Three Manifolds as Branched Covers of S^3

H. Abchir

Hassan II University. Casablanca

GTA seminary 2019-20.

FST-UCA, Marrakech. December 28, 2019

Plan

Problem: Classification of three manifolds.

Problem: Classification of three manifolds.

- Heegaard splitting of three manifolds.

Problem: Classification of three manifolds.

- Heegaard splitting of three manifolds.
- Three manifolds as the surgered three sphere: theorem of Lickorish Wallace (1964).

Problem: Classification of three manifolds.

- Heegaard splitting of three manifolds.
- Three manifolds as the surgered three sphere: theorem of Lickorish Wallace (1964).
- Three manifolds as branched covers of the three sphere.

Problem: Classification of three manifolds.

- Heegaard splitting of three manifolds.
- Three manifolds as the surgered three sphere: theorem of Lickorish Wallace (1964).
- Three manifolds as branched covers of the three sphere.

Remark : in dimensions less than or equal to 3, the P.L. and differentiable classification of manifolds are both equivalent the topological one.

Theorem

Any orientable closed 3-manifold has a Heegaard splitting.

Theorem

Any orientable closed 3-manifold has a Heegaard splitting.

That is if M is a such 3-manifold, there exists an integer $g \geq 0$ such that

$$M = H_1^g \cup_{\partial H^g} H_2^g,$$

where H_i^g, $i = 1, 2$ are two copies of the hadlebody H^g.

Theorem

Any orientable closed 3-manifold has a Heegaard splitting.

That is if M is a such 3-manifold, there exists an integer $g \geq 0$ such that

$$M = H_1^g \cup_{\partial H^g} H_2^g,$$

where H_i^g, $i = 1, 2$ are two copies of the hadlebody H^g.

Proof : this theorem can be easily achieved in the context of P.L. topology. Else, one can use Morse theory.

□

Theorem

Any orientable closed 3-manifold has a Heegaard splitting.

That is if M is a such 3-manifold, there exists an integer $g \geq 0$ such that

$$M = H_1^g \cup_{\partial H^g} H_2^g,$$

where H_i^g, $i = 1, 2$ are two copies of the hadlebody H^g.

Proof : this theorem can be easily achieved in the context of P.L. topology. Else, one can use Morse theory.

□

Examples :

Dehn Twist

Let F be a connected compact oriented surface. Let α a simple closed embedded in F, and let A be an annulus neighbourhood of α.

Dehn Twist

Let F be a connected compact oriented surface. Let α a simple closed embedded in F, and let A be an annulus neighbourhood of α.
The standard annulus is $S^1 \times [0, 1]$ with some fixed orientation.

Dehn Twist

Let F be a connected compact oriented surface. Let α a simple closed embedded in F, and let A be an annulus neighbourhood of α.

The standard annulus is $S^1 \times [0, 1]$ with some fixed orientation.

Definition

A **Dehn twist** of F along the curve α is any homeomorphism isotopic to the homeomorphism $\tau : F \longrightarrow F$ defined such that $\tau_{|F \setminus \overset{\circ}{A}}$ is the identity and, parametrising A as $S^1 \times [0, 1]$ in an orientation-preserving manner, $\tau_{|A}$ is given by $\tau(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$.

Dehn Twist

Let F be a connected compact oriented surface. Let α a simple closed embedded in F, and let A be an annulus neighbourhood of α.
The standard annulus is $S^1 \times [0,1]$ with some fixed orientation.

Definition

A **Dehn twist** of F along the curve α is any homeomorphism isotopic to the homeomorphism $\tau : F \longrightarrow F$ defined such that $\tau_{|F \setminus \overset{\circ}{A}}$ is the identity and, parametrising A as $S^1 \times [0,1]$ in an orientation-preserving manner, $\tau_{|A}$ is given by $\tau(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$.

Remark : A Dehn twists provides nontrivial examples of orientation preserving homeomorphims of a surface of genus $g \geq 1$ not isotopic to the identity.

Theorem (Dehn-Lickorish theorem)

Any orientation-preserving homeomorphism of an oriented 2-manifold (without boundary) is isotopic to the composition of Dehn twists.

Theorem (Dehn-Lickorish theorem)

Any orientation-preserving homeomorphism of an oriented 2-manifold (without boundary) is isotopic to the composition of Dehn twists.

