Z(\mathfrak{g})\subset[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]\subset[\mathfrak{g},\mathfrak{g}],\ \dim Z(\mathfrak{g})=1,\ \dim[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=2\quad\text{and}\quad\dim[\mathfrak{g},\mathfrak{g}]=3.\tag{2.19}$$

To complete the proof of the theorem, we will show that if a five dimensional nilpotent Lie algebra $\mathfrak{g}$ satisfies (2.19) and have an Einstein Lorentzian metric then its center must be degenerate.
Let $\mathfrak{g}$ be a five dimensional Einstein Lorentzian nilpotent Lie algebra satisfying (2.19) such that its center non-degenerate. First note that according to [6, Theorem 4.3], $\mathfrak{g}$ must be Ricci flat. According to Corollary 2.3.1 and Propositions 2.3.1 and 2.3.3, $Z(\mathfrak{g})$ must be Euclidean and $[\mathfrak{g},\mathfrak{g}]$ must be non-degenerate Lorentzian. We distinguish three cases:

1. $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ **is non-degenerate Euclidean**. It is then possible to choose an orthonormal basis (f_1,f_2,f_3,f_4,f_5) of $\mathfrak{g}$ such that $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}\{f_3,f_4,f_5\}$, $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=\mathrm{span}\{f_4,f_5\}$, $Z(\mathfrak{g})=\mathbb{R}f_5$ and $\langle f_3,f_3\rangle=-1$. So:

$$\begin{cases}[f_1,f_2]=af_3+bf_4+cf_5,[f_1,f_3]=df_4+xf_5,[f_1,f_4]=yf_5,\\ [f_2,f_3]=zf_4+tf_5,[f_2,f_4]=uf_5,\ [f_3,f_4]=vf_5,\qquad a\neq0,(z,d)\neq(0,0).\end{cases}$$

This bracket satisfies the Jacobi identity if and only if $v=0$ and $yz-du=0$. The Ricci operator is given by

$$\frac{1}{2}\begin{pmatrix} a^2-b^2-c^2+d^2+x^2-y^2 & dz+xt-yu & zb+ct & cu & 0 \\ dz+xt-yu & a^2-b^2-c^2+z^2+t^2-u^2 & -bd-cx & -cy & 0 \\ -zb-ct & bd+cx & -a^2+d^2+x^2+z^2+t^2 & ab+xy+tu & ac \\ cu & -cy & -ab-xy-tu & b^2-d^2-y^2-z^2-u^2 & bc-dx-zt \\ 0 & 0 & -ac & bc-dx-zt & c^2-x^2+y^2-t^2+u^2 \end{pmatrix}.$$

Since $a \neq 0$ then $c = 0$ and hence the Ricci operator is given by

$$\frac{1}{2}\begin{pmatrix} a^2-b^2+d^2+x^2-y^2 & dz+xt-yu & zb & 0 & 0 \\ dz+xt-yu & a^2-b^2+z^2+t^2-u^2 & -bd & 0 & 0 \\ -zb & bd & -a^2+d^2+x^2+z^2+t^2 & ab+xy+tu & 0 \\ 0 & 0 & -ab-xy-tu & b^2-d^2-y^2-z^2-u^2 & -dx-zt \\ 0 & 0 & & -dx-zt & -x^2+y^2-t^2+u^2 \end{pmatrix}.$$

The couple $(z,d) \neq (0,0)$ otherwise $\dim[\mathfrak{g},\mathfrak{g}] \leq 2$, hence $b = 0$. So

$$\frac{1}{2}\begin{pmatrix} a^2+d^2+x^2-y^2 & dz+xt-yu & 0 & 0 & 0 \\ dz+xt-yu & a^2+z^2+t^2-u^2 & 0 & 0 & 0 \\ 0 & 0 & -a^2+d^2+x^2+z^2+t^2 & xy+tu & \\ 0 & 0 & -xy-tu & -d^2-y^2-z^2-u^2 & -dx-zt \\ 0 & 0 & & -dx-zt & -x^2+y^2-t^2+u^2 \end{pmatrix}.$$

So we must have $\mathrm{Ric}_{4,4} = -d^2-y^2-z^2-u^2 = 0$ and $\mathrm{Ric}_{2,2} = a^2+z^2+t^2-u^2 = 0$, but this implies that $a = 0$ which is impossible.

2. $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ **is nondegenerate Lorentzian**. As in the previous case, we can choose an orthonormal basis $(f_1, f_2, f_3, f_4, f_5)$ such that $\langle f_4, f_4\rangle = -1$ and $Z(\mathfrak{g}) = \mathbb{R}f_5$, $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = \mathrm{span}\{f_4, f_5\}$ and $[\mathfrak{g},\mathfrak{g}] = \mathrm{span}\{f_3, f_4, f_5\}$. So

$$\begin{cases} [f_1,f_2] = af_3+bf_4+cf_5, [f_1,f_3] = df_4+xf_5, [f_1,f_4] = yf_5, \\ [f_2,f_3] = zf_4+tf_5, [f_2,f_4] = uf_5,\ [f_3,f_4] = vf_5, \quad a \neq 0, (z,d) \neq (0,0). \end{cases}$$

The Jacobi identity is given by $bv - ud + yz = av = 0$, hence $v = 0$. Thus the Ricci operator is given by

$$\frac{1}{2}\begin{pmatrix} -a^2+b^2-c^2+d^2-x^2+y^2 & dz-xt+yu & -zb+ct & cu & 0 \\ dz-xt+yu & -a^2+b^2-c^2+z^2-t^2+u^2 & bd-cx & -cy & 0 \\ -zb+ct & bd-cx & a^2+d^2-x^2+z^2-t^2 & -ab-xy-tu & ac \\ -cu & cy & ab+xy+tu & -b^2-d^2+y^2-z^2+u^2 & bc+dx+zt \\ 0 & 0 & ac & -bc-dx-zt & c^2+x^2-y^2+t^2-u^2 \end{pmatrix}$$

So we get $b=c=0$ and hence The Ricci operator is given by

$$\frac{1}{2}\begin{pmatrix} -a^2+d^2-x^2+y^2 & dz-xt+yu & 0 & 0 & 0 \\ dz-xt+yu & -a^2+z^2-t^2+u^2 & 0 & 0 & 0 \\ 0 & 0 & a^2+d^2-x^2+z^2-t^2 & -xy-tu & 0 \\ 0 & 0 & xy+tu & -d^2+y^2-z^2+u^2 & dx+zt \\ 0 & 0 & 0 & -dx-zt & x^2-y^2+t^2-u^2 \end{pmatrix}$$

Now $0=\mathrm{Ric}_{3,3}+\mathrm{Ric}_{4,4}+\mathrm{Ric}_{5,5}=\frac{1}{2}a^2$ and hence $a=0$ which is impossible.

3. $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ **is degenerate** . Then we can choose a basis (f_1,f_2,f_3,f_4,f_5) such that the metric in this basis is given by

$$\mathrm{Diag}\left(1,1,\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},1\right),$$

and $Z(\mathfrak{g})=\mathbb{R}f_5$, $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=\mathrm{span}\{f_4,f_5\}$ and $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}\{f_3,f_4,f_5\}$. So

$$\begin{cases} [f_1,f_2]=af_3+bf_4+cf_5, [f_1,f_3]=df_4+xf_5, [f_1,f_4]=yf_5, \\ [f_2,f_3]=zf_4+tf_5, [f_2,f_4]=uf_5,\ [f_3,f_4]=vf_5, \quad a\neq 0, (z,d)\neq(0,0). \end{cases}$$

The Jacobi identity is given by $bv-ud+yz=av=0$. Hence $v=0$. The Ricci operator is given by

$$\frac{1}{2}\begin{pmatrix} -2ab-c^2-2xy & -yt-xu & az+ct & cu & 0 \\ -yt-xu & -2ab-c^2-2tu & -cx-ad & -cy & 0 \\ cu & -cy & ab-xy-tu & a^2-y^2-u^2 & ac \\ az+ct & -cx-ad & b^2-x^2-t^2 & ab-xy-tu & bc+dy+zu \\ 0 & 0 & bc+dy+zu & ac & c^2+2xy+2tu \end{pmatrix}.$$

So $c=d=z=0$ which is impossible. □

As a consequence of Theorem 1.3.1 and Theorem 2.5.1, we can give the complete classification of Ricci flat Lorentzian metrics on nilpotent Lie algebras of dimension ≤ 5. We will also make use of the following Lemma :

Lemma 2.5.1. *Let $(V,\langle\ ,\ \rangle)$ be a Euclidean vector space and let K and D be two endomorphisms of V such that K is skew-symmetric. Then $KD+D^*K=0$ if and only if there exists a vector subspace $F\subset V$ and linear maps $D_1: F\longrightarrow F$, $D_2: F^\perp\longrightarrow F$, $K_0, S: F^\perp\longrightarrow F^\perp$ where K_0 is skew-symmetric invertible, S symmetric and for any $u\in V$,*

$$Du=\begin{cases} D_1(u) & \text{if} \quad u\in F,\\ D_2(u)+K_0^{-1}S(u) & \text{if} \quad u\in F^\perp\end{cases} \quad \text{and} \quad Ku=\begin{cases} 0 & \text{if} \quad u\in F,\\ K_0(u) & \text{if} \quad u\in F^\perp.\end{cases}$$

Proof. Suppose that $KD+D^*K=0$ and put $F=\ker K$. Obviously $D(F)\subset F$, $K(F^\perp)\subset F^\perp$ and the restriction K_0 of K to $F^\perp$ is skew-symmetric invertible. Denote by D_1 the restriction of D to F and put for any $u\in F^\perp$, $Du=D_2u+D_3u$ where $D_2u\in F$ and $D_3u\in F^\perp$. Then

$$0=K(D_2u+D_3u)+D^*K_0(u)=K_0D_3u+D_3^*K_0(u).$$

Thus $K_0D_3=S$ where $S: F^\perp\longrightarrow F^\perp$ is a symmetric endomorphism and $D_3=K_0^{-1}S$. The converse is obviously true. □

Theorem 2.5.2. *Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be a Ricci-flat nilpotent Lie algebra of dimension ≤ 4. Then:*

(i) If $\dim\mathfrak{g}=3$ then $\mathfrak{g}$ is isomorphic to $(L_{3,2},\langle\ ,\ \rangle_{3,2})$ such that $\langle\ ,\ \rangle_{3,2}=\alpha e_1^*\odot e_3^*+e_2^*\otimes e_2^*$ and $\alpha>0$. This metric is actually flat.

(ii) If $\dim\mathfrak{g}=4$ then $\mathfrak{g}$ is isomorphic to $(L_{4,2},\langle\ ,\ \rangle_{4,2})$ with

$$\langle\ ,\ \rangle_{4,2}=\alpha e_1^*\odot e_3^*+e_2^*\otimes e_2^*+e_4^*\otimes e_4^*+ae_2^*\odot e_4^*,\quad \alpha\neq 0, |a|<1,$$

or to $(L_{4,3},\langle\ ,\ \rangle_{4,3})$ with

$$\langle\ ,\ \rangle_{4,3}=e_1^*\otimes e_1^*+ae_1^*\odot e_2^*+(a^2+b^2)e_2^*\otimes e_2^*+be_2^*\odot e_3^*+\epsilon e_2^*\odot e_4^*+e_3^*\otimes e_3^*,\ a,b\in\mathbb{R},\epsilon=\pm1.$$

The metric $\langle\ ,\ \rangle_{4,2}$ is flat and $\langle\ ,\ \rangle_{4,3}$ is flat if and only if $\epsilon=-1$.

Proof. Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent non abelian Lie algebra of dimension ≤ 5. According Theorems 1.3.1 and 2.5.1, $\mathfrak{g}=\mathbb{R}e\oplus V\oplus\mathbb{R}\bar{e}$, where $(V,\langle\ ,\ \rangle_0)$ is a Euclidean vector space. The Lie brackets are given by:

$$[\bar{e},u]=Du+\langle b,u\rangle_0 e \quad \text{and} \quad [u,v]=\langle Ku,v\rangle_0 e,\quad u,v\in V,$$

such that $b\in V$, $K,D: V\longrightarrow V$ with K skew-symmetric, D is nilpotent, $KD+D^*K=0$ and $\mathrm{tr}(K^2)=-2\mathrm{tr}(D^*D)$ furthermore the metric $\langle\ ,\ \rangle$ satisfies $\langle\ ,\ \rangle_{|V}=\langle\ ,\ \rangle_0$, e and $\bar{e}$ are co-isotropic i.e $\langle e,\bar{e}\rangle=1$ and are orthogonal to V.

1. If $\dim\mathfrak{g}=3$ and $\dim V=1$. Then $K=D=0$ and the Lie algebra $(\mathfrak{g},\langle\ ,\ \rangle)$ is isomorphic to $(L_{3,2},\langle\ ,\ \rangle_{3,2})$ where $\langle\ ,\ \rangle_{3,2}=\alpha e_1^*\odot e_3^*+e_2^*\otimes e_2^*$ and $\alpha>0$. This metric is flat.

2. $\dim\mathfrak{g}=4$ and $\dim V=2$. We distinguish two cases:

- If $K = 0$ then $D = 0$ and there exists a Lorentzian basis $(\bar{e}, e, f_1, f_2)$ of $\mathfrak{g}$ such that:

$$[\bar{e}, f_1] = \alpha e \quad \text{and} \quad [\bar{e}, f_2] = \beta e, \quad \alpha \neq 0.$$

  Put

$$(e_1, e_2, e_3, e_4) = (\epsilon\bar{e}, f_1, |\alpha|e, \mu^{-1}(f_2 - \frac{\beta}{\alpha} f_1)),$$

  where ϵ is the sign of α and $\mu = \|f_2 - \frac{\beta}{\alpha} f_1\|$. The Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is then isomorphic to $(L_{4,2}, \langle\ ,\ \rangle_{4,2})$ with the metric:

$$\langle\ ,\ \rangle_{4,2} = \alpha e_1^* \odot e_3^* + e_2^* \otimes e_2^* + e_4^* \otimes e_4^* + a e_2^* \odot e_4^*, \quad \alpha \neq 0$$

  and $a = \frac{\beta_1}{\sqrt{1+\beta_1^2}}$ where $\beta_1 = \frac{\beta}{\alpha}$. So $|a| < 1$.

- If $K \neq 0$ then, according to Lemma 2.5.1, $D = K^{-1}S$ where S is symmetric. Since D must be nilpotent then the rank of S is equal to 1 and there exists an orthonormal basis $\mathbb{B}_0 = (f_1, f_2)$ of V such that the matrices of K, S and D are given by:

$$M(S, \mathbb{B}_0) = \mathrm{Diag}(0, s),\ M(K, \mathbb{B}_0) = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \quad \text{and} \quad M(D, \mathbb{B}_0) = \begin{pmatrix} 0 & s\alpha^{-1} \\ 0 & 0 \end{pmatrix},\ \alpha > 0.$$

  Put $c = s\alpha^{-1}$. The condition $\mathrm{tr}(K^2) = -2\mathrm{tr}(D^*D)$ gives $c = \epsilon\alpha$ with $\epsilon = \pm 1$. Thus the Lie brackets are given by:

$$[\bar{e}, f_1] = \gamma e,\ [\bar{e}, f_2] = \epsilon\alpha f_1 + \mu e \quad \text{and} \quad [f_1, f_2] = \alpha e.$$

  Put

$$(e_1, e_2, e_3, e_4) = (f_2, -\epsilon\alpha^{-1}\bar{e} + a f_1 + b f_2, f_1, -\alpha e)$$

  with $a = \epsilon\mu\alpha^{-2}$ and $b = -\epsilon\gamma\alpha^{-2}$. Then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is isomorphic to $(L_{4,3}, \langle\ ,\ \rangle_{4,3})$.

□

Theorem 2.5.3. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a Ricci-flat nilpotent Lie algebra of dimension 5. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

(*a*) $(L_{5,2}, \langle\ ,\ \rangle_{5,2})$ *with*

$$\langle\ ,\ \rangle_{5,2} = \alpha e_1^* \odot e_3^* + e_2^* \otimes e_2^* + e_4^* \otimes e_4^* + e_5^* \otimes e_5^* + a e_2^* \odot e_4^* + b e_2^* \odot e_5^* + ab e_4^* \odot e_5^*,\ \alpha \neq 0, |a| < 1, |b| < 1.$$

This metric is flat.

(*b*) $(L_{5,8}, \langle\ ,\ \rangle_{5,8})$ *with*

$$\begin{aligned} \langle\ ,\ \rangle_{5,8} &= e_1^* \otimes e_1^* + a e_1^* \odot e_2^* - yx^{-1} e_1^* \odot e_3^* + (b - ayx^{-1}) e_2^* \odot e_3^* + (a^2 + b^2) e_2^* \odot e_2^* \\ &\quad + \sqrt{x^2 + y^2} e_2^* \otimes e_5^* + (1 + (yx^{-1})^2) e_3^* \otimes e_3^* + x^2 e_4^* \otimes e_4^*,\ (x \neq 0, a, b, y \in \mathbb{R}). \end{aligned}$$

(c) $(L_{5,9}, \langle\ ,\ \rangle_{5,9})$ *with*

$$\begin{aligned}\langle\ ,\ \rangle_{5,9} &= (a^2+b^2)e_1^*\otimes e_1^* + (b-ayx^{-1})e_1^*\odot e_2^* + ae_1^*\odot e_3^* + \epsilon\sqrt{x^2+y^2+1}e_1^*\odot e_5^* \\ &\quad (1+(yx^{-1})^2)e_2^*\otimes e_2^* - yx^{-1}e_2^*\odot e_3^* + e_3^*\otimes e_3^* + x^2e_4^*\otimes e_4^*.\ (x\neq 0, a, b, y\in\mathbb{R}).\end{aligned}$$

(d) $(L_{5,3}, \langle\ ,\ \rangle_{5,3})$ *with*

$$\begin{aligned}\langle\ ,\ \rangle_{5,3} &= e_1^*\otimes e_1^* + ae_1^*\odot e_2^* + (a^2+b^2)e_2^*\otimes e_2^* + be_2^*\odot e_3^* + \epsilon\sqrt{x^2+1}e_2^*\odot e_4^* \\ &\quad +(1+x^2)e_3^*\otimes e_3^* - xe_3^*\odot e_5^* + e_5^*\otimes e_5^*,\ (x, a, b\in\mathbb{R}).\end{aligned}$$

(e) $(L_{5,5}, \langle\ ,\ \rangle_{5,5,1})$ *or* $(L_{5,5}, \langle\ ,\ \rangle_{5,5,2})$ *with*

$$\begin{aligned}\langle\ ,\ \rangle_{5,5,1} &= (a^2+b^2)e_1^*\otimes e_1^* + a\rho^{-1}e_1^*\odot e_2^* + \rho(b-ax^{-1}y)e_1^*\odot e_4^* + \sqrt{x^2+y^2}e_1^*\odot e_5^* \\ &\quad +\rho^{-2}e_2^*\otimes e_2^* - x^{-1}ye_2^*\odot e_4^* + x^2\rho^{-2}e_3^*\otimes e_3^* + \rho^2(1+(x^{-1}y)^2)e_4^*\otimes e_4^*, \\ &\quad (x\neq 0, \rho\neq 0, a, b, y\in\mathbb{R})\end{aligned}$$

or

$$\begin{aligned}\langle\ ,\ \rangle_{5,5,2} &= e_1^*\otimes e_1^* + be_1^*\odot e_2^* + (a^2+b^2)e_2^*\otimes e_2^* + ae_2^*\odot e_3^* + \epsilon\sqrt{x^2+1}e_2^*\odot e_5^* \\ &\quad (1+x^2)e_3^*\otimes e_3^* + x\rho e_3^*\odot e_4^* + \rho^2e_4^*\otimes e_4^*, \quad (\rho\neq 0, x, a, b\in\mathbb{R}).\end{aligned}$$

(f) $(L_{5,6}, \langle\ ,\ \rangle_{5,6})$ *with*

$$\begin{aligned}\langle\ ,\ \rangle_{5,6} &= (a^2+b^2)e_1^*\otimes e_1^* + (b+ax^{-1}y)e_1^*\odot e_2^* + \mu ae_1^*\odot e_3^* + \epsilon\mu^2\sqrt{x^2+y^2+1}e_1^*\odot e_5^* \\ &\quad +(1+x^{-2}y^2)e_2^*\otimes e_2^* + \mu x^{-1}ye_2^*\odot e_3^* + \mu^2e_3^*\otimes e_3^* + \mu^4x^2e_4^*\otimes e_4^*, \\ &\quad \mu\neq 0, \gamma\neq 0, x\neq 0, a, b, y\in\mathbb{R}.\end{aligned}$$

Proof. According to Theorems 1.3.1 and (2.5.1), $\mathfrak{g} = \mathbb{R}e\oplus V\oplus\mathbb{R}\bar{e}$, where $(V, \langle\ ,\ \rangle_0)$ is a 3-dimensional Euclidean vector space. The Lie bracket is given by:

$$[\bar{e}, u] = Du + \langle b, u\rangle_0 e \quad\text{and}\quad [u, v] = \langle Ku, v\rangle_0 e, \quad u, v\in V,$$

with $b\in V$, $K, D : V\longrightarrow V$ with K skew-symmetric, D is nilpotent such that $KD+D^*K=0$ and $\mathrm{tr}(K^2) = -2\mathrm{tr}(D^*D)$ moreover the metric $\langle\ ,\ \rangle$ satisfies $\langle\ ,\ \rangle_{|V} = \langle\ ,\ \rangle_0$, e and $\bar{e}$ are co-isotropic i.e $\langle e, \bar{e}\rangle = 1$ and are orthogonal to V.

• If $K = D = 0$ then there exists a Lorentzian basis $(\bar{e}, e, f_1, f_2, f_3)$ such that:

$$[\bar{e}, f_1] = \alpha e,\ [\bar{e}, f_2] = \beta e \quad\text{and}\quad [\bar{e}, f_3] = \gamma e,\ \alpha\neq 0.$$

Put

$$(e_1, e_2, e_3, e_4, e_5) = (\epsilon\bar{e}, f_1, |\alpha|e, \mu_1^{-1}(f_2-\frac{\beta}{\alpha}f_1), \mu_2^{-1}(f_3-\frac{\gamma}{\alpha}f_1)),$$

where ϵ is the sign of α, $\mu_1 = \|f_2-\frac{\beta}{\alpha}f_1\|$ and $\mu_2 = \|f_3-\frac{\gamma}{\alpha}f_1\|$. Thus $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is isomorphic to $(L_{5,2}, \langle\ ,\ \rangle_{5,2})$ with:

$$\langle\ ,\ \rangle_{5,2} = \alpha e_1^*\odot e_3^* + e_2^*\otimes e_2^* + e_4^*\otimes e_4^* + e_5^*\otimes e_5^* + ae_2^*\odot e_4^* + be_2^*\odot e_5^* + abe_4^*\odot e_5^*,$$

where $\alpha \neq 0$, $a = \frac{\beta_1}{\sqrt{1+\beta_1^2}}$, $b = \frac{\gamma_1}{\sqrt{1+\gamma_1^2}}$, $\beta_1 = \frac{\beta}{\alpha}$ and $\gamma_1 = \frac{\gamma}{\alpha}$. So $|a| < 1$ and $|b| < 1$.

• If $K \neq 0$, Lemma 2.5.1 show that there exists and orthonormal basis $\mathbb{B}_0 = (f_1, f_2, f_3)$ of V in which the matrices of K, S and D are given by:

$$M(S,\mathbb{B}_0) = \mathrm{Diag}(0,a),\ M(K,\mathbb{B}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \quad \text{and} \quad M(D,\mathbb{B}_0) = \begin{pmatrix} 0 & x & y \\ 0 & 0 & a\alpha^{-1} \\ 0 & 0 & 0 \end{pmatrix},\ \alpha > 0.$$

Put $c = a\alpha^{-1}$. The condition $\mathrm{tr}(K^2) = -2\mathrm{tr}(D^*D)$ gives $\alpha = \sqrt{x^2+y^2+c^2}$. Thus the Lie bracket is given by:

$$[\bar{e}, f_1] = \gamma e,\ [\bar{e}, f_2] = xf_1 + \mu e,\ [\bar{e}, f_3] = yf_1 + cf_2 + \beta e \quad \text{and} \quad [f_2, f_3] = \alpha e.$$

Put $a = -\beta\alpha^{-1}$, $b = \mu\alpha^{-1}$, $z = \alpha e$ and $\bar{z} = \bar{e} + af_2 + bf_3$. We have:

$$[\bar{z}, f_1] = \gamma\alpha^{-1} z,\ [\bar{z}, f_2] = xf_1,\ [\bar{z}, f_3] = yf_1 + cf_2 \quad \text{and} \quad [f_2, f_3] = z.$$

Case 1: $\underline{\gamma = 0,\ x \neq 0 \text{ and } c = 0}$. Then:

$$[\bar{z}, f_2] = xf_1 \quad \text{and} \quad [f_2, f_3 - yx^{-1}f_2] = z.$$

Put $(e_1, e_2, e_3, e_4, e_5) = (f_2, \bar{e} + af_2 + bf_3, f_3 - yx^{-1}f_2, -xf_1, \alpha e)$ Thus $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is isomorphic to $(L_{5,8}, \langle\ ,\ \rangle_{5,8})$.

