Polarized symplectic and k-symplectic structures

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Some aspects of theoretical mechanics

Symplectic structures arise naturally in theoretical mechanics, especially during the process of quantization, i.e., in the transition from classical mechanics to quantum mechanics.

1. V. I. Arnold,
2. R. Abraham - J.E. Mardsen,
3. I. Vaisman,
4. N.M.J Woodhouse.

Lagrange equations

The purpose of theoretical mechanics is the discovery of principles which make it possible to describe the temporal evolution of the state of a physical system. In classical mechanics, such a state is given as a point P on a manifold Q of dimension n, Q called configuration space. and P is described by the local coordinates $q_1, \ldots, q_n$, called position variables. The temporal evolution of the system is then described by a curve

$$\gamma : t \longmapsto \gamma(t) = (q_1(t), \cdots, q_n(t)) \mid P(t_0) = P^0.$$

Here, it is necessary to find physical principles which make it possible to determine this trajectory as solution of a differential equation.

Lagrange equations

The starting point for this determination **is the classical mechanical principle of least action**. Each physical system is governed by a function (called Lagrange)

$$\mathscr{L} = \mathscr{L}(q, \dot{q}, t).$$

Hamilton's least action principle decrees that, between two times t_0 and t_1, Motions of the mechanical system coincide with extremals of the functional

$$S = \int_{t_0}^{t_1} \mathscr{L}(\mathrm{q},\dot{q},\mathrm{t})dt. \quad (1)$$

Euler-Lagrange equations

The curve

$$\gamma(t) = (q_1(t), \cdots, q_n(t))$$

is an extremal of (1) if and only if, it is a solution of the following differential system :

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) = \frac{\partial \mathscr{L}}{\partial q}. \tag{2}$$

It is a system of ordinary differential equations in the bundle TQ, with local coordinates

$$q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n,$$

The desired curve γ on Q is the projection of the solution curve $\widetilde{\gamma}$ of (2) on TQ.

Classical mechanics takes the following formulation: for a Lagrange given by a function $\mathscr{L}$, he coordinates position and velocity, $(q, \dot{q})$, are replaced by the coordinates position and momentum (q, p) made possible by the transformation

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \, (i = 1, \cdots, n) \,. \tag{3}$$

The basis of this concept is the transformation of Legendre. Between the tangent bundle and the cotangent bundle

$$\begin{array}{ccc} TQ & \longrightarrow & T^*Q \\ \left(q, \dot{q}\right) & \longmapsto & (q, p) \,. \end{array}$$

Hamilton Equations

The temporal evolution described on TQ by the function of Lagrange $\mathscr{L} = \mathscr{L}(q, \dot{q}, t)$,, is replaced by the function of Hamilton H on the phase space T^*Q defined by

$$H(p, q, t) := p\,\dot{q} - \mathscr{L}(q, \dot{q}, t), \ldots p = \frac{\partial \mathscr{L}}{\partial \dot{q}}. \tag{4}$$

The Lagrange equations (2) are here translated into Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q}; \quad \dot{q} = \frac{\partial H}{\partial p}. \tag{5}$$

Symplectic interpretation

The Hamiltonian function H defines Hamiltonian vector field X_H on the phase space T^*Q.
Compared to the usual coordinates (q,p), the vector field X_H is written:

$$X_H := \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \sum \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Integral curves $\gamma(t) = (q(t), p(t))$ of X_H :

$$\gamma'(t) = X_H(\gamma(t))$$

lead to Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q}; \quad \dot{q} = \frac{\partial H}{\partial p}. \tag{6}$$

Symplectic interpretation

By introducing the differential 2-form

$$\theta = \sum dp_i \wedge dq_i$$

on the cotangent bundle T^*Q, we see that Hamilton's equations (6) are equivalent to:

$$i(X_H)\theta = \theta\,(X_H, .) = -dH. \qquad (7)$$

Polarized linear spaces

A symplectic manifold (M, θ) is said to be polarized if it is equipped with a Lagrangian foliation $\mathfrak{F}$. The notion of polarized symplectic manifold plays an important role in the geometric quantization of Kostant-Souriau ([**?**, **?**]). Interesting properties of the geometry of Lagrangian foliations are given by A. Weinstein ([**?**]) and P. Dazord ([**?**]). The natural model of polarized symplectic manifold is the cotangent bundle T^*M (phase space), equipped with the Liouville form and the real polarization defined by the vertical foliation of the fibration $\pi_M : T^*M \longrightarrow M$.

Polarized linear spaces

Let E be an $\mathbb{R}-$linear space of dimension $2n$, θ be an exterior $2-$form on E and let F be a linear subspace of E of codimension n.

Definition

We say that (θ, F) is a polarized symplectic structure on the space E if:

1. θ is nondegenerate ;
2. $\forall x, y \in F; \theta(x, y) = 0_{\mathbb{R}}$.

The following theorem gives the classification of linear polarized symplectic structures.

Polarized linear spaces

Theorem

If (θ, F) is a polarized symplectic structure on E, then there is a basis $(e_i, e'_i)_{1\leq i\leq n}$ of E^* such that

$$\theta = \sum_{j=1}^{n} \omega^j \wedge \omega'^j, \quad F = \ker \omega'^1 \cap \cdots \cap \ker \omega'^k,$$

where $(\omega^i, \omega'^i)_{1\leq i\leq n}$ is the dual basis of $(e_i, e'_i)_{1\leq i\leq n}$.
$(e_i, e'_i)_{1\leq i\leq n}$ is called polarized symplectic basis.

