Cohomology of divergence forms and Proper Lie Algebra actions

Abdelhak Abouqateb

University Cadi Ayyad Marrakesh

Workshop on Differential Geometry and Nonassociative Algebras

CIRM - Marseille Luminy
12-15 November 2019

Initial motivation

Theorem (A.W. Wadsly, 1975, J. Diff. Geom)

Let $\mu : \mathbb{R} \times M \to M$ be a C^∞ action of $(\mathbb{R}, +)$ with every orbit a circle. Then there exists a C^∞ action $\rho : S^1 \times M \to M$ with the same orbits as μ if and only if there exists a Riemannian metric on M with respect to which the orbits of μ are embedded totally geodesic submanifolds of M.

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According to that, I asked the following question:

"Let $X_1, \cdots, X_p$ be a commuting vector fields on a compact manifold M. Does there exists cohomological obstructions to the existence of a C^∞ toral action $\mathbb{T}^p \times M \to M$ which is generated by $X_1, \cdots, X_p$?"

Example: The hyperbolic torus $\mathbb{T}^3_A$

Let $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $\mathrm{tr}(A) > 2$. Then we can write $A = PDP^{-1}$ for some $P \in \mathrm{GL}(2,\mathbb{R})$ and $D = \mathrm{diag}(\lambda, \lambda^{-1})$. Since by hypothesis $\lambda + \lambda^{-1} > 2$ then $\lambda > 0$ and $\lambda \neq 1$. Define $D^t = \mathrm{diag}(\lambda^t, \lambda^{-t})$ and put $A^t = PD^tP^{-1}$ for any $t \in \mathbb{R}$. This operation defines a Lie group homomorphism:

$$\rho : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{R}^2), \quad t \mapsto A^t$$

Then put:

$$\mathbb{T}^3_{\mathbf{A}} := \mathbb{R}^2 \rtimes_\rho \mathbb{R}/\mathbb{Z}^2 \rtimes_\rho \mathbb{Z}$$

We have a fiber bundle structure

$$\mathbb{T}^2 \hookrightarrow \mathbb{T}^{\mathbf{A}}_3 \to \mathbf{S}^1$$

and a parallelism on $\mathbb{T}^A_3$ is given by three vector fields X, Y and Z with brackets:

$$[X,Y] = 0,\ [X,Z] = X,\ [Y,Z] = -Y$$

Outline

1. Actions of Lie algebras on manifold
2. Proper actions and integrability
3. Averaging process
4. Invariant current on manifolds
5. Cohomology of divergence forms

Actions of Lie algebras on a manifold

Let $\mathcal{G}$ be a finite dimensional Lie algebra and let M be a smooth manifold.

- We say that $\mathcal{G}$ acts on M or that M is a $\mathcal{G}$-manifold if there is a Lie algebra homomorphism

$$\tau : \mathcal{G} \to \mathcal{V}(M)$$

- This is equivalent to the given of a finite family $\{X_1, \ldots, X_p\}$ of vector fields on M such that:

$$[X_i, X_j] = \sum_k C_{ij}^k X_k$$

where C_{ij}^k are the structure constants of $\mathcal{G}$ relatively to a basis.

Terminology

An action $\tau : \mathcal{G} \to \mathcal{V}(M)$ may have the following properties:

- It is called **free** if for each $x \in M$ the mapping $h \mapsto \tau(h)_x$ is injective.
- It is said to be of constant rank k if for each $x \in M$ the mapping $h \mapsto \tau(h)_x$ is of constant rank k.
- It is called **complete** if each fundamental vector field $\tau(h)$ is complete, i.e., it generates a global flow.

The orbits (Singular foliation)

If a Lie algebra $\mathcal{G}$ acts on a manifold M, then it spans an integrable distribution on M, which need not be of constant rank. So through each point x of M there is a unique maximal leaf of that distribution; we also call it the $\mathcal{G}$-orbit through that point. This is the connected embedded submanifold given by: $y \in \mathcal{G}(x)$ if and only if there exists $X_{i_1}, \ldots, X_{i_r} \in \tau(\mathcal{G})$ and $t_{i_1}, \ldots, t_{i_r} \in \mathbb{R}$ such that

$$y = \varphi_{t_{i_1}}^{X_{i_1}} \circ \cdots \circ \varphi_{t_{i_r}}^{X_{i_r}}(x)$$

where φ^X is the flow of the vector field X.