Case 2: $\underline{\gamma = 0,\ x \neq 0 \text{ and } c \neq 0}$. Then:

$$[\bar{z}, f_2] = xf_1,\ [\bar{z}, f_3 - yx^{-1}f_2] = cf_2 \quad \text{and} \quad [f_2, f_3 - yx^{-1}f_2] = z.$$

Put

$$(e_1, e_2, e_3, e_4, e_5) = (c^{-1}(\bar{e} + af_1 + bf_2), f_3 - yx^{-1}f_2, f_2, c^{-1}xf_1, -\alpha e)).$$

After the change of parameters $c^{-1}(a, b, x, y)$ to (a, b, x, y), we get that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is isomorphic to $(L_{5,9}, \langle\ ,\ \rangle_{5,9})$.

Case 3: $\underline{\gamma = 0,\ x = 0,\ c = 0}$. Put:

$$(e_1, e_2, e_3, e_4, e_5) = (f_3, \bar{e} + af_2 + bf_3, -f_2, -yf_1, \alpha e).$$

Thus $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is isomorphic to $(L_{5,8}, \langle\ ,\ \rangle_{5,8})$ with $b = 0$ and $y = 0$.

Case 4: $\underline{\gamma = 0,\ x = 0,\ c \neq 0}$. Put:

$$(e_1, e_2, e_3, e_4, e_5) = (f_3, c^{-1}(\bar{e} + af_2 + bf_3), -f_2 - c^{-1}yf_1, \alpha e, f_1).$$

After the change of parameters $c^{-1}(a, b, y)$ to (a, b, y) we get that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is isomorphic to $(L_{5,3}, \langle\ ,\ \rangle_{5,3})$.

Case 5: $\underline{\gamma \neq 0}$. Put $g_1 = \alpha\gamma^{-1}f_1$, then:

$$[\bar{z}, g_1] = z,\ [\bar{z}, f_2] = x\alpha^{-1}\gamma g_1,\ [\bar{z}, f_3] = y\alpha^{-1}\gamma g_1 + cf_2 \quad \text{and} \quad [f_2, f_3] = z.$$

$\underline{\gamma \neq 0 \text{ and } c = 0}$. Then $(x,y) \neq (0,0)$ and we can suppose that $x \neq 0$. Then:

$$[\bar{z}, g_1] = z,\ [\bar{z}, f_2] = x\alpha^{-1}\gamma g_1,\ [\bar{z}, f_3 - x^{-1}yf_2] = 0 \quad \text{and} \quad [f_2, f_3 - x^{-1}yf_2] = z.$$

Put

$$(e_1,e_2,e_3,e_4,e_5) = (\bar{e}+af_2+bf_3, x^{-1}\alpha\gamma^{-1}f_2, \alpha\gamma^{-1}f_1, x\alpha^{-1}\gamma(f_3-x^{-1}yf_2), \alpha e) \quad \text{and} \quad \rho = x\alpha^{-1}\gamma.$$

Then $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is isomorphic to$(L_{5,5},\langle\ ,\ \rangle_{5,5,1})$.

$\underline{\gamma \neq 0,\ c \neq 0 \text{ and } x = 0}$. Then:

$$[\bar{z}, g_1] = z,\ [\bar{z}, f_3] = y\alpha^{-1}\gamma g_1 + cf_2 \quad \text{and} \quad [f_2, f_3] = z.$$

Put

$$(e_1,e_2,e_3,e_4,e_5) = (-f_3, c^{-1}(\bar{e}+af_2+bf_3), f_2 + c^{-1}y\alpha^{-1}\gamma g_1, cg_1, \alpha e).$$

After the change of parameters $c^{-1}(a,b,y)$ to (a,b,x) and $\rho = c\alpha\gamma^{-1}$ we get that $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is isomorphic to $(L_{5,5},\langle\ ,\ \rangle_{5,5,2})$.

$\underline{\gamma \neq 0,\ c \neq 0 \text{ and } x \neq 0}$. Then:

$$[c^{-1}\bar{z}, cg_1] = z,\ [c^{-1}\bar{z}, f_2] = c^{-1}x\alpha^{-1}\gamma g_1,\ [c^{-1}\bar{z}, f_3 - x^{-1}yf_2] = f_2 \quad \text{and} \quad [f_2, f_3 - x^{-1}yf_2] = z.$$

Put

$$(e_1,e_2,e_3,e_3,e_5) = (-c^{-1}(\bar{e}+af_2+bf_2), f_3 - x^{-1}yf_2, -f_2, -cg_1, \alpha e).$$

Then

$$[e_1,e_2] = e_3, [e_1,e_3] = ke_4, [e_1,e_4] = e_5, [e_2,e_3] = e_5.$$

We can always suppose that $k > 0$ (otherwise replace e_3 by $-e_3$ and e_2 by $-e_2$). We then put $e_1' = \mu e_1$ and $e_3' = \mu e_3$, $e_5' = \mu e_5$ and $\mu^2 = \frac{1}{k}$. After an adequate change of parameters one can see that $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is isomorphic to $(L_{5,6},\langle\ ,\ \rangle_{5,6})$. □

Example 1.

1. *Example of a six dimensional Ricci flat Lorentzian nilpotent Lie algebra with nondegenerate center.*

   $$[e_1,e_3] = e_6,\ [e_1,e_5] = e_6,\ [e_2,e_3] = -e_6, [e_2,e_4] = e_6, [e_3,e_4] = e_1, [e_3,e_5] = e_2 \quad \textit{and} \quad [e_4,e_5] = e_1+e_2.$$

   $\mathbb{B} = (e_1,\ldots,e_6)$ *is an orthonormal basis with* $\langle e_1,e_1\rangle = -1$.

2. *Example of a seven dimensional Ricci flat Lorentzian nilpotent Lie algebra with nondegenerate center.*

   $$[e_1,e_3] = \sqrt{2}e_7,\ [e_2,e_4] = \sqrt{2}e_7,\ [e_4,e_5] = -e_1, [e_4,e_6] = -e_1, [e_3,e_5] = -e_2, [e_3,e_6] = -e_2.$$

   $\mathbb{B} = (e_1,\ldots,e_7)$ *is an orthonormal basis with* $\langle e_1,e_1\rangle = -1$.

3. *Example of an eight dimensional Einstein Lorentzian nilpotent Lie algebra with non vanishing scalar curvature. This example was given in [6].*

$$\begin{cases}[e_1,e_2]=-4\sqrt{3}e_3,\ [e_1,e_3]=\sqrt{\frac{5}{2}}e_4,\ [e_1,e_4]=-2\sqrt{3}e_8,\ [e_1,e_5]=3\sqrt{\frac{7}{2}}e_6,;\\ [e_1,e_6]=-4\sqrt{2}e_7,\ [e_2,e_3]=-\sqrt{\frac{5}{2}}e_5,\ [e_2,e_4]=-3\sqrt{\frac{7}{2}}e_6,\ [e_2,e_5]=-2\sqrt{3}e_7,\\ [e_2,e_6]=-4\sqrt{2}e_8,\ [e_3,e_4]=-\sqrt{21}e_7,\ [e_3,e_5]=-\sqrt{21}e_8.\end{cases}$$

$\mathbb{B}=(e_1,\dots,e_8)$ *is an orthonormal basis with* $\langle e_6,e_6\rangle=-1$.

Lie Algebra	Lie brackets	Non Trace-free Derivation
$L_{3,2}$	$[e_1,e_2]=e_3$	$e^1\otimes e_1+e^3\otimes e_3$
$L_{4,2}$	$[e_1,e_2]=e_3$	$e^1\otimes e_1+e^3\otimes e_3$
$L_{4,3}$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$	$2e^2\otimes e_2-e^1\otimes e_1+e^3\otimes e_3$
$L_{5,2}$	$[e_1,e_2]=e_3$	$e^1\otimes e_1+e^3\otimes e_3$
$L_{5,3}$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$	$2e^2\otimes e_2-e^1\otimes e_1+e^3\otimes e_3$
$L_{5,4}$	$[e_1,e_2]=e_5$, $[e_3,e_4]=e_5$	$e^1\otimes e_1+e^3\otimes e_3+e^5\otimes e_5$
$L_{5,5}$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_2,e_4]=e_5$	$e^3\otimes e_3+2e^2\otimes e_2+2e^5\otimes e_5-e^1\otimes e_1$
$L_{5,6}$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_1,e_4]=e_5$, $[e_2,e_3]=e_5$	$e^1\otimes e_1+2e^2\otimes e_2+3e^3\otimes e_3+4e^4\otimes e_4+5e^5\otimes e_5$
$L_{5,7}$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_1,e_4]=e_5$	$e^1\otimes e_1-2e^2\otimes e_2-e^3\otimes e_3+e^5\otimes e_5$
$L_{5,8}$	$[e_1,e_2]=e_4$, $[e_1,e_3]=e_5$	$e^1\otimes e_1-e^2\otimes e_2+e^5\otimes e_5$
$L_{5,9}$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_2,e_3]=e_5$	$2e^1\otimes e_1-e^2\otimes e_2+e^3\otimes e_3+3e^4\otimes e_4$

Table 2.1: Table of nilpotent Lie algebras of dimension ≤ 5 with non null trace derivation

Lie algebra $\mathfrak{g}$	Nonzero commutators
$L_{5,2}=L_{3,2}\oplus\mathbb{R}^2$	$[e_1,e_2]=e_3$
$L_{5,3}=L_{4,3}\oplus\mathbb{R}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4$
$L_{5,4}$	$[e_1,e_2]=e_5,[e_3,e_4]=e_5$
$L_{5,5}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_5,[e_2,e_4]=e_5$
$L_{5,6}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_5$
$L_{5,7}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5$
$L_{5,8}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5$
$L_{5,9}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_2,e_3]=e_5$

Table 2.2: List of five-dimensional nilpotent Lie algebras

3

Einstein Lorentzian 3-nilpotent Lie groups

3.1 Introduction

The study of left-invariant Einstein Riemannian metrics on Lie groups is a research area that had made huge progress in the last decades (see [10, 12, 13]). However, the indefinite case remains unexplored in comparison and only few significant results had been published in this matter with many questions that are still open (see [18, 6, 21]).
In [21], the authors began an inspection of Einstein Lorentzian nilpotent Lie algebras following guidelines from previous studies of the 2-step nilpotent case (see [14] and [18]). The main Theorem of [21] states that Einstein nilpotent Lie algebras with degenerate center are exactly Ricci-flat and are obtained by a double extension process starting from a Euclidean vector space (see [21, Theorem 4.1] and [3] for the original definition of the double extension). This class of Lie algebras includes all Einstein Lorentzian nilpotent Lie algebras that are either 2-step or of dimension less than 5, in fact as a concrete application of the main Theorem, the authors were able to give a full classification of the latter.
Dimension 6 however falls outside the context of this result as the authors presented the first example in this situation of an Einstein nilpotent Lie algebra with non-degenerate center, which also happens to be 3-step nilpotent. Einstein nilpotent Lie algebras that are non Ricci-flat has been shown to exist in the Lorentzian setting (see [6]) and according to [21, Theorem 4.1] these must have non-degenerate center as well. So the study of Einstein Lorentzian nilpotent Lie algebras with non-degenerate center becomes a natural and challenging problem and the present chapter can be seen as a first attempt to find a general pattern for these Lie algebras. We start by the 3-step nilpotent case and we develop a new approach which can be used later in the general case. Let us give a brief summary of our method and state our main result.
Let $(\mathfrak{h}, [\ ,\ ])$ be a k-nilpotent Lie algebra and $\langle\ ,\ \rangle$ an Einstein Lorentzian metric on $\mathfrak{h}$ such that the center of $\mathfrak{h}$ is non-degenerate. Then $Z(\mathfrak{h})$ is non-degenerate Euclidean (see [21])

and, naturally, we get the orthogonal spitting

$$\mathfrak{h} = Z(\mathfrak{h}) \overset{\perp}{\oplus} \mathfrak{g}.$$

The Lie bracket on $\mathfrak{h}$ splits accordingly as $[u,v] = \omega(u,v) + [u,v]_0$ for any $u,v \in \mathfrak{g}$, and it can be shown that $[\ ,\ ]_0$ is a Lie bracket on $\mathfrak{g}$ and $\omega : \mathfrak{g}\times\mathfrak{g} \longrightarrow Z(\mathfrak{h})$ is a 2-cocycle of the Lie algebra $(\mathfrak{g},[\ ,\ ]_0)$. It turns out that $(\mathfrak{g},[\ ,\ ]_0,\langle\ ,\ \rangle_{|\mathfrak{g}\times\mathfrak{g}})$ is a Lorentzian $(k-1)$-nilpotent Lie algebra and the Einstein equation on $\mathfrak{h}$ can be expressed entirely by means of the Lie algebra $\mathfrak{g}$ as a sort of compatibility condition between ω and the Ricci curvature $\mathrm{Ric}_{\mathfrak{g}}$ of $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}},[\ ,\ ]_0)$ (see Proposition 3.2.2). This shift in perspective is especially useful when the Lie algebra $\mathfrak{h}$ is 3-step nilpotent since $\mathfrak{g}$ is 2-nilpotent and, for instance, we can show that every Einstein Lorentzian 3-step nilpotent Lie algebra with non-degenerate center has positive scalar curvature (Theorem 3.2.1). It also gives rise to the notion of ω-quasi Einstein Lie algebras (see Definition 3.2.2). A careful study of ω-quasi Einstein 2-nilpotent Lie algebras leads to our main result, namely the classification of Einstein Lorentzian 3-step nilpotent Lie algebras with 1-dimensional non-degenerate center. Surprisingly enough, these are shown to only exist in dimensions 6 and 7.

Theorem 3.1.1. *Let $\mathfrak{h}$ be a 3-step nilpotent Lie algebra with $\dim Z(\mathfrak{h}) = 1$. Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathfrak{h}$ such that $Z(\mathfrak{h})$ is non-degenerate, then $\langle\ ,\ \rangle$ is Einstein if and only if it is Ricci-flat and $(\mathfrak{h},\langle\ ,\ \rangle)$ has one of the following forms :*

(i) $\dim\mathfrak{h} = 6$ and $\mathfrak{h}$ is isomorphic to $L_{6,19}(-1)$, i.e., $\mathfrak{h}$ has a basis $(f_i)_{i=1}^6$ such that the non vanishing Lie brackets are

$$[f_1,f_2] = f_4, [f_1,f_3] = f_5, [f_2,f_4] = f_6, [f_3,f_5] = -f_6$$

and the metric is given by :

$$\langle\ ,\ \rangle := f_1^*\otimes f_1^* + 2f_2^*\otimes f_2^* + 2f_3^*\otimes f_3^* + 4\alpha^4 f_6^*\otimes f_6^* - 2\alpha^2 f_4^*\odot f_5^*,\quad \alpha \neq 0. \tag{3.1}$$

(ii) $\dim\mathfrak{h} = 7$ and $\mathfrak{h}$ is isomorphic to the nilpotent Lie algebras $147E$ found in the classification given in [8](p. 57). In precise terms, there exists a basis $\{f_i\}_{i=1}^7$ of $\mathfrak{h}$ where the non vanishing Lie brackets are given by :

$$[f_1,f_2] = f_5,\ [f_1,f_3] = f_6,\ [f_2,f_3] = f_4,\ [f_6,f_2] = (1-r)f_7,\ [f_5,f_3] = -rf_7,\ [f_4,f_1] = f_7, \tag{3.2}$$

with $0 < r < 1$, and the metric has the form:

$$\langle\ ,\ \rangle = f_1^*\otimes f_1^* + f_2^*\otimes f_2^* + f_3^*\otimes f_3^* - af_4^*\otimes f_4^* + arf_5^*\otimes f_5^* + a(1-r)f_6^*\otimes f_6^* + a^2 f_7^*\otimes f_7^*,\quad a > 0. \tag{3.3}$$

Outline We shall adopt the notations and results introduced in Chapter 2 as the content of this chapter is an extension of the previous study. In Section 3.2, we describe an Einstein Lorentzian nilpotent Lie algebra $\mathfrak{h}$ with non-degenerate center by means of its center, a nilpotent Lorentzian Lie algebra $\mathfrak{g}$ of lower order, and a 2-cocycle $\omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h}))$, these are called *the attributes* of $\mathfrak{h}$ (see Definition 3.2.1). The main result of this section is Theorem 3.2.1 in which we prove that any Einstein Lorentzian 3-step nilpotent Lie algebra of non-degenerate center has positive scalar curvature, at the end of the section we introduce the notion of *ω-quasi Einstein Lie algebra*. The remainder of the chapter i.e Section 3.3 is then devoted for the proof of the central results. As the reader can see, the proof of Theorem 3.1.1 turns out to be difficult and it is based on a sequence of Lemmas (Lemma 3.3.1, 3.3.2 and 3.3.3). This suggests that the complete study of Einstein Lorentzian nilpotent Lie algebras with nondegenerate center is a challenging mathematical problem.

3.2 Lorentzian nilpotent Einstein Lie algebras with nondegenerate center

In [21], we studied Lorentzian nilpotent Einstein Lie algebras with degenerate center and gave the first example of a Lorentzian 3-step nilpotent Ricci-flat Lie algebra with non-degenerate center. We also showed that an Einstein Lorentzian nilpotent Lie algebra with non zero scalar curvature must have a non-degenerate center. A first example of such algebras was given in [6]. A 2-step nilpotent Einstein Lorentzian Lie algebra must be Ricci-flat with degenerate center so it is natural to start by studying 3-step nilpotent Einstein Lorentzian Lie algebras with non-degenerate center which must be Euclidean according to Corollary 2.3.1.

Any nilpotent Lie algebra can be obtained by Skjelbred-Sund's method, namely, by an extension from a nilpotent Lie algebra of lower dimension and a 2-cocycle with values in a vector space (see [9]). We will adapt this method to our study.

Let $(\mathfrak{h},\langle\ ,\ \rangle_{\mathfrak{h}})$ be a Lorentzian k-step nilpotent Lie algebra of dimension n with non-degenerate Euclidean center $Z(\mathfrak{h})$ of dimension $p \geq 1$. The restriction of $\langle\ ,\ \rangle$ to $Z(\mathfrak{h})$ is denoted $\langle\ ,\ \rangle_z$, we also set $\mathfrak{g} := Z(\mathfrak{h})^{\perp}$ and let $\langle\ ,\ \rangle_{\mathfrak{g}}$ be the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{g}$. Then:

$$\mathfrak{h} = \mathfrak{g} \overset{\perp}{\oplus} Z(\mathfrak{h}),$$

where $(Z(\mathfrak{h}),\langle\ ,\ \rangle_z)$ is a Euclidean vector space and $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}})$ is a Lorentzian vector space. Moreover, for any $u,v \in \mathfrak{g}$, we have:

$$[u,v] = [u,v]_{\mathfrak{g}} + \omega(u,v), \tag{3.4}$$

where $[u,v]_{\mathfrak{g}} \in \mathfrak{g}$ and $\omega(u,v) \in Z(\mathfrak{h})$. The Jacobi identity applied to $[\ ,\ ]$ is easily seen equivalent to $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}})$ being a Lie algebra and $\omega : \mathfrak{g}\times\mathfrak{g} \longrightarrow Z(\mathfrak{h})$ a 2-cocycle of $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ in $Z(\mathfrak{h})$ (see Appendix A), namely for any $u,v,w \in \mathfrak{g}$,

$$\omega([u,v]_{\mathfrak{g}},w) + \omega([v,w]_{\mathfrak{g}},u) + \omega([w,u]_{\mathfrak{g}},v) = 0.$$

The following properties can be derived immediately from (3.4):

$$Z(\mathfrak{g}) \cap \ker\omega = \{0\} \quad \text{and} \quad \mathscr{C}^{\ell}(\mathfrak{h}) := [\mathscr{C}^{\ell-1}(\mathfrak{h}),\mathfrak{h}] = \mathscr{C}^{\ell}(\mathfrak{g}) + \omega(\mathscr{C}^{\ell-1}(\mathfrak{g}),\mathfrak{g}), \tag{3.5}$$

for any $\ell \in \mathbb{N}^*$. As a result $(\mathfrak{h}, [\ ,\ ])$ is k-step nilpotent if and only if $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ is a $(k-1)$-step nilpotent Lie algebra such that $\mathscr{C}^{k-2}(\mathfrak{g}) \not\subset \ker \omega$.

Definition 3.2.1. *Let $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$ be a Lorentzian nilpotent Lie algebra with nondegenerate Euclidean center. We call the triple $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$, $(Z(\mathfrak{h}), \langle\ ,\ \rangle_z)$ and $\omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h}))$ the attributes of $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$.*

We can now proceed to the important step, which is to express the Ricci curvature of $\mathfrak{h}$ in terms of its attributes $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}}, [\ ,\ ]_{\mathfrak{g}})$, $(Z(\mathfrak{h}), \langle\ ,\ \rangle_z)$ and $\omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h}))$. For any $u \in \mathfrak{g}$, we consider $\omega_u : \mathfrak{g} \longrightarrow Z(\mathfrak{h})$, $v \longrightarrow \omega(u,v)$ and its transpose $\omega_u^* : Z(\mathfrak{h}) \longrightarrow \mathfrak{g}$ given by:

$$\langle \omega_u^*(x), v\rangle_{\mathfrak{g}} = \langle \omega(u,v), x\rangle_z.$$

For any $x \in Z(\mathfrak{h})$, we define $S_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ by:

$$S_x(u) = \omega_u^*(x). \tag{3.6}$$

It is clear that S_x is skew-symmetric. Recall that, for any $u \in \mathfrak{g}$, we denote by $J_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ the skew-symmetric endomorphism given by $J_u(v) = \mathrm{ad}_v^*(u)$. On the other hand, define the endomorphism $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ by:

$$\langle Du, v\rangle_{\mathfrak{g}} = \mathrm{tr}(\omega_u^* \circ \omega_v). \tag{3.7}$$

It is clear that D is symmetric with respect to $\langle\ ,\ \rangle_{\mathfrak{g}}$. Let $(z_1, \ldots, z_p)$ be a basis of $Z(\mathfrak{h})$. Since $\langle\ ,\ \rangle$ is non-degenerate, there exists a unique family $(S_1, \ldots, S_p)$ of skew-symmetric endomorphisms such that, for any $u, v \in \mathfrak{g}$,

$$\omega(u,v) = \sum_{i=1}^{p} \langle S_i u, v\rangle_{\mathfrak{g}} z_i. \tag{3.8}$$

This family will be called *ω-structure endomorphisms* associated to $(z_1, \ldots, z_p)$. A direct computation using (3.7) and (3.8) shows that

$$D = -\sum_{i,j} \langle z_i, z_j\rangle_z S_i \circ S_j. \tag{3.9}$$

This operator has an interesting property.

Proposition 3.2.1. *Let $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle)$ be a Lorentzian nilpotent Lie algebra with Euclidean center and attributes $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$ and $\omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h}))$. Assume that ω satisfies:*

$$\omega(\mathrm{ad}_u^* v, w) + \omega(v, \mathrm{ad}_u^* w) = 0 \tag{3.10}$$

for any $u, v, w \in \mathfrak{g}$, then D given in (3.7) is a derivation of $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$.