Polarized symplectic group

Let E be a linear space of dimension $2n$ equipped with a polarized symplectic structure (θ, F).
The automorphisms of E which preserve (θ, F) is a Lie group, denoted by $Sp(1, n; E)$, and called polarized symplectic group of E.
Let $Sp(1, n; \mathbb{R})$ be the group of matrices of polarized symplectic automorphisms of E expressed in the polarized symplectic base $(\mathrm{e}_i, e'_i)_{1\leq i\leq n}$ of E. The group $Sp(1, n; \mathbb{R})$ consists of the matrices of the type

$$\begin{pmatrix} A & S \\ 0 & \left(A^{-1}\right)^T \end{pmatrix}$$

where A, S are matrices $n \times n$ with entries in $\mathbb{R}$, A is invertible and $A\,S^T = SA^T$.

Polarized symplectic group

We denote by $\mathfrak{sp}(1, n; E)$ the Lie algebra of the polarized symplectic group $Sp(1, n; E)$. $\mathfrak{sp}(1, n; E)$ is identified with the tangent space of the Lie group $Sp(1, n; E)$ in the identity mapping Id_E of E; it consists of all endomorphisms u of E satisfying the relation

$$(\forall x, y \in E) \ \ (u(F) \subseteq F, \ \ \theta(u(x), y) + \theta(x, u(y)) = 0) \, .$$

In terms of set of matrices, we denote by $\mathfrak{sp}(1, n; \mathbb{R})$ the Lie algebra of the polarized symplectic group $Sp(1, n; \mathbb{R})$.
The Lie algebra $\mathfrak{sp}(1, n; \mathbb{R})$ consists of all matrices of the type

$$\begin{pmatrix} A & S \\ 0 & -A^T \end{pmatrix}$$

where A, S are $n \times n$ real matrices with S symmetric.
We observe that $Sp(1, n; \mathbb{R})$ is of dimension $\frac{n(3n+1)}{2}$.

Polarized symplectic manifolds

$$(M_{2n}, \theta, \mathfrak{F}_{\mathfrak{n}})$$

$$E = T\mathfrak{F}$$

Definition

We say that (θ, E) is a polarized symplectic structure on M, if:

1. θ is closed. $(d\theta = 0)$;
2. θ is nondegenerate.
3. $\theta(X, Y) = 0$ for all $X, Y \in \Gamma(E)$.

The cotangent bundle T^*M

Let

$$\pi_M : T^*M \longrightarrow M$$

the cotangent bundle. T^*M; provided with the form

$$\theta = d\lambda$$

is a symplectic manifold, λ being the Liouville form on the cotangent bundle :

$$\langle X_u, \lambda_u \rangle = \langle (\pi_M)_* (X_u), \omega_x \rangle$$

for all $u = (x, \omega_x) \in T^*M$, $X \in \Gamma(T(T^*M))$. With respect to a local coordinate system $\left(\overline{U} = (q^1, \ldots, q^n, p^1, \ldots, p^n)\right)$ of T^*M over $(U, \varphi = (q^1, \ldots, q^n))$, we have

$$\lambda_{\overline{U}} = \sum_{i=1}^{n} p^i dq^i \ , \ \ \theta_{\overline{U}} = \sum_{i=1}^{n} dp^i \wedge dq^i .$$

$(\theta; \mathfrak{F})$ where $\mathfrak{F} = \ker \pi_*$ is a polarized symplectic structure on the cotangent bundle T^*M.

The sphere S^{2n}

The spheres S^{2n} don't admit polarized symplectic structures for all $n \geq 1$.

The space $\hom\left(\mathcal{G},\mathbb{R}^2\right) = \mathcal{G}^* \otimes \mathbb{R}^2$

Symplectic polarized structure on $\hom\left(\mathcal{G},\mathbb{R}^2\right)$. Let $\mathcal{G}$ be a real Lie algebra of dimension n. Let $(e_i)_{1 \leq i \leq n}$ be a basis of $\mathcal{G}$ and $(\omega^i)_{1 \leq i \leq n}$ be its dual basis and let $\hom\left(\mathcal{G},\mathbb{R}^2\right) = \mathcal{G}^* \otimes \mathbb{R}^2$ be the linear space of linear mappings from $\mathcal{G}$ with values in $\mathbb{R}^2$.

$$\left\{\hom\left(\mathcal{G},\mathbb{R}^2\right) = \omega^i \otimes \overline{e},\, \omega^i \otimes \overline{f} \quad (1 \leq i \leq n)\right\}$$

where $\left(\overline{e}, \overline{f}\right)$ is the canonical basis of $\mathbb{R}^2$. Any element u of $\hom\left(\mathcal{G},\mathbb{R}^2\right)$ is written in a unique form $u = \sum_{i=1}^n \left(x_i\omega^i \otimes \overline{e} + y_i\omega^i \otimes \overline{f}\right)$ and can be represented by the matrix

$$\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

where x_i, y_i are real numbers.

The space hom $\left(\mathcal{G},\mathbb{R}^2\right) = \mathcal{G}^* \otimes \mathbb{R}^2$

We equip the space hom $(\mathcal{G},\mathbb{R}^2)$ with the coordinate system $(x_i, y_i)_{1\leq i\leq n}$. It is clear that hom $(\mathcal{G},\mathbb{R}^2)$ is a differentiable manifold of dimension $2n$. We endow naturally this space with the polarized symplectic structure $(\theta, \mathfrak{F})$, where

$$\theta = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

and the foliation $\mathfrak{F}$ is given by equations $dy_1 = 0, \cdots, dy_n = 0$. Note that this structure does not depend on the Lie algebra law of $\mathcal{G}$. The law of $\mathcal{G}$ will be appear in the study of polarized Poisson manifolds on hom $\left(\mathcal{G},\mathbb{R}^2\right)$.