Corollary (Rokhlin's theorem)

Any closed orientable 3-manifold bounds a 4-manifold.

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3.

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3. Let N_i be a regular neighborhood of the component K_i in S^3.

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3. Let N_i be a regular neighborhood of the component K_i in S^3.
Let J_i be a simple closed curve on ∂N_i.

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3. Let N_i be a regular neighborhood of the component K_i in S^3.
Let J_i be a simple closed curve on ∂N_i. The pair $(L, \cup_i^n J_i)$ is called a **framed link.**

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3. Let N_i be a regular neighborhood of the component K_i in S^3.
Let J_i be a simple closed curve on ∂N_i. The pair $(L, \cup_i^n J_i)$ is called a **framed link.**
For each i we consider a homeomorphism

$$\begin{array}{rccc} h_i : & \partial N_i & \longrightarrow & \partial N_i \\ & m_i & \longmapsto & h_i(m_i) = J_i \end{array}$$

where m_i is the meridian curve of ∂N_i.

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3. Let N_i be a regular neighborhood of the component K_i in S^3.
Let J_i be a simple closed curve on ∂N_i. The pair $(L, \cup_i^n J_i)$ is called a **framed link.**
For each i we consider a homeomorphism

$$\begin{array}{cccc} h_i : & \partial N_i & \longrightarrow & \partial N_i \\ & m_i & \longmapsto & h_i(m_i) = J_i \end{array}$$

where m_i is the meridian curve of ∂N_i.
We consider the manifold

$$M_L = S^3 \setminus \cup_{i=1}^{n} \overset{\circ}{N}_i \cup_h \cup_{i=1}^{n} N_i, \;\; h = \cup_i h_i.$$

Let $L = \cup_{i=1}^{n} K_i$ be a link with n components in S^3. Let N_i be a regular neighborhood of the component K_i in S^3.
Let J_i be a simple closed curve on ∂N_i. The pair $(L, \cup_i^n J_i)$ is called a **framed link.**
For each i we consider a homeomorphism

$$\begin{array}{rccc} h_i: & \partial N_i & \longrightarrow & \partial N_i \\ & m_i & \longmapsto & h_i(m_i) = J_i \end{array}$$

where m_i is the meridian curve of ∂N_i.
We consider the manifold

$$M_L = S^3 \setminus \cup_{i=1}^{n} \overset{\circ}{N}_i \cup_h \cup_{i=1}^{n} N_i, \quad h = \cup_i h_i.$$

Definition

The 3-manifold M_L is said to be obtained by surgery on S^3 along the framed link $(L, \cup_i^n J_i)$

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$.

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$. Then the rational $\frac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.**

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$. Then the rational $\dfrac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.** An **integer surgery** is a surgery where $q_i = 1$.

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$. Then the rational $\frac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.**
An **integer surgery** is a surgery where $q_i = 1$.
Examples : if L is the trivial knot then:

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$. Then the rational $\dfrac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.** An **integer surgery** is a surgery where $q_i = 1$.

Examples : if L is the trivial knot then:

1. The surgery ∞ is the identity

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$. Then the rational $\dfrac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.** An **integer surgery** is a surgery where $q_i = 1$.

Examples : if L is the trivial knot then:

1. The surgery ∞ is the identity
2. the 0 surgery gives $S^1 \times S^2$.

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$.
Then the rational $\frac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.**
An **integer surgery** is a surgery where $q_i = 1$.
Examples : if L is the trivial knot then:

1. The surgery ∞ is the identity
2. the 0 surgery gives $S^1 \times S^2$.
3. the (p, q) surgery gives $L(p, q)$.

Remark : We consider the meridian m_i of ∂N_i and chose a prefered parallel l_i on ∂N_i. Each isotpy class of a simple closed curve on ∂N_i is completely determined by a pair (p, q) of coprime integers. Then, for each i, $1 \leq i \leq n$, $J_i = (p, q)$. Then the rational $\dfrac{p_i}{q_i}$ is called the surgery instruction on the component K_i or the **surgery coefficient** or the **framing index.** An **integer surgery** is a surgery where $q_i = 1$.

Examples : if L is the trivial knot then:

1. The surgery ∞ is the identity
2. the 0 surgery gives $S^1 \times S^2$.
3. the (p, q) surgery gives $L(p, q)$.

Theorem (Lickorish)

Any closed connected orientable 3-manifold can be obtained from S^3 by integer surgery along a framed link L.

