Hyperbolic Geometry (Poincaré half-plane)

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Seminar GTA

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- 1829 N. Lobachevsky (non-euclidean geometry)
- 1832 J. Bolyai
- 1868 Riemann - Beltrami (pdeudosphere)
- 1882 Poincaré
- 1955 Blanusa (isometric immersion in $\mathbb{R}^6$)
- 1982 Thurston
- Gromov

The hyperbolic metric

The upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} \ / \ \operatorname{Im} z > 0\}$$

Equipped with the metric :

$$\frac{dx^2 + dy^2}{y^2}$$

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Equipped with the metric :

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More precisely, for all $z \in \mathbb{H}$ and $u, v \in T_z\mathbb{H}$, we have :

$$< u, v >_z := \frac{1}{(\operatorname{Im} z)^2} < u, v >$$

where $<, >$ is the usual Euclidean product for $\mathbb{C} = \mathbb{R}^2$.

The hyperbolic length

For a curve $\gamma : [a, b] \to \mathbb{H}$, $\gamma(t) := x(t) + iy(t)$, the length $h(\gamma)$ is given by

$$h(\gamma) := \int_a^b \frac{\| \gamma'(t) \|}{\operatorname{Im} \gamma(t)} dt = \int_0^1 \frac{\sqrt{(x'(t)^2) + (y'(t)^2}}{y(t)} dt$$

Example

Let $\gamma(t) := it, \;\; t \in [a, b]$. Then

$$h(\gamma) = \int_a^b \frac{1}{t} dt = \ln(\frac{b}{a})$$

Distance and angles

- The hyperbolic distance $\rho(z_0, z_1)$ between two points $z_0, z_1 \in \mathbb{H}$ is defined by the formula

$$\rho(\mathbf{z_0}, \mathbf{z_1}) = \inf \mathbf{h}(\gamma)$$

where the infinimum is taken over all γ joining z_0 and z_1 in $\mathbb{H}$.

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- Hyperbolic angles in $\mathbb{H}$ are the same as Euclidean angles

$$\frac{< \mathbf{u}, \mathbf{v} >_{\mathbf{z}}}{\| \mathbf{u} \|_{\mathbf{z}} \| \mathbf{v} \|_{\mathbf{z}}} = \frac{< \mathbf{u}, \mathbf{v} >}{\| \mathbf{u} \| \| \mathbf{v} \|}$$

Möbius transformations of $\mathbb{H}$

For $a, b, c, d \in \mathbb{R}$, such that $ad - bc = 1$, let

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- ϕ is a diffeomorphism of $\mathbb{H}$:

$$\phi^{-1}(z) = \frac{dz - b}{-cz + a}$$

Action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$

A smooth action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$ is given by

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We can identify $PSL(2,\mathbb{R})$ with the group of Möbius transformations of $\mathbb{H}$:

$$\{\phi : z \mapsto \frac{az+b}{cz+d} /\ ad-bc=1,\ a,b,c,d \in \mathbb{R}\}$$

Remarks

- Möbius transformations include the fractional transformations $F(z) := \dfrac{az+b}{cz+d}$ with $ad - bc > 0$ and $a, b, c, d \in \mathbb{R}$.

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  - **Translations** of the form $z \mapsto z + \alpha$ for $\alpha \in \mathbb{R}$.
  - **Dilatations** of the form $z \mapsto rz$ for $z \in \mathbb{R}$
  - **Inversion** $z \mapsto -\dfrac{1}{z}$

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  - **Inversion** $z \mapsto -\dfrac{1}{z}$
- A generic Möbius transformation $\phi(z) = \dfrac{az+b}{cz+d}$ $(c \neq 0)$ can be decomposed into the following four maps :

$$f_1(z) = z + \frac{d}{c},\ \ f_2(z) = -\frac{1}{z},\ \ f_3(z) = \frac{1}{c^2}z,\ \ f_4(z) = z + \frac{1}{c}$$

