Fuchsian Groups

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Seminar GTA

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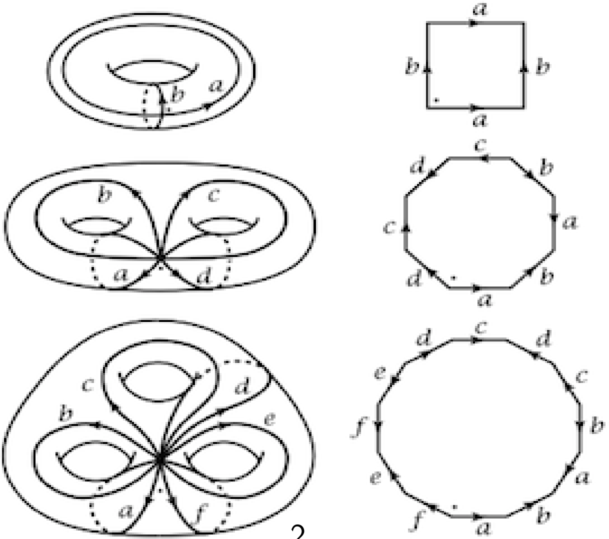
Σ_g: Riemann surfaces ($g \geqslant 1$) and Hyperbolic surfaces ($g \geqslant 2$)



The surface of genus $g = 2$: $\Sigma_g = \mathbb{H}/\Gamma$

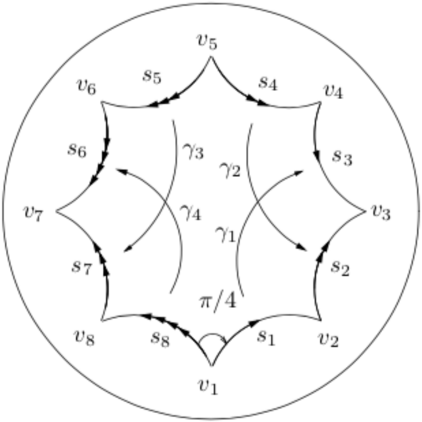
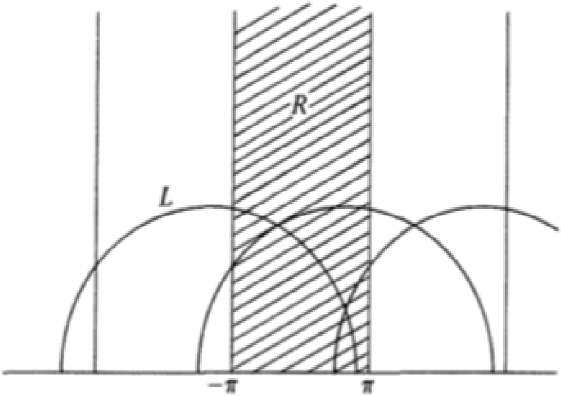


Figure: Hyperbolic octagon

$$\Gamma = < \gamma_1, \gamma_2, \gamma_3, \gamma_4 | \gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1 = e >$$

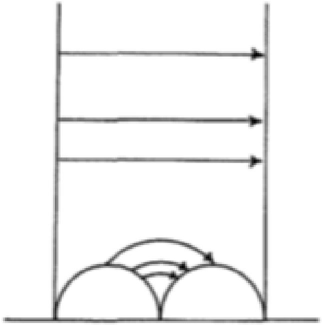
Hyperbolic surfaces : $\mathbb{H}/\Gamma$

Example 1: The **pseudosphere** $S = \mathbb{H}/\Gamma$ where $\Gamma = <\tau_{2\pi}>$ $(\tau(z) = z + 2\pi)$. The lines on $\mathbb{H}/\Gamma$ are defined to be the image of $\mathbb{H}$-lines under $\mathbb{H} \to \mathbb{H}/\Gamma$



The punctured Sphere :

Example 2: The sphere minus three points $S = \mathbb{H}/\Gamma$ where $\Gamma =< \gamma_1, \gamma_2 >$ ($\gamma_1(z) = z + 2$ and $\gamma_2(z) = \frac{z}{2z+1}$). The lines on $\mathbb{H}/\Gamma$ are defined to be the image of $\mathbb{H}$-lines under $\mathbb{H} \to \mathbb{H}/\Gamma$

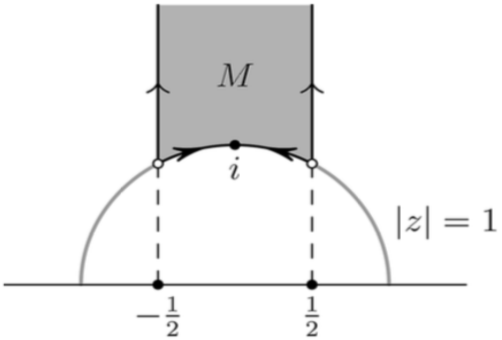


The Modular surface (an orbifold)

Example 3: The modular surface

$$S = \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{H}/ < \alpha_1, \alpha_2 >$$

where $\alpha_1(z) = -\frac{1}{z}$ and $\alpha_2(z) = -\frac{1}{z+1}$. This is an orbifold with two conic points !