The Darboux Theorem

The symplectic geometry is based on the Darboux which states that every symplectic manifold (M, θ) admits an atlas whose the coordinates changes belong to the pseudogroup of local diffeomorphisms of $\mathbb{R}^{2n}$ leaving the canonical form

$$\theta_0 = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

invariant. Where $(x^i, y^i)_{1 \leq i \leq n}$ is the Cartesian coordinates.

Of course, the first proof of the Darboux theorem is due to G.Darboux himself by using induction reasoning. An other proof is given by A. Weinstein (in 1977). This last proof is based on the Moser Lemma.
The Darboux theorem for symplectic manifolds equipped with Lagrangian foliations, is given by I. Vaisman in (1989?2005) in the context of Poisson structures on foliated manifolds.
In this talk, I will reproduce the demonstration of Darboux's theorem for $k-$symplectic structures for $k = 1$, i.e., for symplectic manifolds equipped with Lagrangian, using only quadratures (in 1984 by myself).

Theorem

Let $(M, \theta, \mathfrak{F})$ be a polarized manifold of dimension $2n$. Then, for every point p of M there is an open U of M containing p equipped with local coordinates $(x^i, y^i)_{1 \leq i \leq n}$ such that the differential forms θ is represented on U by

$$\theta = \sum_{i=1}^{n} dx^i \wedge dy^i$$

and the foliation $\mathfrak{F}$ is defined by the equations

$$dy^1 = 0, ..., dy" = 0.$$

The Darboux theorem: Proof

It follows from the Frobenius theorem that there exists a system of local coordinates $(x, y) = (x_1, \cdots, x_n, y^1, \cdots, y^n)$ defined on an open neighbourhood U of M containing p such that the derivatives

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$$

generate the tangent space of the leaves at every point of U.

The Darboux theorem: Proof

The problem is of a local nature, therefore we can assume that U is an open neighbourhood of $\mathbb{R}^{2n}$ and $p = 0$. The two form θ is locally exact (Poincare's lemma), then we can assume that the differential forms θ can be written on the open set U in the form

$$\theta = d\left(\sum_{u=1}^{n} f^u dx_u + \sum_{s=1}^{n} g_s dy^s\right)$$

where f^u and g_s are smooth functions on U; thus

$$\begin{aligned} \theta &= \sum_{u,v} \frac{\partial f^u}{\partial x_v} dx_v \wedge dx_u + \sum_{u,t}^{n} \frac{\partial f^u}{\partial y^t} dy^t \wedge dx_u \\ &+ \sum_{v,s}^{n} \frac{\partial g_s}{\partial x_v} dx_v \wedge dy^s + \sum_{t,s}^{n} \frac{\partial g_s}{\partial y^t} dy^t \wedge dy^s \end{aligned}$$

The Darboux theorem: Proof

so,

$$\begin{aligned}\theta &= \sum_{u<v}\left(\frac{\partial f^v}{\partial x_u}-\frac{\partial f^u}{\partial x_v}\right)dx_u\wedge dx_v+\\ &+ \sum_{v,s}^n\left(\frac{\partial g_s}{\partial x_u}-\frac{\partial f^u}{\partial y^s}\right)dx_u\wedge dy^s+\sum_{t<s}^n\left(\frac{\partial g_s}{\partial y^t}-\frac{\partial g_t}{\partial y^s}\right)dy^t\wedge dy^s\end{aligned}$$

$\mathfrak{F}$ is Lagrangian, then

$$\frac{\partial f^u}{\partial x_v}=\frac{\partial f^v}{\partial x_u},$$

for all $u, v \in [\![1, n]\!]$.
For each $i = 1, \cdots, n$, we take:

$$x^i = g_i - \sum_{u=1}^{n}\int_0^{x_u}\frac{\partial f^u}{\partial y^i}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\,d\xi.$$

Detail for calculation

$$
\begin{aligned}
x^i &= g_i - \int_0^{x_1} \frac{\partial f^1}{\partial y^i} (\xi, x_2, \cdots, x_n, y)\, d\xi \\
&- \int_0^{x_2} \frac{\partial f^2}{\partial y^i} (0, \xi, x_3, \cdots, x_n, y)\, d\xi \\
&- \int_0^{x_3} \frac{\partial f^3}{\partial y^i} (0, 0, \xi, x_4, \cdots, x_n, y)\, d\xi \\
&- \cdots \\
&- \int_0^{x_n} \frac{\partial f^n}{\partial y^i} (0, 0, \cdots, 0, \xi, y)\, d\xi
\end{aligned}
$$

Then we have

$$\frac{\partial x^i}{\partial x_v} = \frac{\partial g_i}{\partial x_v} - \sum_{u=1}^{v-1} \frac{\partial}{\partial x_v} \int_0^{x_u} \frac{\partial f^u}{\partial y^i}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\,d\xi - \frac{\partial}{\partial x_v}\int_0^{x_v} \frac{\partial f^v}{\partial y^i}(0,\cdots,0,\xi,x_{v+1},\cdots,x_n,y)\,d\xi$$