Transitivity

$\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}$:

$$\forall z \in \mathbb{H},\ z = x + iy = \begin{pmatrix} \frac{y}{\sqrt{y}} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i$$

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which gives :

$$\mathrm{SL}(2,\mathbb{R})/SO(2) \cong \mathbb{H}$$

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Corollary

$$\forall \phi \in \mathrm{PSL}(2, \mathbb{R}), \ h(\gamma) = h(\phi(\gamma))$$

Geodesics

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The geodesics in $\mathbb{H}$ *are the hyperbolic lines*

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• **Step 1:** Suppose first that $z_1 = ia,\ z_2 = ib\ (b > a)$. Let γ be any piecewise differentiable path joining ia and ib, with $\gamma(t) = x(t) + iy(t)$

$$h(\gamma) = \int_a^b \frac{\parallel \gamma'(t) \parallel}{y(t)} dt \geqslant \int_a^b \frac{y'(t)}{y(t)} dt = \ln(\frac{b}{a})$$

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• **Step 2:** For arbitrary z_1, z_2 $(\mathrm{Re}\, z_1 > \mathrm{Re}\, z_2)$. There exists $\phi \in \mathrm{PSL}(2, \mathbb{R})$ which maps L into the imaginary axis:

$$\phi(z) = \frac{z - (c + r)}{z - (c - r)}$$

where c is the center of the semicircle L and r the radius of L.

$\widehat{z_0 z_1}$ is parametrised by $\alpha(\theta) = c + re^{i\theta}$, $\theta \in [\theta_1, \theta_2]$. Then

$$h(\alpha) = \int_{\theta_1}^{\theta_2} \frac{1}{\sin(\theta)} d\theta = \ln\left(\frac{\tan(\frac{\theta_2}{2})}{\tan(\frac{\theta_1}{2})}\right)$$

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$$\phi(z_1) = \frac{z_1 - (c + r)}{z_1 - (c - r)} = \frac{e^{i\theta_1} - 1}{e^{i\theta_1} + 1} = i \tan(\frac{\theta_1}{2})$$

and $\phi(z_2) = i \tan(\frac{\theta_2}{2})$.

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and $\phi(z_2) = i\tan(\frac{\theta_2}{2})$. Then (step 1)

$$h(\gamma) = h(\phi \circ \gamma) \geqslant \ln\left(\frac{\tan(\frac{\theta_2}{2})}{\tan(\frac{\theta_1}{2})}\right) = h(\alpha)$$

Hyperbolic distance

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Proof. Suppose $\mathrm{Re}\, z_1 > \mathrm{Re}\, z_2$, with $z_1 = c + re^{i\theta_1}$ and $z_2 = c + re^{i\theta_2}$. Let $\rho = \rho(z_1, z_2)$, then

$$\begin{aligned}\tanh(\frac{\rho}{2}) &= \frac{e^\rho - 1}{e^\rho + 1}\\ &= \frac{\tan(\frac{\theta_2}{2}) - \tan(\frac{\theta_1}{2})}{\tan(\frac{\theta_2}{2}) + \tan(\frac{\theta_1}{2})}\\ &= \frac{\sin(\frac{\theta_2 - \theta_1}{2})}{\sin(\frac{\theta_2 + \theta_1}{2})}\end{aligned}$$

On the other hand

$$\begin{aligned} \mid z_2 - z_1 \mid^2 &= r^2((\cos\theta_2 - \cos\theta_1)^2 + (\sin\theta_2 - \sin\theta_1)^2 \\ &= 2r^2(1 - (\cos(\theta_2 - \cos\theta_1) \\ &= 4r^2 \sin^2(\frac{\theta_2 - \theta_1}{2}) \end{aligned}$$

and similarly

$$\mid z_2 - \overline{z_1} \mid^2 = 4r^2 \sin^2(\frac{\theta_2 + \theta_1}{2})$$

hence

$$\tanh(\frac{\rho}{2}) = \frac{\mid z_2 - z_1 \mid}{\mid z_2 - \overline{z_1} \mid}$$

Proposition

$$\sinh\left(\frac{\rho(z_1,z_2)}{2}\right) = \frac{\mid z_2 - z_1 \mid}{2\,\mathrm{Im}\,z_1\,\mathrm{Im}\,z_2} \tag{1}$$

Exercise.