Elements $\mathrm{PSL}(2, \mathbb{R})$

$\mathrm{PSL}(2, \mathbb{R})$ is identified with the group of Möbius transformations of $\mathbb{H}$:

$$\{T : z \mapsto \frac{az+b}{cz+d} \ / \ ad - bc = 1,\ a, b, c, d \in \mathbb{R}\}$$

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There are three types of elements $T \in PSL(2, \mathbb{R})$

- T is elliptic: $tr(T) := | \ a + d \ | < 2$.
- T is parabolic: $tr(T) := | \ a + d \ | = 2$.
- T is hyperbolic: $tr(T) := | \ a + d \ | > 2$.

Reduction and Fixed points

- T is **elliptic** $(tr(T) := \mid a + d \mid < 2)$. Then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is conjugaite to a matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. There is one fixed point in $\mathbb{H}$.

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- T is **parabolic** $(tr(T) := \mid a+d \mid = 2)$. Then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is conjugate to a matrix $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$. There is one fixed point in $\mathbb{R} \cup \{\infty\}$ (No fixed point in $\mathbb{H}$)

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- T is **hyperbolic** $(tr(T) :=\mid a+d \mid> 2)$.Then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is conjugate to a matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$. There is two fixed points in $\mathbb{R} \cup \{\infty\}$ (No fixed point in $\mathbb{H}$).

Discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$

Definition

A subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is called a **Fuchsian group** if it is a discrete subgroup i.e. if the induced topology of $\mathrm{PSL}(2, \mathbb{R})$ on Γ is a discrete topology.

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Definition

Let X be a metric space and G be a group of homeomorphisms of X. We say that G acts properly discontinuously on X if the G-orbit of any point $x \in X$ is locally finite (i.e. for any compact subset $K \subset X$, $\{g \in G \ / \ g \cdot x \in K\}$ is finite).

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Lemma. G acts properly discontinuously on X if and only if each orbit $G \cdot x$ is discrete subset of $\mathbb{H}$ and the stabiliser G_x of each point is finite.

Proposition

G acts properly discontinuously on X if and only if each point $x \in X$ has a neighborhood V such that $\{g \in G \ / \ g(V) \cap V \neq \emptyset\}$ is finite.

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Example

1. $\mathrm{PSL}(2, \mathbb{Z})$ is a Fuchsian group.
2. $\{z \mapsto z + n \;/\; n \in \mathbb{Z}\}$ and $\{z \mapsto 2^n z \;/\; n \in \mathbb{Z}\}$ are Fuchsian groups (All parabolic and hyperbolic cyclic subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are Fuchsian groups).
3. If $\theta \in \mathbb{Q}$ then $\{z \mapsto e^{i2\pi n\theta} z \;/\; n \in \mathbb{Z}\}$ is a finite subgroup. If $\theta \notin \mathbb{Q}$ then $\{z \mapsto e^{i2\pi n\theta} z \;/\; n \in \mathbb{Z}\}$ is not a Fuchisian group. (An elliptic cyclic subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian group if and only if it is finite).

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Theorem

Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Then Γ is Fuchsian group if and only if Γ acts properly discontinuously on $\mathbb{H}$.

The limit set of a Fuchsian group

Let Γ be a Fuchsian group acting on the Poincaré disc $\mathbb{D}$. We will be interested in the orbit $\Gamma(z)$ of a point $z \in \mathbb{D}$. We shall view $\Gamma(z)$ as a subset of $\mathbb{D} \cup \partial\mathbb{D}$ (with the Euclidean topology). Let $\bigwedge(\Gamma(z))$ denote the set of **limit points** (accumulation points) in $\mathbb{D} \cup \partial\mathbb{D}$ of the orbit $\Gamma(z)$.

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Remark. $\Gamma \subset \partial\mathbb{D}$.

$\bigwedge(\Gamma(z_1)) = \bigwedge(\Gamma(z_2))$?

