Left-symmetric algebras (I), (II), (III)

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Let $(\mathcal{A}, .)$ be a non-associative algebras. On the underlying vector space $\mathcal{A}$, we define the two following new products:

$$x \circ y := \frac{1}{2}(x.y + y.x) \text{ and } [x, y] := \frac{1}{2}(x.y - y.x), \ \forall x, y \in \mathcal{A}.$$

Recall that $x \circ y$ (resp. $[x, y]$is called the anticommutator (resp. the commutator) of the elements x and y of A. We denote by $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) the algebra $(\mathcal{A}, +, \circ)$ (resp.$(\mathcal{A}, [\ ,\ ])$.

Definition

a. Let $(\mathcal{A}, .)$ be a non-associative algebras. $(\mathcal{A}, .)$ is called a Lie algebra if

1. $x.y = -(y.x), \ \forall x, y \in \mathcal{A}$ (Anti-commutativity); In this case, $x.y = [x, y], \ \forall x, y \in \mathcal{A}$.
2. $\mathcal{J}(x, y, z) := x.(y.z) + y.(z.x) + z.(x.y) = 0, \ \forall x, y, z \in \mathcal{A}$ (Jacobi identity).

a. The non-associative algebra $(\mathcal{A}, .)$ is called a Lie-admissible algebra) $\mathcal{A}^-$ is Lie algebra.

Definition Let $(A, .)$ be a non-associative algebra. The associator is the trilinear map Asso : $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$$\text{Asso}(x, y, z) = (x.y).z - x.(y.z), \quad \forall\, x, y, z \in \mathcal{A}.$$

1. The algebra $(\mathcal{A}, .)$ is called an associative algebra if the associator is identically zero.
2. The algebra $(\mathcal{A}, .)$ is called a left-symmetric algebra (resp. right-symmetric algebra) if
$$\text{Asso}(x, y, z) = \text{Asso}(y, x, z), \quad \forall\, x, y, z \in \mathcal{A}.$$
$$(\text{resp. } \text{Asso}(x, y, z) = \text{Asso}(x, z, y), \quad \forall\, x, y, z \in \mathcal{A}).$$

Let $\mathcal{S}_3$ be the symmetric group of degree 3 and G a sub-group of $\mathcal{S}_3$.
Let $(\mathcal{A}, .)$ be a non-associative algebra. $(\mathcal{A}, +, .)$ is called $G-$associative if

$$\sum_{\sigma \in G} \epsilon(\sigma)\text{Asso}(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) = 0, \;\; \forall\, x_1, x_2, x_3 \in \mathcal{A}, \quad (1)$$

where ϵ is The sign map. Therefore,

1. $(\mathcal{A}, +, .)$ is $\{\text{id}\}-$associative $\iff$ $(\mathcal{A}, .)$ is an associative algebra.
2. $(\mathcal{A}, .)$ is $\{\text{id}, (1\ 2)\}-$associative $\iff$ $(\mathcal{A}, .)$ is left-symmetric algebra.
3. $(\mathcal{A}, .)$ is $\{\text{id}, (2\ 3)\}-$associative $\iff$ $(\mathcal{A}, .)$ is a right-symmetric algebra.

Let us denote by E the set $\mathcal{A}\times\mathcal{A}\times\mathcal{A}$ and by $\mathcal{S}(E)$ the group of all bijections from E to E. Let us consider the anti-morphism of groups $\Phi:\mathcal{S}_3\to\mathcal{S}(E)$ defined by

$$\Phi(\sigma)(x_1,x_2,x_3) := \left(x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},x_{\sigma^{-1}(3)}\right),$$

$\forall\,\sigma\in\mathcal{S}_3,\ \forall\,(x_1,x_2,x_3)\in E.$
Therefore if G is a sub-group of $\mathcal{S}_3$, then $(\mathcal{A},+,.)$ is called $G-$associative if and only if

$$\sum_{\sigma\in G}\epsilon(\sigma)\text{Asso}\circ\Phi(\sigma) = 0.$$

Proposition

Let $(\mathcal{A},.)$ be a non-associative algebara.
$(A,+,.)$ is Lie-admissible if and only if $(\mathcal{A},+,.)$ is a $\mathcal{S}_3-$associative algebra.

Corollary

Let G be a sub-group of $\mathcal{S}_3$.
Any G-associative algebra is a Lie-admissible algebra.

Preuve. Let us consider the left cosets $\mathcal{S}_3/G := \{\sigma G \ / \ \sigma \in \mathcal{S}_3\}$ of G in $\mathcal{S}_3$. Let $\mathcal{C}$ be a set of representatives of all the cosets that means $\mathcal{S}_3/G := \{\sigma G \ / \ \sigma \in \mathcal{C}\}$ and for all σ, σ' in $\mathcal{C}$ such that $\sigma \neq \sigma'$ we have $\sigma G \cap \sigma' G = \emptyset$.

Therefore $\sum_{\sigma \in \mathcal{S}_3} \epsilon(\sigma)\text{Asso} \circ \Phi(\sigma) = \sum_{\sigma \in \mathcal{C}} \sum_{\sigma' \in \sigma G} \epsilon(\sigma')\text{Asso} \circ \Phi(\sigma')$.

If $\sigma \in \mathcal{S}_3$, let us remark that $\sum_{\sigma' \in G} \epsilon(\sigma')\text{Asso} \circ \Phi(\sigma') = 0$ is equivalent to $\sum_{\sigma' \in \sigma G} \epsilon(\sigma')\text{Asso} \circ \Phi(\sigma') = 0$.

Indeed $\sum_{\sigma' \in G} \epsilon(\sigma')\text{Asso} \circ \Phi(\sigma') = \epsilon(\sigma^{-1}) \sum_{\sigma' \in \sigma G} \epsilon(\sigma')\text{Asso} \circ \Phi(\sigma^{-1}\sigma') =$
$\epsilon(\sigma^{-1})\Big(\sum_{\sigma' \in \sigma G} \epsilon(\sigma')\text{Asso} \circ \Phi(\sigma')\Big) \circ \Phi(\sigma^{-1})$.
Consequently if $(\mathcal{A}, .)$ is a $G-$associative algebra, then
$\sum_{\sigma \in \mathcal{S}_3} \epsilon(\sigma)\text{Asso} \circ \Phi(\sigma) = \sum_{\sigma \in \mathcal{C}} \sum_{\sigma' \in \sigma G} \epsilon(\sigma)'\text{Asso} \circ \Phi(\sigma') = 0$. We conclude that $(\mathcal{A}, +, .)$ is a Lie-admissible algebra. $\square$

Proposition.
Let $(\mathcal{A}, .)$ be an associative and commutative algebra. If a is an element of $\mathcal{A}$ and D is a derivation of this algebra (i.e. D is an endomorphism of vector space $\mathcal{A}$ such that $D(x.y) = D(x).y + x.D(y), \ \forall x, y \in \mathcal{A}$), then
the vector space $\mathcal{A}$ endowed with the following new product $\star_a$ is a left symmetric algebra:

$$x \star_a y := \ x.D(y) + a.(x.y), \quad \forall\, x, y \in \mathcal{A}.$$

Example. Les us consider the associative commutative algebra $(\mathcal{A} : C^\infty(\mathbb{R}, \mathbb{R}), +, .)$ and $\varphi \in \mathcal{A}$ then $(\mathcal{A} : C^\infty(\mathbb{R}, \mathbb{R}), +, \star_\varphi)$ is a left symmetric algebra where the product $\star_\varphi$ is defined by:

$$f \star_\varphi g := f.\frac{dg}{dx} + \varphi.(f.g), \quad \forall f, g \in \mathcal{A};$$