- **The first Kirby move** It consists in adding to (or deleting from) the given framed link $L \in S^3$ an unknotted circle with framing ± 1 provided that is unlinked with the other components of L.

- **The first Kirby move** It consists in adding to (or deleting from) the given framed link $L \in S^3$ an unknotted circle with framing ± 1 provided that is unlinked with the other components of L.
- **The second Kirby move**

- **The first Kirby move** It consists in adding to (or deleting from) the given framed link $L \in S^3$ an unknotted circle with framing ± 1 provided that is unlinked with the other components of L.
- **The second Kirby move**

Theorem (Kirby)

Two links in S^3 with integer framings produce the same 3-manifold if and only if they can be obtained from each other by a finite sequence of first and second Kirby moves and isotopies.

Plan

Let $L = \cup_{i=1}^{n} K_i$ be a link in S^3. For each i, let N_i be a regular neighborhood of the component K_i in S^3. We denote by X the exterior of the link, that is

$$X = S^3 \setminus \cup_{i=1}^{n} \mathring{N}_i.$$

Let $L = \cup_{i=1}^{n} K_i$ be a link in S^3. For each i, let N_i be a regular neighborhood of the component K_i in S^3. We denote by X the exterior of the link, that is

$$X = S^3 \setminus \cup_{i=1}^{n} \mathring{N}_i.$$

We have $H_1(X) \simeq \mathbb{Z}^n$, where the generators are the meridians of the N_i.

Let $L = \cup_{i=1}^{n} K_i$ be a link in S^3. For each i, let N_i be a regular neighborhood of the component K_i in S^3. We denote by X the exterior of the link, that is

$$X = S^3 \setminus \cup_{i=1}^{n} \overset{\circ}{N}_i.$$

We have $H_1(X) \simeq \mathbb{Z}^n$, where the generators are the meridians of the N_i.

Definition

Let K_1 and K_2 be two simple closed oriented curves in S^3. We denote by $lk(K_1, K_2)$ the number of ones that K_2 describes the meridian of a tubular neighborhood of K_1 (i.e. the number of ones that K_2 turns around K_1). It is an invariant of the link $L = K_1 \cup K_2$.

Let $L = \cup_{i=1}^{n} K_i$ be a link in S^3. For each i, let N_i be a regular neighborhood of the component K_i in S^3. We denote by X the exterior of the link, that is

$$X = S^3 \setminus \cup_{i=1}^{n} \mathring{N}_i.$$

We have $H_1(X) \simeq \mathbb{Z}^n$, where the generators are the meridians of the N_i.

Definition

Let K_1 and K_2 be two simple closed oriented curves in S^3. We denote by $lk(K_1, K_2)$ the number of ones that K_2 describes the meridian of a tubular neighborhood of K_1 (i.e. the number of ones that K_2 turns around K_1). It is an invariant of the link $L = K_1 \cup K_2$.

If $L = \cup_{i=1}^{n} K_i$, we denote by $lk(L) = \sum_{i<j} lk(K_i, K_j)$.

The Hurewicz homomorphism gives the epimorphism

$$\begin{array}{cccccc} \varphi : & \pi_1(X) & \twoheadrightarrow & H_1(X) \simeq \mathbb{Z}^n & \rightarrow & \mathbb{Z} \\ & \alpha & \longmapsto & lk(\alpha, L) = \sum_i lk(\alpha, K_i) & & \end{array}$$

The Hurewicz homomorphism gives the epimorphism

$$\begin{array}{rcccc} \varphi : & \pi_1(X) & \twoheadrightarrow & H_1(X) \simeq \mathbb{Z}^n & \rightarrow & \mathbb{Z} \\ & \alpha & \longmapsto & lk(\alpha, L) = \sum_i lk(\alpha, K_i) & & \end{array}$$

where the second map sends each meridian to $1 \in \mathbb{Z}$.

