Introduction to Differential Topology

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Etude de propriétés globales des variétés différentielles.

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- Etude de propriétés globales des variétés différentielles.
- Irouver et étudier des invariants topologiques des variétés.

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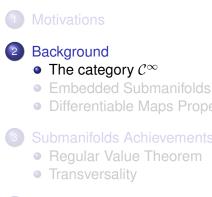
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- Etude de propriétés globales des variétés différentielles.
- Irouver et étudier des invariants topologiques des variétés.
- Riemann, Weyl, Whitney, Milnor.

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The category \mathcal{C}^{∞} Embedded Submanifolds Differentiable Maps Properties

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The category \mathcal{C}^{∞} Embedded Submanifolds Differentiable Maps Properties

Objects

Definition

Let *A* be a subset of \mathbb{R}^n . A map $f : A \to \mathbb{R}^k$ is said to be differentiable at $x \in A$ if there exist an open subset *U* of \mathbb{R}^n and a differential map $F : U \to \mathbb{R}^k$ such that $F_{|U \cap A} = f$.

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$$H^n=\{(x_1,\ldots,x_n)\in\mathbb{R}^n|x_n\geq 0\},\ n\geq 1.$$

Definition

A smooth manifold of dimension n is a topological Haussdorff space M with a countable basis such that

 each point x in M has an open neighborhood U which is homeomorphic to an open subset of the space Hⁿ by a homeomorphism φ:

$$\varphi: U \to \varphi(U) \subset H^n.$$

② if $(U_i)_{i \in I}$ is the set of such neighborhoods, each homeomorphism $\varphi_j \circ \varphi_i^{-1}$ is *C*[∞], where

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_i).$$

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$$H^{n} = \{(x_{1}, \dots, x_{n}) \in \mathbb{R}^{n} | x_{n} \geq 0\}, \quad n \geq 1.$$

$$\partial H^{n} = \{(x_{1}, \dots, x_{n}) \in \mathbb{R}^{n} | x_{n} = 0\}, \quad n \geq 1.$$

$$\partial M = \{x \in M | \exists \text{ a chart } (\varphi, U), \ x \in U, \ \varphi(x) \in \partial H^{n}\}.$$

$$\text{int}M = \{x \in M | \exists \text{ a chart } (\varphi, U), \ x \in U, \ \varphi(x) \in \text{int}H^{n}\}.$$

$$M = \partial M \sqcup \text{int}M.$$

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Examples -:

- $D^n, \, \partial D^n = S^{n-1}.$
- The Moebius band *M* whose boundary is a circle.
- The exterior X of a knot in S^3 . $\partial X = S^1 \times S^1$.

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Morphisms

Let M and N two smooth manifolds respectively of dimension m and n.

A map $f : M \to N$ is *smooth* at a point $x \in M$ if there exist chart (U, φ) of M at x and a chart (ψ, V) of N at f(x) such that the composition $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from an open subset of H^m to a subset of H^n . The map is smooth if it is smooth at each point $x \in M$.

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From local to global

Definition

Let *M* be a smooth manifold, and let $\mathcal{U} = (U_i)_{i \in I}$ be an arbitrary open cover of *M*. A *partition of unity subordinate to* \mathcal{U} is a collection of smooth functions $\{\psi_i : M \to [0, 1]\}_{i \in I}$ with the following properties:

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(i) supp \psi_i \subset U_i.
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(ii) The set {supp ψ_i } $_{i \in I}$ is locally finite.

(iii)
$$\sum_{i \in I} \psi_i(x) = 1$$
 for all $x \in M$.

Theorem

If *M* is a smooth manifold and $\mathcal{U} = (U_i)_{i \in I}$ is any open cover of *M* there exists a smooth partition of unity subordinate to \mathcal{U} .

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Proposition

If *M* is a smooth manifold and $\{U_i\}$ is an open covering of *M* then there is an open covering $\{V_i\}$ such that, $\forall i \ , \overline{V_i} \subset U_i$.

Definition

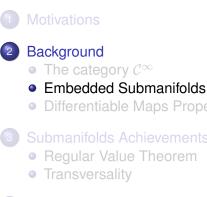
If *M* is a smooth manifold, $A \subset M$ is a closed subset, and *U* an open subset of *M* containing *A*, a continuous function $\psi : M \to [0, 1]$ is called a *bump function* for *A* supported in *U* if $\psi = 1$ on *A* and supp $\psi \subset U$

Theorem (Existence of Bump Functions)

Let M be a smooth manifold. For any closed subset $A \subset M$ and any open set U containing A, there exists a smooth bump function for A supported in U.