Proposition.
Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra. If R is an endomorphism of the vector space $\mathfrak{g}$ satisfying:

$$[R(x), R(y)] = R\Big([R(x), y] + [x, R(y)]\Big), \quad \forall\, x, y \in \mathfrak{g},$$

Then the vector space $\mathfrak{g}$ with the following multiplication "$\star$" is a left symmetric algebra:

$$x \star y := [R(x), y], \quad \forall\; x, y \in \mathfrak{g}.$$

Corollary.
Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra with an invertible deivation D.
Then the vector space $\mathfrak{g}$ with the following multiplication "$\star$" is a left symmetric algebra:

$$x \star y := [D^{-1}(x), y], \quad \forall\; x, y \in \mathfrak{g}.$$

Example.

Let us consider a vector space $\mathcal{A}$ of dimension n with a basis $\{e_1, dots, e_n\}$. On this vector space, we considère the bilinear form $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$$e_i \circ e_j = e_{i+j} \text{ if } i + j \leq n \quad \text{or } 0 \text{ if } i + j > n.$$

$(\mathcal{A}, \circ)$ is a nilpotent associative commutative algebra isomorphic to $X\mathbb{K}[X]/X^{n+1}\mathbb{K}[X]$. It is clear that the endomorphism δ of $\mathcal{A}$ defined by: $\delta(e_i) := ie_i, \ \forall\, i \in \{1, \ldots, n\}$ is an invertible derivation of $(\mathcal{A}, \circ)$. Now if $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ is a Lie algebra, then the vector space $\mathfrak{g} \otimes \mathcal{A}$ with the product (bilinear map) defind by :

$$[x \otimes a, y \otimes b] = [x, y]_{\mathfrak{g}} \otimes (a \circ b), \ \forall\, x, y \in \mathfrak{g}, a, b \in \mathcal{A},$$

is a nilpotent Lie algebra and the endomorphism D of the vector space $\mathfrak{g} \otimes \mathcal{A}$ defined by

$$D(x \otimes a) := x \otimes \delta(a) \ \forall\, x \in \mathfrak{g}, a \in \mathcal{A},$$

Is an invertible derivation of the Lie algebra $(g \otimes \mathcal{A}, [\ ,\ ])$. Consequently, the vector space $g \otimes \mathcal{A}$ with the following product (bilinear map) $\star$ is a Left symmetric algebra:

$$(x \otimes a) \star (y \otimes b) = [D^{-1}(x \otimes a), y \otimes b], \ \forall\, x, y \in \mathfrak{g}, a, b \in \mathcal{A},$$

Proposition.
Let $(\mathfrak{g}, [\ ,\ ], \omega)$ be a symplectic Lie algebra that means $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra and $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a non-degenerate skew-symmetric bilinear form such that:

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \quad \forall\ x, y, z \in \mathfrak{g}.$$

Then the vector space $\mathfrak{g}$ with the multiplication "$\star$", defined by

$$\omega(x \star y, z) = -\omega(y, [x, z]), \quad \forall\ x, y, z \in \mathfrak{g},$$

is a left symmetric algebras. Moreover, $[x, y] = x \star y - y \star x, \ \ \forall\ \ x, y \in \mathfrak{g}$.

If $(\mathfrak{g}, [\ ,\ ], \omega)$ is endowed, in addition, with a non degenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that $B([x, y], z) = B(x, [y, z]), \ \ \forall\ x, y, z \in \mathfrak{g}$, then there exists D an invertible derivation of $\mathfrak{g}$ such that $\omega(x, y) = B(D(x), y), \ \ \forall x, y, z \in \mathfrak{g}$. Consequently,

$$x \star y := D^{-1}([x, D(y)]), \quad \forall x, y \in \mathfrak{g}.$$

Remark.
The Levi-Civita product of a flat pseudo-Euclidean Lie algebra $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ define a left symmetric algebra strucure on the underlying vector space of $\mathfrak{g}$.

Let $((\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a Lie algebra, V a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a linear map. ρ is called a representation of $\mathfrak{g}$ or $V =: V_\rho$ is called a $\mathfrak{g}-$module if $\rho([x, y]_{\mathfrak{g}} = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x),\ \forall x, y \in \mathfrak{g}$. We denote $\rho(x)(v) =: x.v,\ \ \forall\, x \in \mathfrak{g}, v \in V.$

If V_ρ is $\mathfrak{g}-$module, the vector space of 1-cocycle is given by:

$$Z^1(\mathfrak{g}, V_\rho) := \{f : \mathfrak{g} \to V_\rho \ \text{ linear map}/f([x, y]_{\mathfrak{g}}) = x.f(y) - y.f(x),\ \forall x, y \in \mathfrak{g}\};$$

and the vetor space of $1-$ coboundaries is defined by:

$$B^1(\mathfrak{g}, V_\rho) := \{f : \mathfrak{g} \to V_\rho \ \text{ linear map}/\exists v \in V_\rho,\ \ f(x) = x.v,\ \forall x \in \mathfrak{g}\}.$$

Proposition.
Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra.
There exists a left symmetric algebra $(\mathcal{A}, .)$ such that $\mathcal{A}^- = [\mathfrak{g}, [\,,\,])$ if and only if there is a $\mathfrak{g}-$module V_ρ of dimension dim$\mathfrak{g}$ such that $Z^1(\mathfrak{g}, V_\rho)$ contains an invertible $1-$cocycle.

Proof. Suppose that there exists a left symmetric algebra $(\mathcal{A}, .)$ such that $\mathcal{A}^- = [\mathfrak{g}, [\,,\,])$, then for all $x, y, z \in \mathcal{A}$ we have $(x.y).z - x.(y.z) = (y.x).z - y.(x.z)$. It follows that $L_{[x,y]}(z) = (L_x L_y - L_y L_x)(z)$, (where, for all $a \in \mathcal{A}$, the left multiplication by a is Ł$_a$: $\mathcal{A} \to \mathcal{A}$ defined by $L_a(b) := a.b, \ \forall\, b \in \mathcal{A}$.

Therefore, $(A, .)$ is a left symmetric algebra $\iff$ $L : \mathcal{A}^- \to \mathfrak{g}l(\mathcal{A})$ defined by: $L(a) := L_a, \forall\, a \in \mathcal{A},$ is a representation of $\mathcal{A}^-$. This representation define a structure of $\mathcal{A}^-$module on the underlying vector space of $\mathcal{A}_L := \mathcal{A}$.

It is clear that id$_\mathcal{A}$ is invertible element of $Z^1(\mathcal{A}^-, \mathcal{A}_L)$.