Proof. Since ω is a 2-cocycle then $\omega_{[u,v]_{\mathfrak{g}}} = \omega_u \circ \mathrm{ad}_v - \omega_v \circ \mathrm{ad}_u$, thus for any $u, v, w \in \mathfrak{g}$:

$$\begin{aligned}\langle D[u,v]_{\mathfrak{g}}, w\rangle &= \mathrm{tr}(\omega_{[u,v]_{\mathfrak{g}}} \circ \omega_w^*)\\ &= \mathrm{tr}(\omega_u \circ \mathrm{ad}_v \circ \omega_w^*) - \mathrm{tr}(\omega_v \circ \mathrm{ad}_u \circ \omega_w^*).\end{aligned}$$

On the other hand, we get in view of (3.10):

$$\begin{aligned}\langle [Du,v]_{\mathfrak{g}},w\rangle_{\mathfrak{g}}+\langle [u,Dv]_{\mathfrak{g}},w\rangle_{\mathfrak{g}} &= -\mathrm{tr}(\omega_{\mathrm{ad}_v^* w}\circ\omega_u^*)+\mathrm{tr}(\omega_{\mathrm{ad}_u^* w}\circ\omega_v^*),\\ &= \mathrm{tr}(\omega_w\circ\mathrm{ad}_v^*\circ\omega_u^*)-\mathrm{tr}(\omega_w\circ\mathrm{ad}_u^*\circ\omega_v^*),\end{aligned}$$

This proves the claim □

Proposition 3.2.2. *Let $(\mathfrak{h},[\ ,\ ],\langle\ ,\ \rangle)$ be a Lorentzian nilpotent Lie algebra with Euclidean center and attributes $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\langle\ ,\ \rangle_{\mathfrak{g}})$ and $\omega\in Z^2(\mathfrak{g},Z(\mathfrak{h}))$. Its Ricci curvature $\mathrm{ric}_{\mathfrak{h}}$ is given by:*

$$\begin{aligned}\mathrm{ric}_{\mathfrak{h}}(u,v)&=\mathrm{ric}_{\mathfrak{g}}(u,v)-\frac{1}{2}\mathrm{tr}(\omega_u^*\circ\omega_v),\quad u,v\in\mathfrak{g},\\ \mathrm{ric}_{\mathfrak{h}}(x,y)&=-\frac{1}{4}\mathrm{tr}(S_x\circ S_y),\quad x,y\in Z(\mathfrak{h}),\\ \mathrm{ric}_{\mathfrak{h}}(u,x)&=-\frac{1}{4}\mathrm{tr}(J_u\circ S_x),\quad x\in Z(\mathfrak{h}),u\in\mathfrak{g},\end{aligned}$$

where $\mathrm{ric}_{\mathfrak{g}}$ is the Ricci curvature of $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\langle\ ,\ \rangle_{\mathfrak{g}})$ and $S_x:\mathfrak{g}\longrightarrow\mathfrak{g}$ is the endomorphism defined in (3.6).

Proof. According to (1.3), for any $a,b\in\mathfrak{h}$,

$$\mathrm{ric}_{\mathfrak{h}}(a,b)=-\frac{1}{2}\mathrm{tr}(\mathrm{ad}_a^{\mathfrak{h}}\circ(\mathrm{ad}_b^{\mathfrak{h}})^*)-\frac{1}{4}\mathrm{tr}(J_a^{\mathfrak{h}}\circ J_b^{h}),$$

where $\mathrm{ad}_a^{\mathfrak{h}}:\mathfrak{h}\longrightarrow\mathfrak{h}$, $b\mapsto[a,b]$ and $J_a^{\mathfrak{h}}:\mathfrak{h}\longrightarrow\mathfrak{h}$, $b\mapsto(\mathrm{ad}_b^{\mathfrak{h}})^*(a)$. The desired formula will be a consequence of this one and the following relations. For any $u\in\mathfrak{g}$, $x\in Z(\mathfrak{h})$, with respect to the splitting $\mathfrak{h}=\mathfrak{g}\oplus Z(\mathfrak{h})$, we have:

$$\mathrm{ad}_u^{\mathfrak{h}}=\begin{pmatrix}\mathrm{ad}_u^{\mathfrak{g}} & 0\\ \omega_u & 0\end{pmatrix},\ J_u^{\mathfrak{h}}=\begin{pmatrix}J_u^{\mathfrak{g}} & 0\\ 0 & 0\end{pmatrix},\ J_x^{\mathfrak{h}}=\begin{pmatrix}S_x & 0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad \mathrm{ad}_x^{\mathfrak{h}}=0.$$

The claim is then a matter of simple computation. □

Corollary 3.2.1. $(\mathfrak{h},[\ ,\ ],\langle\ ,\ \rangle_{\mathfrak{h}})$ *is λ-Einstein if and only if for any $u,v\in\mathfrak{g}$ and $x,y\in Z(\mathfrak{h})$,*

$$\mathrm{ric}_{\mathfrak{g}}(u,v)=\lambda\langle u,v\rangle_{\mathfrak{g}}+\frac{1}{2}\mathrm{tr}(\omega_u^*\circ\omega_v),\ \mathrm{tr}(J_u\circ S_x)=0\quad\textit{and}\quad\mathrm{tr}(S_x\circ S_y)=-4\lambda\langle x,y\rangle_z.\tag{3.11}$$

Let us derive some consequences of Proposition 3.2.2 and Corollary 3.2.1. In what follows $\mathfrak{h}$ will be an Einstein Lorentzian nilpotent Lie algebra with nondegenerate center, we denote $[\ ,\ ]_{\mathfrak{h}}$ its Lie bracket, $\langle\ ,\ \rangle_{\mathfrak{h}}$ its Lorentzian product and $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\langle\ ,\ \rangle_{\mathfrak{g}})$, $(Z(\mathfrak{h}),\langle\ ,\ \rangle_z)$ and $\omega\in Z^2(\mathfrak{g},Z(\mathfrak{h}))$ its attributes.
Recall that a pseudo-Euclidean Lie algebra $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ is called ***Ricci-soliton*** if there exists a constant $\lambda\in\mathbb{R}$ and derivation D of $\mathfrak{g}$ such that $\mathrm{Ric}_{\mathfrak{g}}=\lambda\mathrm{Id}_{\mathfrak{g}}+D$. By combining Corollary 3.2.1 and Proposition 3.2.1 we get the following result.

Proposition 3.2.3. *Let $\mathfrak{h}$ be an Einstein Lorentzian nilpotent Lie algebra with Euclidean nondegenerate center. If ω satisfies (3.10) then $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\langle\ ,\ \rangle_{\mathfrak{g}})$ is Ricci-soliton.*

Proposition 3.2.4. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian nilpotent Lie algebra with non-degenerate center, let $\mathfrak{g}$ and ω be its attributes (*cf. *Definition 3.2.1). If $\lambda \neq 0$ then the cohomology class of the attribute ω is non trivial. In particular, $H^2(\mathfrak{g}, Z(\mathfrak{h})) \neq \{0\}$.*

Proof. Suppose that there exists $\alpha \in \mathfrak{g}$ such that, for any $u, v \in \mathfrak{g}$, $\omega(u,v) = -\alpha([u,v]_{\mathfrak{g}})$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ with $\langle e_1, e_1 \rangle = -1$. For any $x \in Z(\mathfrak{h})$, we have :

$$\begin{aligned}
\operatorname{tr}(S_x^2) &= \langle S_x(e_1), S_x(e_1)\rangle_{\mathfrak{g}} - \sum_{i=2}^{n} \langle S_x(e_i), S_x(e_i)\rangle_{\mathfrak{g}} \\
&= \langle \omega_{e_1}^*(x), S_x(e_1)\rangle_{\mathfrak{g}} - \sum_{i=2}^{n} \langle \omega_{e_i}^*(x), S_x(e_i)\rangle_{\mathfrak{g}} \\
&= -\langle \operatorname{ad}_{e_1}^* \circ \alpha^*(x), S_x(e_1)\rangle_{\mathfrak{g}} + \sum_{i=2}^{n} \langle \operatorname{ad}_{e_i}^* \circ \alpha^*(x), S_x(e_i)\rangle_{\mathfrak{g}} \\
&= -\langle J_{\alpha^*(x)}(e_1), S_x(e_1)\rangle_{\mathfrak{g}} + \sum_{i=2}^{n} \langle J_{\alpha^*(x)}(e_i), S_x(e_i)\rangle_{\mathfrak{g}} \\
&= -\operatorname{tr}(J_{\alpha^*(x)} \circ S_x).
\end{aligned}$$

By virtue of Corollary 3.2.1, we get that $\lambda\langle x, x\rangle_z = 0$ for any $x \in Z(\mathfrak{h})$ and hence $\lambda = 0$. □

Proposition 3.2.5. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian nilpotent Lie algebra with non-degenerate center, denote $\mathfrak{g}$ and ω its attributes (*cf. *Definition 3.2.1). Then $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$ is a non-degenerate Lorentzian subspace of $\mathfrak{g}$. Moreover, if $\mathfrak{h}$ is 3-step nilpotent and $\lambda \geq 0$ then $\mathrm{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$.*

Proof. According to Corollary 2.3.3, $[\mathfrak{h}, \mathfrak{h}]$ is non-degenerate Lorentzian and one can easily see that $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}^{\perp} = [\mathfrak{h}, \mathfrak{h}]^{\perp} \cap \mathfrak{g}$. Thus $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}^{\perp}$ is non-degenerate Euclidean and hence $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$ is non-degenerate Lorentzian.
Suppose now that $\mathfrak{h}$ is 3-step nilpotent. Then $\mathfrak{g}$ is 2-step nilpotent and so $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}} \subset \mathrm{Z}(\mathfrak{g})$. Let $x \in \mathrm{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}^{\perp}$. Since $\operatorname{ad}_x = 0$ and $J_x = 0$, by virtue of (1.3) , $\operatorname{Ric}_{\mathfrak{g}}(x) = 0$. If $\lambda \geq 0$, the first equation of system (3.11) gives that :

$$0 \leq \lambda\langle x, x\rangle = -\frac{1}{2}\operatorname{tr}(\omega_x^* \circ \omega_x) := Q.$$

Since ω is a 2-cocycle, $\omega(Z(\mathfrak{g}), [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}) = 0$ and hence

$$Q = -\frac{1}{2}\sum_{i=1}^{m} \langle \omega(x, f_i), \omega(x, f_i)\rangle \leq 0$$

where $\{f_1, \ldots, f_m\}$ is an orthonormal basis of $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}^{\perp}$. It follows that $x \in \mathrm{Z}(\mathfrak{g}) \cap \ker \omega$ and hence $x = 0$ by virtue of (3.5). Thus $\mathrm{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$. □

Theorem 3.2.1. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian 3-step nilpotent Lie algebra with nondegenerate center. Then $\lambda \geq 0$.*

Proof. According to (3.11), since $\mathfrak{h}$ is λ-Einstein then:

$$\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}} + \frac{1}{2}D \quad \text{and} \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda\langle x,y\rangle_z, \tag{3.12}$$

for any $x,y \in Z(\mathfrak{h})$, where $\mathfrak{g}$ and ω are the attributes of $\mathfrak{h}$ (*cf.* Definition 3.2.1) and S_x is the operator defined in (3.6). By virtue of Proposition 3.2.5, $[\mathfrak{g},\mathfrak{g}]$ is nondegenerate Lorentzian and hence $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus [\mathfrak{g},\mathfrak{g}]^{\perp}$. We choose an orthonormal basis $\mathbb{B}_0 = (e_1,\ldots,e_s)$ of $[\mathfrak{g},\mathfrak{g}]$ such that $\langle e_1,e_1\rangle_{\mathfrak{g}} = -1$ and an orthonormal basis $\mathbb{B}_1 = (z_1,\ldots,z_p)$ of $Z(\mathfrak{h})$ and we consider the Lie structure endomorphisms $(J_1,\ldots,J_s)$ associated to $\mathbb{B}_0$ and given by (2.7) and $(S_1,\ldots,S_p)$ the ω-structure endomorphisms associated to $\mathbb{B}_1$ and given by (3.8).

Since $\mathfrak{g}$ is 2-step nilpotent then $[\mathfrak{g},\mathfrak{g}] \subset Z(\mathfrak{g})$, hence $J_i([\mathfrak{g},\mathfrak{g}]) = 0$ for any $i = 1,\ldots,s$. Furthermore, J_i is skew-symmetric so it must leave $[\mathfrak{g},\mathfrak{g}]^{\perp}$ invariant, we shall denote its restriction to $[\mathfrak{g},\mathfrak{g}]^{\perp}$ by J_i as well. Next, since ω is a 2-cocycle then $\omega(Z(\mathfrak{g}),[\mathfrak{g},\mathfrak{g}]) = 0$, hence by virtue of (3.8) we get that $S_i([\mathfrak{g},\mathfrak{g}]) \subset [\mathfrak{g},\mathfrak{g}]^{\perp}$ for any $i = 1,\ldots,p$, we denote $B_i : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{g},\mathfrak{g}]^{\perp}$ the resulting linear map. Since S_i is skew-symmetric, then for any $u \in [\mathfrak{g},\mathfrak{g}]^{\perp}$, $S_i u = -B_i^* u + D_i u$ where $D_i : [\mathfrak{g},\mathfrak{g}]^{\perp} \longrightarrow [\mathfrak{g},\mathfrak{g}]^{\perp}$ is skew-symmetric. Using (1.5), (2.9) and (3.9), we deduce that (3.12) is equivalent to:

$$\begin{cases} -\frac{1}{2}J_1^2 + \frac{1}{2}\sum_{i=2}^{s} J_i^2 + \frac{1}{2}\sum_{i=1}^{p}(D_i^2 - B_i B_i^*) = \lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]^{\perp}}. \\ \sum_{i,j=1}^{s} \langle e_i, \,.\,\rangle \mathrm{tr}(J_i \circ J_j) e_j + 2\sum_{i=1}^{p} B_i^* B_i = -4\lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]}. \\ \mathrm{tr}(D_i D_j) - 2\mathrm{tr}(B_i^* B_j) = -4\lambda\delta_{ij}, \ i,j = 1,\ldots,p. \end{cases} \tag{3.13}$$

By taking the trace of the first two equations and using the third one we obtain that:

$$\sum_{i=1}^{p} \mathrm{tr}(D_i^2) = -4(2s + m + 3p)\lambda, \quad m = \dim[\mathfrak{g},\mathfrak{g}]^{\perp}.$$

But $[\mathfrak{g},\mathfrak{g}]^{\perp}$ is a Euclidean vector space and $D_i : [\mathfrak{g},\mathfrak{g}]^{\perp} \longrightarrow [\mathfrak{g},\mathfrak{g}]^{\perp}$ is skew-symmetric and hence $\mathrm{tr}(D_i^2) \leq 0$ which completes the proof. □

To sum up the results of this section, we reduced the study of Einstein Lorentzian k-step nilpotent Lie algebras to the study of a class of Lorentzian $(k-1)$-step nilpotent Lie algebras endowed with a 2-cocycle with values in a Euclidean vector space, which in some cases can be Ricci-soliton. It is natural to give a name to this class of Lie algebras.

Definition 3.2.2. *A pseudo-Euclidean Lie algebra* $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\langle\ ,\ \rangle_{\mathfrak{g}})$ *will be called ω-quasi Einstein of type p if there exists $\lambda \in \mathbb{R}$ and a 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow V$ with values in a Euclidean vector space $(V,\langle\ ,\ \rangle_z)$ of dimension p such that* $\ker\omega \cap Z(\mathfrak{g}) = \{0\}$ *and:*

$$\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}} + \frac{1}{2}D, \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda\langle x,y\rangle_z$$

where $S_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the ω-structure endomorphism corresponding to $x \in V$ i.e (3.6) and D is the linear operator given by:

$$\langle Du, v\rangle_{\mathfrak{g}} = \mathrm{tr}(\omega_u^* \circ \omega_v)$$

such that $\omega_u : \mathfrak{g} \longrightarrow V$, $v \mapsto \omega(u,v)$.

3.3 Type 1 quasi-Einstein Lorentzian 2-nilpotent Lie algebras

In this section, having in mind Proposition 3.2.5 and Theorem 3.2.1, we give a complete description of ω-quasi Einstein Lorentzian 2-step nilpotent Lie algebras of type 1 with non-degenerate Lorentzian derived ideal and Einstein constant $\lambda \geq 0$ as an important step towards the determination of Einstein Lorentzian 3-step nilpotent Lie algebras with nondegenerate 1-dimensional center.

Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$ be a 2-step nilpotent Lie algebra such that $Z(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]$ is non-degenerate Lorentzian. Put $n = \dim[\mathfrak{g},\mathfrak{g}]$ and $m = \dim[\mathfrak{g},\mathfrak{g}]^{\perp}$.

Suppose that $\mathfrak{g}$ is ω-quasi Einstein of type 1 with Einstein constant $\lambda \geq 0$. Denote $S : \mathfrak{g} \to \mathfrak{g}$ the skew-symmetric endomorphism given by $\omega(u,v) = \langle Su, v\rangle_{\mathfrak{g}}$. Since ω is a 2-cocycle and $[\mathfrak{g},\mathfrak{g}] \subset Z(\mathfrak{g})$ then $S([\mathfrak{g},\mathfrak{g}]) \subset [\mathfrak{g},\mathfrak{g}]^{\perp}$, this gives rise to a linear map $B : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{g},\mathfrak{g}]^{\perp}$. The condition $Z(\mathfrak{g}) \cap \ker\omega = \{0\}$ implies that B is injective. On the other hand, the skew-symmetry of S gives that, for any $u \in [\mathfrak{g},\mathfrak{g}]^{\perp}$, $Su = -B^*u + Lu$ where L is a skew-symmetric endomorphism of $[\mathfrak{g},\mathfrak{g}]^{\perp}$. Now consider the endomorphism D associated to ω and given by formula (3.7). According to (3.9), $D = -S^2$ and hence

$$Du = \begin{cases} B^*Bu - LBu \text{ if } u \in [\mathfrak{g},\mathfrak{g}], \\ B^*Lu + BB^*u - L^2u \text{ if } u \in [\mathfrak{g},\mathfrak{g}]^{\perp}. \end{cases}$$

The fact that $\mathfrak{g}$ is ω-quasi Einstein is equivalent to

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 - \frac{1}{2}D = \lambda \mathrm{Id}_{\mathfrak{g}}, \quad \mathrm{tr}(S^2) = -4\lambda, \tag{3.14}$$

where, by virtue of (1.5), $\mathrm{Ric}_{\mathfrak{g}} = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2$.

Let us proceed now to a crucial step which is not possible to perform when ω has its values in a vector space of dimension ≥ 2.

We consider the symmetric endomorphism on $[\mathfrak{g},\mathfrak{g}]$ given by $A = B^*B$. Since B is injective and $[\mathfrak{g},\mathfrak{g}]^{\perp}$ is non-degenerate Euclidean, we have $\langle Au, u\rangle_{\mathfrak{g}} > 0$ for any $u \in \mathfrak{g} \setminus \{0\}$. There are only two categories of nondiagonalizable symmetric endomorphisms on a Lorentzian vector space (see Appendix B, Theorem B.4.2). Those which have an isotropic eigenvector or those which have two linearly orthogonal vectors (e,f) such that $\langle e,e\rangle = 1$, $\langle f,f\rangle = -1$ with $T(e) = ae - bf$ and $T(f) = be + af$. The fact that A is positive definite prevents it to be of these types and hence A is diagonalizable in an orthonormal basis $\mathbb{B}_1 = (e_1, \ldots, e_n)$ of $[\mathfrak{g},\mathfrak{g}]$ such that $\langle e_1, e_1\rangle_{\mathfrak{g}} = -1$. Let $(J_1, \ldots, J_n)$ be the structure endomorphisms associated to $\mathbb{B}_1$. Note that the J_i vanishes on $[\mathfrak{g},\mathfrak{g}] \subset Z(\mathfrak{g})$ and hence leaves invariant $[\mathfrak{g},\mathfrak{g}]^{\perp}$. We denote the restriction of J_i to $[\mathfrak{g},\mathfrak{g}]^\perp$ by J_i as well. Using (1.5) and (2.9), we get that (3.14) is equivalent to:

$$\begin{cases} -\frac{1}{2}J_1^2 + \frac{1}{2}\sum_{j=2}^{n} J_j^2 + \frac{1}{2}(L^2 - BB^*) = \lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]^\perp}, \\ -2B^*B - \sum_{i,j=1}^{n} \langle e_i, u\rangle \mathrm{tr}(J_i \circ J_j) e_j = 4\lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]}, \\ \mathrm{tr}(L^2) - 2\mathrm{tr}(BB^*) = -4\lambda, \\ LB = 0. \end{cases} \tag{3.15}$$

Taking the trace of the first two equations and using the the third equation of (3.15) we get that :

$$\mathrm{tr}(L^2) = -4(2n+m+3)\lambda, \quad n = \dim[\mathfrak{g},\mathfrak{g}],\ m = \dim[\mathfrak{g},\mathfrak{g}]^\perp.$$

When $m = n$, $B : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{g},\mathfrak{g}]^\perp$ is an isomorphism, therefore $LB = 0$ leads to $L = 0$ and by the previous equation $\lambda = 0$. We will show that this fact is still true in the general setting. Put $\mathbb{B}_2 = (f_1,\ldots,f_n) = \left(\frac{B(e_1)}{|B(e_1)|},\ldots,\frac{B(e_n)}{|B(e_n)|}\right)$ which is obviously an orthonormal basis of $\mathrm{Im}(B)$. Since $LB = 0$, L vanishes on $\mathrm{Im}(B)$ and leaves invariant $\mathrm{Im}(B)^\perp = \ker BB^*$. Thus $L(f_i) = 0$ and there exists an orthonormal basis $\mathbb{B}_3 = (g_1,h_1,\ldots,g_r,h_r,p_1,\ldots,p_s)$ of $\ker BB^*$ such that

$$L(g_i) = \mu_i h_i,\ L(h_i) = -\mu_i g_i, L(p_j) = 0.$$

The basis $\mathbb{B}_1$ consists of eigenvectors of B^*B and hence the second relation in (3.15) is equivalent to

$$B^*B(e_i) = -\left(2\lambda + \frac{1}{2}\langle e_i, e_i\rangle_{\mathfrak{g}} \mathrm{tr}(J_i^2)\right)e_i,\ \mathrm{tr}(J_i \circ J_j) = 0,\ i,j = 1,\ldots,n, j \neq i.$$

On the other hand, we also have,

$$BB^*(f_i) = -\left(2\lambda + \frac{1}{2}\langle e_i, e_i\rangle_{\mathfrak{g}} \mathrm{tr}(J_i^2)\right)f_i, \quad i = 1,\ldots,n. \tag{3.16}$$

Summing up the above remarks, if M_i denotes the matrix of the restriction of J_i to $[\mathfrak{g},\mathfrak{g}]^\perp$ in the basis $\mathbb{B}_2 \cup \mathbb{B}_3$ then (3.14) implies that

$$M_1^2 - \sum_{k=2}^{n} M_k^2 = \mathrm{Diag}\left(-\frac{1}{2}\mathrm{tr}(M_1^2), \frac{1}{2}\mathrm{tr}(M_2^2),\ldots,\frac{1}{2}\mathrm{tr}(M_n^2), -(2\lambda+\mu_1^2),\ldots,-(2\lambda+\mu_r^2), -2\lambda,\ldots,-2\lambda\right). \tag{3.17}$$

To study this equation, we need matrix analysis of Hermitian square matrices (see [20]). Let us recall one of the main theorems of this theory. A $m \times m$ Hermitian matrix A has real eigenvalues which can be ordered

$$\lambda_1(A) \leq \ldots \leq \lambda_m(A).$$

Theorem 3.3.1 ([20]). *Let $A, B \in \mathcal{M}_m(\mathbb{C})$ be two Hermitian matrices. Then for all $1 \leq k \leq m$:*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_m(B).$$

Based on this theorem, the following lemma is a breakthrough in our study.

Lemma 3.3.1. *Let $M_1,\dots,M_n$ be a family of skew-symmetric $m\times m$ matrices with $2\le n\le m$ and let $(v_1,\dots,v_{m-n})$ be a family of nonpositive real numbers such that :*

$$M_1^2-\sum_{l=2}^{n}M_l^2=\mathrm{Diag}\left(-\frac{1}{2}\mathrm{tr}(M_1^2),\frac{1}{2}\mathrm{tr}(M_2^2),\dots,\frac{1}{2}\mathrm{tr}(M_n^2),v_1,\dots,v_{m-n}\right). \tag{3.18}$$

Then

$$(v_1,\dots,v_{m-n})=(0,\dots,0),\ \lambda_1\left(\sum_{l=2}^{n}M_l^2\right)=\sum_{l=2}^{n}\lambda_1(M_l^2).$$

Moreover, for any $i\in\{2,\dots,n\}$, $\mathrm{rank}(M_i)\le 2$.