But

The Darboux theorem: Proof

$$
\begin{aligned}
& \frac{\partial}{\partial x_v}\int_0^{x_u}\frac{\partial f^u}{\partial y^i}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\,d\xi \\
= & \int_0^{x_u}\frac{\partial^2 f^u}{\partial x_v \partial y^i}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\,d\xi \\
= & \int_0^{x_u}\frac{\partial^2 f^u}{\partial y^i \partial x_v}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\,d\xi \\
= & \int_0^{x_u}\frac{\partial}{\partial x_u}\frac{\partial f^v}{\partial y^i}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\,d\xi \\
= & \left[\frac{\partial f^v}{\partial y^i}(0,\cdots,0,\xi,x_{u+1},\cdots,x_n,y)\right]_0^{x_u} \\
= & \left[\frac{\partial f^v}{\partial y^i}(0,\cdots,0,x_u,\cdots,x_n,y)-\frac{\partial f^v}{\partial y^i}(0,\cdots,0,x_{u+1},\cdots,x_n,y)\right]
\end{aligned}
$$

Then

$$\begin{aligned}
\frac{\partial x^i}{\partial x_v} &= \frac{\partial g_i}{\partial x_v} - \sum_{u=1}^{v-1} \left[\frac{\partial f^v}{\partial y^i}(0,\cdots,0,x_u,\cdots,x_n,y) - \frac{\partial f^v}{\partial y^i}(0,\cdots,0,x_{u+1}\right. \\
&\qquad - \frac{\partial f^v}{\partial y^i}(0,\cdots,0,x_v,x_{v+1},\cdots,x_n,y) \\
&= \frac{\partial g_i}{\partial x_v} - \frac{\partial f^v}{\partial y^i}(x_1,\cdots,x_n,y) \\
&= \frac{\partial g_i}{\partial x_v}(x,y) - \frac{\partial f^v}{\partial y^i}(x,y)
\end{aligned}$$

The Darboux theorem: Proof

On the other hand, we have

$$\frac{\partial x^s}{\partial y^t} - \frac{\partial x^t}{\partial y^s} = \frac{\partial g_s}{\partial y^t} - \left(\sum_{u=1}^n \int_0^{x_u} \frac{\partial^2 f^u}{\partial y^t \partial y^s}(0, \cdots, 0, \xi, x_{u+1}, \cdots, \right.$$
$$- \frac{\partial g_t}{\partial y^s} + \left(\sum_{u=1}^n \int_0^{x_u} \frac{\partial^2 f^u}{\partial y^s \partial y^t}(0, \cdots, 0, \xi, x_{u+1}, \cdots \right.$$
$$= \frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s}.$$

By the relationship

$$\theta = \sum_{u,s}^n \left(\frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s}\right) dx_u \wedge dy^s + \sum_{t<s}^n \left(\frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s}\right) dy^t \wedge dy^s$$

and

$$\frac{\partial x^i}{\partial x_v} = \frac{\partial g_i}{\partial x_v}(x, y) - \frac{\partial f^v}{\partial y^i}(x, y); \frac{\partial x^s}{\partial y^t} - \frac{\partial x^t}{\partial y^s} = \frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s}.$$

The Darboux theorem: Proof

We deduce that

$$\begin{aligned}
\theta &= \textstyle\sum_{v,i}^{n} \left(\frac{\partial g_i}{\partial x_v} - \frac{\partial f^v}{\partial y^i} \right) dx_v \wedge dy^i + \sum_{t<s}^{n} \left(\frac{\partial x^s}{\partial y^t} - \frac{\partial x^t}{\partial y^s} \right) dy^t \wedge dy^s \\
&= \textstyle\sum_{i=1}^{n} \sum_{v=1}^{n} \frac{\partial x^i}{\partial x_v} dx_v \wedge dy^i + \sum_{s,t}^{n} \frac{\partial x^s}{\partial y^t} dy^t \wedge dy^s \\
&= \textstyle\sum_{s=1}^{n} \left(\frac{\partial x^s}{\partial x_v} dx_v + \frac{\partial x^s}{\partial y^t} dy^t \right) \wedge dy^s \\
&= \textstyle\sum_{s=1}^{n} dx^s \wedge dy^s
\end{aligned}$$

this proves that

$$\theta = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

The Darboux theorem: Proof

it remains to show that the local Pfaffian forms dx^i and dy^i are linearly independent at each point of U. For this, it suffices to show that the Pfaffian forms

$$\omega_s = \sum_{u=1}^{n} \left(\frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s} \right) dx_u$$

$(s = 1, \cdots, n)$ are linearly independent at each point of U. Let us show for this purpose that the matrix $B = (b_s^u)$ is invertible where $b_s^u = \frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s}$.

Let $X = (X_1, \cdots, X_n) \in \mathbb{R}^n$ such that $BX^t = 0$. Then the local vector field

$$\overline{X} = X_1 \frac{\partial}{\partial x_1} + \cdots + X_n \frac{\partial}{\partial x_n}$$

belongs the characteristics subspace $C_x(\theta)$ at each point of U, i.e. $i(\overline{X})\theta_x = 0$; the non degeneracy of θ proves that $\overline{X} = 0$, consequently, $X = (0, \cdots, 0)$; and we deduce that the matrix B is invertible.

The Darboux theorem

Definition

The local coordinates systems $(x^i, y^i)_{1\leq i\leq n}$ constitute an atlas of M, called the Darboux's atlas, and the local coordinates systems $(x^i, y^i)_{1\leq i\leq n}$ are called adapted coordinates systems.

