Introduction to Differential Topology

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2. Trouver et étudier des invariants topologiques des variétés.
3. Riemann, Weyl, Whitney, Milnor.

Plan

Objects

Definition

Let A be a subset of $\mathbb{R}^n$. A map $f : A \to \mathbb{R}^k$ is said to be differentiable at $x \in A$ if there exist an open subset U of $\mathbb{R}^n$ and a differential map $F : U \to \mathbb{R}^k$ such that $F_{|U \cap A} = f$.

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}, \quad n \geq 1.$$

Definition

A smooth manifold of dimension n is a topological Haussdorff space M with a countable basis such that

1. each point x in M has an open neighborhood U which is homeomorphic to an open subset of the space H^n by a homeomorphism φ:
$$\varphi : U \to \varphi(U) \subset H^n.$$
2. if $(U_i)_{i \in I}$ is the set of such neighborhoods, each homeomorphism $\varphi_j \circ \varphi_i^{-1}$ is C^∞, where
$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_i).$$

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}, \quad n \geq 1.$$

$$\partial H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n = 0\}, \quad n \geq 1.$$

$$\partial M = \{x \in M | \exists \text{ a chart } (\varphi, U),\ x \in U,\ \varphi(x) \in \partial H^n\}.$$

$$\text{int}M = \{x \in M | \exists \text{ a chart } (\varphi, U),\ x \in U,\ \varphi(x) \in \text{int}H^n\}.$$

$$M = \partial M \sqcup \text{int}M.$$

Examples -:

1. D^n, $\partial D^n = S^{n-1}$.
2. $S^1 \times D^2$, $\partial(S^1 \times D^2) = S^1 \times S^1$.
3. The Moebius band M whose boundary is a circle.
4. The exterior X of a knot in S^3. $\partial X = S^1 \times S^1$.

Morphisms

Let M and N two smooth manifolds respectively of dimension m and n.
A map $f : M \to N$ is *smooth* at a point $x \in M$ if there exist chart (U, φ) of M at x and a chart (ψ, V) of N at $f(x)$ such that the composition $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from an open subset of H^m to a subset of H^n. The map is smooth if it is smooth at each point $x \in M$.

From local to global

Definition

Let M be a smooth manifold, and let $\mathcal{U} = (U_i)_{i\in I}$ be an arbitrary open cover of M. A *partition of unity subordinate to* $\mathcal{U}$ is a collection of smooth functions $\{\psi_i : M \to [0,1]\}_{i\in I}$ with the following properties:

(i) $\mathrm{supp}\ \psi_i \subset U_i$.

(ii) The set $\{\mathrm{supp}\ \psi_i\}_{i\in I}$ is locally finite.

(iii) $\sum_{i\in I} \psi_i(x) = 1$ for all $x \in M$.

Theorem

If M is a smooth manifold and $\mathcal{U} = (U_i)_{i\in I}$ is any open cover of M there exists a smooth partition of unity subordinate to $\mathcal{U}$.

Proposition

If M is a smooth manifold and $\{U_i\}$ is an open covering of M then there is an open covering $\{V_i\}$ such that, $\forall i$, $\overline{V_i} \subset U_i$.

Definition

If M is a smooth manifold, $A \subset M$ is a closed subset, and U an open subset of M containing A, a continuous function $\psi : M \to [0, 1]$ is called a *bump function* for A supported in U if $\psi = 1$ on A and $\mathrm{supp}\, \psi \subset U$

Theorem (Existence of Bump Functions)

Let M be a smooth manifold. For any closed subset $A \subset M$ and any open set U containing A, there exists a smooth bump function for A supported in U.

Plan

Submanifolds

Definition

A subset V of $\mathbb{R}^n$ is a submanifold of dimension k if each point of V belongs to the domain of a chart $\varphi : U \to \mathbb{R}^n$ of $\mathbb{R}^n$ such that

$$V \cap U = \varphi^{-1}(H^k)$$

Definition

Let M be an n-manifold with or without boundary. A subset S of M is called an *embedded submanifold* of dimension k if each point $x \in S$ belongs to the domain of a chart $\varphi : U \to \mathbb{R}^n$ of M such that $\varphi(U \cap S)$ is a submanifold of $\mathbb{R}^n$ of dimension k in the sens just defined.

Definition

A submanifold S of a manifold M is said a *neat submanifold* if $\partial S = S \cap \partial M$.

Plan

Let M be a smooth manifold. We assume that the notion of tangent bundle $TM = \cup_{x \in M} T_x M$ is known.
Let M and N be two smooth manifolds. Let $f : M \to N$ be a smooth map, we denote by $T_x f$ its derivative at x.
We recall the following definitions:

Definition

The *rank* of f at x is the rank of the linear map $T_x f : T_x M \to T_{f(x)} N$. The map f is said to have constant rank if it has the same rank at every point $x \in M$.