Proof. Denote by M the right-hand side of equation (3.18). By taking the trace of (3.18) we get :

$$\mathrm{tr}(M_1^2)-\sum_{l=2}^{n}\mathrm{tr}(M_l^2)=\frac{2}{3}\sum_{i=1}^{m-n}v_i\le 0. \tag{3.19}$$

For $i=1,\dots,n$, M_i^2 is the square of a skew-symmetric matrix so its eigenvalues are real non-positive and satisfies:

$$\lambda_{2k-1}(M_i^2)=\lambda_{2k}(M_i^2),\quad k\in\left\{1,\dots,\left[\frac{m}{2}\right]\right\}. \tag{3.20}$$

Clearly $-\frac{1}{2}\mathrm{tr}(M_1^2)$ is the only non-negative eigenvalue of M and thus $\lambda_m(M)=-\frac{1}{2}\mathrm{tr}(M_1^2)$. Theorem 3.3.1 applied to (3.18) gives that:

$$\underbrace{\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)}_{a}\le\lambda_m(M_1^2)\le\underbrace{\lambda_m(M)+\lambda_m\left(\sum_{l=2}^{n}M_l^2\right)}_{b}. \tag{3.21}$$

and

$$\underbrace{\lambda_{m-1}(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)}_{c}\le\lambda_{m-1}(M_1^2)\le\underbrace{\lambda_{m-1}(M)+\lambda_m\left(\sum_{l=2}^{n}M_l^2\right)}_{d}. \tag{3.22}$$

Suppose that m is odd. In this case $\lambda_m(M_1^2)=0$ and, by applying Theorem 3.3.1 inductively and using (3.20), we get that :

$$\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)\le\frac{1}{2}\sum_{l=2}^{n}(\lambda_1(M_l^2)+\lambda_2(M_l^2))=\sum_{l=2}^{n}\lambda_1(M_l^2)\le\lambda_1\left(\sum_{l=2}^{n}M_l^2\right).$$

As a consequence of this inequality and the fact that $\lambda_m(M)=-\frac{1}{2}\mathrm{tr}(M_1^2)$, we get

$$-\frac{1}{2}\mathrm{tr}(M_1^2)+\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)\le\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\overset{(3.21)}{\le}\lambda_m(M_1^2)\le 0,$$

This combined with (3.19) gives that $(v_1,\dots,v_{m-n})=(0,\dots,0)$. Suppose now that m is even. In this case, $\lambda_{m-1}(M_1^2)=\lambda_m(M_1^2)$ and it follows from (3.21) and (3.22) that $[a,b]\cap[c,d]\neq\emptyset$. But, we have obviously that $c\leq a$ and $d\leq b$ therefore $a\leq d$. Thus

$$\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\leq\lambda_{m-1}(M)+\lambda_m\left(\sum_{l=2}^{n}M_l^2\right). \tag{3.23}$$

Since $\lambda_m(M)=-\frac{1}{2}\mathrm{tr}(M_1^2)$ then by using (3.19) we get that

$$\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)=-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right).$$

On the other hand, Theorem 3.3.1 once more shows that $\lambda_m(\sum_{l=2}^{n}M_l^2)\leq\sum_{l=2}^{n}\lambda_m(M_l^2)\leq 0$, moreover $\lambda_{m-1}(M)\leq 0$, so (3.23) implies that:

$$-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\leq 0, \tag{3.24}$$

Theorem 3.3.1 also implies that $\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\geq\sum_{l=2}^{n}\lambda_1(M_l^2)$ and hence:

$$\begin{aligned}
-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right) &\geq -\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\sum_{l=2}^{n}\lambda_1\left(M_l^2\right)\\
&\geq -\frac{1}{2}\sum_{l=2}^{n}\sum_{k=1}^{m}\lambda_k(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\sum_{l=2}^{n}\lambda_1(M_l^2)\\
&\overset{(3.20)}{\geq} -\sum_{l=2}^{n}\sum_{k=1}^{[\frac{m}{2}]}\lambda_{2k-1}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\sum_{l=2}^{n}\lambda_1(M_l^2)\\
&\geq -\sum_{l=2}^{n}\sum_{k=2}^{[\frac{m}{2}]}\lambda_{2k-1}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i\geq 0.
\end{aligned}$$

Again we get that $v_i=0$ for all $1\leq i\leq m-n$. To conclude, without any assumption on m, equation (3.21) gives:

$$\begin{aligned}
0\geq\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right) &= -\frac{1}{2}\mathrm{tr}(M_1^2)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\\
&= -\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\\
&= \lambda_1\left(\sum_{l=2}^{n}M_l^2\right)-\frac{1}{2}\sum_{l=2}^{n}\sum_{k=1}^{m}\lambda_k(M_l^2)\\
&= \lambda_1\left(\sum_{l=2}^{n}M_l^2\right)-\sum_{l=2}^{n}\lambda_1(M_l^2)-\frac{1}{2}\sum_{l=2}^{n}\sum_{k=3}^{m}\lambda_k(M_l^2)\geq 0
\end{aligned}$$

As a result $\lambda_1(\sum_{l=2}^{n}M_l^2)=\sum_{l=2}^{n}\lambda_1(M_l^2)$ and $\lambda_k(M_l^2)=0$ for all $k=3,\dots,m$ and $l=2,\dots,n$, which completes the proof. □

If we apply this lemma to our study, we get that $\lambda = 0$, $L = 0$ and $(J_2,\ldots,J_n)$ have rank 2 and satisfy $\lambda_1(\sum_{i=2}^n J_i^2) = \sum_{i=2}^n \lambda_1(J_i^2)$. The following lemma will give us a precise description of the endomorphisms $(J_2,\ldots,J_n)$.

Lemma 3.3.2. *Let V be an m-dimensional Euclidean vector space and $K_1,\ldots,K_n : V \longrightarrow V$ be skew-symmetric endomorphisms with $n < m$. Assume that* $\mathrm{rank}(K_i) = 2$ *and* $\mathrm{tr}(K_i \circ K_j) = 0$ *for all $i \neq j$ and that:*

$$\lambda_1(K) = \sum_{i=1}^n \lambda_1\left(K_i^2\right) \quad \text{with} \quad K := \sum_{i=1}^n K_i^2.$$

Then we can find an orthonormal basis $\{u_0,\ldots,u_n,v_1,\ldots,v_{m-n-1}\}$ such that for all $1 \le i,j \le n$ and all $1 \le l \le m-n-1$:

$$K_i(u_0) = \alpha_i u_i,\ K_i(u_j) = -\delta_{ij}\alpha_i u_0 \quad \text{and} \quad K_i(v_l) = 0.$$

Proof. Consider $E := \ker(K - \lambda_1(K)\mathrm{Id}_V)$ and denote $E_i := \mathrm{Im}(K_i)$, for all $i = 1,\ldots,n$. Note that E_i is a 2-plane and there exists a $\alpha_i \in \mathbb{R}\setminus\{0\}$ such that for any $u \in E_i$, $K_i^2(u) = -\alpha_i^2 u$ and $\lambda_1(K_i^2) = -\alpha_i^2$. We claim that $E \subset \bigcap_{i=1}^n E_i$. Indeed, let $u \in E$ and for each $i = 1,\ldots,n$ choose an orthonormal basis (e_i, f_i) of E_i and write:

$$u = \langle u, e_i\rangle e_i + \langle u, f_i\rangle f_i + v_i \quad \text{and} \quad v_i \in E_i^\perp.$$

Since $\lambda_1(K) = -\alpha_1^2 - \ldots - \alpha_n^2$, we get

$$-\sum_{i=1}^n \alpha_i^2 \langle u,u\rangle = \langle K^2(u), u\rangle = \sum_{i=1}^n \langle K_i^2(u), u\rangle.$$

But $K_i^2(u) = -\alpha_i^2(\langle u,e_i\rangle e_i + \langle u,f_i\rangle f_i)$ and hence

$$\langle K_i^2(u), u\rangle = -\alpha_i^2\left(\langle u,e_i\rangle^2 + \langle u,f_i\rangle^2\right).$$

So

$$0 = \sum_{i=1}^n \alpha_i^2(\langle u,u\rangle - \langle u,e_i\rangle^2 - \langle u,f_i\rangle^2) = \sum_{i=1}^n \alpha_i^2 \langle v_i, v_i\rangle = 0.$$

Thus $v_1,\ldots,v_n = 0$ and the claim follows.

Choose $u_0 \in E$ such that $\langle u_0, u_0\rangle = 1$. Then clearly $(u_0, K_i(u_0))$ is an orthogonal basis of E_i. Complete this basis in order to get an orthonormal basis $(u_0, u_i, f_1, \ldots, f_{m-2})$ of V with $u_i = \frac{1}{|K_i(u_0)|}K_i(u_0)$. We have $K_i(f_k) = 0$ for $k = 1,\ldots,m-2$ and hence for $i,j \in \{1,\ldots,n\}$ such that $i \neq j$:

$$\begin{aligned}
0 = \mathrm{tr}(K_i \circ K_j) &= -\langle K_j(u_0), K_i(u_0)\rangle - \langle K_j(u_i), K_i(u_i)\rangle \\
&= -\langle K_j(u_0), K_i(u_0)\rangle + \frac{\alpha_i^2}{|K_i(u_0)|}\langle K_j(u_i), u_0\rangle \\
&= -\left(1 + \frac{\alpha_i^2}{|K_i(u_0)|^2}\right)\langle K_j(u_0), K_i(u_0)\rangle.
\end{aligned}$$

So the family $(u_0, K_1(u_0), \ldots, K_n(u_0))$ is orthogonal, we orthonormalize it and complete it to get the desired basis. □

The relevance of the following lemma will appear later.

Lemma 3.3.3. *Consider the following system of matrix equations on* $\mathbb{R}^{2k}$ *:*

$$\begin{cases} K^2 = P^{-1}AP + A \\ \alpha K = AP - P^{-1}A \end{cases} \tag{3.25}$$

where K *is an invertible skew-symmetric matrix,* P *an orthogonal matrix,* $A = \mathrm{diag}(-\alpha_1^2, \ldots, -\alpha_{2k}^2)$ *with* $\alpha_i \neq 0$ *and* $\alpha = \pm\sqrt{\alpha_1^2 + \cdots + \alpha_{2k}^2}$. *Then* $k = 1$*, in which case we get that :*

$$A = \begin{pmatrix} -\alpha_1^2 & 0 \\ 0 & -\alpha_2^2 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \epsilon\sqrt{\alpha_1^2+\alpha_2^2} \\ -\epsilon\sqrt{\alpha_1^2+\alpha_2^2} & 0 \end{pmatrix} \quad \textit{and} \quad P = \begin{pmatrix} 0 & \mp\epsilon \\ \pm\epsilon & 0 \end{pmatrix}, \quad \epsilon = \pm 1. \tag{3.26}$$

Proof. To prove the Lemma we reason by contradiction and assume that (K, A, P) is a solution of (3.25) and $k > 1$. To get a contradiction, we prove first that K^2 and A commute and hence A and $P^{-1}AP$ commute as well.
Let $\lambda_1 < \ldots < \lambda_r < 0$ be the different eigenvalues of K^2 and $E_1, \ldots, E_r$ the corresponding vector eigenspaces. Since K is skew-symmetric invertible and $\mathrm{tr}(K^2) = -2\alpha^2$, we have:

$$\mathbb{R}^{2k} = E_1 \oplus \ldots \oplus E_r, \dim E_i = 2p_i \quad \text{and} \quad 2\sum_{i=1}^{r} p_i\lambda_i = -2\alpha^2. \tag{3.27}$$

According to (3.25), $P^{-1}AP + A$ and $AP - P^{-1}A$ commutes and hence:

$$A(P + P^{-1})A = P^{-1}A(P + P^{-1})AP.$$

Moreover the first equation of system (3.25) implies that:

$$K^4 = P^{-1}A^2P + A^2 + AP^{-1}AP + P^{-1}APA$$

and the second equation of (3.25) along with the preceding remarks give that:

$$\begin{aligned} \alpha^2K^2 &= APAP + P^{-1}AP^{-1}A - A^2 - P^{-1}A^2P \\ &= APAP + P^{-1}AP^{-1}A + AP^{-1}AP + P^{-1}APA - K^4 \\ &= (AP + AP^{-1})AP + P^{-1}A(P^{-1}A + PA) - K^4 \\ &= A(P + P^{-1})AP + P^{-1}A(P^{-1} + P)A - K^4 \\ &= A(P + P^{-1})A(P + P^{-1}) - K^4. \end{aligned}$$

Therefore we get that $K^2(K^2 + \alpha^2\mathrm{Id}) = A(P + P^{-1})A(P + P^{-1})$ which leads to:

$$A^{-1}K^2(K^2 + \alpha^2\mathrm{Id}) = (P + P^{-1})A(P + P^{-1}). \tag{3.28}$$

But $P^{-1} = P^t$ and the endomorphism at the right hand side of the previous equality is symmetric. This implies that A^{-1} and therefore A commutes with $K^2(K^2 + \alpha^2\mathrm{Id})$.

We now show that A commutes with K^2. If K^2 is proportional to Id this is obviously true. Suppose that K^2 has at least two distinct eigenvalues, i.e., $r \geq 2$. For any $i, j \in \{1, \dots, r\}$ and for any $v \in E_i, w \in E_j$, we have:

$$\begin{aligned}\langle AK^2(K^2 + \alpha^2\mathrm{Id})(v), w\rangle &= \lambda_i(\lambda_i + \alpha^2)\langle Av, w\rangle \\ &= \langle K^2(K^2 + \alpha^2\mathrm{Id})A(v), w\rangle \\ &= \langle K^2(K^2 + \alpha^2\mathrm{Id})w, A(v)\rangle \\ &= \lambda_j(\lambda_j + \alpha^2)\langle Av, w\rangle.\end{aligned}$$

Thus $(\lambda_i - \lambda_j)(\lambda_i + \lambda_j + \alpha^2)\langle Av, w\rangle = 0$. But from (3.27), we get:

$$2(\lambda_i + \lambda_j + \alpha^2) = -2(p_i - 1)\lambda_i - 2(p_j - 1)\lambda_j - 2\sum_{l \neq i, l \neq j} p_l \lambda_l \geq 0.$$

If $k > 2$ then the last two relations implies that $\langle A(E_i), E_j\rangle = 0$ for $i \neq j$ and hence $A(E_i) = E_i$ for $i = 1, \dots, r$. So A commutes with K^2.
If $k = 2$ then $r = 2$, $\dim E_1 = \dim E_2 = 2$ and $\lambda_1 + \lambda_2 = -\alpha^2$. From $\mathbb{R}^{2k} = E_1 \oplus E_2$ one can deduce easily that $K^2(K^2 + \alpha^2\mathrm{Id}) = -\lambda_1\lambda_2\mathrm{Id}$ and by replacing in (3.28) we get:

$$A(P + P^{-1}) = -\lambda_1\lambda_2(P + P^{-1})^{-1}A^{-1}.$$

Now for any $u \in \mathbb{R}^{2k}$ we get that:

$$0 \geq \langle A(P+P^{-1})(u), (P+P^{-1})(u)\rangle = -\lambda_1\lambda_2\langle (P+P^{-1})^{-1}A^{-1}(u), (P+P^{-1})(u)\rangle = -\lambda_1\lambda_2\langle A^{-1}(u), u\rangle \geq 0,$$

this means that $\langle A^{-1}(u), u\rangle = 0$ which is impossible since A is negative definite.
In conclusion A commutes with K^2 and hence A commutes with $P^{-1}AP$ so that there exists an orthonormal basis $\{v_1, \dots, v_{2k}\}$ of $\mathbb{R}^{2k}$ in which both A and $P^{-1}AP$ are diagonal. For any $i \in \{1, \dots, 2k\}$ we can therefore write:

$$Av_i = -\alpha_i^2 v_i \quad \text{and} \quad P^{-1}AP(v_i) = -\alpha_{\sigma(i)}^2 v_i$$

for some permutation σ of $\{1, \dots, 2k\}$. The second equation of (3.25) gives that:

$$\alpha K(v_i) = AP(v_i) - P^{-1}A(v_i) = -\alpha_{\sigma(i)}^2 P(v_i) + \alpha_i^2 P^{-1}(v_i),$$

for any $i \in \{1, \dots, 2k\}$. Thus:

$$\alpha^2\langle K(v_i), K(v_i)\rangle = \alpha_{\sigma(i)}^4 + \alpha_i^4 - 2\alpha_{\sigma(i)}^2\alpha_i^2\langle P^2(v_i), v_i\rangle. \tag{3.29}$$

Assume that $\sigma(i) = i$ for some $i \in \{1, \dots, 2k\}$. It follows from the first equation of (3.25) that $-2\alpha_i^2$ should be an eigenvalue of K^2 and so it must have multiplicity greater than 2, but since $k > 1$ we deduce that $\mathrm{tr}(K^2) < -4\alpha_i^2$. On the other hand, equation (3.29) and the first equation of (3.25) imply that:

$$\alpha^2\langle K(v_i), K(v_i)\rangle = 2\alpha_i^4(1 - \langle P^2(v_i), v_i\rangle) \quad \text{and} \quad -\langle K(v_i), K(v_i)\rangle = \langle K^2(v_i), v_i\rangle = -2\alpha_i^2.$$

Combining these equations we obtain that $\alpha^2 = \alpha_i^2(1-\langle P^2(v_i), v_i\rangle)$ and the Cauchy-Schwarz inequality $|\langle P^2(v_i), v_i\rangle| \leq \|v_i\| \, \|P^2 v_i\| = 1$ implies that $0 \leq 1 - \langle P^2(v_i), v_i\rangle \leq 2$ which in turn gives that $0 \leq \alpha^2 \leq 2\alpha_i^2$. Finally using that $\mathrm{tr}(K^2) = -2\alpha^2$ we conclude that $-4\alpha_i^2 \leq \mathrm{tr}(K^2)$, and we get a contradiction. Thus $\sigma(i) \neq i$ for all $i = 1, \ldots, 2k$.

From $\sum_{i=1}^{2k} \langle K(v_i), K(v_i)\rangle = -\mathrm{tr}(K^2) = 2\alpha^2$ and equation (3.29) we get:

$$2\alpha^4 = 2\sum_{i=1}^{2k} \alpha_i^4 - 2\sum_{i=1}^{2k} \alpha_{\sigma(i)}^2 \alpha_i^2 \langle P^2(v_i), v_i\rangle.$$

Now:

$$\begin{aligned} \alpha^4 - \sum_{i=1}^{2k} \alpha_i^4 &= (\alpha_1^2 + \ldots + \alpha_{2k}^2)^2 - \sum_{i=1}^{2k} \alpha_i^4 \\ &= \sum_{i \neq j} \alpha_i^2 \alpha_j^2 \\ &= \sum_{i=1}^{2k} \alpha_i^2 \alpha_{\sigma(i)}^2 + \sum_{j \neq i, j \neq \sigma(i)} \alpha_i^2 \alpha_j^2. \end{aligned}$$

So we obtain that:

$$0 \leq \sum_{j \neq i, \sigma(i)} \alpha_i^2 \alpha_j^2 = -\sum_{i=1}^{2k} \alpha_i^2 \alpha_{\sigma(i)}^2 (\langle P^2(v_i), v_i\rangle + 1) \leq 0,$$

the right hand side of the previous equality is negative as a consequence of the Cauchy-Schwarz inequality $|\langle P^2(v_i), v_i\rangle| \leq \|v_i\| \, \|P^2 v_i\| = 1$ which implies that $0 \leq \langle P^2(v_i), v_i\rangle + 1 \leq 2$.

Thus $\sum_{j \neq i, \sigma(i)} \alpha_i^2 \alpha_j^2 = 0$, but this contradicts the fact that A is invertible. We conclude that $k = 1$ and in this case we can put:

$$A = \begin{pmatrix} -\alpha_1^2 & 0 \\ 0 & -\alpha_2^2 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

We get that system (3.25) is equivalent to :

$$\begin{cases} \beta^2 - \alpha_1^2 - (\alpha_1^2 \cos^2(\theta) + \alpha_2^2 \sin^2(\theta)) = 0 \\ \\ \beta^2 - \alpha_2^2 - (\alpha_1^2 \sin^2(\theta) + \alpha_2^2 \cos^2(\theta)) = 0 \\ \\ \cos\theta \sin\theta (\alpha_2^2 - \alpha_1^2) = 0 \\ \pm\beta\sqrt{\alpha_1^2 + \alpha_2^2} - (\alpha_1^2 + \alpha_2^2)\sin\theta = 0. \end{cases}$$

By summing over the first two equations in the previous system and replacing in the last equation we obtain that $\beta = \epsilon\sqrt{\alpha_1^2 + \alpha_2^2}$, $\sin\theta = \pm\epsilon$ and $\cos\theta = 0$ with $\epsilon = \pm 1$, which ends the proof. □

We are now in possession of all the necessary ingredients to characterize ω-quasi Einstein Lorentzian 2-step nilpotent Lie algebras of type 1 as a key step towards the proof of Theorem 3.1.1.

Theorem 3.3.2. *Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be a Lorentzian 2-step nilpotent Lie algebra then suppose that $Z(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]$ is non-degenerate Lorentzian and let $\omega\in Z^2(\mathfrak{g},\mathbb{R})$. Then $\mathfrak{g}$ is ω-quasi Einstein of type 1 with positive Einstein constant λ if and only if $\lambda=0$ and, up to a Lie algebra isomorphism, $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle,\omega)$ has one of the following forms :*

1. $\dim\mathfrak{g}=5$ *and there exists an orthonormal basis $\{e_1,e_2,u_1,u_2,u_3\}$ of $\mathfrak{g}$ with $\langle e_1,e_1\rangle=-1$ such that the non vanishing Lie brackets and ω-products are given by :*

$$[u_1,u_2]=\alpha e_2,\ [u_2,u_3]=\pm\alpha e_1,\ \omega(e_2,u_3)=\epsilon\alpha,\ \omega(e_1,u_1)=\mp\epsilon\alpha,\quad \alpha\neq 0,\epsilon=\pm 1. \tag{3.30}$$

2. $\dim\mathfrak{g}=6$ *and there exists an orthonormal basis $\{e_1,e_2,e_3,u_1,u_2,u_3\}$ of $\mathfrak{g}$, $\langle e_1,e_1\rangle=-1$, such that the non-vanishing Lie brackets and ω-products are given by :*

$$\begin{cases}[u_1,u_2]=\alpha_2e_2,\ [u_1,u_3]=\alpha_3e_3,\ [u_2,u_3]=\epsilon\alpha e_1,\\ \omega(e_2,u_3)=\mp\epsilon\alpha_2,\ \omega(e_3,u_2)=\pm\epsilon\alpha_3,\ \omega(e_1,u_1)=\pm\alpha,\end{cases} \tag{3.31}$$

where $\alpha_2,\alpha_3\neq 0$, $\epsilon=\pm1$ and $\alpha=\sqrt{\alpha_2^2+\alpha_3^2}$.

Proof. We keep the notations from the beginning of section 3.3. The structure endomorphisms $(J_2,\dots,J_n)$ have been shown to satisfy the hypothesis of Lemma 3.3.2, therefore we can find an orthonormal basis $(u_1,u_2,\dots,u_n,v_1,\dots,v_{m-n})$ of $[\mathfrak{g},\mathfrak{g}]^\perp$ and $(\alpha_2,\dots,\alpha_n)\in\mathbb{R}^n$ such that, for all $2\leq i,j\leq n$ and all $1\leq k\leq m-n$:

$$J_i(u_1)=\alpha_iu_i,\ J_i(u_j)=-\delta_{ij}\alpha_iu_1,\ \alpha_i\neq 0\quad\text{and}\quad J_i(v_k)=0.$$

Put $J=\sum_{i=2}^n J_i^2$, it is clear that for all $2\leq i\leq n$ and all $1\leq k\leq m-n$:

$$J(u_1)=-(\alpha_2^2+\ldots+\alpha_n^2)u_1,\ J(u_i)=-\alpha_i^2u_i,J(v_k)=0,\ \mathrm{tr}(J_1^2)=-2(\alpha_2^2+\ldots+\alpha_n^2)\quad\text{and}\quad\mathrm{tr}(J_i^2)=-2\alpha_i^2. \tag{3.32}$$

Consider $\mathbb{B}_2=(f_1,\dots,f_n):=\left(\frac{B(e_1)}{|B(e_1)|},\dots,\frac{B(e_n)}{|B(e_n)|}\right)$. By virtue of equation (3.17), we get that for any $i=2,\dots,n$ and any $v\in\{f_1,\dots,f_n\}^\perp$:

$$J_1^2(f_1)=J(f_1)-\frac{1}{2}\mathrm{tr}(J_1^2)f_1,\ J_1^2(f_i)=J(f_i)+\frac{1}{2}\mathrm{tr}(J_i^2)f_i\quad\text{and}\quad J_1^2(v)-J(v)=0. \tag{3.33}$$

Since $\lambda_1(J)=\frac{1}{2}\mathrm{tr}(J_1^2)$, we deduce that:

$$\langle J_1(f_1),J_1(f_1)\rangle=-\langle J(f_1),f_1\rangle+\lambda_1(J)\leq 0$$

and hence:

$$J_1(f_1)=0\quad\text{and}\quad J(f_1)=\lambda_1(J)f_1.$$

But (3.32) shows that the multiplicity of $\lambda_1(J)$ is equal to one and hence $f_1 = \pm u_1$. Let us show that the restriction of J_1 to $f_1^\perp$ is invertible. We have from (3.15) that:

$$J_1^2 = J - BB^*$$

and from (3.16) the restriction of BB^* to $f_1^\perp$ is positive so if $u \in f_1^\perp$ and $J_1 u = 0$ we get:

$$\sum_{i=2}^{n} \langle J_i u, J_i u\rangle + \langle BB^*(u), u\rangle = 0$$

therefore $u \in \cap_{i=1}^n \ker J_i = Z(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]$ and so $u = 0$. It follows that $J_1 : f_1^\perp \longrightarrow f_1^\perp$ is invertible and thus m must be odd. In view of the last equation of (3.33) along with the fact that $f_1 = \pm u_1$, we obtain that $J_1^2(\{f_1,\ldots,f_n\}^\perp) \subset \mathrm{span}\{u_2,\ldots,u_n\}$, the preceding remark then leads to $m - n \leq n - 1$ thus $m \leq 2n - 1$.