Let $(x^i, y^i)_{1\leq i\leq n}$ and $(\overline{x}^i, \overline{y}^i)_{1\leq i\leq n}$ are be two local adapted coordinate systems defined on an open neighbourhood W of M such that

$$\theta_W = \sum_{j=1}^{n} dx^j \wedge dy^j = \sum_{i=1}^{n} d\overline{x}^i \wedge d\overline{y}^i.$$

We have:

$$\overline{x}^i\left(x, y\right), \overline{y}^i\left(y\right)$$

for all i, because theses charts are foliated with respect to the foliation $\mathfrak{F}$. Therefore,

The Darboux theorem: Proof

$$\begin{aligned}
\theta_W &= \sum_{i=1}^{n} d\overline{x}^i \wedge d\overline{y}^i \\
&= \sum_{i=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} dx^j + \frac{\partial \overline{x}^i}{\partial y^j} dy^j \right) \wedge \sum_{r=1}^{n} \frac{\partial \overline{y}^i}{\partial y^r} dy^r \\
&= \sum_{i=1}^{n} \sum_{j,r=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} dx^j dy^r \\
&+ \sum_{i=1}^{n} \sum_{j,r=1}^{n} \left(\frac{\partial \overline{x}^i}{\partial y^j} \frac{\partial \overline{y}^i}{\partial y^r} \right) dy^j dy^r \\
&= \sum_{r=1}^{n} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} dx^j \right) dy^r \\
&+ \sum_{i=1}^{n} \sum_{j<r} \left(\frac{\partial \overline{x}^i}{\partial y^j} \frac{\partial \overline{y}^i}{\partial y^r} - \frac{\partial \overline{x}^i}{\partial y^r} \frac{\partial \overline{y}^i}{\partial y^j} \right) dy^j dy^r \\
&= \sum_{r=1}^{n} dx^r \wedge dy^r
\end{aligned}$$

Then

$$\frac{\partial \overline{x}^i}{\partial y^j} \frac{\partial \overline{y}^i}{\partial y^r} = \frac{\partial \overline{x}^i}{\partial y^r} \frac{\partial \overline{y}^i}{\partial y^j}; \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} dx^j = dx^r$$

The Darboux theorem: Proof

so

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{y}^i}{\partial y^r} &= \qquad 1 \\ \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} &= 0 \quad \text{for } j \neq r \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial}{\partial x^r}\left(\sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r}\right) &= \qquad 1 \\ \frac{\partial}{\partial x^j}\left(\sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r}\right) &= 0 \;\; \text{for} \quad j \neq r, \end{aligned}$$

so,

$$\begin{aligned} \left(\sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r}\right) &= x^r + \varphi^r(y) \\ \sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r} &= \psi(x^r, y) \end{aligned}$$

The Darboux theorem: Proof

But

$$\overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r} \quad \frac{\partial y^r}{\partial \overline{y}^s} = \frac{\partial y^r}{\partial \overline{y}^s} \quad x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r (y)$$

then

$$\overline{x}^i \frac{\partial \overline{y}^i}{\partial \overline{y}^s} \quad = \frac{\partial y^r}{\partial \overline{y}^s} \quad x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r (y)$$

then

$$\overline{x}^i \delta \quad {}^i_s = \frac{\partial y^r}{\partial \overline{y}^s} \quad x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r (y)$$

so,

$$\begin{cases} \overline{x}^s = & \frac{\partial y^r}{\partial \overline{y}^s} x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r (y) = \frac{\partial y^r}{\partial \overline{y}^s} x^r + \phi^r (y) \\ \overline{y}^s = & \overline{y}^s (y) \end{cases}$$

The Lagrangian foliation $\mathfrak{F}$ is affine

The expressions of these change of coordinates in this atlas allow to deduce the following theorem:

Theorem

The Lagrangian foliation $\mathfrak{F}$ is affine.

This means that any leaf of the foliation F is equipped with a structure of locally affine manifold.
This theorem has been proved by several authors through the connection of R. Bott ([11] [5]).

The Darboux theorem: Proof

The non degeneracy of θ allows us to see that the mapping

$$\zeta : TM \longrightarrow T^*M, \;\; v \longmapsto i(v)\theta$$

is an isomorphism of vector bundles over M, and consequently, ζ defines an isomorphism from $\mathfrak{X}(M)$ onto $\mathscr{A}^1(M)$. We denote by $\mu : \mathscr{A}^1(M) \longrightarrow \mathfrak{X}(M)$ the inverse isomorphism of ζ, and for each $\alpha \in \mathscr{A}^1(M)$, we denote by X_α, the vector field on M associated with α by this isomorphism : $\mu(\alpha) = X_\alpha$.
Let TM/E be the quotient bundle

$$TM/E = \bigcup_{x\in M} T_xM/E_x \quad , \quad \nu : TM \longrightarrow TM/E = \nu E$$

and let ν^*E be the dual bundle of νE:

$$\nu^*E = \bigcup_{x\in M} \nu^*E_x = \bigcup_{x\in M} (T_xM/E_x)^*.$$

The mapping ζ induces an isomorphism of vector bundles from E onto ν^*E.

In terms of local coordinates, $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, $\nu^* E$ is spanned by the Pfaffian forms $dy_1, \cdots, dy_n$ and ζ expresses the duality $\frac{\partial}{\partial x_i} \longmapsto dy_i$ between the geometry along the leaves and the transverse geometry of $\mathfrak{F}$.