Conversely, let us assume that there is a $\mathfrak{g}-$module V_ρ of dimension dim$\mathfrak{g}$ such that $Z^1(\mathfrak{g}, V_\rho)$ contains an invertible $1-$cocycle C. For all $x \in \mathfrak{g}$, we consider the endomorphism $L(x)$ of the vector space $\mathfrak{g}$ defined by: $L(x) := C^{-1} \circ \rho(x) \circ C$.
Let $x, y \in \mathfrak{g}$, $[L(x), L(y)] = L(x) \circ L(y) - L(y) \circ L(x) =$
$C^{-1} \circ \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) \circ C = C^{-1} \circ \rho[x, y] \circ C = L([x, y])$, so L is a representation of $\mathfrak{g}$ and $\mathfrak{g}$ is the vector space of this representation.
Let us consider the new product $\star$, on the underlying vector space of $\mathfrak{g}$ defined by:
$x \star y := L(x)(y), \ \ \forall x, y \in \mathfrak{g}$.
The fact that $C \in Z^1(\mathfrak{g}, V_\rho)$ implies that for all $x, y \in \mathfrak{g}$,
$C([x, y] = \rho(x) \circ C(y) - \rho(y) \circ C(x)$. Since C is invertible, then
$[x, y] = l(x)(y) - L(y)(x) = x \star y - y \star x$. We conclude that the non-associative algabra $\mathcal{A} := (\mathfrak{g}, \star)$ is a left symmetric algebra such that $\mathcal{A}^- = (\mathfrak{g}, [\ ,\ ])$.

Remark.

Let us remark that if $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra with an invertible derivation D, then the vector space $\mathfrak{g}$ with the following product $\star$ is a left symmetric algebra :

$$x \star y := D^{-1}([x, D(y)]), \ \ \forall \ x, y \in \mathfrak{g}.$$

Moreover, $[x, y] = x \star y - y \star x, \ \ \forall \ x, y \in \mathfrak{g}$.

Proposition.
Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra.
There exists $(\mathcal{A}, .)$ a left-symmetric such that $\mathcal{A}^- = (\mathfrak{g}, [\,,\,])$ if and only if there exists C an isomorphism of the underlying vector space of $\mathfrak{g}$ and $\pi : \mathfrak{g} \to \mathfrak{g}l(\mathfrak{g})$ a linear map such that the linear map $\Phi : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g}) := \mathfrak{g}l(\mathfrak{g}) \ltimes \mathfrak{g}$, defined by:
$\Phi(x) := (\pi(x), C(x)), \ \forall x \in \mathfrak{g}$, is a morphism of Lie algebras.

In this case, π is a representation of the Lie algebra $\mathfrak{g}$, $C \in Z^1(\mathfrak{g}, \mathfrak{g}_\pi)$ invertible and

$$\forall x, y \in \mathfrak{g}, \ x.y := \Big(C^{-1} \circ \pi(x) \circ C\Big)(y).$$

Proof. Suppose that there exists $(\mathcal{A}, .)$ a left-symmetric such that $\mathcal{A}^- = (\mathfrak{g}, [\,,\,])$, then $\pi : \mathfrak{g} \to \mathfrak{g}l(\mathfrak{g})$ defined by $\pi(x) := L_x, \ \forall x \in \mathfrak{g}$ is a representation of $\mathfrak{g}$, $\mathrm{id}_{\mathfrak{g}} \in Z^1(\mathfrak{g}, \mathfrak{g}_\pi)$ invertible and $\Phi : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g}) := \mathfrak{g}l(\mathfrak{g}) \ltimes \mathfrak{g}$, defined by: $\Phi(x) := (L_x, x), \ \forall x \in \mathfrak{g}$, is a morphism of Lie algebras.
Conversely, let us assume that there exists C an isomorphism of the underlying vector space of $\mathfrak{g}$ and $\pi : \mathfrak{g} \to \mathfrak{g}l(\mathfrak{g})$ a linear map such that the linear map $\Phi : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g}) := \mathfrak{g}l(\mathfrak{g}) \ltimes \mathfrak{g}$, defined by: $\Phi(x) := (\pi(x), C(x)), \ \forall x \in \mathfrak{g}$, is a morphism of Lie algebras.

Therefore, for all $x, y \in \mathfrak{g}$, we have:

$$\forall x, y \in \mathfrak{g},\ (\pi([x,y], C([x,y]) = ([\pi(x), \pi(y)], \pi(x)(C(y)) - \pi(y)(C(x))),$$

which proves that π is a representation of $\mathfrak{g}$ and C is an invertible $1-$cocycle. Consequently, the following product $''.''$, defined by:

$$\forall x, y \in \mathfrak{g},\ x.y := \Big(C^{-1} \circ \pi(x) \circ C\Big)(y),$$

define a left-symmetric algebra $(\mathcal{A}, .)$ on the underlying vector space of $\mathfrak{g}$ such that $\mathcal{A}^{-} = (\mathfrak{g}, [\ ,\ ])$.

Corollary.

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra of dimension n. Suppose that there exists a left symmetric algebra $(\mathcal{A}, .)$ such that $\mathcal{A}^{-} = (\mathfrak{g}, [\ ,\ ])$.
Then, $(\mathfrak{g}, [\ ,\ ])$ possesses a faithful representation of dimension $n+1$

Proof. If there exists a left symmetric algebra $(\mathcal{A}, .)$ such that $\mathcal{A}^- = [\mathfrak{g}, [\ ,\ ])$, then, by the last proposition, there is a $\mathfrak{g}-$module V_ρ of dimension n such that $Z^1(\mathfrak{g}, V_\rho)$ contains an invertible $1-$cocycle C.
Let us consider the linear map $\pi : \mathfrak{g} \to \mathfrak{g}l(V \times \mathbb{K})$ defined by :

$$\pi(x)((v,k)) := (\rho(x)(v) + kC(x), 0), \quad \forall\ (x, v, k) \in \mathfrak{g} \times V \times \mathbb{K}.$$

Let $x, y \in \mathfrak{g}$ and let $(v, k) \in V \times \mathbb{K}$.
$\pi([x,y])((v,k)) := (\rho([x,y])(v) + kC([x,y]), 0) =$
$([\rho(x), \rho(y)])(v) + k(\rho(x)(C(y)) - \rho(y)(C(x))), 0)$.

$\pi(x)\pi(y)((v,k)) = \pi(x)((\rho(y)(v) + kC(y), 0) = (\rho(x)\rho(y)(v) + k\rho(x)(C(y)), 0)$, and
$\pi(y)\pi(x)((v,k)) = \pi(y)((\rho(x)(v) + kC(x), 0) = (\rho(y)\rho(x)(v) + k\rho(y)(C(x)), 0)$.
Consequently,
$[\pi(x), \pi(y)]((v,k)) = ([\rho(x), \rho(y)](v) + k(\rho(x)(C(y)) - \rho(y)(C(x))), 0) =$
$\pi([x,y])((v,k))$.
We conclude that π is a representation of $\mathfrak{g}$.
Let $x \in \mathrm{Ker}(\pi)$. Then $\pi(x)((0,1)) = (0,0)$, so $(C(x), 0) = (0,0)$, which implies that $x = 0$ because C is an invertible linear map. Therefore π is a faithful representation.

Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra. Suppose that there exists a left symmetric algebra $(\mathcal{A}, .)$ such that $\mathcal{A}^{-} = [\mathfrak{g}, [\,,\,])$, then
$L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by $L(x) := L_x, \ \forall x \in \mathfrak{g}$, is a representation of the Lie algebra $\mathfrak{g}$.
Let us consider $L^\star$ the dual representation of L. Recall that $L^\star : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^\star)$ and defined by: $L^\star(x)(f) := -f \circ L(x), \ \forall (x, f) \in \mathfrak{g} \times \mathfrak{g}^\star$.

Proposition.
There exists a non-degenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ such that $B(x.y, z) = -B(y, x.z), \ \forall x, y, z \in \mathfrak{g}$ then the representations L and $L^\star$ are equivalent (ie. the modules $\mathfrak{g}_L$ and $\mathfrak{g}_{L^\star}$ are isomorphic via an isomorphism φ) such that

$$\forall x \in \mathfrak{g}, \forall y \in \mathfrak{g}, \quad \varphi(x)(y) = \varphi(y)(x).$$

Proof. Let us assume that There exists a non-degenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ such that $B(x.y, z) = -B(y, x.z), \ \forall x, y, z \in \mathfrak{g}$. The the linear map $\varphi : \mathfrak{g} \to \mathfrak{g}^\star$, defined by $\varphi(x) := B(x.), \ \forall x \in \mathfrak{g}$, is invertible which satisfies $\varphi \circ L(x) = L^\star(x) \circ \varphi, \ \forall x \in \mathfrak{g}$. So L and $L^\star$ are equivalent.
Since B is symmetric, then $\varphi(x)(y) = \varphi(y)(x)$, for all $x, y \in \mathfrak{g}$.

Conversely, Let us remark that if L and $L^\star$ are equivalent, then there exists the invertible linear map $\varphi : \mathfrak{g} \to \mathfrak{g}^\star$ such that $\varphi \circ L(x) = L^\star(x) \circ \varphi, \ \forall x \in \mathfrak{g}$.

Let us consider $T : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by: $T(x,y) := \varphi(x)(y),\ \forall x, y \in \mathfrak{g}$. So T is a bilinear non-degenerate form and $T(x.y,z) = -T(y, x.z),\ \forall\ x,y,z \in \mathfrak{g}$.
Now, we consider the following two bilinear foms T_S and T_a defined from T by:

$$T_S(x,y) = \frac{1}{2}(T(x,y) + T(y,x)) \quad T_a(x,y) = \frac{1}{2}(T(x,y) - T(y,x)),\ \forall\, x,y \in \mathfrak{g}.$$

It is clear that $T = T_a + T_s$ and
$T_s(x.y,z) = -T_s(y,x.z)$ and $T_a(x.y,z) = -T_a(y,x.z)\ \forall\ x,y,z \in \mathfrak{g}$.
Let us denote I (resp. J) the radical of T_s (resp. T_a.) The fact that T is non-degenrate implies that $I \cap J = \{0\}$.
Since $\varphi(x)(y) = \varphi(y)(x)$. for all $x \in \mathfrak{g}.\mathfrak{g}$ and for all $y \in \mathfrak{g}$, then J contains $\mathfrak{g}.\mathfrak{g}$. Now we consider M a sub-vector space of $\mathfrak{g}$ such that $\mathfrak{g} = J \oplus M$, and ψ a symmetric non-degenerate bilinear form on M.
Now, it is clear that the bilinear form B on $\mathfrak{g}$, defined by:

$$B_{|_{J\times J}} := (T_s)_{|_{J\times J}},\ \ B_{|_{M\times M}} := \psi, \quad B(J,M) = B(M,J) = \{0\},$$

is symmetric, non-degenerate such that $B(x.y,z) = -B(y,x.z)$, for all $x,y,z \in \mathfrak{g}$.

Some definitions (I).

Let $(\mathcal{A}, .)$ be a non-associative algebras.

1. $\mathcal{A}$ is simple if $\mathcal{A}.\mathcal{A} \neq \{0\}$ and the only ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$.
2. $\mathcal{A}$ is semi-simple if $\mathcal{A} = \{0\}$ or $\mathcal{A}$ is a direct sum of simple algebras.
3. $\mathcal{A}^1 := \mathcal{A} =: \mathcal{A}^{(0)}$, and then by induction

$$\mathcal{A}^{n+1} := \sum_{i+j=n+1} \mathcal{A}^i.\mathcal{A}^j, \quad \mathcal{A}^{(n+1)} := \mathcal{A}^{(n)}.\mathcal{A}^{(n)}, \quad \forall n \in \mathbb{N}^*.$$

   Let us remak that $\mathcal{A}^n$ is the linear span of all products $x_1.\dots.x_n$ of any elements$x_1, \dots, x_n$ of $\mathcal{A}$ in all possible associations.
4. $\mathcal{A}$ is nilpotent (resp. solvable) algebra if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^n = \{0\}$ (resp. $\mathcal{A}^{(n)} = \{0\}$.
   The smallest n with this property is the nilpotency (resp. solvability) of $\mathcal{A}$.
5. $x \in \mathcal{A}$ is nilpotent if the sub-algebra it generate is nilpotent.
6. $\mathcal{A}$ is nil-algebra (resp. I is a nil-ideal) if every element of $\mathcal{A}$ (resp. I) is nilpotent.
7. $\mathcal{A}$ is power-associative if every elemnt of $\mathcal{A}$ generates an associative subalgebra. More precisely, if $x \in \mathcal{A}$, we put $x^1 := x$ and $x^{n+1} := x^n.x$, for all $n \in \mathbb{N}^*$. $\mathcal{A}$ is called power-associative if

$$\forall x \in \mathcal{A}, \forall m, n \in \mathbb{N}^*, \quad x^m.x^n = x^{n+m}.$$

Some definitions (II).

Let $(\mathcal{A}, .)$ be a non-associative algebras.

1. The nucleus (or the associative center) of $\mathcal{A}$, denoted by Nuc$(\mathcal{A})$, is defined by

$$\text{Nuc}(\mathcal{A}) := \{x \in \mathcal{A} \ / \ \text{Asso}(x, \mathcal{A}, \mathcal{A}) = \text{Asso}(\mathcal{A}, x, \mathcal{A}) = \text{Asso}(\mathcal{A}, \mathcal{A}, x) = 0\}.$$

2. The center $Z(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$Z(\mathcal{A}) := \{x \in \text{Nuc}(\mathcal{A}) \ / \ x.y = y.x, \ \ \forall x, y \in \mathcal{A}\}.$$

3. The centroïd of $\mathcal{A}$ denoted cent$(\mathcal{A})$ is defined by

$$\text{cent}(\mathcal{A}) := \{T \in \text{End}(\mathcal{A}) \ / \ TR_x = R_xT = R_{T(x)}, \ \ TL_x = L_xT = L_{T(x)}, \ \ \forall x \in \mathcal{A}\}.$$

4. $\mathcal{A}$ is called central if

$$\text{cent}(\mathcal{A}) := \{\lambda \text{id}_{\mathcal{A}} \ / \ \lambda \in \mathbb{K}\}.$$

Solvable Radical.