Theorem (Rank Theorem for Manifolds)

Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $f : M \to N$ be a smooth map of constant rank k. For each $x \in M$, there exist coordinate charts φ aroud x and ψ around $f(x)$ so that the map $\psi f \varphi^{-1}$ is represented by

$$\begin{array}{rccc} \psi f \varphi^{-1} : & \mathbb{R}^m & \to & \mathbb{R}^n \\ & (x_1, \ldots, x_m) & \longmapsto & (x_1, \ldots, x_k, 0, \ldots, 0) \end{array}$$

Definition

1. The map f is said to be a *submersion* at a point $x \in M$ if the linear map $T_x f$ is surjective, i.e. if $\mathrm{rank}\ T_x f = \dim N$.
2. The map f is said to be an *immersion* at a point $x \in M$ if the linear map $T_x f$ is injective, i.e. if $\mathrm{rank}\ T_x f = \dim M$.
3. The map f is said to be an *embedding* if it is an immersion at each point $x \in M$ that is also a homeomorphism onto its image $f(M)$.

Examples -

Remark -A topological embedding which is smooth is not necesseraly a smooth embedding.

Example -

Proposition

Let $f : M \longrightarrow N$ be an injective immersion. If either of the following conditions holds, then f is an embedding with closed image:

(i) *M is compact.*

(ii) *f is a proper map.*

Example -the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding.

Plan

Definition

Let $f : M \to N$ be a smooth map. A point $x \in M$ is said to be a *regular point* of f if $T_x f$ is surjective. It is a critical point otherwise.
A point $y \in N$ is said to be a *regular value* of f if the set $f^{-1}(y)$ consists of regular points or if $f^{-1}(y)$ is an emptyset. It is a *critical value* otherwise.

Theorem

Let M and N be a manifolds with boundary. Let $f : M \to N$ be a smooth map. If $y \in N \setminus \partial N$ is a regular value for both f and $f_{|\partial M}$, then $f^{-1}(y)$ is a neat submanifold of M.

Theorem (Sard's theorem.)

If $f : M \longrightarrow N$ is any smooth map, the set of critical values of f has measure zero in N.

Remark -: The set of regular values of a differential map $f : M \to N$ is dense in N.

Example -:

1. $O(n)$.
2. Each knot bounds a surface in S^3.

Plan

Definition

Let M and N be two differential manifolds.Let S be a k-dimensional submanifold of N. A differentiable map $f : M \longrightarrow N$ is called transverse to S if

$$T_x f(T_x M) + T_{f(x)} S = T_{f(x)} N, \quad \forall x \in f^{-1}(S).$$

Theorem

Let B be a submanifold of N and $f : M \longrightarrow N$ is a smooth map. Suppose $\partial B = \emptyset$ and f, $f_{|\partial M}$ are both transverse to B. Then $f^{-1}(B)$ is a submanifold with boundary $f^{-1}(B) \cap \partial M$.

Proof.

Show that $f^{-1}(S)$ is locally a submanifold.
Let $x \in f^{-1}(S)$, there exists a chart (V, ψ) of N at $y = f(x)$ such that $V \cap S$ is a submanifold of N and $\psi(V \cap S) = W \cap \mathbb{R}^{n-k}$ where W is an open subset of $\mathbb{R}^n$ containing 0 and $\mathbb{R}^{n-k} = \mathbb{R}^{n-k} \times \{0\}$.
Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection on the last coordinates .
Then $\pi \circ \psi(V \cap S) = 0$. Now if $U = f^{-1}(V) \cap f^{-1}(S)$, and $g = \pi \circ \psi \circ f$, then $U = g^{-1}(0)$.
Since $dg_x = d(\pi \circ \psi)_y \circ df_x$ and $d(\pi \circ \psi)_y$ is surjective, if $v \in \mathbb{R}^k$, then there exists $u \in T_yN$ such that $d(\pi \circ \psi)_y(u) = v$.
Since $f \pitchfork Z$, $u = d_xf(w_1) + w_2$ where $w_1 \in T_xM$ and $w_2 \in T_yZ$.
Hence $d(\pi \circ \psi)_y(u) = d(\pi \circ \psi)_y \circ df_x(u) + 0$ because $T_yZ = T_y(Z \cap U) = \ker d(\pi \circ \psi)_y$. □

Corollary

Let A and B be two submanifolds of a manifold M such that

$$T_xA + T_xB = T_xM, \ \ \forall x \in A \cap B.$$

Then $A \cap B$ is a submanifold of M such that $codim A \cap B = codim A + codim B$. One says that A and B are transverse to each other.