Polarized Hamiltonian Vector Fields

Recall that, a real function $f \in \mathscr{C}^{\infty}(M)$ is said to be basic, if for any vector field Y tangent to $\mathfrak{F}$, the function $Y(f)$ is identically zero. We denote by $\mathscr{A}_b^0(M, \mathfrak{F})$ the subring of $\mathscr{C}^{\infty}(M)$ which consists of all basic functions.
We recall also, that a vector field $X \in \mathfrak{X}(M)$ is said to be foliate, or that it is an infinitesimal automorphism of $\mathfrak{F}$, if in the neighborhood of any point of M, the local one parameter group associated to X leaves the foliation $\mathfrak{F}$ invariant. We denote by $\mathscr{I}(M, \mathfrak{F})$ the space of all foliate vector fields.
For each a vector field X tangent to $\mathfrak{F}$, the Pfaffian form $\alpha = \zeta(X)$ belongs to the annihilator $Ann(E)$ of E.

Definitions

A vector field $X \in \mathfrak{X}(M)$ is said to be locally polarized Hamiltonian if:

1. X is foliate;
2. the Pfaffian form $\zeta(X)$ is closed.
   We denote by $H^0(M, \mathfrak{F})$ the real linear space of locally polarized Hamiltonian vector fields

$$H^0(M, \mathfrak{F}) = \{X \in \mathscr{I}(M, \mathfrak{F}) \mid d(\zeta(X)) = 0\}.$$

An element $X \in \mathscr{I}(M, \mathfrak{F})$ is called a polarized Hamiltonian vector field if the Pfaffian form $\zeta(X)$ is exact. We denote by $H(M, \mathfrak{F})$ the real linear space which consists of all polarized Hamiltonian vector fields.

The image $\zeta(H(M, \mathfrak{F}))$ is a linear subspace of $\mathscr{A}_1(M)$. We take

$$\mathfrak{H}(M, \mathfrak{F}) = d^{-1}(\zeta(H(M, \mathfrak{F}))),$$

Polarized Hamiltonian Vector Fields

where d is the exterior differentiation operator.

Theorem

For all $H \in \mathscr{C}^\infty(M)$, the following are equivalent:

1. $H \in \mathfrak{H}(M, \mathfrak{F})$.
2. *there is a unique polarized vector field $X_H \in H(M, \mathfrak{F})$ such that: $i(X_H)\theta = \zeta(X_H) = -dH$.*

Let X be a locally polarized Hamiltonian vector field. In a neighborhood U of an arbitrary point of M, equipped with a local coordinate system $(x^i, y^i)_{1 \leq i \leq n}$, there is a mapping $H \in C^\infty(U)$ such that $\zeta(X) = -dH$. And consequently, the equations of the motion of X are given by the following differential system, called the Hamilton's equations of X:

$$\begin{cases} \frac{dx_i}{dt} = -\frac{\partial H}{\partial y_i} \\ \frac{dy_i}{dt} = \frac{\partial H}{\partial x_i} \\ \frac{\partial H}{\partial \cdots} \in \mathscr{A}^0_{\cdots}(M). \end{cases}$$

Polarized Hamiltonian Vector Fields

Locally, the expressions of H and X are

$$H = \sum_{j=1}^{n} a^j(y_1, \ldots, y_n)x_j + b(y_1, \ldots, y_n)$$

and

$$X = -\sum_{s=1}^{n} \left(\sum_{j=1}^{n} x_j \frac{\partial a^j}{\partial y_s} + \frac{\partial b}{\partial y_s} \right) \frac{\partial}{\partial x_s} + \sum_{j=1}^{n} a^j \frac{\partial}{\partial y_j}$$

respectively, where $a^j, b \in \mathscr{A}_b^0(U, \mathfrak{F}_U)$.
A real function on M is said to be locally affine on the foliation $\mathfrak{F}$ if its restriction on each leaf of $\mathfrak{F}$ is locally affine function.
From the Hamilton equations we deduce the following proposition:

Theorem

For each $H \in C^\infty(M)$, the following properties are equivalent:

1. $H \in \mathfrak{H}(M; \mathfrak{F})$;
2. H *is locally affine function on the foliation* $\mathfrak{F}$.

Corollary

$\mathfrak{H}(M;\mathfrak{F})$ is the set $\mathfrak{a}(M;\mathfrak{F})$ of all smooth real functions on M which are locally affine functions on the foliation $\mathfrak{F}$:

$$\mathfrak{H}(M;\mathfrak{F}) = \mathfrak{a}(M;\mathfrak{F}).$$

Each element of $\mathfrak{H}(M,\mathfrak{F})$ is called a polarized Hamiltonian mapping and X_H is called the polarized Hamiltonian vector field associated with the polarized Hamiltonian H.

So, we have a mapping,

$$\rho : \mathfrak{H}(M,\mathfrak{F}) \longrightarrow H(M,\mathfrak{F});\ H \longmapsto X_H.$$

The following commutative diagram:

$$\begin{array}{ccccc} H(M,\mathfrak{F}) & & \overset{\zeta}{\longrightarrow} & & \mathscr{A}^1(M) \\ & \nwarrow & & \nearrow & \\ \rho & & \mathfrak{H}(M,\mathfrak{F}) & & -d \end{array}$$

Polarized Poisson structure subordinate to a polarized symplectic structure

Let $H, K \in \mathfrak{H}(M, \mathfrak{F})$ and \$X_{H}\$, \$X_{K}\$ the associated polarized Hamiltonian vector fields. Then the Lie bracket $[X_H, X_K]$ is a polarized Hamiltonian vector field and it is associated with $\{K, H\} = \theta\,(X_H, X_K)$ i.e. $[X_H, X_K] = X_{\{K,H\}}$.
The mapping $(H, K) \longmapsto \{H, K\}$ from $\mathfrak{H}(M, \mathfrak{F}) \times \mathfrak{H}(M, \mathfrak{F})$ into $\mathfrak{H}(M, \mathfrak{F})$, defines a real Lie algebra structure on $\mathfrak{H}(M, \mathfrak{F})$. and satisfies in addition the Leibniz identity with respect to polarized Hamiltonian mappings. $\{H, K\}$ is called the polarized Poisson bracket of the polarized Hamiltonians H and K and the Lie algebra $(\mathfrak{H}(M, \mathfrak{F}), \{,\})$ is called polarized Poisson structure subordinate to the polarized symplectic structure (θ, E).