- Show that the Euclidean circle $C^E(ib,r)$ is the hyperblolic circle $C^{\mathbb{H}}(ic,R)$ wher $c=\sqrt{b^2-r^2}$ and $R=\frac{1}{2}\ln\left(\frac{b+r}{b-r}\right)$.
- Show that every hyperbolic disc in $\mathbb{H}$ is an Euclidean disc (with different center of course), and vice versa.
- Show that the topology induced by the hyperbolic metric is the same as the topology induced by the Euclidean metric.
- show that $(\mathbb{H},\rho)$ is a complete metric space.

Isometry group

Theorem

- $$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H})$$
- *The group* $\mathrm{Isom}(\mathbb{H})$ *is generated by transformations in* $\mathrm{PSL}(2,\mathbb{R})$ *together with* $z \mapsto -\overline{z}$*.*

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Proof. Let ψ be any isometry of $\mathbb{H}$. Then $L = \psi(\mathbb{R}_+^* i)$ is a hyperbolic line. There exists an element of $\mathrm{PSL}(2,\mathbb{R})$ which maps L to the imaginary axis $\mathbb{R}_+^* i$.

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$$\phi \circ \psi(i) = i \quad \text{and} \quad \phi \circ \psi(2i) = 2i$$

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We will show that $f := \phi \circ \psi$ fixes each point of $\mathbb{R}_+^* i$.

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and by 1,

$$(x^2 + (y - t)^2)v = (u^2 + (v - t)^2)y$$

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- If $f(z) = z$, then $\psi \in \mathrm{PSL}(2, \mathbb{R})$
- If $f(z) = -\bar{z}$, then $\psi(z) = \dfrac{a\bar{z} + b}{c\bar{z} + d}$ with $ad - bc = -1$.

The unit tangent bundle $S\mathbb{H}$

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$$\phi[z,u] := [\phi(z), d\phi_z(u)]$$

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This action is transitive and free (exercise):

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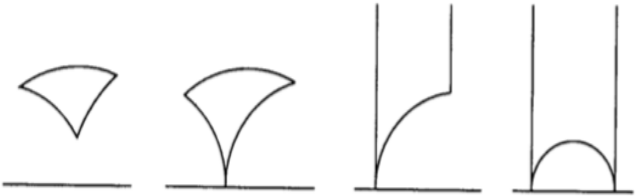
$$\forall [z,u],\ \exists!\ \phi \in \mathrm{PSL}(2,\mathbb{R}), \phi(i) = z \ \text{ and } \ d\phi_i(0,1) = u$$

Theorem

There is a $\mathrm{PSL}(2,\mathbb{R})$*-equivariant diffeomorphism between* $\mathrm{PSL}(2,\mathbb{R})$ *and the unit tangent bundle* $S\mathbb{H}$

$$: \Psi : \mathrm{PSL}(2,\mathbb{R}) \to S\mathbb{H}, \quad \Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left[\frac{ai+b}{ci+d}, \frac{i}{(ci+d)^2}\right]$$

Gauss-Bonnet Theorem



Theorem

Let ABC be a hyperbolic triangle with internal angles α, β, γ. Then

$$\mathit{Area}(ABC) = \pi - (\alpha + \beta + \gamma)$$

Proof

We can assume that one side AC of a triangle ABC is a vertical line and we express a such triangle as the difference of two generalised triangles $AB\infty$ and $CB\infty$.

$$Area(AB\infty) = \int\int_{AB\infty} \frac{dxdy}{y^2} = \int_{\cos(\pi-\alpha)}^{\cos\beta} (\int_{\sqrt{1-x^2}}^{+\infty} \frac{1}{y^2} dy) dx$$