The Hurewicz homomorphism gives the epimorphism

$$\begin{array}{rcccl} \varphi: & \pi_1(X) & \twoheadrightarrow & H_1(X) \simeq \mathbb{Z}^n & \rightarrow \mathbb{Z} \\ & \alpha & \longmapsto & lk(\alpha, L) = \sum_i lk(\alpha, K_i) & \end{array}$$

where the second map sends each meridian to $1 \in \mathbb{Z}$.Then, for each integer $k \geq 2$, we get an epimorphism

$$\varphi_k : \pi_1(X) \twoheadrightarrow \mathbb{Z}_k.$$

The Hurewicz homomorphism gives the epimorphism

$$\begin{array}{rccccc} \varphi: & \pi_1(X) & \twoheadrightarrow & H_1(X) \simeq \mathbb{Z}^n & \to & \mathbb{Z} \\ & \alpha & \longmapsto & lk(\alpha, L) = \sum_i lk(\alpha, K_i) & & \end{array}$$

where the second map sends each meridian to $1 \in \mathbb{Z}$.Then, for each integer $k \geq 2$, we get an epimorphism

$$\varphi_k : \pi_1(X) \twoheadrightarrow \mathbb{Z}_k.$$

So, for each integer $k \geq 2$ we have a covering space $\tilde{X}_k$ of X with k sheets, called the ***k*-fold cyclic covering** space of X.

Remark : *Ker*φ induces a covering $\tilde{X}_\infty$ of X with fiber $\mathbb{Z}$ called the abelian covering or the infinite cyclic covering.
$H_1(X_\infty)$ is called the **Alexander module** of K which is a $A = \mathbb{Z}[t, t^{-1}]$-module. We call a presentation of X, an exact sequence

$$A^p \to A^n \to H_1(\tilde{X}_\infty) \to 0.$$

If we consider a Seifert surface of the link. If V is the associated matrix called the Seifert matrix, then $tV - V^T$ is a presentation of the Alewander module.
The polynomial

$$\Delta_L(t) = \det(tV - V^T)$$

is the very known interesting invariant called the Alexander polynomial of the link L.

Now for each $k \geq 2$, the epimorphism φ_k induces a covering space $\tilde{X}_k$ of X such that

$$\pi_1(\tilde{X}_k) = \ker \varphi_k \text{ and } Aut(\tilde{X}_k) \simeq \pi_1(X)/\ker \varphi_k \simeq \mathbb{Z}_k.$$

Remark :

$$\pi_1(\tilde{X}_\infty) = \ker \varphi = [\ker \varphi, \ker \varphi].$$

$$\pi_1(\tilde{X}_k) = \ker \varphi_k = [\ker \varphi_k, \ker \varphi_k].$$

Geometric achievement of the cyclic coverings

Let L be a link in S^3.
Let F be a Seifert surface of L.
Let X the exterior of L in S^3.
Let Y be the exterior X cut along F. We get two copies of F denoted by F^- and F^+.
We make a coutable copies of Y, so we get a family $(Y_i)_{i\in\mathbb{Z}}$.
The copy Y_i contains F_i^- and F_i^+ respectively copies of F^- and F^+.

1. For each i, we glue Y_i to Y_{i+1} by the identification of F_i^+ with F_{i+1}^-. Then we get a space X_∞.
2. For each i, $0 \leq i \leq k-1$, we glue Y_i to Y_{i+1} by the identifying F_i^+ with F_{i+1}^-. Furthermore, we glue Y_k to Y_0 by identifying F_k^+ to F_0^-.

Proposition

We have the following commutative diagram of coverings

$$\begin{array}{ccc} X_\infty & \rightarrow & X_k \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Branched coverings of surfaces

Definition

A continuous map $p : M^2 \longrightarrow N^2$ is said to be a **branched covering** if there exists a finite set of points $\{x_1, \ldots, x_n\} \subset N^2$ such that the set $p^{-1}(\{x_1, \ldots, x_n\})$ is discrete and the restriction of p to the set $M^2 \setminus p^{-1}(\{x_1, \ldots, x_n\})$ is a covering. The points $\{x_1, \ldots, x_n\}$ are called the **branched points** of p.

Branched coverings of surfaces

Definition

A continuous map $p : M^2 \longrightarrow N^2$ is said to be a **branched covering** if there exists a finite set of points $\{x_1, \ldots, x_n\} \subset N^2$ such that the set $p^{-1}(\{x_1, \ldots, x_n\})$ is discrete and the restriction of p to the set $M^2 \setminus p^{-1}(\{x_1, \ldots, x_n\})$ is a covering. The points $\{x_1, \ldots, x_n\}$ are called the **branched points** of p.