Polarized Poisson structure subordinate to a polarized symplectic structure

Theorem

We have the following properties:

1. $\mathscr{A}_b^0(M)$ *is an abelian Lie subalgebra of* $\mathfrak{H}(M,\mathfrak{F})$.
2. $H(M,\mathfrak{F})$ *is a real Lie algebra.*
3. $\left[H^0(M,\mathfrak{F}), H^0(M,\mathfrak{F})\right] \subset H(M,\mathfrak{F})$.
4. $H(M,\mathfrak{F})$ *is an ideal of* $H^0(M,\mathfrak{F})$.
5. *The sequence of Lie algebras:*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{H}(M,\mathfrak{F}) \xrightarrow{-\rho} H(M,\mathfrak{F}) \hookrightarrow H^0(M,\mathfrak{F}) \longrightarrow H^0(M,\mathfrak{F})/H($$

is exact.

Let $(\mathfrak{H}(M,\mathfrak{F}),\{,\})$ be the polarized Poisson structure subordinate to the polarized symplectic structure (θ, E). Let P be the natural Poisson tensor associated with symplectic form θ:

Polarized Poisson structure subordinate to a polarized symplectic structure

Theorem

We have the following properties:

1. $P(dH, dK) = \{H, K\}, \forall H, K \in \mathfrak{H}(M, \mathfrak{F})$.
2. $P(dH, dK) = -X_H(K), \forall H, K \in \mathfrak{H}(M, \mathfrak{F})$.
3. P *vanishes on the annihilator of* E *in the space* $\mathscr{A}^1(M)$.
4. P *is nondegenerate.*

So, we see that at every point of U *we have*

$$P = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

Polarized Poisson structure

With respect to a local coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, we have

$$P = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

Let $H \in C^{\infty}(M)$ such that $X_H \in \mathscr{I}(M, \mathfrak{F})$ then with respect to the coordinates (x_i, y_i) we have :

$$\begin{array}{rcl} X_H & = & X^i(x,y)\frac{\partial}{\partial x_i} + Y^i(y)\frac{\partial}{\partial y_i} \\ & = & P(dH,.) \\ & = & P\left(\frac{\partial H}{\partial x_i}dx_i + \frac{\partial H}{\partial y_i}dy_i;.\right) \\ & = & \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i} \end{array}$$

then

$$\frac{\partial H}{\partial y_i} = X^i(x,y) \ \text{ and } \ \frac{\partial H}{\partial x_i} = -Y^i(y)$$

so,

$$H = -\sum Y^j(y)\, x^j + b(y).$$

and

$$X_H = -\sum x^j \frac{\partial Y^j(y)}{\partial y_i}_j + Y^i(y)\frac{\partial}{\partial y_i}$$

We deduce that

Theorem

$\mathfrak{H}(M,\mathfrak{F})$ is the set of differentiable mappings $H \in C^\infty(M)$ such that the associated vector field X_H is foliate:

$$\mathfrak{H}(M,\mathfrak{F}) = \{H \in C^\infty(M) \mid X_H \in \mathscr{I}(M,\mathfrak{F})\}$$

Associated Poisson structures with the polarized symplectic structure on hom $(\mathcal{G},\mathbb{R}^2)$

Let $(\mathcal{G},[,])$ be a real Lie algebra of dimension n endowed with a basis $(e_i)_{1\leq i\leq n}$. Let $(\omega^i)_{1\leq i\leq n}$ its dual basis.
We denote by C_{ij}^k the structural constants of $\mathcal{G}$: $[e_i,e_j]=C_{ij}^k e_k$.
We endow hom $(\mathcal{G},\mathbb{R}^2)$ with the natural polarized symplectic structure $(\theta,\mathfrak{F})$ defined by the differential $2-$form $\theta=\sum_{i=1}^n dx^i\wedge dy^i$ and the foliation $\mathfrak{F}$ defined by the equations $dy^1=0,\cdots,dy^n=0$.
Every element X of hom $(\mathcal{G},\mathbb{R}^2)$ can be written in the following form:

$$X=\sum_{i=1}^n\left(x^i\omega^i\otimes\overline{e}+y^i\omega^i\otimes\overline{f}\right)=\begin{pmatrix} x^1 & \dots & x^n \\ y^1 & \dots & y^n \end{pmatrix}.$$

Associated Poisson structures with the polarized symplectic structure on hom $(\mathcal{G},\mathbb{R}^2)$

The linear mapping $X : \mathcal{G} \longrightarrow \mathbb{R}^2$ transforms $u = \sum_{i=1}^{n} \left(u^j e_j\right)$ into

$$X(u) = \sum_{i=1}^{n} \left(x^i u_i\right) \overline{e} + \sum_{i=1}^{n} \left(y^i u_i\right) \overline{f}.$$

In terms of matrices we have

$$X(u) = \begin{pmatrix} x^1 & \dots & x^n \\ y^1 & \dots & y^n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

The polarized Hamiltonians of the polarized symplectic structure are the differentiable functions $H \in \mathscr{C}^\infty\left(\text{hom}\left(\mathcal{G},\mathbb{R}^2\right)\right)$ defined at X by expressions of the type

$$H(X) = \sum_{j=1}^{n} a_j(y^1, ..., y^n)x^j + b(y^1, ..., y^n),$$

where $a_1, \dots, a_n, b$ are basic functions.