The space of orbits $M/\mathcal{G}$

Let M be a $\mathcal{G}$-manifold. Obviously if x and x' lie in the same orbit, then $\mathcal{G}(x) = \mathcal{G}(x')$. We will denote the set of orbits by $M/\mathcal{G}$. The action is said to be **transitive** if there is only one orbit $M = \mathcal{G}(x)$, then $M/\mathcal{G}$ contains only one point. In general, $M/\mathcal{G}$ contains many points. We will equip with $M/\mathcal{G}$ the quotient topology. This topology might be non-Hausdorff.

Invariant submanifolds

Let $\tau : \mathcal{G} \to \mathcal{V}(M)$ be an action. A submanifold $N \subset M$ is said to be $\mathcal{G}$-invariant if for each $x \in N$ the orbit $\mathcal{G}(x)$ is included in N, which is equivalent to say that any fundamental vector field $X \in \tau(\mathcal{G})$ is tangent to N. Hence we get a canonical induced action of $\mathcal{G}$ on N.

Remark

Any open subset $U \subset M$ can be endowed with an induced action $\tau_U : h \mapsto \tau(h)_U$, but this doesn't mean that the open set U is invariant by the initial action τ.

Lie group actions $\rightleftharpoons$ Lie algebra actions

Definition

An action of a Lie group G on a manifold M is a group homomorphism

$$\rho : G \to \mathrm{Diff}(M), \quad g \mapsto \rho(g)$$

such that the action map: $(g, x) \mapsto g \cdot x := \rho(g)(x)$ from $G \times M$ to M, is smooth.

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Proposition

Given an action of a Lie group $\rho : G \to \mathrm{Diff}(M)$, one obtains a complete Lie algebra action $\rho' : \mathcal{G} \to \mathcal{V}(M)$, where $\mathcal{G} = Lie(G)$ and the flow of the vector field $\rho'(h)$ is given by

$$\Phi^{\rho'(h)}(t, x) = \rho(\exp(-th))(x)$$

Integrability

Definition

Let $\tau : \mathcal{G} \to \mathcal{V}(M)$ be a Lie algebra action. We will say that this action integrates to an action of a Lie group G with Lie algebra $\mathcal{G}$ if there exists an action $\rho : G \to \mathrm{Diff}(M)$ such that $\rho' = \tau$. The action ρ will be said a *primitive* of τ.

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R. Palais' Integrability Theorem asserts that any complete Lie algebra action can be integrated to a Lie group action.

We will be called a **proper Lie algebra action** any action $\tau : \mathcal{G} \to \mathcal{V}(M)$ which can be integrated to a **proper Lie group action** $\rho : G \to \mathrm{Diff}(M)$.

Proper Lie group action

Definition

A G-action on a manifold M is called **proper action** if for all pairs of compact sets (C_1, C_2) in M the set

$$\{g \in G / gC_1 \cap C_2 \neq \emptyset\}$$

is compact.

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is compact.

- For instance, the action of a group on itself by left or right multiplication is proper.
- Actions of compact groups are always proper.
- Given a proper G-action, the induced action of a closed subgroup is also proper.
- The irrational flow on a 2-torus is a non-proper $\mathbb{R}$-action.
- Let G be a closed subgroup of a Lie group H and K a compact subgroup of H, then the homogeneous action of G on H/K is proper.

Properties of proper Lie algebra action

- Let A be a skew-symmetric 3×3 matrix, then the associated linear vector field A^* is tangent to the sphere S^2. This generates a proper action of the abelian one dimensional Lie algebra $\mathbb{R}$ on S^2.
- For any Riemannian manifold (M, g), the canonical action of the Lie algebra of Killing vector fields $\mathrm{Kill}_g(M)$ on M is proper.
- Let M be a proper $\mathcal{G}$-manifold, then the $\mathcal{G}$-orbits are closed in M and the space of orbits $M/\mathcal{G}$ is a Hausdorff topological space.

- Let $U := \mathsf{S}^+(n)$ be the space of $n \times n$ symmetric positive definite matrices (this is an open subset of the vector space $S(n)$ of symmetric matrices).

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$$\mathrm{GL}^+(n, \mathbb{R}) \times U \to U, \quad g.B := gBg^\top$$

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 This is a proper action. Hence the induced infinitesimal action:

$$\tau : M(n, \mathbb{R}) \to \mathcal{V}(U), \quad \tau(h)(B) := -hB - Bh^\top$$

is a proper Lie algebra action.