Let $(\mathcal{A}, .)$ be a non-associative algebra

1. Let I be an ideal of $\mathcal{A}$. If $\mathcal{A}$ is solvable if and only if I and A/I are solvable.

2. If I and J two solvable ideal of $\mathcal{A}$, then $I + J$ is a solvable ideal of $\mathcal{A}$ because $(I + J)/(I \cap J)$ is isomorphic to $I/(I \cap J) \times J/I \cap J$.

3. There exists a unique maximal solvable ideal of $\mathcal{A}$ denoted $\mathcal{R}ad(\mathcal{A})$.

4. If $\mathcal{A}$ is semi-simple, then $\mathcal{R}ad(\mathcal{A}) = \{0\}$.

Indeed, if $\mathcal{A} \neq \{0\}$ is semi-simple, then $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ where $A_1, \ldots, A_n$ are simple ideals od $\mathcal{A}$. Therefore $Rad(\mathcal{A}).\mathcal{A} = \mathcal{A}.Rad(\mathcal{A}) = \{0\}$ because $Rad(\mathcal{A}) \cap \mathcal{A}_i = \{0\}$ for all $i \in \{1, \ldots, n\}$. Now let $x \in Rad(\mathcal{A})$ then $x = x_1 + \cdots + x_n$ such that $x_i.\mathcal{A}_i = \mathcal{A}_i.x_i = \{0\}$ for all $i \in \{1, \ldots, n\}$. So $x_1 = \cdots = x_n = 0$, then $x = 0$.

It is well known that if $\mathcal{A}$ is Lie or Jordan or alternative of finite dimension (and charcteristic of $\mathbb{K}$ is zero), we have $\mathcal{A}$ is a semi-simple if and only if $\mathcal{R}ad(\mathcal{A}) = \{0\}$. The following example shows that this equivalence is false in the case of the Left-symmetric algebras.

$\mathcal{A} :=$ span$\{x, y, z\}$ with the multiplication:

$$x.x = x, \qquad x.y = y, \qquad x.z = x + \frac{1}{2}y,$$

$$y.x = 0, \qquad y.y = 0, \qquad y.z = x + y,$$

$$z.x = x, \qquad z.y = x + 2y, \qquad z.z = x + y + z,$$

is a Left-symmetric algebra denoted $\mathfrak{E}_1$.
$I :=$ span$\{x, y\}$ is the only proper ideal of $\mathfrak{E}_1$ and span$\{y\}$ is an ideal of I, consequently $\mathfrak{E}_1$ is not semi-simple and $\mathcal{R}ad(\mathfrak{E}_1) = \{0\}$.

Remark. In this Left-symmetric algebra $\mathfrak{E}_1$, let us remark that $(z.z).z \neq z.(z.z)$. Then, $\mathfrak{E}_1$ is not power-associative.

Albert Radical.

Let $(\mathcal{A}, .)$ be a non-associative algebra. The **Albert radical** is defined to be the intersection of all maximal ideals M of $\mathcal{A}$ such that $\mathcal{A}.\mathcal{A} \not\subseteq M$.
If there do not exist such maximal ideals of $\mathcal{A}$, then $\alpha(\mathcal{A}) = \mathcal{A}$ (for example if $\mathcal{A}.\alpha = \{0\}$ then $\alpha(\mathcal{A}) = \mathcal{A}$.

Proposition.

Suppose that $\mathcal{A}$ is finite-dimensional non-associative algebra.
1. $\mathcal{R}ad(\mathcal{A}) \subseteq \alpha(\mathcal{A})$;
2. $\mathcal{A}/\alpha(\mathcal{A})$ is semi-simple;
3. $\mathcal{A}$ is semi-simple if and only if $\alpha(\mathcal{A}) = \{0\}$.

Proof. 1. If M is a maximal $\mathcal{A}$ such that $\mathcal{A}.\mathcal{A} \not\subseteq M$, the $\mathcal{A}/M$ is a simple algebra. Consider the canonical surjection $S : \mathcal{A} \to \mathcal{A}/M$ which is a morphism of algebras. It is clear that $S(\mathcal{R}ad(\mathcal{A}))$ is a solvable ideal of $\mathcal{A}/M$, so $S(\mathcal{R}ad(\mathcal{A})) = \{0\}$. Then $S(\mathcal{R}ad(\mathcal{A})) \subseteq M$. We conclude that $\mathcal{R}ad(\mathcal{A}) \subseteq \alpha(\mathcal{A})$.

2. Suppose that there exist at least a maximal ideal of $\mathcal{A}$ such $\mathcal{A}.\mathcal{A} \not\subseteq M$ and denote P the set of such ideals. Let us consider Q the set of intersections of a finite number of element of P and U the set of the dimensions of underlying vector spaces on the elements of Q. Note m_0 the smallest element of U. Then there exist $M_1, \dots, M_r$ elements of P such that dim$(\cap_{i\in\{1,\dots,r\}} M_i) = m_0$, so $\alpha(\mathcal{A}) = (\cap_{i\in\{1,\dots,r\}} M_i)$. Therefore, the algebra $\mathcal{A}/\alpha(\mathcal{A})$ is isomorphic to the semi-simple algebra $\Pi_{i\in\{1,\dots,r\}} \mathcal{A}/M_i$.

3. It is clear that $\mathcal{A}$ is semi-simple if $\alpha(\mathcal{A}) = \{0\}$.
Now, let us assume that $\mathcal{A} \neq \{0\}$ is semi-simple, then $\mathcal{A} = I_1 \oplus \cdots \oplus I_n$, where I_k is a simple idea lof $\mathcal{A}$ for all $k \in \{1, \dots, n\}$. Since for all $k \in \{1, \dots, n\}$, $M_k := \oplus_{j\in\{1,\dots,n\}\setminus\{k\}} I_j$ is a maximal ideal of $\mathcal{A}$ because $\mathcal{A}/M_k$ is isomorphic to I_k which is simple, then $\alpha(\mathcal{A}) = \{0\}$. because $\alpha(\mathcal{A}) \subseteq \cap_{k\in\{1,\dots,n\}} M_k = \{0\}$.

Left nilpotent radical of Left-symmetric algebras.

Let $(\mathcal{A}, .)$ be a non-associative algebra and I is an ideal of $\mathcal{A}$.

1. I is called nilpotent if there exit $k \in \mathbb{N}^*$ such that $I^k = \{0\}$.

2. For $k \in \mathbb{N}^*$, denote ${}^{<k>}I$ (resp. $I^{<k>}$) the linear span of $L_{x_1} \dots L_{x_{k-1}}(x_k)$ (resp. $R_{x_k} \dots R_{x_2}(x_1)$), for all $x_1 \dots, x_k \in I$.

If ${}^{<k>}I = \{0\}$ (resp. $I^{<k>} = \{0\}$) for some $k \in \mathbb{N}^*$, then I is said to be left nilpotent (resp. right nilpotent).

It is clear that if I is nilpotent, then I is both left and right nilpotent. But the converse is false.

Unlike solvability, the existence of a unique maximal nilpotent or left or right nilpotent ideal in $\mathcal{A}$ is not guaranteed even in finite-dimensional case.

The existence of such radicals depends on the variety of algebras considered (i.e. the identities that define these algebras). For example, in the case of alternative algebra or the case of Jordan algebras these radicals coincide with the solvable radical.