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Submanifolds

Definition

A subset V of \mathbb{R}^n is a submanifold of dimension k if each point of V belongs to the domain of a chart $\varphi : U \to \mathbb{R}^n$ of \mathbb{R}^n such that

$$V \cap U = \varphi^{-1}(H^k)$$

Definition

Let *M* be an *n*-manifold with or without boundary. A subset *S* of *M* is called an *embedded submanifold* of dimension *k* if each point $x \in S$ belongs to the domain of a chart $\varphi : U \to \mathbb{R}^n$ of *M* such that $\varphi(U \cap S)$ is a submanifold of \mathbb{R}^n of dimension *k* in the sens just defined.

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Definition

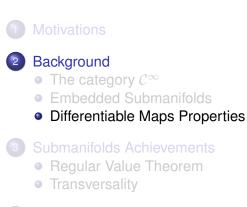
A submanifold *S* of a manifold *M* is said a *neat submanifold* if $\partial S = S \cap \partial M$.



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Embedding theorem

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Let *M* be a smooth manifold. We assume that the notion of tangent bundle $TM = \bigcup_{x \in M} T_x M$ is known. Let *M* and *N* be two smooth manifolds. Let $f : M \to N$ be a smooth map, we denote by $T_x f$ its derivative at *x*. We recall the following definitions:

Definition

The *rank* of *f* at *x* is the rank of the linear map $T_x f : T_x M \to T_{f(x)} N$. The map *f* is said to have constant rank if it has the same rank at every point $x \in M$.

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Theorem (Rank Theorem for Manifolds)

Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $f : M \to N$ be a smooth map of constant rank k. For each $x \in M$, there exist coordinate charts φ aroud x and ψ around f(x) so that the map $\psi f \varphi^{-1}$ is represented by

$$\begin{array}{cccc} \psi f \varphi^{-1} : & \mathbb{R}^m & \to & \mathbb{R}^n \\ & & (x_1, \dots, x_m) & \longmapsto & (x_1, \dots, x_k, 0, \dots, 0) \end{array}$$

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Definition

- The map *f* is said to be a *submersion* at a point $x \in M$ if the linear map $T_x f$ is surjective, i.e. if rank $T_x f = \dim N$.
- 2 The map *f* is said to be an *immersion* at a point $x \in M$ if the linear map $T_x f$ is injective, i.e. if rank $T_x f = \dim M$.
- ③ The map *f* is said to be an *embedding* if it is an immersion at each point *x* ∈ *M* that is also a homeomorphism onto its image f(M).

Examples -

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Remark -A topological embedding which is smooth is not necesserally a smooth embedding.

Example -

Proposition

Let $f: M \longrightarrow N$ be an injective immersion. If either of the following conditions holds, then f is an embedding with closed image:

(i) M is compact.

(ii) f is a proper map.

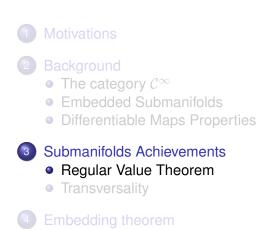
Example -the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding.

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Regular Value Theorem Transversality

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Regular Value Theorem Transversality

Definition

Let $f : M \to N$ be a smooth map. A point $x \in M$ is said to be a *regular point* of *f* if $T_x f$ is surjective. It is a critical point otherwise.

A point $y \in N$ is said to be a *regular value* of f if the set $f^{-1}(y)$ consists of regular points or if $f^{-1}(y)$ is an emptyset. It is a *critical value* otherwise.

Theorem

Let M and N be a manifolds with boundary. Let $f : M \to N$ be a smooth map. If $y \in N \setminus \partial N$ is a regular value for both f and $f_{|\partial M}$, then $f^{-1}(y)$ is a neat submanifold of M.

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Regular Value Theorem Transversality

Theorem (Sard's theorem.)

If $f: M \longrightarrow N$ is any smooth map, the set of critical values of f has measure zero in N.

Remark -: The set of regular values of a differential map $f: M \to N$ is dense in *N*.

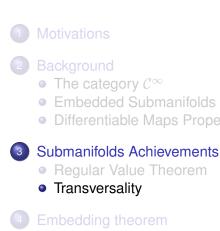
Example -:

- O(n).
- 2 Each knot bounds a surface in S^3 .

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Regular Value Theorem Transversality

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Regular Value Theorem Transversality

Definition

Let *M* and *N* be two differential manifolds.Let *S* be a *k*-dimensional submanifold of *N*. A differentiable map $f: M \longrightarrow N$ is called transverse to *S* if

$$T_x f(T_x M) + T_{f(x)} S = T_{f(x)} N, \quad \forall x \in f^{-1}(S).$$

Theorem

Let B be a submanifold of N and $f : M \longrightarrow N$ is a smooth map. Suppose $\partial B = \emptyset$ and f, $f_{|\partial M}$ are both transverse to B. Then $f^{-1}(B)$ is a submanifold with boundary $f^{-1}(B) \cap \partial M$.