Criterion of $\mathcal{G}$-properness

Theorem

Let (M, g) be a comlete Riemannian manifold and let $\mathcal{G} \subset Kill_g(M)$ be a Lie subalgebra. We will denote by $Iso(M, g)$ the group of isometries of (M, g) and $T(\mathcal{G}) \subset Iso(M, g)$ the subgroup generated by the flows φ_t^X for $t \in \mathbb{R}$ and $X \in \mathcal{G}$. Then:
The action of $\mathcal{G}$ on M is proper if and only if there exists $x \in M$ such that the following two statements are satisfied:

1. *The $\mathcal{G}$-orbit $\mathcal{G}(x)$ is closed in M.*
2. *The isotropy group $T(\mathcal{G})_x$ is closed (hence compact) in $Iso(M, g)$.*

The proof of this theorem is based on a two previous results of Kulkarni and Palais.

G-vector bundles

Let $E \xrightarrow{\pi} M$ be a G-vector bundle (i.e we have a representation of G in $Aut(E)$ the group of the automorphisms of the bundle).
Example: The pth exterior of the cotangent bundle $\bigwedge^p T^*M \to M$ of a G-manifold.

• An induced action of G on the space of smooth sections with compact support $C_c^\infty(E)$ (and on $C^\infty(E)$) is defined by:

$$(g\sigma)(x) = g\sigma(g^{-1}x) \qquad \forall x \in M$$

Now if X_h is the fundamental vector field on M associated to $h \in \mathcal{G}$ and $\sigma \in C^\infty(E)$, the Lie derivative $L_{X_h}\sigma$ of a smooth section σ by X_h is given by

$$(L_{X_h}\,\sigma)(x) = \frac{d}{dt}\,|_{t=0}\ ((\exp th)\sigma)(x) \qquad \forall x \in M$$

Averaging process

Let $(E \stackrel{\pi}{\to} M)$ be a **proper G-vector bundle** (which means that the action of G on the base M is proper). For any section $\sigma \in C_c^\infty(E)$ and $x \in M$, the set $\{g \in G \ / \ g^{-1}x \in supp(\sigma)\}$ is compact outside which the function $g \longmapsto (g \cdot \sigma)(x)$ vanishes.

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$$\begin{array}{rccc} m: & C_c^\infty(E) & \longrightarrow & C^\infty(E) \\ & \sigma & \longmapsto & m\sigma \end{array}$$

given by the following formula:

$$(m\sigma)(x) = \int_G (g \cdot \sigma)(x) dg$$

Consider the group $G := \mathrm{GL}(n, \mathbb{R})$ which is a subset of $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. In view of this, we write $g = (g_{ij})$ for $g \in \mathrm{GL}(n, \mathbb{R})$. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, the restriction of the Lebesgue measure $\prod_{i,j=1}^{n} dg_{ij}$ to $\mathrm{GL}(n, \mathbb{R})$ provides us a Radon measure on $\mathrm{GL}(n, \mathbb{R})$. The Haar measure of the unimodular group $\mathrm{GL}(n, \mathbb{R})$ is given by:

$$dg := \frac{1}{|\det g|^n} \prod_{i,j=1}^{n} dg_{ij}$$

Hence for a proper $\mathrm{GL}(n, \mathbb{R})$-vector bundle, we obtain

$$(m\sigma)(x) = \int_{\mathrm{GL}(n,\mathbb{R})} \frac{1}{|\det g|^n} g.\sigma(g^{-1}.x) \prod_{i,j=1}^{n} dg_{ij}$$

The kernel and the image of the averaging operator

Let $E \to M$ be a G vector bundle. A smooth section σ will be called a G-**semi-invariant** if for every $g \in G$

$$g\sigma = \triangle_{G}(g)\sigma$$

We should point out that the support of such section is G-invariant. We will denote by $\overline{C}^{\infty}_{Gs}(E)$ the space of G-semi-invariant sections τ such that $supp(\tau)/G$ is compact.

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Theorem

- *Image*$(m) = \overline{C}^\infty_{Gs}(E)$.
- *The kernel* $\ker(m)$ *is the space spanned by* $L_X\tau$ *where* $\tau \in C^\infty_c(E)$ *and* X *is a fundamental vector field.*

Reference: A. Abouqateb, *Courants invariants par une action propre.* Manuscripta math. 98, 349-362 (1999).