Thus

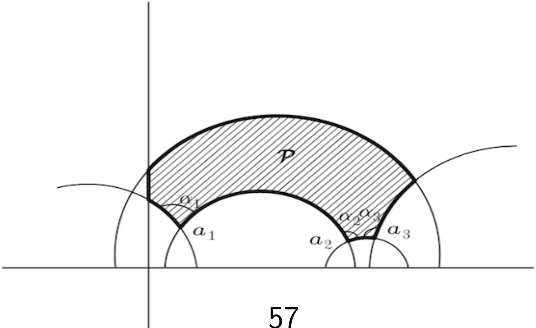
$$Area(AB\infty) = \pi - \alpha - \beta.$$

Aera of a hyperbolic polygon

Corollary

The area of a hyperbolic n-gon $\mathcal{P}$ (sides hyperbolic line segments) is given by the formula

$$(n-2)\pi - (\alpha_1 + \cdots + \alpha_n)$$



Hyperbolic trigonometry

Definition

Two triangles ABC and $A'B'C'$ are congruent if there exists $\psi \in \mathrm{Isom}(\mathbb{H})$ such that $\psi(A) = A'$, $\psi(B) = B'$ and $\psi(C) = C'$

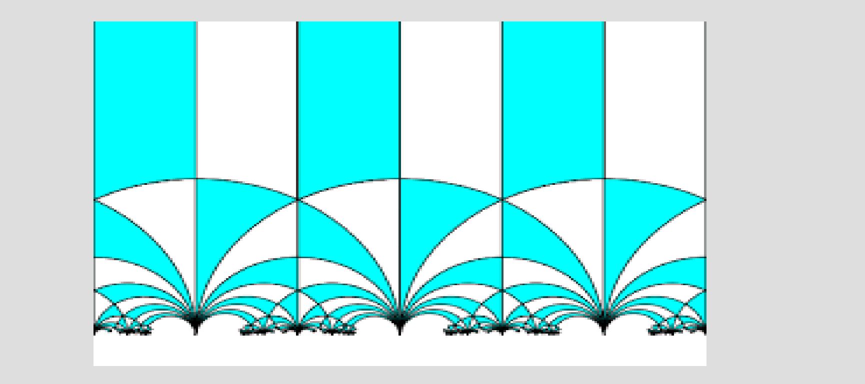
Proposition

Two triangles with same internal angles are congruent

Proposition

Let Δ be a triangle with angles $0, \frac{\pi}{2}, \alpha$ and the finite side a. Then

$$\tan(\alpha) = \frac{1}{\sinh(a)}, \quad \sin(\alpha) = \frac{1}{\cosh(a)}, \quad \cos(\alpha) = \frac{1}{\coth(a)}$$



Proposition

Let Δ *be a triangle with angles* α, β, γ *and opposite sides of lenghts* a, b, c. *Then*

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

$$\cos \gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$$

$$\cos c = \frac{\cos \gamma + \cos \alpha \cos \beta}{\sin \alpha \sin \beta}$$

The Poincaré disc model

The Poincaré disc

$$\mathbb{D} := \{z \in \mathbb{C} \ / \ \mid z \mid < 1\}$$

Equipped with the metric :

$$g_{\mathbb{D}} := 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

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More precisely, for all $z \in \mathbb{D}$ and $u, v \in T_z\mathbb{D}$, we have :