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Regular Value Theorem Transversality

Proof.

Show that $f^{-1}(S)$ is locally a submanifold. Let $x \in f^{-1}(S)$, there exists a chart (V, ψ) of N at y = f(x) such that $V \cap S$ is a submanifold of N and $\psi(V \cap S) = W \cap \mathbb{R}^{n-k}$ where W is an open subset of \mathbb{R}^n containing 0 and $\mathbb{R}^{n-k} = \mathbb{R}^{n-k} \times \{\mathbf{0}\}.$ Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection on the last coordinates. Then $\pi \circ \psi(V \cap S) = 0$. Now if $U = f^{-1}(V) \cap f^{-1}(S)$, and $g = \pi \circ \psi \circ f$, then $U = g^{-1}(0)$. Since $dg_x = d(\pi \circ \psi)_V \circ df_X$ and $d(\pi \circ \psi)_V$ is surjective, if $v \in \mathbb{R}^k$, then there exists $u \in T_v N$ such that $d(\pi \circ \psi)_v(u) = v$. Since $f \oplus Z$, $u = d_x f(w_1) + w_2$ where $w_1 \in T_x M$ and $w_2 \in T_y Z$. Hence $d(\pi \circ \psi)_{\nu}(u) = d(\pi \circ \psi)_{\nu} \circ df_{\kappa}(u) + 0$ because $T_{\nu}Z = T_{\nu}(Z \cap U) = \ker d(\pi \circ \psi)_{\nu}.$

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Regular Value Theorem Transversality

Corollary

Let A and B be two submanifolds of a manifold M such that

$$T_x A + T_x B = T_x M, \ \forall x \in A \cap B.$$

Then $A \cap B$ is a submanifold of M such that $codimA \cap B = codimA + codimB$. One says that A and B are transverse to each other.

Proof.

apply the last theorem to the inclusion map $i_A : A \hookrightarrow M$.

Example -

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The major task of this section is to show the following theorem

Theorem (Whitney)

Let M be a compact n-manifold. Then there is an embedding of M in \mathbb{R}^{2n+1} .

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Theorem

Let M be a compact n-manifold. Then there is an embedding of M in $\mathbb{R}^{m(n+1)}$ for some positive integer m.

Proof -We will start by proving the theorem in the special case of a compact n-manifold M. Then M has a finite set of charts. We denote

$$B^n(0,r) = \{x \in \mathbb{R}^n \mid |x| < r\}$$

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Lemma

Let M be a smooth manifold. Every open cover (U_i) of M has a regular refinement (W_i) , i.e.

- (i) the cover (W_i) is countable and locally finite.
- (ii) Each W_i is open and is the domain of a smooth coordinate map $\varphi_i : W_i \to \mathbb{R}^n$ whose image is $B^n(0,3) \subset \mathbb{R}^n$.
- (iii) The collection (U_i) still covers M, where $U_i = \varphi_i^{-1}(B^n(0, 1))$

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Then, starting with an atlas of M, we associate it with a finite regular refinement as in the lemma: $(\varphi_i, U_i)_{1 \le i \le m}$. Let

$$\lambda: \mathbb{R}^n \to [0,1]$$

be a bump function for $\overline{B^n(0,1)}$ supported in $B^n(0,2)$. Define smooth maps

$$\lambda_i: M \rightarrow [0,1]$$

 $x \longmapsto \lambda_i(x) = \begin{cases} \lambda \circ \varphi_i & \text{on } U_i \\ 0 & \text{on } M \setminus U_i \end{cases}$

It follows that the closed sets $B_i = \lambda^{-1}(1) \subset U_i$ cover M.

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Define maps

$$\begin{array}{rcccccc} f_{i}: & \mathcal{M} & \to & \mathbb{R}^{n} \\ & x & \longmapsto & f_{i}(x) = \left\{ \begin{array}{ccccccc} \lambda_{i}(x)\varphi_{i}(x) & \text{if} & x \in U_{i} \\ & 0 & \text{if} & x \in \mathcal{M} \setminus U_{i} \end{array} \right. \end{array}$$

Put

$$g_i = (f_i, \lambda_i) : \mathbf{M} \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}.$$

and

$$g = (g_1, \ldots, g_m) : M \to \mathbb{R}^{m(n+1)}$$

We note that g is an injective immersion. Then g is an embedding.

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From the first step, it follows that we can consider *M* as a submanifold of some ℝ^p.



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- From the first step, it follows that we can consider *M* as a submanifold of some ℝ^p.
- If p ≤ 2n + 1, there is nothing more to prove: hence we assume p > 2n + 1.