The proof of the above theorem rely on a previous result of A. Haefliger (*Some remarks on foliations with minimal leaves.* J. of Diff. Geo. Vol. 15, 269-284 (1980)) and the following proposition:

Proposition (Existence of smooth Cutoff functions)

Let $G \times M \to$ be a proper action of a connected Lie group G. Then there exists a nonnegative function $\varphi \in C^\infty(M)$ on M satisfying:

- *For each compact $C \subset M$, the restriction $\varphi_{|_{GC}}$ of φ to GC has a compact support.*
- *For all $x \in M$,*

$$\int_G \varphi(g^{-1}x)\triangle_G(g)dg = 1.$$

Application from a global analysis approach

The space of sections τ with G-invariant support such that $supp(\tau)/G$ is compact will be denoted by $\overline{C}^{\infty}(E)$.

Let denote by $\mathcal{B}$ the family of closed subsets B of M such that B is G-invariant and B/G is compact. For any $B \in \mathcal{B}$ the space $C_B^{\infty}(E)$ of sections with support in B will be endowed with the topology C_B^{∞} (i.e. which incuced from the C^{∞} topology). This is a Frechet space (it is closed in $C^{\infty}(E)$).

We will endow the space

$$\overline{C}^{\infty}(E) = \bigcup_{B \in \mathcal{B}} C_B^{\infty}(E)$$

with the topology of strict inductive limit of the topologies C_B^{∞}; we obtain an L.F. space (this is a Hausdorff complete locally convexe topological vector space) : a sequence $(\tau_n)_n$ in $\overline{C}^{\infty}(E)$ convergs to τ if and only ifthere exists $B \in \mathcal{B}$ such that $supp(\tau_n) \in B$, $supp(\tau) \in B$ and $\tau_n \to \tau$ in the meaning of the C^{∞} topology.
Now the spaces $\overline{C}_{Gs}^{\infty}(E)$ and $\overline{C}_{G}^{\infty}(E)$ of sections τ respectively G-semi-invariant and G-invariant with $supp(\tau)/G$ compact, will be endowed with L.F. topology induced from $\overline{C}^{\infty}(E)$.

In what follows G will be a connected Lie group and $E \xrightarrow{\pi} M$ a G-proper vector bundle. Then the averaging operator $m : C_c^\infty(E) \to C^\infty(E)$ satisfies the following properties:

Proposition

As a mapping from $C_c^\infty(E)$ to $\overline{C}_{Gs}^\infty(E)$, the averaging operator m is continous and open.

Invariant Schwartz distributions

Let $E \overset{\pi}{\to} M$ be a vector bundle. A continous linear form $T : C_c^\infty(E) \to \mathbb{R}$ is called a **section-distribution** of the bundle $E \overset{\pi}{\to} M$. In the particular case of the pth cotangent bundle $\bigwedge^{n-p} T^*M$, we obtain the notion of a p-**current** (de Rham) on M or **distribution** on M (for $p = n = \dim M$).

Proposition

Let $G \times M \to$ be a proper action of a connected Lie group G.Then there exists a smooth function $\triangle_M : M \to \mathbb{R}_+^*$ which satisfies: $\triangle_M(g.x) = \triangle_G(g)\triangle_M(x)$.

Theorem

Let $E \xrightarrow{\pi} M$ be G-proper vector bundle with G a connected Lie group. Then the vector space $(C_c^\infty(E))'_G$ of G-invariant continous linear forms on the Schwartz topological vector space $C_c^\infty(E)$ is isomorphic to the topological dual of $\overline{C}_G^\infty(E)$.

Corollary

Let M be a G-proper manifold of dimension n. Then

1. *The space $C_G^p(M)$ of p-invariant currents is isomorphic to the topological dual $\overline{\Omega}_G^{n-p}(M)$.*
2. *If the orbit space M/G is compact, then the space $C_G^p(M)$ of p-invariant currents is isomorphic to the topological dual of $\Omega_G^{n-p}(M)$ the space of $(n-p)$-invariant forms endowed with the C^∞-topology.*

Example

Consider G a connected Lie group and H a connected closed subgroup.

Proposition

1. *The space $(\Omega^k(G/H))^G$ of G-invariant forms is isomorphic to the space $(\bigwedge^k(\mathcal{G}/\mathcal{H})^*)^H$. The action of H on $\bigwedge^k(\mathcal{G}/\mathcal{H})^*$ is given by:*

$$(a \cdot \lambda)(u_1 + \mathcal{H}, \cdots, u_k + \mathcal{H}) =$$

$$\lambda(Ad_{(a^{-1})}(u_1) + \mathcal{H}, \cdots, Ad_{(a^{-1})}(u_k) + \mathcal{H}).$$

2. *If the group H is* **compact***, the space of G-invariant p-currents on G/H is isomorphic to $(\bigwedge^{m-p}(\mathcal{G}/\mathcal{H})^*)^H$.*

Cohomology of divergence forms

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is stable under the usual de Rham differential operator d and hence define a cohomology

$$\mathrm{H}^r_\tau(M) := \frac{ker\{d : C^r_\tau(M) \to C^{r+1}_\tau(M)\}}{d(C^{r-1}_\tau(M))}$$

we will call it the *cohomology of $\mathcal{G}$-divergence forms*.