Proof.

apply the last theorem to the inclusion map $i_A : A \hookrightarrow M$. □

Example -

The major task of this section is to show the following theorem

Theorem (Whitney)

Let M be a compact n-manifold. Then there is an embedding of M in $\mathbb{R}^{2n+1}$.

First step

Theorem

Let M be a compact n-manifold. Then there is an embedding of M in $\mathbb{R}^{m(n+1)}$ for some positive integer m.

Proof -We will start by proving the theorem in the special case of a compact n-manifold M. Then M has a finite set of charts. We denote

$$B^n(0,r) = \{x \in \mathbb{R}^n \mid |x| < r\}$$

Lemma

Let M be a smooth manifold. Every open cover (U_i) of M has a regular refinement (W_i), i.e.

(i) *the cover (W_i) is countable and locally finite.*

(ii) *Each W_i is open and is the domain of a smooth coordinate map $\varphi_i : W_i \to \mathbb{R}^n$ whose image is $B^n(0,3) \subset \mathbb{R}^n$.*

(iii) *The collection (U_i) still covers M, where $U_i = \varphi_i^{-1}(B^n(0,1))$*

Then, starting with an atlas of M, we associate it with a finite regular refinement as in the lemma: $(\varphi_i, U_i)_{1 \leq i \leq m}$.
Let

$$\lambda : \mathbb{R}^n \to [0, 1]$$

be a bump function for $\overline{B^n(0, 1)}$ supported in $B^n(0, 2)$.
Define smooth maps

$$\begin{array}{rccc} \lambda_i : & M & \to & [0, 1] \\ & x & \longmapsto & \lambda_i(x) = \left\{ \begin{array}{ccc} \lambda \circ \varphi_i & \text{on} & U_i \\ 0 & \text{on} & M \setminus U_i \end{array} \right. \end{array}$$

It follows that the closed sets $B_i = \lambda^{-1}(1) \subset U_i$ cover M.

Define maps

$$\begin{array}{rcl} f_i : \quad M & \to & \mathbb{R}^n \\ x & \longmapsto & f_i(x) = \begin{cases} \lambda_i(x)\varphi_i(x) & \text{if} \quad x \in U_i \\ 0 & \text{if} \quad x \in M \setminus U_i \end{cases} \end{array}$$

Put

$$g_i = (f_i, \lambda_i) : M \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}.$$

and

$$g = (g_1, \ldots, g_m) : M \to \mathbb{R}^{m(n+1)}.$$

We note that g is an injective immersion. Then g is an embedding.

Second Step

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- If $p \leq 2n + 1$, there is nothing more to prove: hence we assume $p > 2n + 1$.
- It is sufficient to prove that such an M embeds in $\mathbb{R}^{p-1}$. By repeating the argument, the manifold M will eventually embed in $\mathbb{R}^{2n+1}$.

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Are there some vectors v for which $\pi_{v_{|M}}$ is an embedding?

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Are there some vectors v for which $\pi_{v_{|M}}$ is an embedding?
Since M is compact, it suffices to find vectors v such that π_v is an injective immersion.
We limit our search to unit vectors.

- For π_v to be injective, means that if x and y are in M, $x \neq y$, then

$$v \neq \frac{x-y}{\mid x-y \mid} \tag{1}$$

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- The requirement that π_v be an immersion: $\forall z \in TM$, $z \neq 0$,

$$v \neq \frac{z}{\mid z \mid} \tag{2}$$

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$$\begin{array}{rccc} \sigma: & M \times M \setminus \Delta & \to & S^{p-1} \\ & (x,y) & \longmapsto & \sigma(x,y) = \frac{x-y}{|x-y|} \end{array}$$

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Then v satisfies (1) iff v is not in the image of σ!
Note that $\dim(M \times M \setminus \Delta) = 2n < \dim S^{p-1}$.

Lemma (Corollary of Sard Theorem)

Let $g : A \to B$ be a smooth map. If $\dim B > \dim A$ then the complement of the image of g is dense in B.

By applying the lemma to σ, we get that every nonempty open subset of S^{p-1} contains v which is not in the image of σ.

The condition 2 holds for all $z \in TM$ provided it holds whenever $|z| = 1$. Let

$$T_1 M = \{z \in TM : |z| = 1\}$$

which is a compact submanifold of TM of dimension $2n - 1$. To see that one can consider the map

$$\begin{array}{rccc} \nu : & TM & \to & \mathbb{R} \\ & z & \longmapsto & |z|^2 \end{array}$$

and then notice that $T_1 M = \nu^{-1}(1)$ where 1 is a regular value of ν.

We then apply the last lemma to the restriction to $T_1 M$ of the projection of $M \times \mathbb{R}^p$ onto S^{p-1}.

$$\tau ; \quad T_1 M \quad \to \quad S^{p-1}$$

Theorem

Let M be an dimensional manifold which is compact. Then there is a neat embedding of M into H^{2n+1}.