Some Algebraic Structures and Links

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Plan

Definition

A link L with n components is a submanifold of $\mathbb{R}^3$ or S^3 which is homeomorphic to the disjoint union of n circles $S^1 \sqcup ... \sqcup S^1$. If $n = 1$, we call L a knot and denote it by K.

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Notation: We denote by $L = K_1 \sqcup ... \sqcup K_n$ a link with n components.

$$S^1 \sqcup ... \sqcup S^1 \hookrightarrow \mathbb{R}^3 \text{ or } S^3$$

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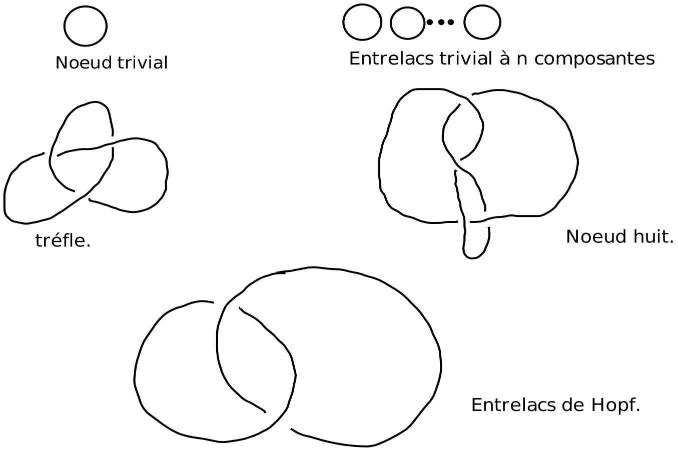

Figure : Exemples d'entrelacs.

Isotopy equivalence

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Two knots K_1 and K_2 are **ambient isotopic** if there exists a family of diffeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0, 1]$ such that $h_0 = id_{\mathbb{R}^3}$ and $h_1(K_1) = K_2$. If they are so, we say that K_1 and K_2 are equivalent.

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Proposition

Two knots K_1 and K_2 are equivalent if and only if there exists a preserving orientation diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = h(K_2)$.

Example -:

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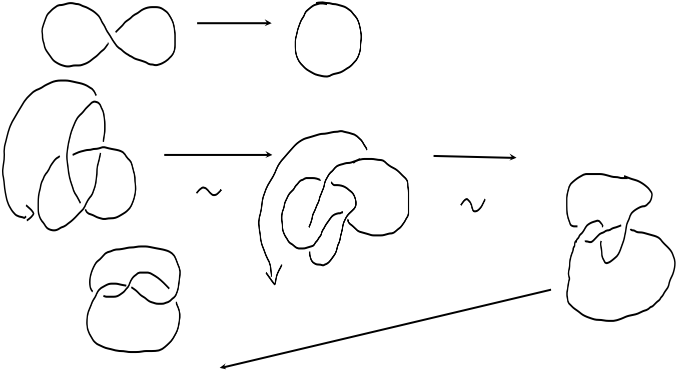

Figure : Nœuds isotopes.

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Definition

Let K be a knot. Let $\mathcal{P}$ be a plane in the space. Let p be a perpendicular projection on $\mathcal{P}$. We say that $p(K)$ is a **regular projection** of K on $\mathcal{P}$ if it satisfies

1. the tangent lines to the knot at all points are projected onto lines on the plane. (i.e. the projections of the tangents never degenerate into points);
2. No more than two distinct points of the knot are projected on one and the same point of the plane;
3. The set of **crossing points** (those on which two points project) is finite and at each crossing point the projections of the two tangents do not coincide.

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Definition

A diagram D of a knot K is its image by a regular projection

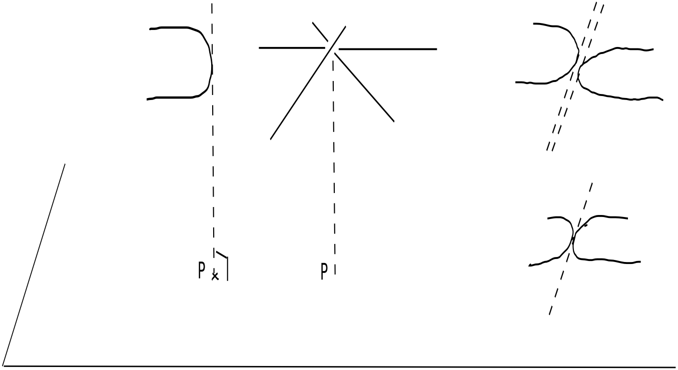

Figure : Forbidden projections.