Calcul de $\mathrm{H}^0_\tau(M)$

It is clear that $\mathrm{H}^0_\tau(M) = 0$ if M is a non compact manifold.

Proposition

Let M be a compact connected $\mathcal{G}$-manifold.

- *If there exists a $\mathcal{G}$-invariant volume form on M, we have $\mathrm{H}^0_\tau(M) = 0$.*
- *If there is not a $\mathcal{G}$-invariant volume form on M and the $\mathcal{G}$-action is transitive, we have $\mathrm{H}^0_\tau(M) = \mathbb{R}$.*

The complex of $\mathcal{G}$-semi-invariant forms

Let M be a $\mathcal{G}$-manifold.

Definition

A differential form ω will be said a $\mathcal{G}$-semi-invariant form if for any vector field $X \in \tau(\mathcal{G})$ we have

$$\mathsf{L}_X \omega = \mathsf{Tr}(\mathsf{ad}_X)\omega.$$

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We will denote by

$$(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}s})$$

the space of $\mathcal{G}$-semi-invariant differential forms with $\mathcal{G}$-compact support (i.e. the orbit space $supp(\omega)/\mathcal{G}$ is compact). This is also a graded vector space stable under the usual de Rham differential operator d and hence define a cohomology

$$\mathrm{H}(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}s})$$

When the Lie algebra $\mathcal{G}$ is unimodular (i.e. $\mathrm{Tr}(\mathrm{ad}_X) = 0$ for every $X \in \mathcal{G}$), then we have

$$\mathrm{H}(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}s}) = \mathrm{H}(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}})$$

the cohomology of $\mathcal{G}$-invariant differential forms with $\mathcal{G}$-compact support. If moreover the $\mathcal{G}$-orbit space $M/\mathcal{G}$ is compact, we will have

$$\mathrm{H}(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}s}) = \mathrm{H}(\Omega(M)^{\mathcal{G}})$$

Now what is the relationships between this cohomology and the cohomology of divergence forms ?

Consider a G-proper manifold and let $E := \bigwedge^p T^*M$ the pth exterior of the cotangent bundle which is a G-proper bundle.

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More explicitely, for $X^1, \ldots, X^k \in \mathcal{V}(M)$ and $x \in M$ we have

$$m(\omega)_x(X^1, \ldots, X^k) = \int_G \omega_{g^{-1}x}(g_*^{-1}X^1_x, \ldots, g_*^{-1}X^k_x)dg.$$

Proposition

The operator m commutes with the differential d: $m \circ d = d \circ m$, and we obtain a short exact sequence of graded differential vector spaces:

$$0 \to C_{\mathcal{G}}(M) \to \Omega_c(M) \xrightarrow{m} \Omega_{\mathcal{G}c}(M)^{\mathcal{G}s} \to 0 \qquad (\epsilon)$$

Theorem

Let $\tau : \mathcal{G} \to \mathcal{V}(M)$ *be a* $\mathcal{G}$*-proper action.*
Then there exists a a long exact sequence in cohomology:

$$0 \to \mathrm{H}^0_c(M) \to \mathrm{H}^0(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}s}) \to \mathrm{H}^1_\tau(M) \to \mathrm{H}^1_c(M) \to \cdots$$

$$\cdots \to \mathrm{H}^r_\tau(M) \to \mathrm{H}^r_c(M) \to \mathrm{H}^r(\Omega_{\mathcal{G}c}(M)^{\mathcal{G}s}) \to \mathrm{H}^{r+1}_\tau(M) \to \cdots$$

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Theorem

Let $\mathcal{G}$ *be a unimodular Lie algebra and* M *a* $\mathcal{G}$*-proper manifold with* $M/\mathcal{G}$ *compact.*
Then there exists a a long exact sequence in cohomology:

$$0 \to \mathrm{H}^0_c(M) \to \mathrm{H}^0(\Omega(M)^{\mathcal{G}}) \to \mathrm{H}^1_\tau(M) \to \cdots$$

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Corollary

Let M be a $\mathcal{G}$-proper compact manifold. Then: $\mathrm{H}_\tau(M) = 0$.