$$< u, v >_z := \frac{4}{(1 - \mid z \mid^2)^2} < u, v >$$

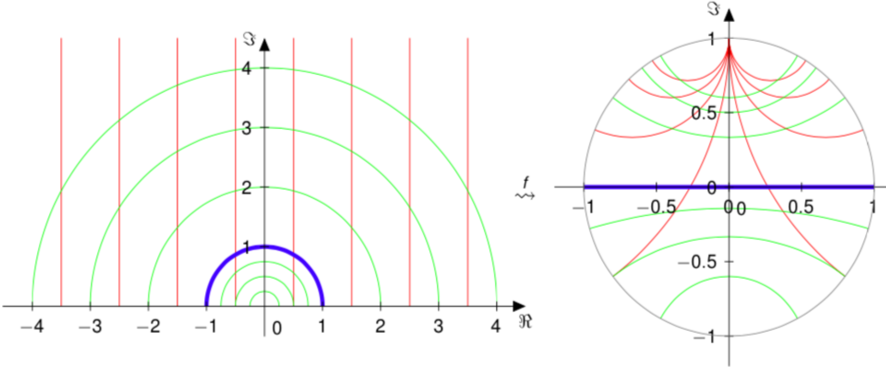
where $<, >$ is the usual Euclidean product for $\mathbb{C} = \mathbb{R}^2$.

Isometry between $\mathbb{H}$ and $\mathbb{D}$

$$f : \mathbb{H} \to \mathbb{D}, \quad f(z) := \frac{iz+1}{z+i}$$

with inverse

$$f : \mathbb{D} \to \mathbb{H}, \quad f^{-1}(\omega) := \frac{-i\omega+1}{\omega-i}$$


ℑ
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−0.5
0
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1
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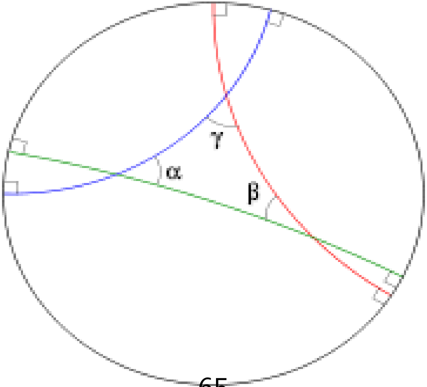
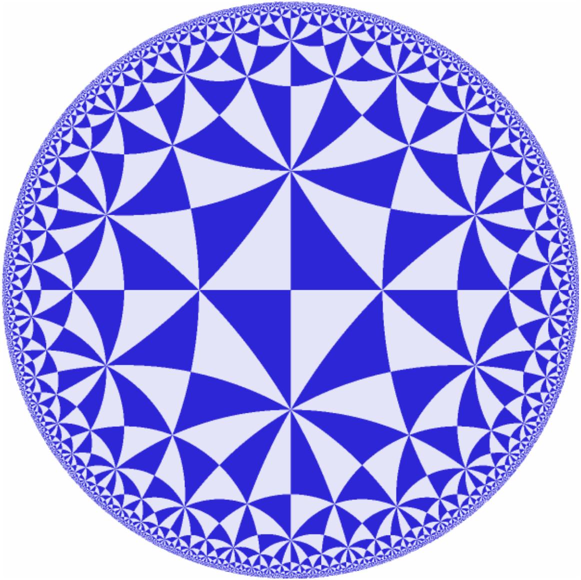
−1

geodesics in $\mathbb{D}$

Proposition

The geodesics in $\mathbb{D}$ are the diameters of $\mathbb{D}$ and the arcs of circles in $\mathbb{D}$ that meet $\partial\mathbb{D}$ at right-angles.
They have equations of the form $\alpha z\bar{z} + \beta z + \overline{\beta}\bar{z} + \alpha = 0$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$.





Area and distance in $\mathbb{D}$

- If $A \subset \mathbb{D}$, then

$$AreaA = \int\int_A \frac{4}{(1-x^2-y^2)^2} dxdy$$

- The distance is given by

$$\rho(a,b) = 2\,\mathrm{Argth}\mid \frac{b-a}{1-\bar{a}b} \mid$$

Isometry group of $\mathbb{D}$

The group

$$SU(1,1) := \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ / \ a\bar{a} - b\bar{b} = 1 \}$$

acts on $\mathbb{D}$ by Möbius transformations. The induced action of the group

$$\mathrm{PSU}(1,1) := \mathrm{SU}(1,1)/\mp I_2$$

on $\mathbb{D}$ is effective, hence

$$\mathrm{PSU}(1,1) \cong \text{Möb}(\mathbb{D}) := \{ \psi : \omega \mapsto e^{i\theta} \frac{\omega - a}{1 - \bar{a}\omega} \ / \ \theta \in \mathbb{R}, \ a \in \mathbb{C} \}$$