Associated Poisson structures with the polarized symplectic structure on $\hom(\mathcal{G},\mathbb{R}^2)$

The Polarized Poisson bracket of Polarized Hamiltonians

$$H = \sum_{j=1}^{n} a_j(y^1, ..., y^n)x^j + b(y^1, ..., y^n) \;\; ; \;\; K = \sum_{j=1}^{n} a'_j(y^1, ..., y^n)x^j + b'(y^1, ..$$

is given by

$$\begin{aligned}\{H, K\} &= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y^i} \frac{\partial H}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial K}{\partial y^i} \right) \\ &= \left(x^j \frac{\partial a_j}{\partial y^i} + \frac{\partial b}{\partial y^i} \right) a'_i - a_i \left(x^j \frac{\partial a'_j}{\partial y^i} + \frac{\partial b'}{\partial y^i} \right) \\ &= \left(a'_i \frac{\partial a_j}{\partial y_i} - a_i \frac{\partial a'_j}{\partial y^i} \right) x^j + a'_i \frac{\partial b}{\partial y^i} - a_i \frac{\partial b'}{\partial y^i}.\end{aligned}$$

We use here the Einstein summation convention. The bracket, so defined, allows to provide $\mathfrak{H}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right), \mathfrak{F}\right)$ with a polarized Poisson structure subordinate to the real polarization $(\theta, \mathfrak{F})$.

Associated Poisson structures with the polarized symplectic structure on hom $(\mathcal{G},\mathbb{R}^2)$

In addition to the Poisson structure subordinate to the real natural polarization on hom $(\mathcal{G},\mathbb{R}^2)$, we can define another polarized Poisson structure $\left(\mathfrak{a}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right);\{,\}^L\right)$, so-called the linear polarized Poisson structure of $(\mathcal{G},[,])$.
Let $H \in \mathfrak{a}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right)$, $X \in \hom\left(\mathcal{G},\mathbb{R}^2\right)$ and $j_1 : \mathcal{G}^* \longrightarrow \hom\left(\mathcal{G},\mathbb{R}^2\right)$ be the mapping defined by

$$j_1(\omega^i) = \omega^i \otimes \overline{e}.$$

The compososed mappings

$$\mathcal{G}^* \xrightarrow{j_1} Hom\left(\mathcal{G},\mathbb{R}^2\right) \xrightarrow{dH_X} \mathbb{R}$$

is completely defined by

$$(dH_X \circ j_1)\left(\omega^i\right) = dH_X\left(\omega^i \otimes \overline{e}\right) = \frac{\partial H}{\partial x^i}(X) = a_i,$$

consequently, $dH_X \circ j_1 = \sum_{i=1}^n a_i e_i$.

Associated Poisson structures with the polarized symplectic structure on hom $(\mathcal{G},\mathbb{R}^2)$

We define

$$\{H, K\}^L (X) = pr_1 \left\langle [dH_X \circ j_1, dK_X \circ j_1] , X \right\rangle$$

pr_1 being the first projection $(x, y) \longmapsto x$, $\mathbb{R}^2 \longrightarrow \mathbb{R}$.

Associated Poisson structures with the polarized symplectic structure on $\hom(\mathcal{G},\mathbb{R}^2)$

Then,

$$\begin{aligned}
\{H,K\}^L(X) &= pr_1\langle [dH_X\circ j_1, dK_X\circ j_1], X\rangle \\
&= \sum_{i,j=1}^n pr_1\langle [a^i e_i, a'^j e_j], X\rangle \\
&= \sum_{i,j=1}^n pr_1\left\langle a_i a'_j C_{ij}^k e_k, X\right\rangle \\
&= \sum_{i,j=1}^n a_i a'_j \sum_{m=1}^n C_{ij}^m x^m \\
&= \sum_{1\le i<j\le n}\sum_{m=1}^n C_{ij}^m\left(a^i a'^j - a^j a'^i\right)x^m.
\end{aligned}$$

Theorem

$\left(\mathfrak{H}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right);\{,\}^L\right)$ *is a polarized Poisson structure on the foliated manifold* $\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right)$, *called the linear polarized Poisson structure of the Lie algebra* $\mathcal{G}$.

We give here the linear polarized Poisson structures corresponding to simple examples.

1. $\mathcal{G}$ is abelian Lie algebra. In this case $\{,\}^L = 0$; Consequently, $\left(\mathfrak{a}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right);\{,\}^L\right)$ is the abelian polarized Poisson structure.
2. $\mathcal{G}$ is the Heisenberg's Lie algebra $\mathcal{H}_1$ of dimension 3. The Lie algebra law of $\mathcal{H}_1$ is given by $[e_1, e_2] = e_3$. And so for all $H, K \in \mathfrak{a}\left(\hom\left(\mathcal{H}_1,\mathbb{R}^2\right),\mathfrak{F}\right)$ $X \in \hom\left(\mathcal{H}_1,\mathbb{R}^2\right)$ where, $H(X) = a_i\left(y^1, y^2, y^3\right)x^i + b\left(y^1, y^2, y^3\right)$ and $K(X) = a'_i\left(y^1, y^2, y^3\right)x^i + b'\left(y^1, y^2, y^3\right)$, we have

$$\{H, K\}^L(X) = \left(a_1 a'_2 - a_2 a'_1\right)x^3.$$