Proof. Let $z_1, z_2 \in \mathbb{D}$. Recall the formula

$$\sinh^2\left(\frac{d_{\mathbb{D}}(z_1, z_2)}{2}\right) = \frac{|\, z_1 - z_2 \,|^2}{(1 - |\, z_1 \,|^2)(1 - |\, z_2 \,|^2)}$$

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Let $\gamma \in \Gamma$. As γ is an isometry, it follows that

$$\sinh^2(\frac{d_{\mathbb{D}}(z_1,z_2)}{2}) = \frac{\mid \gamma(z_1) - \gamma(z_2) \mid^2}{(1- \mid \gamma(z_1) \mid^2)(1- \mid \gamma(z_2) \mid^2)}$$

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Hence for any $\gamma \in \Gamma$ we have

$$|\, \gamma(z_1) - \gamma(z_2) \,| \leqslant (1 - |\, \gamma(z_1) \,|^2)^{\frac{1}{2}} \sinh\left(\frac{d_{\mathbb{D}}(z_1, z_2)}{2}\right)$$

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From this formula, we can show that if $\omega = \lim \gamma_n(z_1)$ (where $(\gamma_n) \in \Gamma$) then $\omega = \lim \gamma_n(z_2)$...$\square$

Precisions about the definition of $\bigwedge(\Gamma)$ with $\mathbb{H}$

We have $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

- $\omega \in \mathbb{R} \subset \mathbb{H}$ is a limit point of $\Gamma(z)$ if there exists $(\gamma_n) \in \Gamma$, $\lim \mid \omega - \gamma_n(z) \mid = 0$.
- ∞ is a limit point of $\Gamma(z)$ if for all $R > 0$ there exists $\gamma \in \Gamma$, $\mid \gamma(z) \mid > R$.

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Example

1. $\Gamma = \{z \mapsto 2^n z \ / \ n \in \mathbb{Z}\}$, we can show that

$$\bigwedge(\Gamma) = \{0, \infty\}$$

2. $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, we can show that

$$\bigwedge(\Gamma) = \partial\mathbb{H}$$

Properties of $\bigwedge(\Gamma)$

- $\bigwedge(\Gamma)$ is compact and Γ-invariant.
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Theorem

A Fuchsian group is of the first kind if a fundamental domain has finite area.

Fundamental regions

Definition

Let Γ be a Fuchsian group. A fundamental domain D for Γ is an open subset of $\mathbb{H}$ such that

1. $$\bigcup_{\gamma \in \Gamma} \gamma(\overline{D}) = \mathbb{H}$$
2. For all $\gamma \in \Gamma \backslash \{Id\}$,
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The family $\{\gamma(D) \ / \ \gamma \in \Gamma$ is called a tesselation of $\mathbb{H}$. We can show that any Fuchsian group possesses a nice (convex domain) fundamental domain.

Properties

Example

1. $\Gamma = \{z \mapsto z + n \ / \ n \in \mathbb{Z}\}$,
   $D = \{z \in \mathbb{H} \ / \ 0 < Re(z) < 1\}$.
2. $\Gamma = \{z \mapsto 2^n z \ / \ n \in \mathbb{Z}\}$, $D = \{z \in \mathbb{H} \ / \ 1 <| z |< 2\}$.

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Remark

- Fundamental domains are not necessarily unique.
- If D_1 and D_2 are two fundamental domains for a Fuchsian group Γ, with $Area(D_1) < \infty$ and $Area(\partial D_1) = 0 = Area(\partial D_2)$, then we can show that $Area(D_1) = Area(D_2)$.

Perpendicular bissector

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A perpendicular bissector of the geodesic segment $[z_1, z_2]$ is the unique geodesic through the mid-point of $[z_1, z_2]$.

Proposition

The line determined by the equation $d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)$ is the perpendicular bissector of the line $[z_1, z_2]$.

Dirichlet polygons

Let Γ be a Fuchsina group a,d $p \in \mathbb{H}$. The Dirichlet domain for Γ centered at p is the set

$$D_p(\Gamma) := \{z \in \mathbb{H} \ / \ d_{\mathbb{H}}(z, p) < d_{\mathbb{H}}(z, \gamma(p)) \text{ for all } \gamma \in \Gamma \backslash \{Id\}\}$$

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Theorem (Siegel's Theorem)

$D_p(\Gamma)$ is a fundamental domain of Γ. Moreover, if $Area(\mathbb{H}/\Gamma) < \infty$ then $D_p(\Gamma)$ is a convex hyperbolic polygon.