Proposition.
Let $(\mathcal{A},.)$ be a non-associative algebra. Then any left or right nilpotent ideal of $\mathcal{A}$ is solvable

Proof. Let I be a vector sub-space of $\mathcal{A}$ and $k \in \mathbb{N}^*$. Put $I_k :=^{<k>} I \cap I^{<k>}$. Let us remark that $I_k.I_k \subseteq I_{k+1}$. By reasoning by induction on $k \in \mathbb{N}^*$, it is shown that $I^{(k)} \subset I_{k+1}$. Indeed, $I^{(1)} = I.I =^{<2>} I = I^{<2>}$. Suppose that $I^{(k)} \subset I_{k+1}$. $I^{(k+1)} = I^{(k)}.I^{(k)} \subseteq I_{k+1}.I_{k+1} \subseteq I_{k+2}$.

Proposition.
Let $(\mathcal{A},.)$ be a Left-symmetric algebra. If I and J are left nilpotent ideals of $\mathcal{A}$ then so is $I+J$.

Proof. If $V_1,\dots,V_m$ are subspaces of $\mathcal{A}$. Let us denote $V_1\dots V_m := V_1(V_2(\dots V_m)\dots)$. I and J are left nilpotent, the there exist $m \in \mathbb{N}^*$ such that $^{<m>}I =^{<m>} I = \{0\}$. $^{<2m>}(I+J)$ is expanted to a sum of the form $V_1 \dots V_{2m}$ where $V_i = I$ or J for $i \in \{1,\dots,2m\}$ and I or J occurs at least m times in the product $V_1\dots V_{2m}$. It suffices to verify that each product in the sum vanishes, so that $^{<2m>}(I+J) = \{0\}$.
If $V_1\dots V_{2m}$ contains at least m copies of J, then the fact that $\mathcal{A}$ is a left-symmetric algebra implies that $V_1\dots V_{2m}$ is a sum of subspaces of the form $W_1\dots W_q.(^{<r>}J)\dots)$ where $r \geq m$. Indeed, since $\mathcal{A}$ is a left-symmetric algebra, then for all $x,y,z \in \mathcal{A}$ we have

By $(\star)$, $V_1 \dots V_i.J.I.V_{i+3} \dots V_{2m} \subseteq V_1 \dots V_i.I.J.V_{i+3} \dots V_{2m} + V_1 \dots V_i.J.V_{i+3} \dots V_{2m}$.
Then $V_1 \dots V_{2m} = \{0\}$.
Similary if $V_1 \dots V_{2m}$ contains at least m copies of I.
We conclude that $I + J$ is a left-nilpotent of $\mathcal{A}$.

Corollary.
Let $(\mathcal{A}, .)$ be a finite-dimensional left-symmetric algebra.
Then $\mathcal{A}$ contains a unique maximal left nilpotent ideal $N(\mathcal{A})$ containing all left nilpotent ideals of $\mathcal{A}$ such that $N(\mathcal{A}) \subseteq \mathcal{R}ad(\mathcal{A})$.

Definition.
Let $(\mathcal{A}, .)$ be a left-symmetric algebra. $N(\mathcal{A})$ is called tje left nilpotent radical $\mathcal{A}$.

Definition.
Let $(\mathcal{A}, .)$ be a left-symmetric algebra.

1. An element x of $\mathcal{A}$ is said to be right-nil if for some $k \in \mathbb{N}^*$, $x^k := (R_x)^{k-1}(x) = 0$.
2. If, for all $x \in \mathcal{A}$, x is right-nil, then $\mathcal{A}$ is called right-nil algebra.
3. An element x of $\mathcal{A}$ is said to be left-nil if for some $k \in \mathbb{N}^*$, $(L_x)^{k-1}(x) = 0$.
4. If, for all $x \in \mathcal{A}$, x is left-nil, then $\mathcal{A}$ is called left-nil algebra.

Remark.
If $\mathcal{A}$ is a left (resp. right) nilpotent algebra, then $\mathcal{A}$ is a left-nil (resp. right-nil) algebra. The converse is not true even when $\mathcal{A}^-$ is nilpotent.

Definition.
A left-symmetric algebra $(\mathcal{A}, .)$ is called complete if, for all $x \in \mathcal{A}$, R_x is nilpotent (which equivalent to $id_{\mathcal{A}} + R_x$ is invertible, for all $x \in \mathcal{A}$).

Interesting proofs of the following two theorems are given in ”H. Kim, J. Diff Geometry $(1986)337 - 394$”.

Theorem. (Scheunmann)
If $\mathcal{A}$ is a complete left-symmetric algebra and $\mathcal{A}^-$ is a nilpotent Lie algebra, then L_x is nilpotent, for all $x \in \mathcal{A}$.

Theorem.
Let $\mathcal{A}$ be a left-symmetric algebra such that L_x is nilpotent, for all $x \in \mathcal{A}$.
Then $\mathcal{A}^-$ is a nilpotent Lie algebra and R_x is nilpotent for all $x \in \mathcal{A}$ (i.e. $\mathcal{A}$ is complete left-symmetric algebra).

Theorem.
Let $(\mathcal{A},.)$ be a left-symmetric algebra over $\mathbb{R}$.

1. $\mathcal{A}$ is complete if and only if $\mathcal{A}$ is right-nil algebra.

2. The following assertions are equivalent

1. $\forall x \in \mathcal{A}, \;\; L_x$ is nilpotent;
2. $\mathcal{A}$ is left nilpotent;
3. $\mathcal{A}$ is complete and $\mathcal{A}^-$ is nilpotent.

Proof. 1. It is clear that if $\mathcal{A}$ is complete, then $\mathcal{A}$ is right-nil algebra.
Conversely, suppose that $\mathcal{A}$ is a right-nil algebra.
Let x an element of $\mathcal{A}$ and $k \in \mathbb{N}^*$. By reasoning by induction on k, we obtain the following formula

$$(R_x)^k = R_{x^k} - [L_{x^{k-1}}, R_x] - R_x[L_{x^{k-2}}, R_x] - \cdots - (R_x)^{k-2}[L_x, R_x].$$

Consequently, $\mathrm{tr}((R_x)^k) = \mathrm{tr}(R_{x^k})$.
Since $\mathcal{A}$ is a right-nil algebra, the there existe $r \in \mathbb{N}^*$ such that $x^r = 0$. So, forall $m \in \mathbb{N}^*$, $x^{mr} = 0$, then $\mathrm{tr}(((R_x)^r)^m) = 0$. Therefore $(R_x)^r$ is nilpotent, which proves that R_x is nilpotent. We conclude that $\mathcal{A}$ is complete.

2. Assume that L_x is nilpotent, for all $x \in A$. Recall that $L : \mathcal{A}^- \rightarrow \mathfrak{gl}(\mathcal{A})$, defined by: $L(x) := L_x, \ \forall x \in \mathcal{A}$, is a representation of the Lie algebra $\mathcal{A}^-$. Then, by Engel's Theorm, there exist a basis B of $\mathcal{A}$ such that the matrice of L_x by respect of B is a strict upper triangular form, for all $x \in \mathcal{A}$. Therefore $\mathcal{A}$ is left nilpotent.Hence 1) implies 2). 2) implies 3) follows from the last Theorem and 3) implies 1) follows from the Sheunmann's Theorem

The Koszul's radical of Left-symmetric algebras.