For convenience we set $w_i := B(e_i)$ for $i = 1,\ldots,n$. From (3.16) we get:

$$\langle w_i, w_i\rangle = -\frac{1}{2}\mathrm{tr}(J_i^2) \quad \text{and} \quad \langle w_i, w_j\rangle = 0, i \neq j.$$

So

$$BB^*(x) = -(\alpha_2^2 + \cdots + \alpha_n^2)\langle x, u_1\rangle u_1 + \sum_{i=2}^{n} \langle x, w_i\rangle w_i. \tag{3.34}$$

The fact that B defines a 2-cocycle is equivalent to:

$$\sum_{i=1}^{n} (\langle J_i u, v\rangle w_i + \langle w_i, u\rangle J_i v - \langle w_i, v\rangle J_i u) = 0, \quad u, v \in [\mathfrak{g},\mathfrak{g}]^\perp.$$

If we apply this equation to $u = u_1$ we get:

$$\langle w_1, u_1\rangle J_1 v = -\sum_{i=2}^{n} (\alpha_i \langle u_i, v\rangle w_i - \alpha_i \langle w_i, v\rangle u_i).$$

From the definition of w_1 we get that $\langle w_1, u_1\rangle = \pm\sqrt{\alpha_2^2 + \cdots + \alpha_n^2}$ and therefore the previous equation gives that:

$$J_1 = \pm\frac{1}{\sqrt{\alpha_2^2 + \ldots + \alpha_n^2}} \sum_{i=2}^{n} \alpha_i u_i \wedge w_i. \tag{3.35}$$

Actually this is equivalent to B being a 2-cocycle. The expression of BB^* given in (3.34) leads to:

$$J_1^2 - \sum_{i=2}^{q} J_i^2 = (\alpha_2^2 + \ldots + \alpha_n^2)\langle x, u_1\rangle u_1 - \sum_{i=2}^{n} \langle ., w_i\rangle w_i. \tag{3.36}$$

Put $a = \pm\frac{1}{\sqrt{\alpha_2^2+\ldots+\alpha_n^2}}$. Equation (3.35) on the other hand gives that:

$$\begin{aligned} J_1 w_l &= a\alpha_l^3 u_l - a\sum_{i=2}^{n}\alpha_i\langle u_i, w_l\rangle w_i, \\ J_1 u_l &= -a\alpha_l w_l + a\sum_{i=2}^{n}\alpha_i\langle w_i, u_l\rangle u_i, \\ J_1 v_k &= a\sum_{i=2}^{n}\alpha_i\langle w_i, v_k\rangle u_i, \end{aligned} \tag{3.37}$$

Now using (3.36) and then (3.37), it is straightforward to check that:

$$\langle J_1^2 v_k, v_k\rangle = -\sum_{l=2}^{n}\langle w_i, v_k\rangle^2 = a\sum_{i=2}^{n}\alpha_i\langle w_i, v_k\rangle\langle J_1 u_i, v_k\rangle = -a^2\sum_{l=2}^{n}\alpha_i^2\langle w_i, v_k\rangle^2.$$

So we conclude that:

$$\sum_{l=2}^{n}(1 - a^2\alpha_i^2)\langle w_i, v_k\rangle^2 = 0.$$

Thus either $n = 2$ or $n \geq 3$ and $\langle w_i, v_k\rangle = 0$ for $i = 1, \ldots, n$ and $v_k = 1, \ldots, m-n$. So we get that either $n = 2$ or $n \geq 3$ and $m = n$.
For $n = 2$, we have $m = 3$, (e_1, e_2) is an orthonormal basis of $[\mathfrak{g}, \mathfrak{g}]$ such that $\langle e_1, e_1\rangle = -1$ and (u_1, u_2, v) is an orthonormal basis of $[\mathfrak{g}, \mathfrak{g}]^\perp$, moreover $B(e_1) = au_1$, $B(e_2) = bv$.

$$J_2 = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_1 = bu_2 \wedge v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & -b & 0 \end{pmatrix} \quad \text{and} \quad a^2 = b^2 = \alpha^2.$$

This automatically leads to (3.30). For $n \geq 3$, we have $n = m = 2k+1$. Recall that:

$$[u,v] = \begin{cases} \sum_{i=1}^{n}\langle J_i(u), v\rangle e_i, & u, v \in [\mathfrak{g},\mathfrak{g}]^\perp \\ 0, & \text{otherwise} \end{cases}, \quad \omega(u,v) = \begin{cases} \langle B(u), v\rangle, & u \in [\mathfrak{g},\mathfrak{g}],\ v \in [\mathfrak{g},\mathfrak{g}]^\perp \\ 0, & \text{otherwise} \end{cases}$$

From what have been shown so far, the only Lie brackets of $\mathfrak{g}$ that do not automatically vanish are:

$$[u_1, u_i] = \langle J_i(u_1), u_i\rangle e_i = \alpha_i e_i \quad \text{and} \quad [u_i, u_j] = \langle J_1(u_i), u_j\rangle e_1 := \beta_{ij} e_1,$$

for $2 \leq i, j \leq n$, furthermore since J_1 is invertible on $u_1^\perp$ it follows that $K := (\beta_{ij})_{i,j}$ is a skew-symmetric invertible matrix. On the other hand, put $\hat{P}(f_i) := u_i$ for $2 \leq i \leq m$ then it is clear that $\hat{P} := (\hat{p}_{ij})_{i,j}$ is an orthogonal matrix, and a straightforward computation shows that $\langle B(e_i), u_j\rangle = \langle B(e_i), \hat{P}(f_j)\rangle = \epsilon_i \hat{p}_{ji}\alpha_i$ with $\epsilon_i = \pm 1$, note that $P = (\epsilon_j \hat{p}_{ij})_{i,j}$ is an orthogonal matrix as well. Next since $f_1 = \pm u_1$ we get that:

$$\langle B(e_1), u_1\rangle = \pm\sqrt{\alpha_2^2 + \cdots + \alpha_n^2}.$$

Finally in these notations notice that $J_1^2 - \sum_{i=2}^n J_i^2 = -BB^*$ is equivalent to $K^2 = P^{-1}AP + A$ with $A = \mathrm{diag}(-\alpha_2^2, \dots, -\alpha_n^2)$ and the cocycle condition $\oint \langle B([u,v]), w\rangle = 0$ is equivalent to the equation $\pm\alpha K = AP - P^{-1}A$ where $\alpha = \sqrt{\alpha_2^2 + \cdots + \alpha_n^2}$. This is exactly the situation of Lemma 3.3.3 and consequently $k = 1$, i.e $n = m = 3$ which means $\dim \mathfrak{g} = 6$, furthermore in view of (3.26) we get that the Lie algebra structure of $\mathfrak{g}$ is given by (3.31). This ends the proof. □

Following the discussion of section 3.3 we get as a consequence of the preceding Theorem that a Lorentzian 3-step nilpotent Lie algebras $(\mathfrak{h}, \langle\ ,\ \rangle)$ with non-degenerate 1-dimensional center is Einstein if and only if it is Ricci-flat and has one of the following forms :

1. Either $\dim \mathfrak{h} = 6$ in which case $\dim[\mathfrak{h},\mathfrak{h}] = \mathrm{codim}[\mathfrak{h},\mathfrak{h}] = 3$ and there exists an orthonormal basis $\{x, e_1, e_2, u_1, u_2, u_3\}$ of $\mathfrak{h}$ with $\langle e_1, e_1\rangle = -1$ such that the Lie algebra structure is given by :

$$[u_1,u_2] = \alpha e_2,\ [u_2,u_3] = \pm\alpha e_1,\ [e_2,u_3] = \alpha x,\ [e_1,u_1] = \mp\alpha x, \quad \alpha \neq 0. \tag{3.38}$$

$$[u_1,u_2] = \alpha e_2,\ [u_2,u_3] = \pm\alpha e_1,\ [e_2,u_3] = -\alpha x,\ [e_1,u_1] = \pm\alpha x, \quad \alpha \neq 0. \tag{3.39}$$

2. $\dim \mathfrak{h} = 7$ in which case $\dim[\mathfrak{h},\mathfrak{h}] = \mathrm{codim}[\mathfrak{h},\mathfrak{h}] + 1 = 4$. Moreover there exists an orthonormal basis $\{x, e_1, e_2, e_3, u_1, u_2, u_3\}$ of $\mathfrak{h}$ such that $\langle e_1, e_1\rangle = -1$ and in which the Lie algebra structure is given by :

$$\begin{aligned}[u_1,u_2] = \alpha_2 e_2,\ [u_1,u_3] = \alpha_3 e_3,\ [u_2,u_3] = \epsilon\alpha e_1,\ [e_2,u_3] = \mp\epsilon\alpha_2 x,\\ [e_3,u_2] = \pm\epsilon\alpha_3 x,\ [e_1,u_1] = \pm\alpha x\end{aligned} \tag{3.40}$$

where $\alpha = \sqrt{\alpha_2^2 + \alpha_3^2}$.

Proof of Main Theorem. In case 1, the Lie algebra structure [,] of $\mathfrak{h}$ has one of the forms given by either (3.38) or (3.39). It is clear that (3.39) can be obtained from (3.38) by replacing u_3 with $-u_3$, for this reason it suffices to treat the case where $\mathfrak{h}$ is given by (3.38). If we now put:

$$f_1 = u_2,\ f_2 = u_3 + u_1,\ f_3 = u_3 - u_1,\ f_4 = \pm\alpha e_1 - \alpha e_2,\ f_5 = \pm\alpha e_1 + \alpha e_2,\ f_6 = 2\alpha^2 x.$$

Then we can easily see that:

$$[f_1,f_2] = f_4,\ [f_1,f_3] = f_5,\ [f_2,f_4] = f_6,\ [f_3,f_5] = -f_6,$$

$$[f_2,f_3] = [f_1,f_4] = [f_1,f_5] = [f_2,f_5] = [f_3,f_4] = [f_4,f_5] = [f_i,f_6] = 0.$$

Thus $\mathfrak{h} \simeq \mathrm{L}_{6,19}(-1)$ and the metric $\langle\ ,\ \rangle$ is represented in the basis $\{f_1,\dots,f_6\}$ of $\mathfrak{h}$ by the expression (3.1). For case 2, when $\mathfrak{h}$ is given by (3.40) we can put:

$$f_1 := u_1,\ f_2 := u_2, f_3 := u_3, f_4 := \epsilon\sqrt{\alpha_2^2 + \alpha_3^2}e_1,\ f_5 := \alpha_2 e_2,\ f_6 = \alpha_3 e_3,\ f_7 := \pm\epsilon(\alpha_2^2 + \alpha_3^2),$$

then the Lie algebra $\mathfrak{h}$ is given by (3.2) with $r = \frac{\alpha_2^2}{\alpha_2^2+\alpha_3^2}$. Moreover if we set $a = \alpha_2^2 + \alpha_3^2$ then we get that $\langle\ ,\ \rangle$ is given by (3.3). □

We end our chapter by some examples of Einstein Lorentzian nilpotent Lie algebras with non-degenerate center of dimension greater that one, the goal is to illustrate that such Lie algebras do occur even in the 3-step nilpotent case. This gives motivation for a future investigation.

Example 2. *Let $\mathfrak{h}$ be the 8-dimensional nilpotent Lie algebra with Lie bracket $[\ ,\ ]$ given in a basis $\mathbb{B} = \{e_1, \dots, e_8\}$ by :*

$$\begin{cases} [e_1,e_2] = -4\sqrt{3}e_3,\ [e_1,e_3] = \sqrt{\frac{5}{2}}e_4,\ [e_1,e_4] = -2\sqrt{3}e_8,\ [e_1,e_5] = 3\sqrt{\frac{7}{2}}e_6, \\ [e_1,e_6] = -4\sqrt{2}e_7,\ [e_2,e_3] = -\sqrt{\frac{5}{2}}e_5,\ [e_2,e_4] = -3\sqrt{\frac{7}{2}}e_6,\ [e_2,e_5] = -2\sqrt{3}e_7, \\ [e_2,e_6] = -4\sqrt{2}e_8,\ [e_3,e_4] = -\sqrt{21}e_7,\ [e_3,e_5] = -\sqrt{21}e_8. \end{cases}$$

One can define a Lorentzian inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}$ by requiring $\mathbb{B}$ to be an orthonormal basis with $\langle e_6, e_6\rangle = -1$. Then it is easy to see that $\mathrm{Z}(\mathfrak{h}) = \mathrm{span}\{e_7, e_8\}$ hence non-degenerate with respect to $\langle\ ,\ \rangle$. Moreover a straightforward computation shows that $(\mathfrak{h}, \langle\ ,\ \rangle)$ is Einstein with nonvanishing scalar curvature. This example was first given in [6].

Example 3. *Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathbb{R}^7$ and $\{e_1, \dots, e_7\}$ an orthonormal basis with respect to $\langle\ ,\ \rangle$ such that $\langle e_1, e_1\rangle = -1$. Define the Lie bracket $[\ ,\ ]$ by setting :*

$$\begin{cases} [e_1,e_3] = \sqrt{2}e_7,\ [e_2,e_4] = \sqrt{2}e_7,\ [e_4,e_5] = -e_1,\ [e_4,e_6] = -e_1, \\ [e_3,e_5] = -e_2,\ [e_3,e_6] = -e_2. \end{cases}$$

Put $\mathfrak{h} := (\mathbb{R}^7, [\ ,\ ])$, then it is straightforward to check that $(\mathfrak{h}, \langle\ ,\ \rangle)$ is a Ricci-flat 3-step nilpotent Lie algebra with $\mathrm{Z}(\mathfrak{h}) = \mathrm{span}\{e_7, e_5 - e_6\}$, therefore $\mathfrak{h}$ has non-degenerate center.

Example 4. *Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathbb{R}^{10}$ and $\{e_1, \dots, e_{10}\}$ an orthonormal basis with respect to $\langle\ ,\ \rangle$ such that $\langle e_5, e_5\rangle = -1$. Choose $p, r \in \mathbb{R}$ such that $p, r \neq 0$ and define on $\mathbb{R}^{10}$ the Lie bracket $[\ ,\ ]$ given by :*

$$\begin{cases} [e_1,e_3] = -\sqrt{p^2+r^2}e_5,\ [e_1,e_4] = -\sqrt{p^2+r^2}e_6,\ [e_2,e_4] = -\sqrt{p^2+r^2}e_5,\ [e_2,e_3] = -\sqrt{p^2+r^2}e_6, \\ [e_5,e_1] = pe_7,\ [e_5,e_2] = pe_8,\ [e_5,e_3] = re_9,\ [e_5,e_4] = re_{10} \\ [e_6,e_1] = pe_8,\ [e_6,e_2] = pe_7,\ [e_6,e_3] = re_{10},\ [e_6,e_4] = re_9. \end{cases}$$

Put $\mathfrak{h} := (\mathbb{R}^{10}, [\ ,\ ])$, then it is straightforward to check that $(\mathfrak{h}, \langle\ ,\ \rangle)$ is a Ricci-flat 3-step nilpotent Lie algebra with $\mathrm{Z}(\mathfrak{h}) = \mathrm{span}\{e_7, e_8, e_9, e_{10}\}$, therefore $\mathfrak{h}$ has non-degenerate center.

A

Generalities on Lie algebras

A.1 Lie algebras and Lie groups

A.1.1 Lie algebras

Recall that a ***real Lie algebra*** is any real vector space $\mathfrak{g}$ endowed with a skew-symmetric bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(u,v) \mapsto [u,v]$ satisfying for any $u,v,w \in \mathfrak{g}$:

$$[[u,v],w] + [[w,u],v] + [[v,w],u] = 0 \quad \textit{(Jacobi Identity).}$$

We shall use the term Lie algebra in order to refer to real Lie algebras. We say that a Lie algebra $\mathfrak{g}$ is finite-dimensional when its underlying vector space is finite-dimensional.

Example 5.

1. *Let V be an arbitrary (real) vector space and let* $\mathrm{End}(V)$ *denote the set of all endomorphisms of V. Given $u,v \in \mathrm{End}(V)$, define:*

$$[u,v] := u \circ v - v \circ u.$$

   It is then straightforward to check that $[\ ,\ ] : \mathrm{End}(V) \times \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$ *defines a Lie bracket on* $\mathrm{End}(V)$. *Furthermore* $(\mathrm{End}(V), [\ ,\ ])$ *is a finite-dimensional Lie algebra if and only if V is a finite-dimensional vector space. When its Lie algebra structure is taken into account, the notation* $\mathfrak{gl}(V)$ *is used instead of* $\mathrm{End}(V)$.

2. *Let M be a smooth manifold let $\mathcal{X}(M)$ be the vector space of smooth vector fields on M. Recall that vector fields can be identified with derivations of $\mathscr{C}^\infty(M)$. For any $X, Y \in \mathcal{X}(M)$, let $[X,Y]$ be the vector field on M given as a derivation of $\mathscr{C}^\infty(M)$ by the expression:*

$$[X,Y](f) := X(Y(f)) - Y(X(f)),$$

   for any $f \in \mathscr{C}^\infty(M)$. One can easily check using local coordinate systems that the operation $[\ ,\ ] : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ defines a Lie bracket on $\mathcal{X}(M)$. Moreover, the Lie algebra $(\mathcal{X}(M), [\ ,\ ])$ is infinite dimensional whenever $\dim M > 0$.

Let $\mathfrak{g}$ be a Lie algebra and consider a vector subspace $\mathfrak{h} \subset \mathfrak{g}$. We say that $\mathfrak{h}$ is a ***Lie subalgebra*** of $\mathfrak{g}$ if it satisfies $[u,v] \in \mathfrak{h}$ for any $u,v \in \mathfrak{h}$. We say that $\mathfrak{h}$ is an ***ideal*** of the Lie algebra $\mathfrak{g}$ if for any $u \in \mathfrak{g}$ and any $v \in \mathfrak{h}$ we have that $[u,v] \in \mathfrak{h}$.

Example 6.

1. *Let* $(\mathfrak{g},[\ ,\ ])$ *be any Lie algebra. Denote* $[\mathfrak{g},\mathfrak{g}] := \mathrm{span}\{[u,v],\ u,v \in \mathfrak{g}\}$ *i.e the vector subspace of* $\mathfrak{g}$ *spanned by all the Lie brackets. It is easy to see that* $[\mathfrak{g},\mathfrak{g}]$ *is an ideal of* $\mathfrak{g}$, *it is called* ***the derived ideal*** *of* $\mathfrak{g}$. *On the other hand, if we set:*

$$\mathrm{Z}(\mathfrak{g}) = \{u \in \mathfrak{g},\ [u,v] = 0, \textit{ for all } v \in \mathfrak{g}\},$$

   then $\mathrm{Z}(\mathfrak{g})$ *is an ideal of* $\mathfrak{g}$, *called* ***the center*** *of* $\mathfrak{g}$. *Finally any vector subspace of* $\mathfrak{g}$ *either containing* $[\mathfrak{g},\mathfrak{g}]$ *or contained in* $\mathrm{Z}(\mathfrak{g})$ *is itself an ideal of* $\mathfrak{g}$

2. *Let* V *be a finite-dimensional vector space and* $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}$ *be a non-degenerate inner product (see Appendix B). Let* $\mathfrak{so}(V,\langle\ ,\ \rangle)$ *be the vector subspace of* $\mathfrak{gl}(V)$ *consisting of all endomorphisms* $u : V \longrightarrow V$ *satisfying*

$$\langle u(x), y\rangle = -\langle x, u(y)\rangle,$$

   for all $x,y \in V$ *i.e* u *is skew-symmetric with respect to* $\langle\ ,\ \rangle$. *One can check that* $\mathfrak{so}(V,\langle\ ,\ \rangle)$ *is a Lie subalgebra of* $\mathfrak{gl}(V)$.

Let $(\mathfrak{g}_1,[\ ,\ ]_1)$ and $(\mathfrak{g}_2,[\ ,\ ]_2)$ be arbitrary Lie algebras. A linear map $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is called a Lie algebra homomorphism if it satisfies that $\varphi[u,v]_1 = [\varphi(u),\varphi(v)]_2$ for any $u,v \in \mathfrak{g}_1$, when $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is bijective, we say that it defines a Lie algebra isomorphism.
For any Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$, it is easy to see that $\ker(\varphi)$ is a Lie subalgebra of $\mathfrak{g}_1$ and $\mathrm{Im}(\varphi)$ is a Lie subalgebra of $\mathfrak{g}_2$.

Example 7.

1. *Let* $(\mathfrak{g},[\ ,\ ])$ *be a Lie algebra and* $\mathfrak{h} \subset \mathfrak{g}$ *a Lie subalgebra of* $\mathfrak{g}$. *By the definition of a Lie subalgebra, the natural inclusion* $\iota : \mathfrak{h} \longrightarrow \mathfrak{g}$ *is a Lie algebra homomorphism.*

2. *Let* $\phi : V_1 \longrightarrow V_2$ *be any vector space isomorphism. The map* $\varphi : \mathfrak{gl}(V_2) \longrightarrow \mathfrak{gl}(V_1)$ *given by* $\varphi(u) := \phi^{-1} \circ u \circ \phi$ *is then a Lie algebra isomorphism.*

3. *Let* $(\mathfrak{g},[\ ,\ ])$ *be a Lie algebra and* $I \subset \mathfrak{g}$ *an ideal of* $\mathfrak{g}$. *There exists a unique Lie algebra structure on the quotient vector space* $\mathfrak{h} := \mathfrak{g}/I$, *denoted* $[\ ,\ ]_q$, *such that the natural projection* $\pi : \mathfrak{g} \longrightarrow \mathfrak{h}$ *is a Lie algebra homomorphism. In other words,* $[\ ,\ ]_q$ *is defined such that:*

$$[\pi(u), \pi(v)]_q := \pi[u,v],$$

   and doesn't depend on the representative elements $u,v \in \mathfrak{g}$.

The Lie bracket $[\ ,\ ]$ of a Lie algebra $\mathfrak{g}$ induces a linear map $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, $u \mapsto \mathrm{ad}_u$ which is given by the expression $\mathrm{ad}_u(v) = [u,v]$. This map is called the ***adjoint representation*** of $\mathfrak{g}$, using the Jacobi identity, one can show that:

$$\mathrm{ad}_{[u,v]} := [\mathrm{ad}_u, \mathrm{ad}_v],$$

this means that $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism. A ***derivation*** of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ is any linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying for any $u, v \in \mathfrak{g}$:

$$D[u,v] = [D(u), v] + [u, D(v)]. \tag{A.1}$$

The vector space of all derivations of a Lie algebra $\mathfrak{g}$ is denote $\mathrm{Der}(\mathfrak{g})$, it is itself a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. The Jacobi identity of $\mathfrak{g}$ can be rewritten as:

$$\mathrm{ad}_u([v,w]) = [\mathrm{ad}_u(v), w] + [v, \mathrm{ad}_u(w)],$$

for any $u, v, w \in \mathfrak{g}$, this shows that in particular $\mathrm{ad}_u \in \mathrm{Der}(\mathfrak{g})$. In fact, the elements of $\mathrm{ad}(\mathfrak{g})$ of this form are called ***inner derivations*** of $\mathfrak{g}$, and one can check using (A.1) that $\mathrm{ad}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.