Proposition

Let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ *and let*

$$p : \begin{array}{ccc} D^2 & \longrightarrow & D^2 \\ z & \longmapsto & z^k. \end{array}$$

Then p is a k-fold branched covering with the branch point $z = 0$.

Remark : If p is a k-fold branched covering and U is a sufficiently small disk neighborhood of a branch point y, then $p^{-1}(U)$ consists of one or several disks D_i such that the restriction of p to each of them is equivalent to the covering $z \longmapsto z^{m_i}$ in the proposition. Each point x of D_i is said to have the **branching index** m_i. Then $k = \sum m_i$.

Remark : If p is a k-fold branched covering and U is a sufficiently small disk neighborhood of a branch point y, then $p^{-1}(U)$ consists of one or several disks D_i such that the restriction of p to each of them is equivalent to the covering $z \longmapsto z^{m_i}$ in the proposition. Each point x of D_i is said to have the **branching index** m_i. Then $k = \sum m_i$.

Example : Consider the map

$$\begin{array}{rccc} f: & \mathbb{C} \setminus \{0\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & f(z) = 2(z + \frac{1}{z}). \end{array}$$

Remark : If p is a k-fold branched covering and U is a sufficiently small disk neighborhood of a branch point y, then $p^{-1}(U)$ consists of one or several disks D_i such that the restriction of p to each of them is equivalent to the covering $z \longmapsto z^{m_i}$ in the proposition. Each point x of D_i is said to have the **branching index** m_i. Then $k = \sum m_i$.
Example : Consider the map

$$\begin{array}{rccc} f: & \mathbb{C} \setminus \{0\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & f(z) = 2(z + \frac{1}{z}). \end{array}$$

This map is a 2-fold branched covering with branch points ± 4. The preimages of these points are the points ± 1, and the branching index of each is 2.

Theorem

For every closed connected orientable surface M^2 there exists a branched covering $p : M^2 \longrightarrow S^2$.

Theorem

For every closed connected orientable surface M^2 there exists a branched covering $p : M^2 \longrightarrow S^2$.

Theorem

For every closed connected orientable surface M^2 with genus $g \geq 1$, there exists a branched covering $p : M^2 \longrightarrow S^2$ with exactly three branch points.

Branched coverings of 3-manifolds

Definition

Let M and N two compact 3-manifolds with proper one dimensional submanifolds $J \subset M$ and $L \subset N$. Then a continous map $p : M \longrightarrow N$ is said to be a **branched covering** with **branch sets** J and L if

1. components of preimages of open sets of N are a basis for the topology of M, and
2. $p(J) = L$, $p(M \setminus J) = N \setminus L$, and $N \setminus L$ is exactly the set of points in N which have neighborhoods U such that p sends each component of $p^{-1}(U)$ homeomorphicaly on U.

Remarks :

Remarks :

1. The restriction $p : M \setminus J \to N \setminus K$ is a covering called the **associated unbranched covering.**

Remarks :

1. The restriction $p : M \setminus J \to N \setminus K$ is a covering called the **associated unbranched covering.**
2. Each branch point x in J has a **branching index** k, meaning that p is k to one near x, and this number is constant on components of J.

Remarks :

1. The restriction $p : M \setminus J \to N \setminus K$ is a covering called the **associated unbranched covering.**
2. Each branch point x in J has a **branching index** k, meaning that p is k to one near x, and this number is constant on components of J.

Examples :

Remarks :

1. The restriction $p : M \setminus J \to N \setminus K$ is a covering called the **associated unbranched covering.**
2. Each branch point x in J has a **branching index** k, meaning that p is k to one near x, and this number is constant on components of J.

Examples :

1. A branched cover $S^3 \longrightarrow S^3$ branched over S^1 may be obtained by suspending the unbranched k-fold cover $S^1 \longrightarrow S^1$ 2 times. figure for S^2

Remarks :

1. The restriction $p : M \setminus J \to N \setminus K$ is a covering called the **associated unbranched covering.**
2. Each branch point x in J has a **branching index** k, meaning that p is k to one near x, and this number is constant on components of J.

Examples :

1. A branched cover $S^3 \longrightarrow S^3$ branched over S^1 may be obtained by suspending the unbranched k-fold cover $S^1 \longrightarrow S^1$ 2 times. figure for S^2
2. $q_k : S^1 \times D^2 \longrightarrow S^1 \times D^2$ wich assigns to each (z_1, z_2) the element (z_1, z_2^k).