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Second Step

- From the first step, it follows that we can consider *M* as a submanifold of some ℝ^p.
- If p ≤ 2n + 1, there is nothing more to prove: hence we assume p > 2n + 1.
- It is sufficient to prove that such an *M* embeds in ℝ^{p-1}. By repeating the argument, the manifold *M* will eventually embed in ℝ²ⁿ⁺¹.

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Suppose $M \subset \mathbb{R}^p$, with p > 2n + 1.

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Suppose $M \subset \mathbb{R}^p$, with p > 2n + 1. Identify \mathbb{R}^{p-1} with $\{(x_1, \ldots, x_p) | x_p = 0\}$.



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Suppose $M \subset \mathbb{R}^{p}$, with p > 2n + 1. Identify \mathbb{R}^{p-1} with $\{(x_1, \ldots, x_p) | x_p = 0\}$. If $v \in \mathbb{R}^{p} \setminus \mathbb{R}^{p-1}$, denote by $\pi_v : \mathbb{R}^{p} \to \mathbb{R}^{p-1}$ the projection parallel to v.

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Suppose $M \subset \mathbb{R}^{p}$, with p > 2n + 1. Identify \mathbb{R}^{p-1} with $\{(x_1, \ldots, x_p) | x_p = 0\}$. If $v \in \mathbb{R}^{p} \setminus \mathbb{R}^{p-1}$, denote by $\pi_v : \mathbb{R}^{p} \to \mathbb{R}^{p-1}$ the projection parallel to v.

Are there some vectors v for which $\pi_{v_{IM}}$ is an embedding?

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Suppose $M \subset \mathbb{R}^p$, with p > 2n + 1. Identify \mathbb{R}^{p-1} with $\{(x_1, \ldots, x_p) | x_p = 0\}$. If $v \in \mathbb{R}^p \setminus \mathbb{R}^{p-1}$, denote by $\pi_v : \mathbb{R}^p \to \mathbb{R}^{p-1}$ the projection parallel to v.

Are there some vectors v for which $\pi_{v_{|M}}$ is an embedding? Since M is compact, it suffices to find vectors v such that π_v is an injective immersion.

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Suppose $M \subset \mathbb{R}^p$, with p > 2n + 1. Identify \mathbb{R}^{p-1} with $\{(x_1, \ldots, x_p) | x_p = 0\}$. If $v \in \mathbb{R}^p \setminus \mathbb{R}^{p-1}$, denote by $\pi_v : \mathbb{R}^p \to \mathbb{R}^{p-1}$ the projection parallel to v.

Are there some vectors v for which $\pi_{v_{|M}}$ is an embedding? Since M is compact, it suffices to find vectors v such that π_v is an injective immersion.

We limit our search to unit vectors.

• For π_v to be injective, means that if x and y are in M, $x \neq y$, then

$$v \neq \frac{x - y}{|x - y|} \tag{1}$$

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• For π_v to be injective, means that if *x* and *y* are in *M*, $x \neq y$, then

$$v \neq \frac{x - y}{|x - y|} \tag{1}$$

• The requirement that π_v be an immersion: $\forall z \in TM, z \neq 0$,

$$v \neq \frac{z}{\mid z \mid} \tag{2}$$

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The condition 1 is studied by means of the map

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The condition 1 is studied by means of the map

$$\begin{array}{cccc} \sigma : & M \times M \setminus \Delta & \to & S^{p-1} \\ & (x,y) & \longmapsto & \sigma(x,y) = \frac{x-y}{|x-y|} \end{array}$$

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The condition 1 is studied by means of the map

Then *v* satisfies (1) iff *v* is not in the image of σ ! Note that dim $(M \times M \setminus \Delta) = 2n < \dim S^{p-1}$.

Lemma (Corollary of Sard Theorem)

Let $g : A \to B$ be a smooth map. If dim $B > \dim A$ then the complement of the image of g is dense in B.

By applying the lemma to σ , we get that every nonempty open subset of S^{p-1} contains v which is not in the image of σ .

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The condition 2 holds for all $z \in TM$ provided it holds whenever |z| = 1. Let

$$T_1M = \{z \in TM : |z| = 1\}$$

which is a compact submanifold of *TM* of dimension 2n - 1. To see that one can consider the map

and then notice that $T_1 M = \nu^{-1}(1)$ where 1 is a regular value of ν .

We then apply the last lemma to the restriction to $T_1 M$ of the projection of $M \times \mathbb{R}^p$ onto S^{p-1} .

$$au; T_1M \rightarrow S^{p-1}$$

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Theorem

Let *M* be an dimensional manifold which is compact. Then there is a neat embedding of *M* into H^{2n+1} .



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