Theorem

$$\mathrm{PSU}(1,1) \cong \mathrm{Isom}^+(\mathbb{D}) \qquad \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PSU}(1,1)$$

The hyprbolic model

Let $\mathcal{H}^+$ be the "upper sheet" $z > 0$ of the two sheeted hyperboloid in $\mathbb{R}^3$ defined by

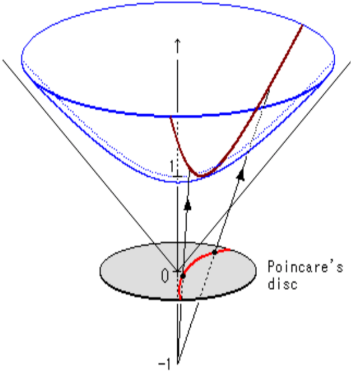
$$z^2 - x^2 - y^2 = 1$$

Equipped with the metric :

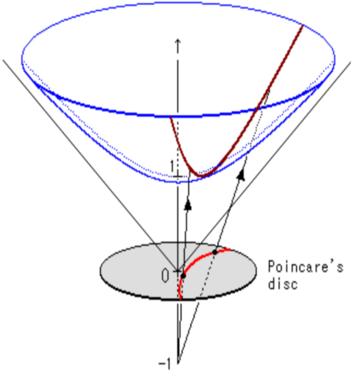
$$g_{\mathcal{H}} := \iota^* m$$

where $\iota : \mathcal{H}^+ \hookrightarrow \mathbb{R}^3$ is the inclusion, and m is the Minkowski metric $dx^2 + dy^2 - dz^2$.

The hyperbolic steregraphic projection

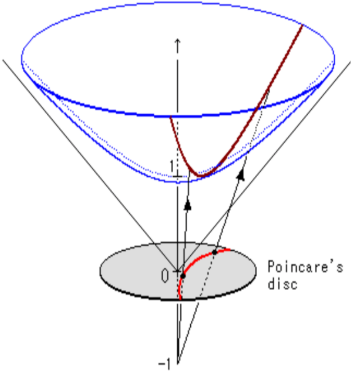


The hyperbolic steregraphic projection



$$p : \mathcal{H}^+ \to \mathbb{D}, \quad p(x, y, z) := \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

The hyperbolic steregraphic projection



$$p : \mathcal{H}^+ \to \mathbb{D}, \quad p(x, y, z) := (\frac{x}{1+z}, \frac{y}{1+z})$$

with inverse $p^{-1} : \mathbb{D} \to \mathcal{H}^+$, given by

$$p^{-1}(X, Y) = (\frac{2X}{1-X^2-Y^2}, \frac{2Y}{1-X^2-Y^2}, \frac{1+X^2+Y^2}{1-X^2-Y^2})$$

Proposition

- *The metric $g_{\mathcal{H}}$ is positive definite*
- *The stereograhic projection p is an isometry*

$$(p^{-1})^*(g_{\mathcal{H}}) = g_{\mathbb{D}}$$

Geodesics and isometry

- Geodesics in $\mathcal{H}^+$ are exactly the intersection of the planes through the origin with $\mathcal{H}^+$.

Geodesics and isometry

• Geodesics in $\mathcal{H}^+$ are exactly the intersection of the planes through the origin with $\mathcal{H}^+$.
• Let $O_+(2,1)$ be the subgroup of $O(2,1)$ which map $\mathcal{H}^+$ to itself, and $SO_+(2,1) := O_+(2,1) \cap \mathrm{SL}(3,\mathbb{R})$. Show that

$$\mathrm{PSL}(2,\mathbb{R}) \cong SO_+(2,1)$$