Definition.

Let $\mathcal{A}$ be a left-symmetric algebra and

$$T(\mathcal{A}) := \{x \in \mathcal{A} \quad / \quad \text{tr}(R_x) = 0\}.$$

The largest left ideal of $\mathcal{A}$ contained in $T(A)$ is called the Koszul's radical of the left-symmetric algebra $\mathcal{A}$ and is denoted by $K(\mathcal{A})$.

Let us reemark that if $\mathcal{A}$ is complete, then $K(\mathcal{A}) = A$.
An example of J. Helmstetter: Let $(A, .)$ be a left-symmetric algebra. On the vector space $\tilde{A} := \text{End}(A) \oplus A$ we define the following product

$$(f + x) \bullet (h + y) := (f \circ h + [L_x, h]) + (x.y + f(y) + h(x),$$

$\forall x, y \in \mathcal{A}$, $\forall f, h \in \text{End}(A)$.
Then $\tilde{A}, \bullet)$ is a left-symmetric algebra.
If $(\mathcal{A}, .)$ is non complete then $K(\tilde{A}) = \{0\}$.
If $(\mathcal{A}, .)$ is complete and $\mathcal{A}.\mathcal{A} \neq \{0\}$, then $K(\tilde{A})$ is not a two-sided ideal of $\tilde{A}$.

An other example: We consider the left-symmetric algebra $(\mathcal{A} := \text{span}\{\epsilon_1, e_2, e_3, e_4\}, .)$ where the product $''.''$ is defined by:

$$e_1.e_3 = e_3;\; e_2.e_2 = 2e_2;\; e_3.e_4 = e_2;\; e_1.e_4 = -e_4;$$

$$e_2.e_3 = e_3;\; e_4.e_3 = e_2;\; e_2.e_4 = e_4;$$

and the other products equal to zero. The simple calculations prove that

$$T(A) := \text{span}\{e_1, e_3, e_4\} \quad \text{and} \quad K(A) = \text{span}\{e_1\}.$$

Let us remak that $K(A)$ is not a right ideal of $\mathcal{A}$.

The Lie algebra $\mathcal{A}^-$ is solvable but non nilpotente because its product is defined by

$$[e_1, e_3] = e_3;\; [e_2, e_3] = e_3;\; [e_1, e_4] = -e_4;\; [e_2, e_4] = e_4;$$

and the other products equal to zero.

Radical of the bilinear form σ of a Left-symmetric algebra.

Let $(\mathcal{A}, .)$ be left-symmetric algebra. We consider the symmetric bilinear forms $\varphi : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ and $\sigma : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ defined by

$$\varphi(x, y) := \text{tr}(L_x L_y) \quad \text{and} \quad \sigma(x, y) := \text{tr}(R_x R_y), \quad \forall x, y \in \mathcal{A}.$$

Let $x, y \in \mathcal{A}$, the fact that $R_x R_y - R_{x.y} = [R_x, L_y]$ implies that

$$\sigma(x, y) = \text{tr}(R_x R_y) = \text{tr}(R_{y.x}) = \text{tr}(R_y R_x) = \text{tr}(R_{x.y}).$$

Theorem.

Let $(\mathcal{A}, .)$ be a left-symmetric algebra over $\mathbb{K}$ and let Der$(\mathcal{A})$ be the set of the derivations of $\mathcal{A}$ algebra which is a Lie algebra.

1. φ is bi-invariant (or associative) in $\mathcal{A}^-$, i.e.

$$\varphi([x, y], z) = \varphi(x, [y, z]), \quad \forall x, y, z \in \mathcal{A}.$$

2. φ and σ are Der$(\mathcal{A})-$ invariant i.e.

$$\varphi(D(x), y) = -\varphi(x, D(y)) \quad \text{and} \quad \sigma(D(x), y) = -\sigma(x, D(y)), \quad \forall x, y, z \in \mathcal{A}.$$

Proof.
1. Let $x, y, z \in \mathcal{A}$.
$\varphi([x,y],z) = \text{tr}(L_{[x,y]}L_z) = \text{tr}(L_xL_yL_z - L_yL_xL_z) = \text{tr}(L_x(L_yL_z - LzL_y)) =$
$\text{tr}(L_xL_{[y,z]}) = \varphi(x,[y,z])$.
2. Let $x, y \in \mathcal{A}$. IIt's easy to see that

$$L_{D(x)} = [D, L_x] \quad \text{and} \quad R_{D(x)} = [D, R_x].$$

Consequently,
$\varphi(D(x),y) + \varphi(x,D(y)) = \text{tr}([D,L_x]L_y + L_x[D,L_y])$
$= \text{tr}(DL_xL_y - DL_yL_x + DL_yL_x - DL_xL_y) = 0 \quad$ and

$\sigma(D(x),y) + \sigma(x,D(y)) = \text{tr}(R_{D(x).y+x.D(y)}) = \text{tr}(R_{D(X.y)}) = \text{tr}([D,R_{x.y}]) = 0.$

Notation. $\text{tr}(R)$ will designate the linear form $\text{tr} \circ R$ of $\mathcal{A}$ defined by:
$\text{tr} \circ (R)(x) := \text{tr}(R_x), \ \forall x \in \mathcal{A}$.
Let us specify that the interest of σ comes from the fact that it is written in the form

$$\sigma(x,y) = f(x.y), \quad \forall x, y \in \mathcal{A},$$

where $f := \text{tr}(R)$ is a linear form.

Proposition.

Let $(\mathcal{A}, .)$ be a left-symmetric algebra over $\mathbb{K}$.
1. If I is a left ideal of $\mathcal{A}$, then $I^{\perp_\sigma}$ (the orthogonal of I with respect to σ) is a subalgebra of $\mathcal{A}^-$.

2. If I is an ideal of $\mathcal{A}$, then $I^{\perp_\sigma}$ is a subalgebra of $\mathcal{A}$.
3. If I is an ideal of A and $I \subseteq$ Ker(tr(R)), then $I^{\perp_\sigma} = \mathcal{A}$.

Proof.

Let $x, y \in I^{\perp_\sigma}$ and $a \in I$.
1. $\sigma([x,y],a) = \sigma(x.y,a) - \sigma(y.x,a) = \text{tr}(R_{(x.y).a-(y.x).a}) = \text{tr}(R_{x.(y.a)-y.(x.a)}) = \sigma(x, y.a) - \sigma(y, x.a) = 0$. Which proves that $[x,y] \in I^{\perp_\sigma}$. Then $I^{\perp_\sigma}$ is a subalgebra of $\mathcal{A}^-$.
2. Let us assume that I is an ideal of $\mathcal{A}$.
$\sigma(x.y,a) = \text{tr}(R_{(x.y).a}) = \text{tr}(R_{x.(y.a)}) + \text{tr}(R_{(y.x).a)}) - \text{tr}(R_{y.(x.a)}) = \text{tr}(R_{(y.x).a)}) =$
$\text{tr}(R_{a.(y.x))}) = \text{tr}(R_{(a.y).x)}) - \text{tr}(R_{(y.a).x)}) + \text{tr}(R_{y.(a.x)}) =$
$\sigma(a.y,x) - \sigma(y.a,x) + \sigma(y,a.x) = 0$. Then $I^{\perp_\sigma}$ is a subalgebra of $\mathcal{A}$.
3. Now assume that that I is an ideal of $\mathcal{A}$ and and $I \subseteq$ Ker(tr(R)).
Let $x \in \mathcal{A}$ and $y \in I$. Then $x.y \in I$. Therefore $x.y \in$ Ker(tr(R)), so $\sigma(x,y) = 0$. We conclude that $I^{\perp_\sigma} = \mathcal{A}$. $\mathcal{A}^{\perp_\sigma}$ is called the radical of σ.