A.1.2 Lie groups

Recall that a ***Lie group*** G is a topological group which is endowed with a structure of a differentiable manifold such that the multiplication and inversion maps:

$$m_G : G \times G \longrightarrow G, \quad (x,y) \mapsto xy \quad \text{et} \quad i_G : G \longrightarrow G, \quad x \mapsto x^{-1}$$

are smooth. Denote $\ell_g, r_g : G \longrightarrow G$ respectively the left and right multiplications by an element $g \in G$, i.e $\ell_g(x) = gx$ and $r_g(x) = xg$.

Example 8.

1. *The group* $\mathrm{GL}(n,\mathbb{R})$ *of* $n \times n$ *real invertible matrices is a Lie group.*

2. *Let* V *be a finite-dimensional vector space and denote* $\mathrm{GL}(V)$ *the group of all automorphisms of* V *with its natural topology. By fixing a basis of* V *one can show that* $\mathrm{GL}(V)$ *is isomorphic to* $\mathrm{GL}(n,\mathbb{R})$ *with* $n := \dim(V)$, *and so* $\mathrm{GL}(V)$ *can be given a Lie group structure using this isomorphism. One also shows that this structure is independent of the choice of a basis of* V.

3. *Let* V *be a finite-dimensional vector space and* $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}$ *a non-degenerate bilinear form. Let* $\mathrm{O}(V, \langle\ ,\ \rangle)$ *be the group of all linear isometries of* $(V, \langle\ ,\ \rangle)$ *i.e*

$$\mathrm{O}(V, \langle\ ,\ \rangle) = \{u \in \mathrm{GL}(V),\ \langle u(x), u(y)\rangle = \langle x, y\rangle, \textit{ for all } x, y \in V\},$$

   One can show that $\mathrm{O}(V, \langle\ ,\ \rangle)$ *is a Lie group.*

A vector field X on a Lie group G is said to be ***left invariant*** if for any $g \in G$, we have:

$$X_{gx} = T_x \ell_g(X_x).$$

Recall that for any diffeomorphism $f : G \longrightarrow G$ and any vector field $X \in \mathcal{X}(G)$, one can define a vector field $f_* X$ on G by the expression $(f_* X)_x := (T_{f^{-1}(x)} f)(X_x)$. This gives rise to a Lie algebra isomorphism $f_* : \mathcal{X}(G) \longrightarrow \mathcal{X}(G)$, and it is clear that X is a left-invariant vector field (resp. right-invariant vector field) if and only if for any $g \in G$:

$$(\ell_g)_* X = X \quad (\text{resp. } (r_g)_* X = X). \tag{A.2}$$

The set of left-invariant vector fields on G is denoted $\mathcal{X}^\ell(G)$, moreover (A.2) shows that it is a Lie subalgebra of $\mathcal{X}(G)$. In fact $\mathcal{X}^\ell(G)$ is a finite-dimensional real vector space and moreover $\dim \mathcal{X}^\ell(G) = \dim G$, this is a consequence of the fact that the evaluation map

$$\phi : \mathcal{X}^\ell(G) \longrightarrow T_e G, \qquad X \mapsto X_e$$

is a vector space isomorphism. The isomorphism ϕ allows to transport the Lie algebra structure of $\mathcal{X}^l(G)$ on $T_e G$ as follows: If we denote for all $v \in T_e G$, $v^\ell := \phi^{-1}(v) \in \mathcal{X}^l(G)$ we obtain a Lie algebra structure on $T_e G$ given by the bracket:

$$[v, w] := [v^\ell, w^\ell]_e.$$

We then call it the ***Lie algebra of* G** and we denote $\mathrm{Lie}(G)$ the couple $(T_e G, [\ ,\ ])$.

Example 9.

1. *The Lie algebra of* $\mathrm{GL}(n, \mathbb{R})$ *is exactly* $\mathcal{M}_n(\mathbb{R})$ *endowed with the Lie bracket* $[\ ,\ ]$ *given by:*

   $$[A, B] = A \cdot B - B \cdot A,$$

   for any $A, B \in \mathcal{M}_n(\mathbb{R})$.

2. *Let* V *be a finite-dimensional vector space, the Lie algebra of* $\mathrm{GL}(V)$ *is exactly* $\mathfrak{gl}(V)$.

3. *Consider a finite-dimensional vector space* V *together with a non-degenerate bilinear form* $\langle\ ,\ \rangle$. *One can check that the Lie algebra of* $\mathrm{O}(V, \langle\ ,\ \rangle)$ *is* $\mathfrak{so}(V, \langle\ ,\ \rangle)$.

As for vector fields, one can define a differential form ω on G to be left invariant (*resp.* right invariant) if it satisfies $\ell_g^* \omega = \omega$ (*resp.* $r_g^* \omega = \omega$) for any $g \in G$. The set of all left-invariant forms is denoted $\Omega(G)^\ell$, and one can show that it defines a differential subcomplex of the de Rham complex $\Omega_{dR}(G)$, i.e it is stable under the de Rham differential and exterior products. Furthermore, if $\Lambda^p \mathfrak{g}^*$ denotes the vector space of all p-forms on the Lie algebra $\mathfrak{g}$ of G, with $p \in \mathbb{N}$, and if we set $\Lambda \mathfrak{g}^* := \oplus_p \Lambda^p \mathfrak{g}^*$, then we get that:

$$\psi : \Omega(G)^\ell \longrightarrow \Lambda \mathfrak{g}^*, \quad \omega \mapsto \omega_e,$$

is a vector space isomorphism. So the study of left-invariant (*resp.* right-invariant) forms on a Lie group, reduces to the study of the exterior forms of its Lie algebra.

A.2 Nilpotent Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The ***descending central series*** of $\mathfrak{g}$ is the family of ideals $(\mathscr{C}^k(\mathfrak{g}))_{k\in\mathbb{N}}$ of $\mathfrak{g}$ defined inductively bt the formula:

$$\mathscr{C}^0(\mathfrak{g}) = \mathfrak{g} \quad \textit{and} \;\; \mathscr{C}^{n+1}(\mathfrak{g}) := [\mathfrak{g}, \mathscr{C}^n(g)],$$

for any $n \in \mathbb{N}$. Notice that $\mathscr{C}^1(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]$ is just the derived ideal of the Lie algebra $\mathfrak{g}$. We say that $\mathfrak{g}$ is ***nilpotent*** if $\mathscr{C}^n(\mathfrak{g}) = \{0\}$ for some interger $n \in \mathfrak{g}$. The smallest integer $k \in \mathbb{N}$ for which $\mathscr{C}^k(\mathfrak{g}) = \{0\}$ is called the ***nilindex*** of $\mathfrak{g}$ or the ***nilpotency order*** of $\mathfrak{g}$, in which case $\mathfrak{g}$ is said to be k***-step nilpotent***, notice that in this case $\mathscr{C}^{k-1}(\mathfrak{g}) \subset \mathrm{Z}(\mathfrak{g})$. Here are some known properties of nilpotent Lie algebras, one can see [11] for a proof:

Proposition A.2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then:*

1. *If $\mathfrak{g}$ is non-trivial, then its center $\mathrm{Z}(\mathfrak{g})$ is also non-trivial.*

2. *Any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is nilpotent with nilindex smaller than the nilindex of $\mathfrak{g}$.*

3. *If $\varphi : \mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$ is a Lie algebra homomorphism then $\varphi(\mathfrak{g})$ is a nilpotent Lie algebra. In particular, for any ideal $I \subset \mathfrak{g}$, we get that the quotient Lie algebra $\mathfrak{g}/I$ is nilpotent.*

4. *The endomorphisms $\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ are nilpotent for all $x \in \mathfrak{g}$.*

A Lie algebra $\mathfrak{g}$ is called ***abelian*** if $[\mathfrak{g},\mathfrak{g}] = \{0\}$, i.e $\mathfrak{g}$ is 1-step nilpotent, it is then clear that this is equivalent to stating that $\mathfrak{g} = \mathrm{Z}(\mathfrak{g})$.

Example 10.

1. *Any vector space V can be given an abelian Lie algebra structure $[\ ,\ ]$ by setting $[u,v] = 0$ for all $u,v \in V$.*

2. *For any Lie algebra $\mathfrak{g}$, the quotient Lie algebra $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian.*

3. *The Heisenberg Lie algebra $\mathfrak{h}_p$ consists of a $2p+1$-dimensional vector space together with a Lie bracket $[\ ,\ ]$ defined in a basis $\{e_1,\dots,e_{2p+1}\}$ by the expression:*

$$[e_{2i-1}, e_{2i}] = e_{2p+1},$$

for all $i = 1,\dots,p$, note that we only write the non-vanishing bracket. It is easy to check that $\mathfrak{h}_p$ is a 2-step nilpotent Lie algebra.

4. *Let $\mathfrak{g}$ be the n-dimensional Lie algebra whose bracket $[\ ,\ ]$ is defined in a basis $\{e_1,\dots,e_n\}$ by the expression:*

$$[e_1, e_i] = e_{i+1} \quad 2 \leq i \leq n-1.$$

It is straightforward to check that $\mathfrak{g}$ is nilpotent with nilindex equal to $n-1$.

5. *Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ be the set of all strictly upper triangular $n \times \mathfrak{n}$ matrices, namely:*

$$\mathfrak{g} := \{A \in \mathfrak{gl}(n, \mathbb{K}),\ A_{ij} = 0 \text{ for all } 1 \le i \le j \le n\}.$$

Then one can check that $\mathfrak{g}$ is a nilpotent Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$.

Here are some further properties of nilpotent Lie algebras that are used throughout Chapter 2 and 3:

Proposition A.2.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra.*

1. *If $\mathfrak{g}$ is nilpotent and non-abelian then $\operatorname{codim}_{\mathfrak{g}}[\mathfrak{g}, \mathfrak{g}] \ge 2$.*
2. *The Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent.*
3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ are nilpotent ideals of $\mathfrak{g}$, then $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} + \mathfrak{b}$ are nilpotent ideals of $\mathfrak{g}$ as well.*

Property 3 in Proposition *A.2.2* is especially important because it shows that any finite-dimensional Lie algebra $\mathfrak{g}$ has a unique maximal nilpotent ideal $\mathfrak{n}$, which can be defined merely by setting

$$\mathfrak{n} := \sum_{\mathfrak{a} \in \mathcal{N}} \mathfrak{a} \quad \text{where} \quad \mathcal{N} := \{\mathfrak{a} \subset \mathfrak{g},\ \mathfrak{a} \text{ is a nilpotent ideal of } \mathfrak{g}\},$$

where the notation simply means that $\mathfrak{n}$ consists of finite linear combinations of elements in the reunion of $\mathfrak{a}$ with $\mathfrak{a} \in \mathcal{N}$. We call $\mathfrak{n}$ the ***nilradical*** of $\mathfrak{g}$, it is usually denoted $\operatorname{nilrad}(\mathfrak{g})$. In the case where the Lie algebra $\mathfrak{g}$ is nilpotent, it is obvious that $\mathfrak{g} = \operatorname{nilrad}(\mathfrak{g})$.

Let us now state a fundamental Theorem in the theory of nilpotent Lie algebras, which is the converse of 4 in Proposition A.2.1:

Theorem A.2.1 (Engel's Theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if the operator ad_x is nilpotent for all $x \in \mathfrak{g}$.*

As a consequence, we get the following important result:

Corollary A.2.1. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. If the quotient Lie algebra $\mathfrak{g}/\mathfrak{a}$ is nilpotent, and if for all $x \in \mathfrak{g}$, the restriction of ad_x to $\mathfrak{a}$ is nilpotent, then $\mathfrak{g}$ is also nilpotent.*

Corollary A.2.2. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$ whose elements are nilpotent. Then $\mathfrak{g}$ is nilpotent.*

We close this paragraph with some important classes of nilpotent Lie algebras:

Two-step nilpotent Lie algebras. A nilpotent Lie algebra $\mathfrak{g}$ is said to be ***2-step nilpotent*** or ***metabelian*** if it satisfies $\mathscr{C}^2(\mathfrak{g}) = \{0\}$. It is clear that any abelian Lie algebra is trivially 2-step nilpotent. Also the Heisenberg Lie algebra $\mathfrak{h}_p$ defined in Example 10-2. In fact $\mathfrak{h}_p$ is a model for a subclass of 2-step nilpotent Lie algebras, this is summarized by the following Proposition:

Proposition A.2.3. *Every Lie algebra $\mathfrak{g}$ satisfying $Z(g) = [\mathfrak{g}, \mathfrak{g}]$ and $\dim Z(\mathfrak{g}) = 1$ is isomorphic to the Heisenberg Lie algebra.*

Filiform Lie algebras. A nilpotent Lie algebra $\mathfrak{g}$ is said to be ***filiform*** if it has maximal nilindex, in precise terms, it means that if $\dim \mathfrak{g} = n$ then $\mathfrak{g}$ has nilindex $n-1$. One can show that this is equivalent to $\dim C^k(\mathfrak{g}) = n-k-1$ for all $k = 1, \ldots, n-1$. This gives in particular that $\dim[\mathfrak{g}, \mathfrak{g}] = n-2$ and $\dim \mathrm{Z}(\mathfrak{g}) = 1$. A concrete case of a filiform Lie algebra was given in Example 10-2, we add two more Examples:

Example 11.

1. *Let $\mathfrak{g}$ be the $(n+1)$-dimensional nilpotent Lie algebra with Lie bracket $[\ ,\ ]$ defined in a basis $\{e_0, \ldots, e_n\}$ as:*

$$[e_0, e_i] = e_{i+1} \quad \textit{and} \quad [e_i, e_{n-i}] = (-1)^i e_n,$$

for all $i = 1 \ldots, n-1$. One can check the Lie algebra $\mathfrak{g}$ is indeed filiform.

2. *Let $\mathfrak{g}$ be a $(n+1)$-dimensional nilpotent Lie algebra with bracket $[\ ,\ ]$ given in a basis $\{e_0, \ldots, e_n\}$ by the expression:*

$$[e_0, e_i] = e_{i+1} \quad \textit{and} \quad [e_1, e_j] = e_{j+2},$$

for all $1 \le i \le n-1$ and $2 \le j \le n-1$. This algebra is filiform as well.

Characteristically nilpotent Lie algebras. Recall that the vector space $\mathrm{Der}(\mathfrak{g})$ of derivations of a finite-dimensional Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. If $\mathfrak{g}$ is nilpotent, then by Engel's Theorem its inner derivations are all nilpotent, furthermore in view of Corollary A.2.2 a necessary condition for $\mathrm{Der}(\mathfrak{g})$ to be nilpotent is to only consist of nilpotent elements i.e every derivation of $\mathfrak{g}$ is nilpotent. Nonetheless, there are many cases where $\mathrm{Der}(\mathfrak{g})$ cannot be nilpotent, as a matter of fact there is a well-known Theorem due to Jacobson which states that "*A Lie algebra which admits an invertible derivation is automatically nilpotent*" and there are concrete examples of such Lie algebras. A Lie algebra $\mathfrak{g}$ is said to be ***characteristically nilpotent*** if $\mathfrak{g}$ is nilpotent and every derivation $D \in \mathrm{Der}(\mathfrak{g})$ is nilpotent.

Example 12. *Let $\mathfrak{g}$ be the 7-dimensional Lie algebra such that its Lie bracket $[\ ,\ ]$ is given in a basis $\{e_1, \ldots, e_7\}$ by:*

$$\begin{aligned} &[e_1, e_2] = e_3 && [e_1, e_6] = e_7 \\ &[e_1, e_3] = e_4 && [e_2, e_3] = e_6 \\ &[e_1, e_4] = e_5 && [e_2, e_4] = [e_2, e_5] = -[e_3, e_4] = e_7 \\ &[e_1, e_5] = e_6 && \end{aligned}$$

Through a direct computation of $\mathrm{Der}(\mathfrak{g})$ *one can prove that $\mathfrak{g}$ is characteristically nilpotent. This example is minimal in the sense that 7 is the smallest dimension where it is possible to find characteristically nilpotent Lie algebras.*

A.3 Lie algebra representations and cohomology

Let $\mathfrak{g}$ be a Lie algebra and V a vector space. ***A representation*** of the Lie algebra $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Such a representation is called ***faithful*** if it satisfies $\ker(\rho) = \{0\}$ and it is called ***trivial*** if $\mathrm{Im}(\rho) = \{0\}$. The couple (V, ρ) is called a $\mathfrak{g}$-***module*** and we shall say that the $\mathfrak{g}$-module (V, ρ) is ***faithful*** or ***trivial*** depending whether the corresponding representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is faithful or trivial.

Example 13.

1. *The most straightforward example of Lie algebra representation is perhaps the adjoint representation* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$.

2. *Denote* $\mathfrak{g} := \mathfrak{X}(M)$ *the Lie algebra of all smooth vector fields on a smooth manifold M and let* $V := \mathscr{C}^\infty(M)$ *be the space of all smooth functions on M. Then one obtains a natural Lie algebra representation* $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *by setting* $\rho(X)(f) := X(f)$ *i.e the derivative of the function f in the direction of the vector field X.*

Let $\mathfrak{g}$ be a Lie algebra, $p \in \mathbb{N}$ and $\omega : \mathfrak{g}^p \longrightarrow V$ a p-linear map. We say that ω is a V-valued p-***cochain*** if it is alternating. The vector space of all p-cochains on $\mathfrak{g}$ shall be denoted $\mathscr{C}^p(\mathfrak{g}, V)$, it is finite-dimensional whenever $\mathfrak{g}$ and V are finite-dimensional, in fact if $n := \dim \mathfrak{g}$ and $k := \dim V$ then $\dim \mathscr{C}^p(\mathfrak{g}, V) := kC_n^p$ with the convention that $C_n^p = 0$ for $p > n$. We set $\mathscr{C}^0(\mathfrak{g}, V) := V$ and $\mathscr{C}(\mathfrak{g}, V) := \oplus_p \mathscr{C}^p(\mathfrak{g}, V)$.
Assume now that $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation, for any $p \in \mathbb{N}$ we can define a linear map $\mathrm{d}_\rho^p : \mathscr{C}^p(\mathfrak{g}, V) \longrightarrow \mathscr{C}^{p+1}(\mathfrak{g}, V)$ be the expression:

$$(\mathrm{d}_\rho^p \omega)(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i \rho(x_i)\Big(\omega(x_0, \ldots, \widehat{x_i}, \ldots, x_p)\Big) + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_p), \tag{A.3}$$

for all $\omega \in \mathscr{C}^p(\mathfrak{g}, V)$ and all $x_0, \ldots, x_p \in \mathfrak{g}$. A simple computation shows that $\mathrm{d}_\rho^{p+1} \circ \mathrm{d}_\rho^p = 0$, this gives rise to a linear map $\mathrm{d}_\rho : \mathscr{C}(\mathfrak{g}, V) \longrightarrow \mathscr{C}(\mathfrak{g}, V)$ satisfying $\mathrm{d}_\rho \circ \mathrm{d}_\rho = 0$, it is called the ***Chevalley-Eilenberg differential*** (relative to the representation ρ).
We say that $\omega \in \mathscr{C}^p(\mathfrak{g}, V)$ is a p-***cocycle*** if $\mathrm{d}_\rho \omega = 0$, and if $p > 1$, we that ω is a p-***coboundary*** of $\omega = \mathrm{d}_\rho \eta$ for some $\eta \in \mathscr{C}^{p-1}(\mathfrak{g}, V)$.

Example 14.

1. *Let* $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(1, \mathbb{R})$ *be the trivial representation of a Lie algebra $\mathfrak{g}$ on $\mathbb{R}$. One can easily check that* $\mathscr{C}(\mathfrak{g}, \mathbb{R}) := \Lambda \mathfrak{g}^*$, *i.e the vector space of all alternating linear forms on $\mathfrak{g}$. Moreover, by formula (A.3), the Chevalley-Eilenberg differential* $\mathrm{d}_\rho^p : \Lambda^p \mathfrak{g}^* \longrightarrow \Lambda^{p+1} \mathfrak{g}^*$ *is given by:*

$$(\mathrm{d}_\rho^p \omega)(x_0, \ldots, x_p) \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_p), \tag{A.4}$$

for all $\omega \in \Lambda^p \mathfrak{g}^*$ *and all* $x_0, \ldots, x_p \in \mathfrak{g}$.

2. *For a smooth manifold M, set $\mathfrak{g} := \mathcal{X}(M)$ and $V := \mathscr{C}^\infty(M)$. Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be the natural Lie algebra representation defined by the action of vector fields on smooth functions* (cf. *Example 13-2). Clearly $\mathscr{C}^p_\rho(\mathfrak{g}, V) := \Omega^p(M)$ i.e the space of all differential p-forms on M. The Chevalley-Eilenberg differential $\mathrm{d}^p_\rho : \mathscr{C}^p_\rho(\mathfrak{g}, V) \longrightarrow \mathscr{C}^{p+1}_\rho(\mathfrak{g}, V)$ in this case coincides with the de Rham differential $\mathrm{d}_{dR} : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ of M, thus:*

$$(\mathrm{d}^p_\rho \omega)(X_0, \ldots, X_p) := \sum_{i=0}^{p} (-1)^i (\mathrm{L}_{X_i}\omega)(X_0, \ldots, \widehat{X_i}, \ldots, X_p),$$

for any $\omega \in \Omega^p(M)$ and any $X_0, \ldots, X_p \in \mathcal{X}(M)$, where $L_X : \Omega(M) \longrightarrow \Omega(M)$ denotes the Lie derivative with respect to the vector field X.

We denote $\mathrm{Z}^p_\rho(\mathfrak{g}, V)$ the vector space of all p-cocycles with values in the $\mathfrak{g}$-module (V, ρ) and $\mathrm{B}^p_\rho(\mathfrak{g}, V)$ the space of all p-coboundaries with values in the $\mathfrak{g}$-module (V, ρ). It is clear from their given definitions that:

$$\mathrm{Z}^p_\rho(\mathfrak{g}, V) := \ker\left(d^p_\rho : \mathscr{C}^p(\mathfrak{g}, V) \longrightarrow \mathscr{C}^{p+1}(\mathfrak{g}, V)\right)$$

and

$$\mathrm{B}^p_\rho(\mathfrak{g}, V) := \mathrm{Im}\left(d^{p-1}_\rho : \mathscr{C}^{p-1}(\mathfrak{g}, V) \longrightarrow \mathscr{C}^p(\mathfrak{g}, V)\right).$$

Moreover, the fact that $\mathrm{d}_\rho : \mathscr{C}(\mathfrak{g}, V) \longrightarrow \mathscr{C}(\mathfrak{g}, V)$ is a differential operator, i.e $\mathrm{d}^p_\rho \circ \mathrm{d}^{p-1}_\rho = 0$ for all $p \in \mathbb{N}^*$ shows that $\mathrm{B}^p_\rho(\mathfrak{g}, V) \subset \mathrm{Z}^p_\rho(\mathfrak{g}, V)$. As a result, we define the ***p-th cohomology group of*** $\mathfrak{g}$, ***with values in the*** $\mathfrak{g}$***-module*** (V, ρ) to be the vector space $\mathrm{H}^p_\rho(\mathfrak{g}, V)$ consisting of all p-cocycles modulo p-coboundaries, namely:

$$\mathrm{H}^p_\rho(\mathfrak{g}, V) := \mathrm{Z}^p_\rho(\mathfrak{g}, V)/\mathrm{B}^p_\rho(\mathfrak{g}, V) = \frac{\ker\left(d^p_\rho : \mathscr{C}^p(\mathfrak{g}, V) \longrightarrow \mathscr{C}^{p+1}(\mathfrak{g}, V)\right)}{\mathrm{Im}\left(d^{p-1}_\rho : \mathscr{C}^{p-1}(\mathfrak{g}, V) \longrightarrow \mathscr{C}^p(\mathfrak{g}, V)\right)}.$$

When (V, ρ) is a trivial $\mathfrak{g}$-module, we shall use the symbol $\mathrm{H}^p(\mathfrak{g}, V)$ to denote the p-th cohomology of $\mathfrak{g}$.

Example 15.

