Research Issues in Geometric Topology

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Plan

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- Geometric Topology as an area may be said to have originated in the 1935 classification of lens spaces by Reidemeister torsion.
- The use of the term Geometric Topology to describe these seems to have originated rather recently.
- Prototypical problem: Poincaré Conjecture.

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- The 1960s and 1970s saw long strides taken in the analysis of the high-dimensional manifolds, including Smale's proof of h-cobordism theorem . . . .
- In the second portion of the 20th century came such results as: the analysis of 4-manifolds, powerfully stoked by the work of Donaldson and Freedman; a variety of results on 3-manifolds and classical knot theory emerging from new invariants as the Jones polynomial; and the emergence of an algebraic-geometric-topological hybrid known as *geometric group theory*.

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Remarque : The distinction is because **surgery theory** works in dimension 5 and above. It works topologically in dimension 4. It doesn't work in dimension 3 and below.

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- **PL**: true in dimensions other than 4; unknown in dimension 4 where it is equivalent to **Diff**.
- **Diff**: false generally, true in some dimensions including , 1, 2, 3, 5 and 6. First known counterexample is in dimension 7. The case of dimension 4 is unsettled.

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- is the same in dimension 3 and below;
- in dimension 4 **PL** and **Diff** agree, but **Top** differs.
- In dimension above 6 they all differ.
- In dimensions 5 and 6 avery **PL** manifold admits an infinitely differential structure that is so-called Whitehead compatible.

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   - Geometric Topology
   - Differences between low and high-dimensional Topology
   - Poincaré Conjecture: status
   - Branches of Geometric Topology
2. Basic Knot Theory
   - Comparison criteria
   - Links up to isotopy
3. The coloring process
   - Rack Colorings of Framed Knots
   - Leibniz algebras and Lie racks
4. Quasi-alternating links
5. Link homotopy

Definition

A link L with n components is a submanifold of $\mathbb{R}^3$ or S^3 which is homeomorphic to the disjoint union of n circles $S^1 \sqcup ... \sqcup S^1$. If $n = 1$, we call L a knot and denote it by K.

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We consider links in S^3. (classical theory)

Exemples :

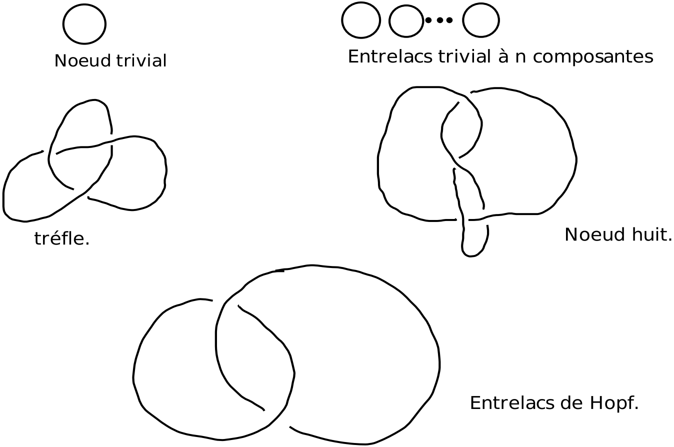

Figure: Some links.

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- One of the main tasks of knot theory is to find efficient computable invariants.

Isotopy equivalence

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Definition

Two links L_1 and L_2 are **ambient isotopic** if there exists a family of diffeomorphisms $h_t : S^3 \to S^3$, $t \in [0, 1]$ such that $h_0 = id_{S^3}$ and $h_1(L_1) = L_2$. If they are so, we say that L_1 and L_2 are equivalent.

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- Diagrams
- Reidemeister Theorem

Theorem

Two links are isotopic if and only if they have two diagrams one of which can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies.

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Theorem

Tri-colorability is a knot invariant.

Definition

A **rack** is a set X with a binary operation

$$\begin{aligned} \triangleright : \quad X \times X &\longrightarrow X \\ (x, y) &\longmapsto x \triangleright y \end{aligned}$$

satisfying:

(ii) $\forall y \in X$, the map $\beta_y : X \to X$ defined by $\beta_y(x) = x \triangleright y$ is invertible.

(iii) $\forall x,\ y,\ z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (self-distributivity).

Exemples :

1. Dihedral racks: $Q = \mathbb{Z}_p$, and for $x, y \in Q$

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3. Lie racks: that is a rack Q which is a manifold such that the binary operation is smooth.

About Lie racks

A pointed rack Q is a rack with a distinguished element 1 such that

$$R_1 = id_Q \text{ and } R_x(1) = 1, \forall x \in Q.$$

or

$$\forall x \in Q, \; x \triangleright 1 = x \text{ and } 1 \triangleright x = 1$$

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A **Lie rack** is a manifold Q with a pointed rack structure such that

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- $\forall y \in X$, the map $\beta_y : X \to X$ is a diffeomorphism.

Leibniz algebras

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Definition

A right (left) Leibniz algebra is an algebra $(\mathcal{L}, \cdot)$ such that for any $u \in \mathcal{L}$, the right (left) multiplication R_u (L_u)is a derivation, that is, for any $v, w \in \mathcal{L}$

$$(v \cdot w) \cdot u = (v \cdot u) \cdot w + v \cdot (w \cdot u)$$

respectively

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w + v \cdot (u \cdot w).$$

An algebra $(\mathcal{L}, \cdot)$ which is both right and left Leibniz algebra is called a **symmetric** Leibniz algebra.

Let $(\mathcal{L}, \cdot)$ be an algebra. We define the products

$$[u, v] = \frac{1}{2}(u.v - v.u)$$

and

$$u \circ v = \frac{1}{2}(u.v + v.u).$$

then

$$u.v = [u, v] + u \circ v.$$

Proposition

Let $(\mathcal{L}, \cdot)$ be an algebra. The following assertions are equivalent:

1. *$(\mathcal{L}, \cdot)$ is a Leibniz algebra,*
2. (i) *$(\mathcal{L}, [,])$ is a Lie algebra,*
   (ii) *For any u and v in $\mathcal{L}$, $u \circ v$ is in the center of $(\mathcal{L}, [,])$,*
   (iii) *For any u, v and w in $\mathcal{L}$, $([u, v]) \circ w = (u \circ v) \circ w = 0$*

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Let $(\mathcal{L}, \cdot)$ be a symmetric Leibniz algebra. Let G be the connected simply connected Lie group whose Lie algebra is $(\mathcal{L}, [,])$.

Theorem

We can endow G with a binary operation $\triangleright$ such that $(G, \triangleright)$ is a Lie rack. Furthermore, theassociated right Leibniz algebra is equal to $(\mathcal{L}, \cdot)$

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link-homotopy