Plan

Definition

Two diagrams D_1 and D_2 of a knot K are said equivalent if we can obtain D_1 from D_2 by a finite sequence of

1. ambient plane isotopies and
2. Ω_1-moves, Ω_2-moves and Ω_3-moves.

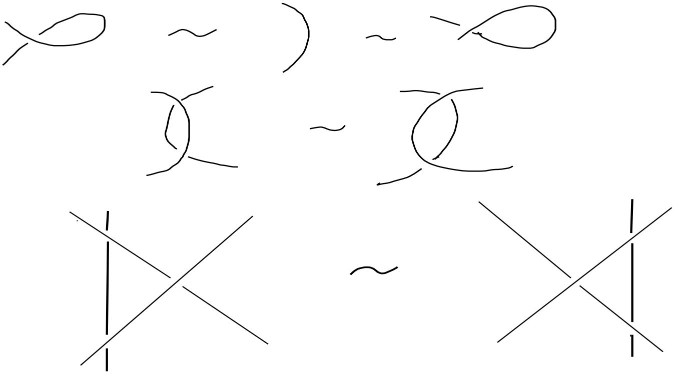

Figure : Reidemeister moves.

Theorem

Two knot diagrams correspond to isotopic knots if and only if one can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies.

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Corollary

There is a one-to-one correspondence between the equivalence classes of knots and the equivalence classes of diagrams.

Some families of links:

1. Alternating knots.

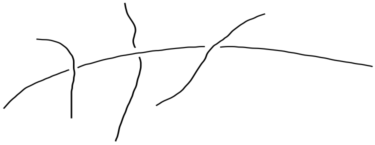

Figure : Alternating links.

2. Torus knot $T_{(p,q)}$ where p and q are coprime integers.

The crossing and the unknotting numbers

1. The crossing number $c(L)$.
   - In fact for each positive integer $c \leq 3$ there exists an (irreductible) knot such that $c(K) = c$.
   - $c(T(p,q)) = (p-1)q$.
2. The unknotting number $u(L)$.
   - $u(T(p,q)) = \frac{1}{2}(p-1)(q-1)$.
3. The 3-coloring number.

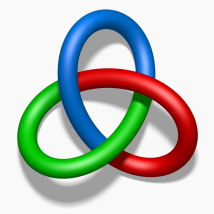

Figure : Colored trefoil

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It turns out that hidden in the simplicity of the tricoloring rules is a new kind of algebra, a powerful algebraic structure which ultimately gives us a complete invariant of knots. Start at the beginning.

Definition

Let X be a set. An operation

$$\begin{aligned} \triangleright : \quad X \times X &\longrightarrow X \\ (x, y) &\longmapsto x \triangleright y \end{aligned}$$

is a *Kei* operation if it satisfies the following three axioms:

(i) $\forall x \in X$, $x \triangleright x = x$ (idempotency).

(ii) $\forall x,\ y \in X$, $(x \triangleright y) \triangleright y) = x$ (involution).

(iii) $\forall x,\ y,\ z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (self-distributivity).

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- ***Takasaki Kei***: let $X = \mathbb{Z}_n$ and $x \triangleright y = 2y - x \pmod{n}$ **??**. It is also sometimes called *cyclic kei* or *dihedral quandle*.

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$\triangleright$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

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  $x, y \in S^2$, connect x to y by the a geodesic of least distance which is unique unless the two points are *antipodes* x and $-x$.
  Define $x \triangleright y$ as the result of finding the geodesic connecting x to y and then going from x to y and then past y along the geodesic by the same distance. In terms of unit vectors we have

$$x \triangleright y = 2(x \cdot y)y - x.$$

Remark -: We can represent a kei operation on a set $X = \{x_1, \ldots, x_n\}$ with n elements with an $n \times n$ matrix M_X which encodes the operation table by dropping the "x"s like in the following example where $n = 3$:

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$\triangleright$	x_1	x_2	x_3
x_1	x_1	x_3	x_2
x_2	x_3	x_2	x_1
x_3	x_2	x_1	x_3

$$\longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = M_X$$

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The set $W_{\mathcal{K}}$ of *kei words* in X is defined recursively by the rules that

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$$x \triangleright x = x,\ \ (x \triangleright y) \triangleright y = x,\ \ (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z),\ \ \forall x, y \in W_{\mathcal{K}}.$$

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Example -: the trefoil knot has fundamental kei presentation **??**:

$$\mathcal{K}(K) = \langle x, y, z \mid x \triangleright y = z,\ y \triangleright z = x,\ z \triangleright x = y$$

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2. Indeed, we can represent the fundamental kei of a knot with a *presentation matrix*.