1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(1, \mathbb{R})$ be the trivial representation* (cf. *Example 14-1). The corresponding cohomology $\mathrm{H}(\mathfrak{g}, \mathbb{R})$ is called the cohomology of left-invariant forms of the Lie group G (see appendix C for more on left-invariant structures).*

2. *Let M be a smooth manifold and denote $\mathfrak{g} := \mathcal{X}(M)$ and $V := \mathscr{C}^\infty(M)$. Example 14-2 shows in particular that the cohomology groups $\mathrm{H}^p_\rho(\mathfrak{g}, V)$ corresponding to the natural representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, $\rho(X)(f) := X(f)$ are exactly the de Rham cohomology groups $\mathrm{H}^p_{dR}(M)$ of the manifold M.*

B

Pseudo-Euclidean vector spaces

B.1 Symmetric bilinear forms and scalar products

Throughout this chapter and unless otherwise stated, we shall use E to denote a real n-dimensional vector space.

Definition B.1.1. *Let $\langle\ ,\ \rangle : E \times E \to \mathbb{R}$ be a symmetric bilinear form, i.e an inner product. We say that $\langle\ ,\ \rangle$ is:*

(i)- ***Positive definite.*** *(resp.* ***Negative definite.****) if for every $v \in E\backslash\{0\}$ we have $\langle v, v\rangle > 0$ (resp. $\langle v, v\rangle < 0$)*

(ii)- ***Positive semi-definite.*** *(resp.* ***Negative semidefinite.****) if $\langle v, v\rangle \geq 0$ (resp. $\langle v, v\rangle \leq 0$) for every $v \in E\backslash\{0\}$,*

(iii)- ***Indefinite.*** *if it is neither positive semi-definite nor negative semi-definite,*

(iv)- ***Non-degenerate.*** *if the condition $\langle v, w\rangle = 0$ for every $w \in E$ implies that $v = 0$. Otherwise, we say that $\langle\ ,\ \rangle$ is degenerate, the subset $N = \{v \in E : \langle v, w\rangle = 0, \forall w \in E\}$ is called the radical of $\langle\ ,\ \rangle$.*

Remark 3. *The fact that $\langle\ ,\ \rangle : E \times E \to \mathbb{R}$ is non-degenerate implies that the map $\theta : E \to E^*$ given by:*

$$\theta(v)(w) = \langle v, w\rangle$$

is one to one, hence a vector space isomorphism. Let $\mathbb{B} = (e_1, \ldots, e_n)$ be a basis of E with dual basis $\mathbb{B}^* = \left(e_1^*, \ldots, e_n^*\right)$, then $\langle\ ,\ \rangle$ is nondegenerate if and only if $\det M \neq 0$, where:

$$M := \mathrm{Mat}\,(\theta, \mathbb{B}, \mathbb{B}^*) = \left(\left\langle e_i, e_j\right\rangle\right)_{1\leq i,j\leq n}$$

Furthermore, if we denote $U := (u_1, \ldots, u_n)^T$ and $V := (v_1, \ldots, v_n)^T$ the coordinates of u and v respectively in the basis $\mathbb{B}$ of E, then $\langle u, v\rangle = U^T M V$.

Definition B.1.2. *A pseudo-Euclidean vector space is a pair $(E, \langle\ ,\ \rangle)$ consisting of a real vector space E together with a non-degenerate inner product $\langle\ ,\ \rangle$.*

Example 16. *Define on $\mathbb{R}^{p+q}$ the bilinear form $\langle\ ,\ \rangle_{p,q} : \mathbb{R}^{p+q}\times\mathbb{R}^{p+q} \longrightarrow \mathbb{R}$ given by:*

$$\langle u,v\rangle_{p,q} = \sum_{i=1}^{p} u_i v_i - \sum_{i=p+1}^{p+q} u_i v_i.$$

It is straightforward to check that $(\mathbb{R}^{p+q}, \langle\ ,\ \rangle_{p,q})$ is a Pseudo-Euclidean vector space.

Example 17. *Recall that for any pseudo-euclidean vector space $(E,\langle\ ,\ \rangle)$ and for any $f \in \mathrm{End}(E)$, there exists a unique endomorphism $f^* \in \mathrm{End}(E)$ satisfying $\langle f(u),v\rangle = \langle u, f^*(v)\rangle$ for $u,v \in E$, it is called the adjoint of f with respect to $\langle\ ,\ \rangle$. This allows one to define a symmetric bilinear form $\langle\ ,\ \rangle_{\#}$ on $\mathrm{End}(E)$ by the expression:*

$$\langle f,g\rangle_{\#} := \mathrm{tr}(f\circ g^*).$$

It is straightforward to check that $\langle\ ,\ \rangle_{\#}$ is a pseudo-Euclidean inner product on $\mathrm{End}(E)$.

Let E be a vector space endowed with an inner product $\langle\ ,\ \rangle$. Two subsets $A,B \subset E$ are said to be ***orthogonal*** (with respect to $\langle\ ,\ \rangle$), if $\langle v,w\rangle = 0$ for any $v \in A$ and $w \in B$. ***The orthogonal of*** A ***in*** E is the vector subspace $A^\perp$ of E given by:

$$A^\perp = \{v \in E, \langle v,a\rangle = 0 \text{ for any } a \in A\}.$$

Proposition B.1.1. *Let $(E,\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space, and let $A \subset E$ be any vector subspace of E, then $\dim A + \dim A^\perp = \dim E$.*

Denote $\mathscr{A}^+$ the set of vector subspaces $A \subset E$ such that the restriction $\langle\ ,\ \rangle_{|A\times A}$ is positive definite. We also denote $\mathscr{A}^-$ the set of vector subspaces $A \subset V$ such that $\langle\ ,\ \rangle_{|A\times A}$ is negative definite. Put:

$$p := \max_{A\in\mathscr{A}^+} \dim A \quad \text{ and } \quad q := \max_{A\in\mathscr{A}^-} \dim A$$

The couple (p,q) is called ***the signature*** of $(E,\langle\ ,\ \rangle)$.

Proposition B.1.2. *Let $(E,\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space with signature (p,q) and let A be a vector subspace of E. Then $p+q = \dim E$ and $\dim(A\cap A^\perp) \le \min(p,q)$.*

Let $(E,\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space, a vector subspace $A \subset E$ is said to be ***non-degenerate*** in $(E,\langle\ ,\ \rangle)$ if $A\cap A^\perp = \{0\}$, or equivalently, the restriction $\langle\ ,\ \rangle_{|A\times A}$ is non-degenerate. In this case, the property $\dim A + \dim A^\perp = \dim E$ implies that $E = A\oplus A^\perp$.

We say that A is ***totally isotropic*** if $\langle\ ,\ \rangle_{|A\times A} = 0$, or equivalently $A \subset A^\perp$. If (p,q) denotes the signature of $\langle\ ,\ \rangle$, the previous Proposition shows that in this case $\dim A \le \min(p,q)$.

Proposition B.1.3. *Let $(E,\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space, and $W \subset V$ a vector subspace, then $(W^\perp)^\perp = W$ and $E = W\oplus W^\perp$ if and only if W is non-degenerate.*

Proposition B.1.4. *Let $(E,\langle\ ,\ \rangle)$ be a Pseudo-Euclidean vector space and A a nondegenerate vector subspace of E. Denote (p_1,q_1) and (p_2,q_2) the signatures of $\langle\ ,\ \rangle_{|A\times A}$ and $\langle\ ,\ \rangle_{|A^\perp\times A^\perp}$ respectively. Then (p_1+p_2, q_1+q_2) is the signature of $(E,\langle\ ,\ \rangle)$.*

We say that two pseudo-Euclidean vector spaces $(E_1, \langle\ ,\ \rangle_1)$ and $(E_2, \langle\ ,\ \rangle_2)$ are ***isometric*** if there exists a vector space isomorphism $\phi : E_1 \longrightarrow E_2$ satisfying for any $u, v \in E_1$:

$$\langle \phi(u), \phi(v) \rangle_2 = \langle u, v \rangle_1.$$

In the case where $E_1 = E_2 := E$ and $\langle\ ,\ \rangle_1 = \langle\ ,\ \rangle_2 := \langle\ ,\ \rangle$, the endomorphism ϕ is then called an ***isometry*** of $(E, \langle\ ,\ \rangle)$

Theorem B.1.1. *Two pseudo-Euclidean vector spaces $(E_1, \langle\ ,\ \rangle_1)$ and $(E_2, \langle\ ,\ \rangle_2)$ are isometric if and only if they have the same signature.*

This shows that the signature is the only invariant for Pseudo-Euclidean vector spaces of the same dimension.

Definition B.1.3. *A Pseudo-Euclidean vector space is called Euclidean if its signature is of the form $(n, 0)$. It is called Lorentzian if its signature is of the form $(n-1, 1)$.*

The class of Lorentzian vector spaces is a very special subclass of pseudo-Euclidean vector spaces.

Proposition B.1.5. *Let $(E, \langle\ ,\ \rangle)$ be a Lorentzian vector space and let A be a non-degenerate vector subspace of E. Then A is either Euclidean or Lorentzian.*

Proof. Denote (p, q) the signature of $\langle\ ,\ \rangle_{|A\times A}$ and $(p^\perp, q^\perp)$ the signature of $\langle\ ,\ \rangle_{|A^\perp\times A^\perp}$, according to Proposition B.1.4, the signature of $(E, \langle\ ,\ \rangle)$ is precisely $(p+p^\perp, q+q^\perp) = (n-1, 1)$ thus either $q = 1$, $q^\perp = 0$ which gives that A is Lorentzian and $A^\perp$ is Euclidean, or $q = 0$ and $q^\perp = 1$ which gives that A is Euclidean and $A^\perp$ is Lorentzian. □

B.2 Orthonormal bases and pseudo-Euclidean bases

Let $(E, \langle\ ,\ \rangle)$ be a n-dimensional pseudo-Euclidean vector space of signature (p, q) and assume without loss of generality that $p \geq q$. A family $(u_1, \ldots, u_s)$ of vectors in E is called ***orthogonal*** if $\langle u_i, u_j \rangle = 0$ for $i \neq j$. It is called ***orthonormal*** if furthermore $\langle u_i, u_i \rangle \in \{-1, 1\}$, any orthonormal family is automatically linearly independent.

Proposition B.2.1. *Let $(E, \langle\ ,\ \rangle)$ be a pseudo-Euclidean n-dimensional vector space with signature (p, q) and choose an orthogonal basis $\mathbb{B} = (e_1, \ldots, e_n)$ of $(E, \langle\ ,\ \rangle)$. Then $\langle e_i, e_i \rangle \neq 0$ and:*

$$p = \mathrm{card}\{i \in \{1, \ldots, n\}, \langle e_i, e_i \rangle > 0\} \quad \textit{and} \quad q = card\{i \in \{1, \ldots, n\}\ \langle e_i, e_i \rangle < 0\}.$$

It is worth to mention that a pseudo-Euclidean vector space $(E, \langle\ ,\ \rangle)$ always admits an orthogonal basis. This is due to the fact that for any non-degenerate vector subspaces $A \subset E$ and $B \subset A^\perp$, $A \oplus B$ is non-degenerate, therefore starting from a non-isotropic vector $e_1 \in E$ one can find a non-isotropic vector $e_2 \in e_1^\perp$ and by the previous observation span$\{e_1, e_2\}$ is non-degenerate. Thus by an inductive argument one obtains an orthogonal basis, which can be made orthonormal through normalization.

Let $(E,\langle\ ,\ \rangle)$ be a Pseudo-Euclidean, non Euclidean vector space. A basis of $(E,\langle\ ,\ \rangle)$ is said to be ***Pseudo-Euclidean*** if it has the form $\mathscr{B} := \{f_1, \bar{f}_1, \dots, f_r, \bar{f}_r, e_1, \dots, e_s\}$ such that the only non-vanishing products $\langle u,v\rangle$ with $u,v \in \mathscr{B}$ are:

$$\langle f_i, \bar{f}_i\rangle = 1, \quad \text{and} \quad \langle e_j, e_j\rangle = 1,$$

for any $i = 1\dots r$, $j = 1\dots s$. Observe that the signature of $(E,\langle\ ,\ \rangle)$ is in this case $(r+s,r)$, this is a consequence of Proposition B.1.1 and the fact that $A := \mathrm{span}\{f_1, \bar{f}_1, \dots, f_r, \bar{f}_r\}$ is non-degenerate with signature (r,r) and $A^\perp = \mathrm{span}\{e_1,\dots,e_s\}$ is Euclidean and so its signature is exactly $(s,0)$. We also note that these bases are as important as orthonormal bases and often play a crucial role in the proof of several results, as they represent a second generalization of orthonormality in Euclidean spaces.

Proposition B.2.2. *Any pseudo-Euclidean vector space admits a Pseudo-Euclidean basis.*

Remark 4. *On a Pseudo-Euclidean, non-Euclidean vector space $(E,\langle\ ,\ \rangle)$, one can always construct an orthonormal basis starting with a Pseudo-Euclidean basis and conversely.*
To see this denote (p,q) the signature of $(E,\langle\ ,\ \rangle)$ with $p \geq q$ and write $(p,q) := (r+s,r)$ such that $r,s > 0$, thus if $\mathscr{B} := \{f_1, \bar{f}_1, \dots, f_r, \bar{f}_r, e_1, \dots, e_s\}$ is any Pseudo-Euclidean basis of E, set :

$$u_i := \frac{1}{\sqrt{2}}(f_i + \bar{f}_i),\ v_i := \frac{1}{\sqrt{2}}(f_i - \bar{f}_i),$$

then one easily checks that $\{u_1, v_1, \dots, u_r, v_r, e_1, \dots, e_s\}$ is an orthonormal basis of E.

B.3 Symmetric and skew-Symmetric Endomorphisms

Let $(E,\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space and $\phi \in \mathrm{End}(E)$. We say that ϕ is ***symmetric*** (with respect to $\langle\ ,\ \rangle$) if $\phi^* = \phi$, and we say that ϕ is ***skew-symmetric*** if $\phi^* = -\phi$.

Proposition B.3.1. *Let $(E\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space and let $\phi \in \mathrm{End}(E)$. For any vector subspace $F \subset E$, $\phi(F) \subset F$ if and only if $\phi^*(F^\perp) \subset F^\perp$. In the case where ϕ is (skew-)symmetric we get that $\phi(F) \subset F$ if and only if $\phi(F^\perp) \subset F^\perp$.*

Definition B.3.1. *Let $(E,\langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space and $\phi : E \longrightarrow E$ an endomorphism. A vector subspace $F \subset E$ is said to be ϕ-indecomposable if it satisfies the following properties :*

(i) *F is nondegenerate.*

(ii) *F is ϕ-invariant, which means that $\phi(F) \subset F$.*

(iii) *The only non-degenerate ϕ-invariant subspaces of F are $\{0\}$ and F.*

The following lemma is the key to reduce the symmetric and the skew-symmetric endomorphisms in a Euclidean or Lorentzian space.

Lemma B.3.1. *Let $\phi : E \longrightarrow E$ be a (skew)-symmetric endomorphism on a pseudo-Euclidean vector space $(E, \langle\ ,\ \rangle)$. There exists a family $\{F_1, \ldots, F_r\}$ of ϕ-indecomposable vector subspaces such that :*

$$V = F_1 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} F_r.$$

Remark 5. *Let $\phi : E \longrightarrow E$ be a (skew)-symmetric endomorphism on a Pseudo-Euclidean vector space $(E, \langle\ ,\ \rangle)$ and let $E = F_1 \oplus \cdots \oplus F_r$ be an orthogonal decomposition of E into ϕ-indecomposable vector subspaces. If (p_i, q_i) denotes the signature of $\langle\ ,\ \rangle_{F_i \times F_i}$ then by applying Proposition B.1.4 inductively we get that $(E, \langle\ ,\ \rangle)$ has signature $(p_1 + \cdots + p_r, q_1 + \cdots + q_r)$. In particular, if $(E, \langle\ ,\ \rangle)$ is Lorentzian then only one of the subspaces F_i is Lorentzian while the rest are Euclidean.*

Let $(E, \langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space, denote (p, q) its signature and fix an orthonormal basis $\mathscr{B} = \left(e_1, \ldots, e_p, f_1, \ldots, f_q\right)$ of E such that $\langle e_i, e_i \rangle = 1$ and $\langle f_j, f_j \rangle = -1$. Next let $\phi : E \longrightarrow E$ be an endomorphism of E, and write :

$$\mathrm{Mat}(\phi, \mathscr{B}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

It is clear that $A_{ij} = \langle \phi(e_j), e_i \rangle$, $B_{ij} = \langle \phi(f_j), e_i \rangle$, $C_{ij} = -\langle \phi(e_j), f_i \rangle$ and $D_{ij} = -\langle \phi(f_j), f_i \rangle$. It follows that ϕ is symmetric if and only if $A^t = A$, $B^t = -C$ and $D^t = D$. Similarly we get that ϕ is skew-symmetric if and only if $A^t = -A$, $B^t = C$ and $D^t = -D$.

Lemma B.3.2. *Let E be a finite dimensional vector space and let $\phi \in \mathrm{End}(E)$. There exists a non-trivial couple of vectors $(u, v) \in E \times E$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that :*

$$\phi(u) = \lambda_1 u - \lambda_2 v \quad \text{and} \quad \phi(v) = \lambda_2 u + \lambda_1 v.$$

In particular, there exists a ϕ-invariant vector subspace $F \subset E$ such that $1 \leq \dim F \leq 2$.

Proof. Denote $E^{\mathbb{C}} = E \oplus iE$ and $\phi^{\mathbb{C}} \in \mathrm{End}(E^{\mathbb{C}})$ the complexification of E and $\phi \in \mathrm{End}(E)$ respectively, i.e $\phi^{\mathbb{C}}(x + iy) = \phi(x) + i\phi(y)$ for any $x, y \in E$. It is clear that $\phi^{\mathbb{C}}$ admits a nonzero eigenvector $w \in E^{\mathbb{C}}$ corresponding to some eigenvalue $\lambda \in \mathbb{C}$, write $w := u + iv$ and $\lambda := \lambda_1 + i\lambda_2$ for some $u, v \in E$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, so that $\phi^{\mathbb{C}}(u + iv) = (\lambda_1 + i\lambda_2)(u + iv)$, then

$$\phi(u) = \lambda_1 u - \lambda_2 v \quad \text{and} \quad \phi(v) = \lambda_2 u + \lambda_1 v,$$

The vector subspace $F = \mathrm{span}\{u, v\}$ is either 1 or 2-dimensional and it is clearly ϕ-invariant. □

B.3.1 Reduction of skew-symmetric endomorphisms in Lorentzian vector spaces

We start by stating the well-known result on the reduction of skew-endomorphisms in a Euclidean vector space.

Theorem B.3.1. *Let $(E,\langle\ ,\ \rangle)$ be a Euclidean vector space and $\phi : E \longrightarrow E$ be a skew-symmetric endomorphism. There exists a family of non-vanishing real numbers $\lambda_1 \leq \ldots \leq \lambda_r$ and an orthonormal basis $(e_1, f_1, \ldots, e_r, f_r, g_1, \ldots, g_s)$ of E such that*

$$\phi(e_i) = \lambda_i f_i,\ \phi(f_i) = -\lambda_i e_i \quad \textit{and} \quad \phi\left(g_j\right) = 0, \quad i = 1, \ldots, r, j = 1, \ldots, s$$

Let us now adress the Lorenzian case.

Lemma B.3.3. *Let $(E,\langle\ ,\ \rangle)$ be a Lorentzian vector space $\phi : E \longrightarrow E$ a skew-symmetric endomorphism. Suppose that there exists $e \in E$ such that $\langle e, e\rangle = 0$ and $\phi(e) = \lambda e$, E is ϕ-indecomposable and $\dim E \geq 3$. Then $\dim E = 3$ and there exists a couple of vectors $(\bar{e}, f)$ such that $(e, \bar{e}, f)$ is a Lorentzian basis of E and*

$$\phi(e) = 0,\ \phi(f) = \alpha e \quad \textit{and} \quad \phi(\bar{e}) = -\alpha f$$

Proof. Denote $F := \mathbb{R}e$ and $V := F^\perp/F$. Endow V with the inner product $\langle\ ,\ \rangle_q$ given by

$$\langle [u], [v]\rangle_q = \langle u, v\rangle, \quad u, v \in F^\perp$$

It is straightforward to check that $\left(V, \langle\ ,\ \rangle_q\right)$ is a Euclidean vector space. Since F is a ϕ-invariant subspace then $F^\perp$ is ϕ-invariant as well and ϕ induces a skew-symmetric endomorphism $\bar{\phi} : V \longrightarrow V$ explicitely given by $\bar{\phi}([u]) = [\phi(u)]$, for any $u \in F^\perp$. Next, Theorem B.3.1 shows that there exists a family of non zero real numbers $\lambda_1 \leq \ldots \leq \lambda_r$ and an orthonormal basis $\hat{\mathscr{B}} := \left(\hat{e}_1, \hat{f}_1, \ldots, \hat{e}_r, \hat{f}_r, \hat{g}_1, \ldots, \hat{g}_s\right)$ such that of $(V, \langle\ ,\ \rangle_q)$ such that :

$$\bar{\phi}(\hat{e}_i) = \lambda_i \hat{f}_i,\ \bar{\phi}\left(\hat{f}_i\right) = -\lambda_i \hat{e}_i \quad \text{and} \quad \bar{\phi}\left(\hat{g}_j\right) = 0, \quad i = 1, \ldots, r, j = 1, \ldots, s \tag{B.1}$$

If we write $\hat{e}_i := [e_i]$, $\hat{f}_j := [f_j]$ and $\hat{g}_i := [g_i]$ for some nonzero vectors $e_i, f_j, g_k \in F^\perp$, then it is clear that $\mathscr{B} := (e_1, f_1, \ldots, e_r, f_r, g_1, \ldots, g_s)$ is an orthonormal family of $F^\perp$ and using (B.1) we get that :

$$\phi(e_i) = a_i e + \lambda_i f_i,\ \phi(f_i) = b_i e - \lambda_i e_i \quad \text{and} \quad \phi\left(g_j\right) = c_j e.$$

Assume that either $\lambda \neq 0$ or $\lambda_l \neq 0$ for some $l \in \{1, \ldots, r\}$ and set $H_l := \operatorname{span}\{e, e_l, f_l\}$, then clearly $\phi(H_l) \subset H_l$ furthermore an easy computation shows that $\phi_l := \phi_{|H_l}$ has characteristic polynomial

$$P(X) = -(X - \lambda)(X - i\lambda_l)(X + i\lambda_l),$$

now for any eigenvector $v_l \in H_l^{\mathbb{C}}$ of $\phi_l^{\mathbb{C}}$ corresponding to the eigenvalue $i\lambda_l$ we have that $\bar{v}_l$ is an eigenvector for the eigenvalue $-i\lambda_l$ therefore if we set $h_l := v_l + \bar{v}_l$ and $\tilde{h}_l := -i(v_l - \bar{v}_l)$ then we get that $h_l, \tilde{h}_l \in H_l$ and $\phi(h_l) = \lambda_l \tilde{h}_l$, $\phi(\tilde{h}_l) = -\lambda_l h_l$. If $\lambda_l = 0$ then $\lambda \neq 0$ and $\ker(\phi_l)$

is non-trivial, it is clearly ϕ-invariant, but since $e \notin \ker(\phi_l)$ then Proposition B.1.5 gives that $\ker(\phi_l)$ is Euclidean, contradicting the fact that E is ϕ-indecomposable.
Thus $\lambda_l \neq 0$ and $\{e, v_l, \bar{v}_l\}$ forms a basis of the $\mathbb{C}$-vector space $H_l^{\mathbb{C}}$, this gives that $\{e, h_l, \tilde{h}_l\}$ is a basis of H_l, but this implies that $\operatorname{span}\{h_l, \tilde{h}_l\}$ is Euclidean (Proposition B.1.5) and ϕ-invariant, a contradiction. Thus, $\lambda_i = \lambda = 0$ for all $1 \leq i \leq r$ and in particular $\phi(F^\perp) \subset \mathbb{R}e$.
Let W be any Euclidean subspace of $F^\perp$ such that $F^\perp = W \oplus \mathbb{R}e$. Since $\ker(\phi_{|W})$ is a ϕ-invariant subspace of E it follows that $\ker(\phi_{|W}) = \{0\}$ and since $\phi(W) \subset \mathbb{R}e$ we get that $\dim(W) \leq 1$, now $\dim W = \dim E - 2 \geq 1$ and so $\dim W = 1$ and $\phi(W) = \mathbb{R}e$, this also shows that $\dim E = 3$.
Finally if we write $W := \mathbb{R}f$ and set $\phi(f) := \alpha e$ then $\alpha \neq 0$, now choose an isotropic vector $\bar{e} \in E$ such that $\langle e, \bar{e}\rangle = 1$ and $\langle e, f\rangle = 0$, then $(e, \bar{e}, f)$ is a Lorentzian basis of E and it is easy to see that :

$$\phi(e) = 0,\ \phi(f) = \alpha e \quad \text{and} \quad \phi(\bar{e}) = -\alpha f,$$

this ends the proof. □

Theorem B.3.2. *Let $\phi : E \longrightarrow E$ be a skew-symmetric endomorphism on a Lorentzian vector space $(E, \langle\ ,\ \rangle)$. Then E can be written as $E = L \oplus V$ such that V is Euclidean and ϕ-invariant and L is ϕ-indecomposable Lorentzian satisfying one of the following properties :*

- (i). $\dim L = 1$ *and* $L \subset \ker\phi$
- (ii). $\dim L = 2$ *and there exists* $\alpha > 0$ *and a Lorentzian basis* $(e, \bar{e})$ *of* L *such that* $\phi(e) = \alpha e$ *and* $\phi(\bar{e}) = -\alpha\bar{e}$.
- (iii). $\dim L = 3$ *and there exists a Lorentzian basis* $(e, \bar{e}, f)$ *of* L *such that*

$$\phi(e) = 0, \phi(\bar{e}) = -\alpha f \quad \textit{and} \quad \phi(f) = \alpha e$$

Proof. We know that E can be written $E = V_1 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} V_r$ as the sum of ϕ-indecomposable subspaces. We also know that exactly one V_i is Lorentzian, denote it L and set $V := \oplus_{j\neq i} V_j$. According to Theorem B.3.1 each V_j for $j \neq i$ is either 1 or 2 dimensional, furthermore we have the following cases :

- (i). $\dim L = 1$, write $L := \mathbb{R}x$ and $\phi(x) = \lambda x$ with $\lambda \in \mathbb{R}$. Then $0 = \langle\phi(x), x\rangle = \lambda\langle x, x\rangle$ and since L is Lorentzian, $\langle x, x\rangle < 0$ and so $\lambda = 0$.
- (ii). $\dim L = 2$. First notice that the restriction $\phi_{|L} \neq 0$ otherwise L would contain a ϕ-invariant, non-degenerate 1-dimensional subspace which contradicts the assumption that L is ϕ-indecomposable. Let $(e, \bar{e})$ be a Lorentzian basis of L, then we can write $\phi(e) = ae + b\bar{e}$ and $\phi(\bar{e}) = ce + d\bar{e}$, since ϕ is skew-symmetric, then:

$$0 = \langle e, \phi(e)\rangle = \langle\bar{e}, \phi(\bar{e})\rangle \quad \text{and} \quad \langle e, \phi(\bar{e})\rangle = -\langle\bar{e}, \phi(e)\rangle$$

which is equivalent to $b = c = 0$ and $a = -d = \alpha \neq 0$.