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4. Let $H_p(\gamma)$ be the half-plane determined by $L_p(\gamma)$ that contains p.
5. Then

$$D_p(\Gamma) = \bigcap_{\gamma \in \Gamma \backslash \{Id\}} H_p(\gamma)$$

Example

Let $\Gamma := \{z \mapsto z + n \ / \ n \in \mathbb{Z}\}$, we can show that $D_i(\Gamma) = \{z \in \mathbb{H} \ / \ -\frac{1}{2} < Re(z) < \frac{1}{2}\}$.

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Example

Let $\Gamma := \mathrm{PSL}(2,\mathbb{Z})$ and $p = 2i$. Let

$$\gamma_1(z) = z+1 \ \text{ and } \ \gamma_2(z) = -\frac{1}{z}$$

then

$$D_{2i}(\mathrm{PSL}(2,\mathbb{Z})) \subseteq F := \bigcap_{\gamma=\gamma_1,\gamma_1^{-1},\gamma_2} H_{2i}(\gamma)$$

From which we can show that

$$D_{2i}(\mathrm{PSL}(2,\mathbb{Z})) = F = \{z \ / \ \mid z \mid > 1, \ -\frac{1}{2} < Re(z) < \frac{1}{2}\}.$$

Side-pairing transformations

Let D be a hyperbolic polygon. A **side** $s \subset \mathbb{H}$ of D is an dge of D in $\mathbb{H}$ equipped with an orientation (i.e. an edge which starts at one vertex and ends at another).

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Example

Let $D = \{z \ / \ \mid z \mid > 1, \ -\frac{1}{2} < Re(z) < \frac{1}{2}\}$. Then $\gamma_1(z) = z + 1$ and $\gamma_2(z) = -\frac{1}{z}$ are two side-pairing transformations.

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Given a side s of a Dirichlet polygon $D_p(\Gamma)$, we can find a side-pairing transformation associated to it :

1. By the way in which $D_p(\Gamma)$ is constructed, we see that s is contained in the perpendicular bissector $L_p(\alpha)$ of the geodesic segment $[p, \alpha(p)]$ for some $\alpha \in \Gamma \backslash \{Id\}$.

Side-pairing transformations of a Dirichlet polygon

Given a side s of a Dirichlet polygon $D_p(\Gamma)$, we can find a side-pairing transformation associated to it :

1. By the way in which $D_p(\Gamma)$ is constructed, we see that s is contained in the perpendicular bissector $L_p(\alpha)$ of the geodesic segment $[p, \alpha(p)]$ for some $\alpha \in \Gamma \setminus \{Id\}$.
2. One can show that the Möbius transformation

$$\gamma_s := \alpha^{-1}$$

   maps s to another side of $D_p(\Gamma)$.

Example

Let $\Gamma := \{z \mapsto z + n \ / \ n \in \mathbb{Z}\}$ with the Dirichlet domain

$$D_i(\Gamma) = \{z \in \mathbb{H} \ / \ -\frac{1}{2} < Re(z) < \frac{1}{2}\}$$

and the side

$$s := \{z \in \mathbb{H} \ / \ Re(z) = -\frac{1}{2}\}$$

Then s is the perpendicular bissector of

$$[i, i-1] = [i, \alpha(i)]$$

where $\alpha(z) = z - 1$, hence $\gamma_s(z) = z + 1$, so that

$$\gamma_s(s) = \{z \in \mathbb{H} \ / \ Re(z) = \frac{1}{2}\}.$$

Theorem

Let Γ be a Fuchsian group and $\{\gamma_i\}$ be the subset of Γ consisting of those elements which pair the sides of some Dirichlet polygon. Then $\{\gamma_i\}$ is a set of generators for Γ.

Theorem

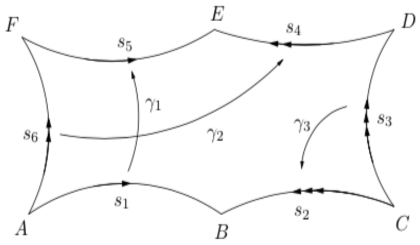
Let Γ be a Fuchsian group with Dirichlet polygon D with all vertices in $\mathbb{H}$. Then there exits a positive integer m such that

$$\sum_i \theta_{v_i} = \frac{2\pi}{m}$$

where θ_{v_i} is the interior of D at the vertex v_i.