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Let X and Y be keis. A **kei homomorphism** is a functio $f : X \to Y$ satisfying

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Example -: $X = \mathbb{Z}$, $x \triangleright y = 2y - x$. Let $l \in \mathbb{Z}$

$$\begin{array}{rccc} f: & X & \longrightarrow & X \\ & x & \longmapsto & lx \end{array}$$

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Example -: If $f : X \to Y$ is a kei homomorphism, then *Im f* is a subkei of Y.

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Example -: The eight knot n-coloring when $n = 3$ and $n = 5$.

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Let $(X, \triangleright)$ be kei. Let K be a knot or link.

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Remark -: The cardinality of the set $Hom(\mathcal{K}(K), X)$ is a computable invariant known as the *kei counting invariant*.

Examples -:

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- L is the Hopf link and $X = \mathbb{Z}_4$. There are two relations R_1 and R_2 are $x \triangleright y = x$ and $y \triangleright x = y$. Then we have

$\triangleright$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

$f(x)$	$f(y)$	R_1?	R_2?	$f(x)$	$f(y)$	R_1?	R_2?
0	0	+	+	2	0	+	+
0	1			2	1		
0	2	+	+	2	2	+	+
0	3			2	3		
1	0			3	0		
1	1	+	+	3	1	+	+
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- Thus, we get $|Hom(\mathcal{K}(K), \mathbb{Z}_4)| = 8$.

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  Thus, we get $|Hom(\mathcal{K}(K), \mathbb{Z}_4)| = 8$.
- The kei counting number for two unlinked circles is $|Hom(\mathcal{K}(K), \mathbb{Z}_4)| = 16$.

Definition

A **quandle** is a set X with a binary operation

$$\begin{array}{rccc} \triangleright : & X \times X & \longrightarrow & X \\ & (x, y) & \longmapsto & x \triangleright y \end{array}$$

satisfying:

(i) $\forall x \in X$, $x \triangleright x = x$ (idempotency).

(ii) $\forall y \in X$, the map $\beta_y : X \to X$ defined by $\beta_y(x) = x \triangleright y$ is invertible.

(iii) $\forall x, \ y, \ z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (self-distributivity).

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We write $x \triangleright^{-1} y$ for $\beta_y^{-1}(x)$.

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3. As with kei, we can understand the quandle axioms in terms of knot diagrams (fig).

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$$x \triangleright y = y^{-1}xy.$$
4. Alexander Quandle: for any module M and an invertible linear transformation $t : M \to M$ of M, define a quandle structure on M by
$$x \triangleright y = tx + (1 - t)y.$$

Plan

1. Fundamental quandle: as for keis, there is a **fundamental quandle** sometimes called *knot quandle*, associated to an oriented knot or link given by a presentation with genarators corresponding to arcs and quandle relations at crossings denoted by $\mathcal{Q}(K)$.

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   Remark -: the third quandle axiom expresses that the right translations β_y are quandle homomorphisms.
3. As with kei, given a finite quandle X we have a counting invariant $\Phi_X^{\mathbb{Z}}$.

Example of a counting invariant

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Consider the knot 8_{19} and the quandle $X = \mathbb{Z}_7[t, t^{-1}]/(t-3)$, i.e; $\mathbb{Z}_7$ with the quandle operation

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X has the operation table

$\triangleright$	0	1	2	3	4	5	6
0	0	5	3	1	6	4	2
1	3	1	6	4	2	0	5
2	6	4	2	0	5	3	1
3	2	0	5	3	1	6	4
4	5	3	1	6	4	2	0
5	1	6	4	2	0	5	3
6	4	2	0	5	3	1	6

$$\begin{bmatrix} 3 & 6 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 3 & 6 & 0 & 5 & 0 & 0 & 0 \\ 0 & 5 & 3 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 & 3 & 6 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 3 & 6 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 3 & 6 \\ 6 & 0 & 0 & 0 & 5 & 0 & 0 & 3 \end{bmatrix}$$

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The resolution of the system gives the counting invariant

$$|Hom(\mathcal{Q}(8_{19}), X)| = 7^2 = 49.$$

Plan

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the quandle operation $x \triangleright y$ in the fundamental quandle is then given by the figure 8

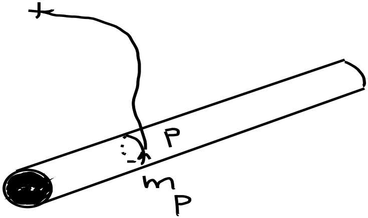

Figure : A path in the fundamental quandle of K.