Remarks :

1. The restriction $p : M \setminus J \to N \setminus K$ is a covering called the **associated unbranched covering.**
2. Each branch point x in J has a **branching index** k, meaning that p is k to one near x, and this number is constant on components of J.

Examples :

1. A branched cover $S^3 \longrightarrow S^3$ branched over S^1 may be obtained by suspending the unbranched k-fold cover $S^1 \longrightarrow S^1$ 2 times. figure for S^2
2. $q_k : S^1 \times D^2 \longrightarrow S^1 \times D^2$ wich assigns to each (z_1, z_2) the element (z_1, z_2^k).

Theorem (Alexander branched cover theorem)

Let M be a closed oriented 3-manifold. Then there exists a branched covering $p : M \to S^3$.

Let L be an n component link in S^3. Consider the k-fold cyclic covering $p_k : \tilde{X}_k \longrightarrow X$ of the exterior X. Recall that $\partial \tilde{X}_k$ is a disjoint union of torus such that the restriction of p_k to each of them is exactly the covering map

$$\begin{array}{rcl} S^1 \times S^1 : & \longrightarrow & S^1 \times S^1 \\ (z_1, z_2) & \longmapsto & (z_1, z_2^k) \end{array}$$

Let L be an n component link in S^3. Consider the k-fold cyclic covering $p_k : \tilde{X}_k \longrightarrow X$ of the exterior X. Recall that $\partial \tilde{X}_k$ is a disjoint union of torus such that the restriction of p_k to each of them is exactly the covering map

$$\begin{array}{rcl} S^1 \times S^1 : & \longrightarrow & S^1 \times S^1 \\ (z_1, z_2) & \longmapsto & (z_1, z_2^k) \end{array}$$

Now if we glue to $\tilde{X}_k$ the cover space of q_k we get a closed connected orientable manifold denoted by M_k which is a branched cover of S^3 associated to the k-fold cyclic covering of S^3.

Let L be an n component link in S^3. Consider the k-fold cyclic covering $p_k : \tilde{X}_k \longrightarrow X$ of the exterior X. Recall that $\partial\tilde{X}_k$ is a disjoint union of torus such that the restriction of p_k to each of them is exactly the covering map

$$\begin{array}{ccc} S^1 \times S^1 : & \longrightarrow & S^1 \times S^1 \\ (z_1, z_2) & \longmapsto & (z_1, z_2^k) \end{array}$$

Now if we glue to $\tilde{X}_k$ the cover space of q_k we get a closed connected orientable manifold denoted by M_k which is a branched cover of S^3 associated to the k-fold cyclic covering of S^3.

Definition

The 3-manifold M_k is called the k-fold cyclic branched cover of S^3 branched over L.

Plan

Theorem (Hilden-Montesinos theorem)

For any closed oriented 3-manifold M, there exists a 3-fold covering $p : M \to S^3$ of the 3-sphere by this manifold branching along a knot.

Theorem (Hilden-Montesinos theorem)

For any closed oriented 3-manifold M, there exists a 3-fold covering $p : M \to S^3$ of the 3-sphere by this manifold branching along a knot.

Proof Let $p : M \to N$ be

Lemma

:

Definition

A 1-submanifold L of the sphere S^3 will be said **universal** if any closed orientable 3-manifold M there exists a covering $p : M \to S^3$ branching along L.

Theorem (Hilden-Montesinos theorem)

For any closed oriented 3-manifold M, there exists a 3-fold covering $p : M \to S^3$ of the 3-sphere by this manifold branching along a knot.

Proof Let $p : M \to N$ be

Lemma

:

Definition

A 1-submanifold L of the sphere S^3 will be said **universal** if any closed orientable 3-manifold M there exists a covering $p : M \to S^3$ branching along L.

Theorem

The Borromean rings constitute a universal link.

Theorem (Hilden-Montesinos theorem)

For any closed oriented 3-manifold M, there exists a 3-fold covering $p : M \to S^3$ of the 3-sphere by this manifold branching along a knot.

Proof Let $p : M \to N$ be

Lemma

⋮

Definition

A 1-submanifold L of the sphere S^3 will be said **universal** if any closed orientable 3-manifold M there exists a covering $p : M \to S^3$ branching along L.

Theorem

The Borromean rings constitute a universal link.