Diagrams

Representing the side-pairing transformations in a diagram to indicate which sides of $D_p(\Gamma)$ are paired and how the side-pairing transformations act :



Poincaré theorem

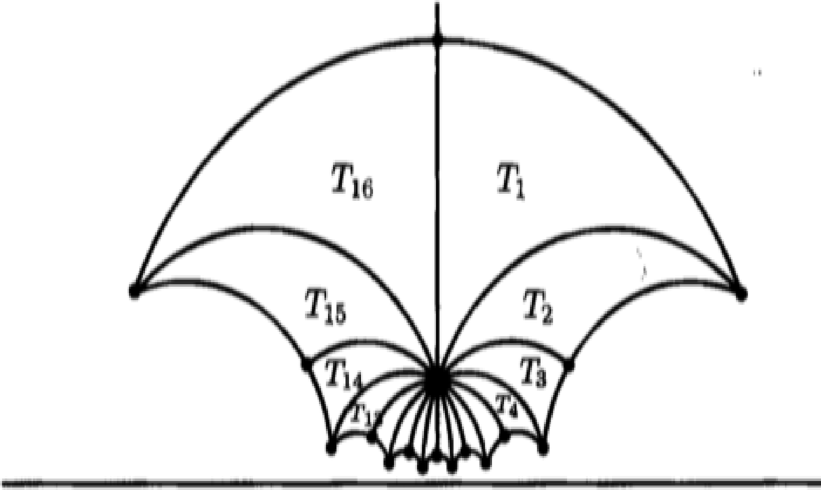
Theorem

Let D be a hyperbolic polygon with finitely many sides. Suppose that all vertices lie inside $\mathbb{H}$ and that D is equipped with a collection $\{\gamma_i\}$ of side-pairing Möbius transformations. Suppose that no side of D is paired with itself. Suppose that the elliptic cycles $\mathcal{E}_1, \cdots, \mathcal{E}_r$ of D satisfies the elliptic condition : For each $\mathcal{E}_j$ there exists an integer $m_j \geqslant 1$ (the order of $\mathcal{E}_j$) such that

$$m_j Sum(\mathcal{E}_j) = 2\pi$$

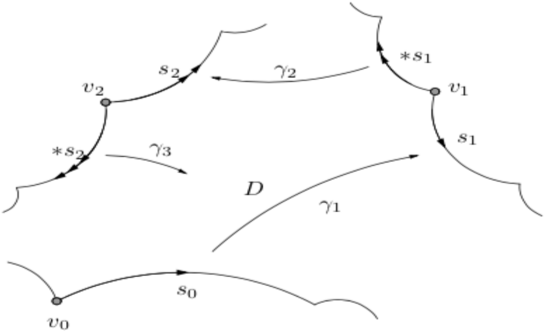
Then the group Γ generated by $\{\gamma_i\}$ is a Fuchsian group and has D as a fundamental domain

Hyperbolic octagon in the half-plan



Procedure

Let Γ be a Fuchsian group and $D_p(\Gamma)$ be a Dirichlet polygon for Γ. We assume that all vertices of $D_p(\Gamma)$ are in $\mathbb{H}$. Consider the following **procedure** :



1. Let $v = v_0$ be a chosen vertex of $D_p(\Gamma)$ and let s_0 be a side with an end point v_0.
2. Let γ_1 be the side-pairing transformation associated to s_0 and let $v_1 = \gamma_1(v_0)$. This gives a new pair (v_1, s_1).
3. Consider the pair

$$*(v_1, s_1) := (v_1, *s_1)$$

   where $*s_1$ is the other side with v_1 as endpoint.
4. Let γ_2 be the side-pairing transformation associated to $*s_1$ and let $s_2 := \gamma_2(*s_1)$.
5. Repeat the above inductively.

Thus we obtain a sequence of vertices

$$(v_0, s_0) \xrightarrow{\gamma_1} (v_1, s_1) \xrightarrow{*} (v_1, *s_1) \xrightarrow{\gamma_2} (v_2, s_2) \to \cdots$$

Elliptic cycle

Definition

An elliptic cycle is a sequence of vertices

$$\mathcal{E} = v_0 \to v_1 \to \cdots \to v_{n-1}$$

such that $(v_n, *s_n) = (v_0, s_0)$.
The transformation

$$\gamma = \gamma_n \gamma_{n_1} \cdots \gamma_2 \gamma_1$$

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There exists an integer $m \geqslant 1$ such that $\gamma^m = Id$. The order of γ is the least such integer.

Theorem

Let Γ be a Fuchsian group with Dirichlet polygon D with all vertices in $\mathbb{H}$ and let $\mathcal{E}$ be an elliptic cycle. Then

$$\sum_i \theta_{v_i} = \frac{2\pi}{m}$$

where θ_{v_i} is the interior of D at the vertex v_i and m is the order of $\mathcal{E}$.

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