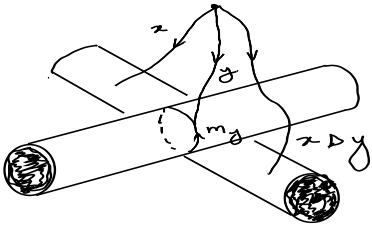

Figure : A path in the fundamental quandle of K.

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The subgroup of $Aut(X)$ generated by the permutations β_x, is called the *inner* automorphism group of X and is denoted by $\mathrm{Inn}(X)$.
Quandle axiom (iii) implies that the map $\beta : X \to \mathrm{Inn}(X)$ sending u to β_u satisfies the equation

$$\beta_z \beta_y = \beta_{y \triangleright z} \beta_z$$

for all $y, z \in X$, which can be rewritten as

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Thus if the group $\mathrm{Inn}(X)$ is considered as a quandle with conjugation then the map β becomes a quandle homomorphism.

Transvection group

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Proposition

1. *$Transv(X) \leq \mathrm{Inn}(X)$ and $\mathrm{Inn}(X) \leq Aut(X)$.*
2. *The quotient group $\mathrm{Inn}(X)/Transv(X)$ is a cyclic group.*

Definition

Let X be a quandle. The *orbit* of an element $x \in X$ is the the subset of elements $y \in X$ which are images of x by an inner automorphism

$$orb(x) = \{y \in X | \exists f \in \mathrm{Inn}(X),\ y = f(x)\}.$$

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- A quandle is **connected** if it has a single orbit.
- A quandle X is **medial** if for all $a, b, c, d \in X$ we have

$$(a \triangleright b) \triangleright (c \triangleright d) = (a \triangleright c) \triangleright (b \triangleright d).$$

It turns out that a quandle is medial if and only if its transvection group is abelian; thus, medial quandles are also called **abelian**. For example, Alexander quandles are medial.

- A quandle is **faithful** if the mapping

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- A quandle X is called **simple** if the only surjective quandle homomorphisms $f : X \to Y$ have trivial images or bijective.

Plan

Definition

A framed knot is a knot K in S^3 with a parallel curve J on the boundary of a tubular neighborhood N of K which wraps around the torus p times meridionaly and once longitudinaly: $[J] = p[m] + [l]$ where m and l are respectively the meridian and theprefered parallel of ∂N.
The integer p is called the framing of the knot K.

Definition

Two framed knot K with the curve J and K' with the curve J' are isotopic if there is a homeomorphism of S^3 preserving the ambient orientation and which sends K on K' and the curve J on the curve J'.

The Reidemeister moves for framed knots

Plan

It turns out that the framed type I move imposes no conditions on the algenraic structure at all. Thus, we have a new definition

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Definition

A **rack** is a set X with two binary operations $\triangleleft, \triangleright^{-1} X \times X :\longrightarrow X$ satisfying

1. $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$ and
2. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

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Going through a kink is a bijective map

$$\begin{array}{cccc} \pi & X & \longrightarrow & X \\ & x & \longmapsto & \pi(x) = x \triangleright x \end{array}$$

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with inverse $\pi^{-1}(x) = x \triangleright^{-1} x$.
We call π the *kink map*.

Remark -: Even if $\pi(x) \neq x$, if we go through enough kinks we eventually run out of new labels.

Definition

Let X be a rack. For any $x \in X$, the **rank** of x is the smallest positive integer n such that $\pi^n(x) = x$.
The least common multiple of all such n for all elements of X is called the **rack characteristic** or **rack rank** of X.

Example -: Every quandle is a rack. We can define a quandle as a rack in which the kink map is the identity, or as a rack of characteristic 1.

Plan

As with quandles, each framed oriented knot K has a *fundamental rack* $\mathcal{R}(K)$ which we can define topologically or combinatorially.

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the quandle operation $x \triangleright y$ in the fundamental quandle is then given by the figure as in 8, in which we add the framing curve and ask that the homtopies move the points along the framing curve and fix the basepoint.