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Let $\mathcal{G}$ be a unimodular Lie algebra and M a $\mathcal{G}$-proper noncompact manifold such that $M/\mathcal{G}$ is compact, then $\mathrm{H}^1_\tau(M) \neq 0$.

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Let M be a $\mathcal{G}$-proper compact manifold. Then: $\mathrm{H}_\tau(M) = 0$.

Corollary

Let $\mathcal{G}$ be a unimodular Lie algebra and M a $\mathcal{G}$-proper noncompact manifold such that $M/\mathcal{G}$ is compact, then $\mathrm{H}^1_\tau(M) \neq 0$.

Corollary

Let $\mathcal{G}$ be a nonunimodular Lie algebra and M a $\mathcal{G}$-proper manifold with $\mathrm{H}^1_c(M) = 0$. *Then* $\mathrm{H}^1_\tau(M) = 0$.

Discussion of the case of the hyperbolic torus

The action $\tau : \mathbb{R}^2 \to \mathcal{V}(\mathbb{T}^3_A)$ of the abelian Lie algebra $\mathbb{R}^2$ on $\mathbb{T}^3_A$ (generated by X, Y) whose orbits are a torus but this is not a proper Lie algebra action. This follow from the fact that $\mathsf{H}^1_\tau(\mathbb{T}^3_A) \neq 0$: Consider α, β and γ the dual 1-forms, a direct computation leds to

$$d\gamma = 0,\ \gamma = L_X\alpha,\ L_Z(\alpha \wedge \beta \wedge \gamma) = 0$$

this enables us to prove that $[\gamma]_\tau \neq 0$ (becaue if we write $\gamma = df$ then $1 = Zf$ which leds to $Vol(\mathbb{T}^3_A) = 0$!)

Let G be a unimodular connected Lie group and K a compact maximal subgroup (example: $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$). Then $M := G/K$ is a contractible manifold which can be endowed with the canonical homogeneous action $\tau : \mathcal{G} \to \mathcal{V}(G/K)$ given by: $\tau(h)_{\overline{g}} := \frac{d}{dt}_{t=0} \overline{\exp(-th)g}$. This is a proper action.

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Theorem

Consider G a unimodular connected Lie group and K a compact maximal subgroup. Then for every $0 \leq r \leq \dim(G/K) - 1$ we have the isomorphisms

$$\delta : \mathrm{H}^r(\Omega(G/K)^{\mathcal{G}}) \xrightarrow{\cong} \mathrm{H}^{r+1}_{\tau}(G/K),$$

given by

$$\delta(\omega) := [\omega \wedge d\lambda]$$

where $\lambda \in C^\infty_c(G/K)$ is function such that $\int_G \lambda(g^{-1} \cdot x)dg = 1$ for every $x \in G/K$ (dg is the Haar measure of the group G).

Now we recall the differentiable cohomology $\mathrm{H}^*_d(G)$ of a Lie group G is defined as the cohomology of the differentiable complex $(C^*(G), d)$ where $C^p(G) = C^\infty(G \times \cdots \times G)$ is the space of p times smooth functions on G and d is the differential operator given by

$$\begin{aligned} dc(g_1, \ldots, g_{p+1}) = {} & c(g_2, \ldots, g_{p+1}) + \\ & \sum_{i=1}^{p} (-1)^i c(g_1 \ldots, g_i g_j, \ldots, g_{p+1}) + \\ & (-1)^{p+1} c(g_1, \ldots, g_p) \end{aligned}$$

By a famous Van Est theorem " *The cohomology of G-invariant forms on G/K is isomorphic to the differentiable cohomology of G*". This allows us to obtain the following:

Corollary

Let G a unimodular connected Lie group and K a compact maximal subgroup. Then for any $p = 1, \cdots, \dim(G/K)$ we have

$$\mathrm{H}^p_\tau(G/K) \cong \mathrm{H}^{p-1}_d(G).$$

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Corollary

Let G be a unimodular contratible Lie group. Then

$$\mathrm{H}^p_\tau(G) \cong \mathrm{H}^{p-1}(\mathcal{G})$$

where $\mathrm{H}(\mathcal{G})$ is the cohomology of the Lie algebra $\mathcal{G}$ and $\tau(h)$ is the right invariant vector field on G associated to $-h \in \mathcal{G}$.

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