(iii). $\dim L \geq 3$. According to Lemma B.3.2, there exists a couple of vectors $(u,v) \in L \times L$ with $(u,v) \neq (0,0)$ and two real numbers λ_1, λ_2 such that

$$\phi(u) = \lambda_1 u - \lambda_2 v \quad \text{and} \quad \phi(v) = \lambda_2 u + \lambda_1 v$$

Since ϕ is skew-symmetric, we have

$$0 = \langle u, \phi(u)\rangle = \langle v, \phi(v)\rangle \quad \text{and} \quad \langle u, \phi(v)\rangle = -\langle v, \phi(u)\rangle.$$

which is equivalent to

$$\begin{pmatrix} \lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 \\ \lambda_2 & 2\lambda_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} \langle u,u\rangle \\ \langle v,u\rangle \\ \langle v,v\rangle \end{pmatrix} = 0 \tag{B.2}$$

We will show that there exists an isotropic vector $e = \alpha u + \beta v$ such that $\mathbb{R}e$ is ϕ-invariant and then apply Lemma B.3.3 to draw a conclusion. For this purpose, there are two cases to consider :

(a). The family $\{u,v\}$ is linearly dependent, i.e $v = au$ and $u \neq 0$. Then :

$$\phi(u) = (\lambda_1 - a\lambda_2)\, u \quad \text{and} \quad a\phi(u) = (\lambda_2 + a\lambda_1)\, u$$

This gives $\lambda_2 + a\lambda_1 = a(\lambda_1 - a\lambda_2)$ so that $\left(1 + a^2\right)\lambda_2 = 0$, thus $\lambda_2 = 0$ and $\phi(u) = \lambda_1 u$, i.e $\mathbb{R}u$ is a proper ϕ-invariant subspace of L, and since L is ϕ-decomposable it follows that $\mathbb{R}u$ is degenerate, therefore $\langle u,u\rangle = 0$. We thus take $e := u$.

(b). The family $\{u,v\}$ is linearly independent, so the vector subspace $\operatorname{span}\{u,v\}$ cannot be totally isotropic. Thus the vector $(\langle u,u\rangle, \langle v,u\rangle, \langle v,v\rangle)$ is non-zero, using (B.2) we get :

$$0 = \begin{vmatrix} \lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 \\ \lambda_2 & 2\lambda_1 & -\lambda_2 \end{vmatrix} = -2\lambda_1\left(\lambda_1^2 + \lambda_2^2\right)$$

If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, then (B.2) gives that $\langle u,v\rangle = 0$ and $\langle u,u\rangle = \langle v,v\rangle \neq 0$. Thus $\operatorname{span}\{u,v\}$ is a ϕ-invariant, non-degenerate proper subspace of L, a contradiction. So necessarily $\lambda_1 = \lambda_2 = 0$ which means that $\phi(x) = 0$ for any $x \in \operatorname{span}\{u,v\}$. It follows that $\operatorname{span}\{u,v\}$ is degenerate and therefore contains an isotropic vector e such that $\phi(e) = 0$.

This ends the proof. □

B.4 Reduction of symmetric endomorphisms on Lorentzian vector spaces

Let us start with the statement of a classical result on the reduction of symmetric endomorphisms in a Euclidean space.

Theorem B.4.1. *Let $(E,\langle\ ,\ \rangle)$ be a Euclidean vector space and $\phi : E \longrightarrow E$ be a symmetric endomorphism. There exists an orthonormal basis $(e_1,\ldots,e_n)$ of E and a family $\lambda_1 \leq \ldots \leq \lambda_n$ of real numbers such that for any $i \in \{1,\ldots,n\}$, we have $\phi(e_i) = \lambda_i e_i$.*

We now focus on the Lorentzian case, let us start by a key Lemma:

Lemma B.4.1. *Let $(E,\langle\ ,\ \rangle)$ be a Lorentzian vector space such that $\dim E \geq 3$ and $\phi : E \longrightarrow E$ a symmetric endomorphism. Then ϕ admits a real eigenvalue.*

Proof. Lemma B.3.2 shows that there exists a nonzero couple of vectors$(u,v) \in E \times E$ and two real numbers λ_1, λ_2 such that $\phi(u) = \lambda_1 u - \lambda_2 v$ and $\phi(v) = \lambda_2 u + \lambda_1 v$, there are two cases to discuss :

1. The family $\{u,v\}$ is linearly dependent, i.e $v = au$ and $u \neq 0$.Then :

$$\phi(u) = (\lambda_1 - a\lambda_2)u \quad \text{and} \quad a\phi(u) = (\lambda_2 + a\lambda_1)u,$$

   so $\lambda_2 + a\lambda_1 = a(\lambda_1 - a\lambda_2)$, this shows that $\lambda_2 = 0$ and $\phi(u) = \lambda_1 u$ which is the desired result.

2. The family $\{u,v\}$ is linearly independent. Since ϕ is symmetric, $\langle\phi(u),v\rangle = \langle u,\phi(v)\rangle$ which is equivalent to

$$\lambda_2(\langle u,u\rangle + \langle v,v\rangle) = 0 \tag{B.3}$$

   If $\lambda_2 = 0$ then $\phi(u) = \lambda_1 u$ and the proof is achieved . Assume that $\lambda_2 \neq 0$, then (B.3) gives that $\langle u,u\rangle = -\langle v,v\rangle$. Denote $P := \mathrm{span}\{u,v\}$, the matrix of $\langle\ ,\ \rangle_{|P\times P}$ with respect to $\{u,v\}$ is given by :

$$M = \begin{pmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle u,v\rangle & -\langle u,u\rangle \end{pmatrix},$$

   furthermore P is nondegenerate if and only if $\det M \neq 0$. Now :

$$\det M = -\langle u,u\rangle^2 - \langle u,v\rangle^2,$$

   this means that $\det M = 0$ if and only if P is totally isotropic, which is impossible as $\dim P > 1$. Therefore P is non-degenerate Lorentzian, so that $P^\perp$ is non-degenerate Euclidean and since ϕ is symmetric and P is ϕ-invariant we get that $P^\perp$ is ϕ-invariant as well. Theorem B.4.1 then shows that the restriction $\phi_{|P^\perp}$ admits a real eigenvalue.

This ends the proof. □

Theorem B.4.2. *Let $(E,\langle\ ,\ \rangle)$ be a n-dimensional Lorentzian vector space such that $n \geq 3$ and let $\phi : E \longrightarrow E$ be a symmetric endomorphism. There exists a basis $\mathbb{B}$ of E in which ϕ and $\langle\ ,\ \rangle$ have the following form :*

1. *Type { diag } :*

$$\mathrm{Mat}(\phi,\mathbb{B}) = \mathrm{diag}(\alpha_1,\ldots,\alpha_n), \quad \mathrm{Mat}(\langle\ ,\ \rangle,\mathbb{B}) = \mathrm{diag}(+1,\ldots,+1,-1)$$

2. *Type* $\{n-2, z\bar{z}\}$ *:*

$$\mathrm{Mat}(\phi,\mathbb{B}) = \mathrm{diag}(\alpha_1,\ldots,\alpha_{n-2}) \oplus \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, b \neq 0, \quad \mathrm{Mat}(\langle\ ,\ \rangle,\mathbb{B}) = \mathrm{diag}(+1,\ldots,+1,-1)$$

3. *Type* $\{n, \alpha 2\}$ *:*

$$\mathrm{Mat}(\phi,\mathbb{B}) = \mathrm{diag}(\alpha_1,\ldots,\alpha_{n-2}) \oplus \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \mathrm{Mat}(\langle\ ,\ \rangle,\mathbb{B}) = I_{n-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

4. *Type* $\{n, \alpha 3\}$ *:*

$$\mathrm{Mat}(\phi,\mathbb{B}) = \mathrm{diag}(\alpha_1,\ldots,\alpha_{n-3}) \oplus \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}, \quad \mathrm{Mat}(\langle\ ,\ \rangle,\mathbb{B}) = I_{n-3} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Proof. According to Lemma B.3.1, we have the orthogonal decomposition $E = V_1 \oplus \ldots \oplus V_r$ of E into ϕ-indecomposable subspaces, since E is Lorentzian we can assume that V_1 is Lorentzian, denote it L, in which case $V := V_2 \oplus \cdots \oplus V_r$ is Euclidean. Since $\phi_{|V} : V \longrightarrow V$ is symmetric then by Theorem B.4.1 the subspaces V_i are 1-dimensional for all $2 \leq i \leq r$. There are only three cases to consider :

1. $\dim L = 1$. Write $L := \mathbb{R}x$, then $\phi(x) = \lambda x$ for some $\lambda \in \mathbb{R}$. Hence ϕ is of type {diag}.

2. $\dim L = 2$. Let $(e, \bar{e})$ be a Lorentzian basis of L then set $\phi(e) = ae + b\bar{e}$ and $\phi(\bar{e}) = ce + a\bar{e}$. The characteristic polynomial of the restriction $\phi_{|L}$ is $P(X) = X^2 - 2aX + a^2 - bc$ with discriminant $\Delta = 4bc$. So either :

   (a). $bc > 0$, hence $\phi_{|L}$ admits two distinct real eigenvalues λ_1, λ_2 with respective eigenvectors $u_1, u_2 \in L$. Since $\phi_{|L}$ is symmetric and $\lambda_1 \neq \lambda_2$, then $\langle u_1, u_2 \rangle = 0$. Now L is non-degenerate and 2-dimensional, therefore $\langle u_i, u_i \rangle \neq 0$ for $i = 1, 2$, in particular $\mathbb{R}u_1$ is a non-degenerate ϕ-invariant, proper subspace of L, a contradiction.

   (b). $bc = 0$, we can suppose without loss of generality that $b = 0$. If $c = 0$ as well then $\mathbb{R} \cdot (e + \bar{e})$ is a non-degenerate, ϕ-invariant proper subspace of L, which is impossible. Thus $c \neq 0$ and by taking $\left\{\epsilon\sqrt{|c|}e, \epsilon\frac{1}{\sqrt{|c|}}\bar{e}\right\}$ where ϵ is the sign of c, we can easily see that ϕ is of type $\{n,\ \alpha 2\}$.

(c). $bc < 0$, take $\alpha = \left(-\frac{c}{b}\right)^{\frac{1}{4}}$, then $\left\{\alpha e, \alpha^{-1}\bar{e}\right\}$ is a basis of L and one can check that ϕ is of type $\{n-2, z\bar{z}\}$.

3. $\dim L \geq 3$. Lemma B.4.1 shows that the restriction $\phi_{|L}$ admits a real eigenvalue, since L is ϕ-indecomposable, then its corresponding eigenvector e must be isotropic. Set $D := \mathbb{R}e$ then put $F := D^{\perp} \cap L$ and $W = F/D$. The quotient space W is naturally endowed with an inner product $\langle\ ,\ \rangle_q$ given by :

$$\langle [u], [v] \rangle_q = \langle u, v \rangle, \quad u, v \in F$$

Let $\bar{\phi} : W \longrightarrow W$ be the endomorphism induced by ϕ on W. Then $\left(W, \langle\ ,\ \rangle_q\right)$ is Euclidean and $\bar{\phi}$ is symmetric, according to Theorem B.4.1 we can find an orthonormal basis $(\widetilde{u}_1, \ldots, \widetilde{u}_r)$ of W and real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ such that $\bar{\phi}(\widetilde{u}_i) = \lambda_i \widetilde{u}_i$. Let $u_i \in F$ such that $[u_i] = \widetilde{u}_i$, then we get that :

$$\phi(u_i) = a_i e + \lambda_i u_i, \quad i = 1, \ldots, r \tag{B.4}$$

It is clear that the family $(u_1, \ldots, u_r)$ is orthonormal with $\langle u_i, u_i \rangle > 0$, furthermore $a_1 \neq 0$ since otherwise $\mathbb{R}u_1$ would be non-degenerate and ϕ-invariant, which is false. On the other hand span$\{a_i u_1 - a_1 u_i\}$ is obviously non-degenerate and ϕ-invariant, it must therefore be 0-dimensional or equivalently $u_i = 0$ and $a_i = 0$ for all $i = 2, \ldots, r$. This shows that $\dim F = 2$ and $\dim L = 3$. Choose $\bar{e} \in L$ co-isotropic to e and such that $(e, \bar{e}, u_1)$ is a Lorentzian basis of L. Using (B.4) and the fact that $\phi_{|L}$ is symmetric, we can write :

$$\phi(e) = \lambda e,\ \phi(\bar{e}) = ae + \lambda\bar{e} + a_1 u_1 \quad \text{and} \quad \phi(u_1) = a_1 e + \lambda_1 u_1, \quad a_1 \neq 0.$$

Necessarily we must have $\lambda = \lambda_1$, otherwise we would get that $v := \frac{a_1}{\lambda_1 - \lambda} e + u_1$ is an eigenvector of $\phi_{|L}$ which is impossible since $\langle v, v \rangle > 0$. In summary :

$$\phi(e) = \lambda e,\ \phi(\bar{e}) = ae + \lambda\bar{e} + a_1 u_1 \quad \text{and} \quad \phi(u_1) = a_1 e + \lambda u_1, \quad a_1 \neq 0$$

Thus ϕ is of type $\{n,\ \alpha 3\}$ with respect to the basis $\left(a_1 e, \frac{a}{a_1} e + u_1, \frac{1}{a_1}\bar{e}\right)$ of L.

This ends the proof. □

C

Preliminaries on Pseudo-Riemannian Lie Groups

C.1 Pseudo-Euclidean Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and $L : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be any bilinear map, denote $L_u := L(u, .)$ then let:

$$K^L : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}), \quad T^L : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

be the bilinear maps given by $K^L(u,v) = [L_u, L_v] - L_{[u,v]}$ and $T^L(u,v) = L_u v - L_v u - [u,v]$, which we shall respectively call the ***curvature*** and ***torsion*** of L. We say that L is ***torsion-free*** if $T^L = 0$.

Proposition C.1.1. *Let $L : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be a bilinear map on a Lie algebra $\mathfrak{g}$ with curvature K. Assume L is torsion-free, then for any $u, v, w \in \mathfrak{g}$:*

1. *$K(u,v) = -K(v,u)$, (Skew-Symmetry).*

2. *$K(u,v)w + K(w,u)v + K(v,w)u = 0$, (Bianchi Identity).*

A pseudo-Euclidean Lie algebra is a couple $(\mathfrak{g}, \langle\ ,\ \rangle)$ consisting of a Lie algebra $\mathfrak{g}$ together with a pseudo-Euclidean inner product $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$. ***The Levi-Civita product*** of a pseudo-Euclidean Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is the bilinear map $L : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by the expression:

$$\langle L_u v, w \rangle := \frac{1}{2}(\langle [u,v], w \rangle + \langle [w,u], v \rangle + \langle [w,v], u \rangle). \tag{C.1}$$

It is straightforward to check that the Levi-Civita product L is torsion-free. We shall define the ***curvature of a pseudo-Euclidean Lie algebra*** to be the curvature of its Levi-Civita product.

Proposition C.1.2. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra and denote L its Levi-Civita product and K its curvature. Then for any $u, v, w, z \in \mathfrak{g}$, we have:*

1. *$\langle L_u v, w \rangle = -\langle v, L_u w \rangle$, i.e L_u is skew-symmetric with respect to $\langle\ ,\ \rangle$.*

2. $\langle K(u,v)w,z\rangle = -\langle K(u,v)z,w\rangle$.

3. $\langle K(u,v)w,z\rangle = \langle K(w,z)u,v\rangle$.

The ***Ricci operator*** of a pseudo-Euclidean Lie algebra $(\mathfrak{g},\langle\ ,\ \rangle)$ is the linear map $\mathrm{Ric}:\mathfrak{g}\longrightarrow\mathfrak{g}$ given by:

$$\langle \mathrm{Ric}(u),v\rangle := \mathrm{tr}(w\mapsto K(u,w)v) = \sum_{i=1}^{n}\epsilon_i\langle K(u,e_i)v,e_i\rangle, \tag{C.2}$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis of $(\mathfrak{g},\langle\ ,\ \rangle)$ and $\epsilon_i := \langle e_i,e_i\rangle = \pm 1$. The bilinear form $\mathrm{ric}:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ given by $\mathrm{ric}(u,v) := \langle \mathrm{Ric}(u),v\rangle$ is called the ***Ricci curvature*** of $(\mathfrak{g},\langle\ ,\ \rangle)$. We say that the pseudo-Euclidean Lie algebra $(\mathfrak{g},\langle\ ,\ \rangle)$ is ***Einstein*** if its Ricci curvature satisfies $\mathrm{ric} = \lambda\langle\ ,\ \rangle$ for some $\lambda\in\mathbb{R}$ i.e $\mathrm{Ric} = \lambda\mathrm{Id}_{\mathfrak{g}}$.

C.2 Pseudo-Riemannian Lie groups

Let us now introduce a particular pseudo-Riemannian structure on Lie groups.

Definition C.2.1. *Let G be a Lie group. A pseudo-Riemannian metric* g *on G is* ***left invariant*** *if for any $g\in G$, we have $l_g^*\mathrm{g} = \mathrm{g}$. In other words for any $x\in G$ and every $v,w\in T_xG$:*

$$\mathrm{g}_{gx}(T_xl_g(v),T_xl_g(w)) = \mathrm{g}_x(v,w).$$

In the same way g *is* ***right invariant*** *if $r_g^*\mathrm{g} = \mathrm{g}$.*

Let G be a Lie group with Lie algebra $\mathfrak{g}$, denote $\mathcal{M}^\ell(G)$ the set of left invariant metrics on the Lie group G and $\mathcal{M}(\mathfrak{g})$ the set of pseudo-Euclidean products on the underlying vector space of $\mathfrak{g}$. The evalution map $\Psi:\mathcal{M}^\ell(G)\longrightarrow\mathcal{M}(\mathfrak{g})$, $\mathrm{g}\longrightarrow\mathrm{g}_e$ is a bijection. This shows that a left invariant metric on a Lie group G can always be obtained by providing an inner product on the Lie algebra $\mathfrak{g}$.

Proposition C.2.1. *Let X,Y be left invariant vector fields on the Lie group G and let* g *be any left invariant metric on G. Then the map $G\longrightarrow\mathbb{R}$, $x\mapsto\mathrm{g}_x(X_x,Y_x)$ is constant.*

The next result is a consequence of the previous Proposition and is frequently used due to its practical importance.

Proposition C.2.2. *Let* g *be a left invariant metric on the Lie group G. If $\{e_1,\dots,e_n\}$ is an orthonormal basis of* $(\mathrm{Lie}(G),\mathrm{g}_e)$*, then $\{e_1^\ell,\dots,e_n^\ell\}$ defines a global orthonormal frame on* (G,g).

We shall call ***pseudo-Riemannian Lie group*** any couple (G,g) consisting of a Lie group G together with a left invariant Riemannian metric g on G.

Definition C.2.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and ∇ an affine connection on G. We say that ∇ is a* **left-invariant connection** *on G if it satisfies $(\ell_g)_*\nabla = \nabla$ for any $g\in G$, which means that for any $x\in G$ and any $X,Y\in\mathcal{X}(G)$:*

$$(\nabla_XY)_{gx} = T_x\ell_g((\nabla_{(\ell_g)_*X}(\ell_g)_*Y)_x).$$

It follows that ∇ is left-invariant if and only if for any $X, Y \in \chi^\ell(G)$, $(\nabla_X Y)_{gx} = T_x\ell_g((\nabla_X Y)_x)$. Any left invariant affine connection ∇ on G gives rise to a bilinear map $L^\nabla : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by $L^\nabla_u v := (\nabla_{u^\ell} v^\ell)_e$, where u^ℓ denotes the (unique) left invariant vector field satisfying $u^\ell_e := u$. In fact if we define $\mathscr{C}^\ell(G)$ to be the set of all left invariant affine connections on G, then the correspondence $\mathscr{C}^\ell(G) \longrightarrow \mathfrak{g}^* \otimes \mathrm{End}(\mathfrak{g})$, $\nabla \mapsto L^\nabla$ is a bijection.

Proposition C.2.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and ∇ a left invariant affine connection on G. Denote R^∇ and T^∇ respectively the curvature and torsion tensor fields corresponding to ∇, and by K and T the curvature and torsion of L^∇. Then for any $u, v, w \in \mathfrak{g}$:*

$$(K(u,v)w)^\ell = R^\nabla(u^\ell, v^\ell)w^\ell \quad \textit{and} \quad (T(u,v))^\ell = T^\nabla(u^\ell, v^\ell).$$

Proposition C.2.4. *Let (G, g) be a pseudo-Riemannian Lie group with Lie algebra $\mathfrak{g}$. Then:*

1. *The Levi-Civita connection ∇ of (G, g) is left invariant and $L^\nabla : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is exactly the Levi-Civita product of $(\mathfrak{g}, \mathrm{g}_e)$.*

2. *Let Ric^∇ be the Ricci tensor of (G, g) and ric^∇ its Ricci curvature, and denote by* Ric *and* ric *respectively the Ricci operator and Ricci curvature of $(\mathfrak{g}, \mathrm{g}_e)$. Then for any $u, v \in \mathfrak{g}$:*

$$\mathrm{ric}^\nabla(u^\ell, v^\ell) = \mathrm{ric}(u,v) \quad \mathrm{Ric}^\nabla(u^\ell) = \mathrm{Ric}(u)^\ell.$$

3. *The scalar curvature s^∇ of (G, g) is a constant function, more precisely $s^\nabla = s$ where s denotes the scalar curvature of $(\mathfrak{g}, \mathrm{g}_e)$ i.e $s := \mathrm{tr}(\mathrm{Ric})$.*

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