Remark. Unlike the alternative or the Jordan case $\mathcal{A}^{\perp_\sigma}$ is not an ideal of $\mathcal{A}$ in general.

Interplay between radicals of Left-symmetric algebras.

Lemma(a). Let I be a left (resp. right) ideal of $\mathcal{A}$ with $I \subseteq$ Ker(tr(R)). Then $I \subseteq \mathcal{A}^{\perp_\sigma}$.

Proof. I is a left ideal of $\mathcal{A}$ such that $I \subseteq$ Ker(tr(R)). Then for all $a \in I$ and for all $x \in \mathcal{A}$, we have $\sigma(x, a) =$ tr$(R_{x.a}) = 0$ because $x.a \in I$ and $I \subseteq$ Ker(tr(R)). Then $I \subseteq \mathcal{A}^{\perp_\sigma}$.

Lemma(b). Let I is a left ideal of $\mathcal{A}$ which is complete as a left-symmetric algebra, Then $I \subseteq \mathcal{A}^{\perp_\sigma}$

Proof.Let $a \in I$. We denote by $\tilde{R}_a$ is a right multiplication in I by a. Since I is a left ideal, then $\tilde{R}_a = R_{a_{|I}}$ and tr$(\tilde{R}_a) =$ tr(R_a) (because $R_a(A) \subseteq I$). Since I is a complete left symmetric algebra, then tr$(\tilde{R}_a) = 0$, for all $a \in I$, so we have tr$(R_a) = 0$, for all $a \in I$. Then $I \subseteq$ Ker(tr(R)) and , by Lemma(a), $I \subseteq \mathcal{A}^{\perp_\sigma}$.

Theorem.
Let $(\mathcal{A}, .)$ be a left-symmetric algebra over $\mathbb{K}$. Then

$$N(\mathcal{A}) \subseteq K(A) \subseteq \mathcal{A}^{\perp_\sigma} \subseteq \mathcal{S} \subseteq \text{Ker}(\text{tr}(R)),$$

where $\mathcal{S} := \{a \in \mathcal{A} \ / \ R_a \text{is nilpotent}\}$.

Proof.
$\mathcal{S} \subseteq \text{Ker}(\text{tr}(R))$ is obvious.
Let $x \in \mathcal{A}^{\perp_\sigma}$ and $m \in \mathbb{N} \backslash \{0, 1\}$.
$\text{tr}(R_{x^m}) = \text{tr}(R_{x^{m-1}.x}) = \sigma(x^{m-1}, x) = 0$. Therefore $(R_x)^2$ is nipotent, so R_x is nilpotent. We conclude that $\mathcal{A}^{\perp_\sigma} \subseteq \mathcal{S}$.
Since $K(\mathcal{A})$ is a left ideal contained in $\text{Ker}(\text{tr}(R))$, then, by Lemma(a), $K(\mathcal{A})$ is contained in $\mathcal{A}^{\perp_\sigma}$.
Since $N(A)$ is a left nilpotent ideal, then $N(\mathcal{A})$ is a complete as a left-symmetric algebra. By Lemma b, $N(\mathcal{A}) \subseteq \mathcal{A}^{\perp_\sigma}$. It follows that $N(\mathcal{A}) \subseteq \text{Ker}(\text{tr}(R))$. Since $K(\mathcal{A})$ is the largest left ideal of $\mathcal{A}$ contained in $\text{Ker}(\text{tr}(R))$, then $N(\mathcal{A}) \subseteq K(A)$.

Theorem.

Let $(\mathcal{A},.)$ be a left-symmetric algebra over $\mathbb{K}$.
(i) $N(\mathcal{A}), K(A)$ and $\mathcal{A}^{\perp_\sigma}$ are complete as left-symmetric algebras
(ii) $K(A)$ is the maximal complete left ideal of $\mathcal{A}$.

Proof. (i) $\mathcal{A}^{\perp_\sigma}$ is a left-symmetric subalgebra of A (because $\mathcal{A}$ is an ideal of $\mathcal{A}$).
If $x \in \mathcal{A}^{\perp_\sigma}$, $\tilde{R}_x : \mathcal{A}^{\perp_\sigma} \to \mathcal{A}^{\perp_\sigma}$ denote the right multiplication in $\mathcal{A}^{\perp_\sigma}$ by x, then $\tilde{R}_x = R_{x|_{\mathcal{A}^{\perp_\sigma}}}$. By last theorem $\mathcal{A}^{\perp_\sigma} \subseteq \mathcal{S}$, il follows that R_x is nilpotent for all $x \in \mathcal{A}^{\perp_\sigma}$. Therefore $\tilde{R}_x$ is nilpotent for all $x \in \mathcal{A}^{\perp_\sigma}$. Consequently, $\mathcal{A}^{\perp_\sigma}$ is acomplete left-symmetric algebra.
$N(\mathcal{A})$ and $K(\mathcal{A})$ follows similary from the last theorem.

(ii) Let I be a complete left ideal of $\mathcal{A}$. Then, by Lemma(b), $I \subseteq \mathcal{A}^{\perp_\sigma} \subseteq$. Since $\mathcal{A}^{\perp_\sigma} \subseteq \mathcal{S}$ and $K(\mathcal{A})$ is the largest left ideal contained in Ker(tr(R)), then $I \subseteq K(\mathcal{A})$.

Theorem.

Let $(\mathcal{A}, .)$ be a left-symmetric algebra over $\mathbb{K}$. The following assertions are equivalent:

1. $\mathcal{A}$ is complete;
2. $\mathcal{A} = K(A) = \mathcal{A}^{\perp_\sigma} = \mathcal{S} = \text{Ker}(\text{tr}(R))$;
3. σ is identically zero;
4. $\mathcal{A}^2 \subseteq \text{Ker}(\text{tr}(R))$.

Proof.$((1) \implies (2))$. If $\mathcal{A}$ is complete, then $\mathcal{A} = K(\mathcal{A})$. It follows that
$\mathcal{A} = K(A) = \mathcal{A}^{\perp_\sigma} = \mathcal{S} = \text{Ker}(\text{tr}(R))$.
$((2) \implies (1))$. $A = \text{Ker}(\text{tr}(R))$, then A is complete.
$((3) \iff (4))$. By the definition of σ.
$((2) \iff (3))$. By the last theorem and by the definition of $K(\mathcal{A})$.

In general, any of the inclusions can not be replaced by equality for a Nonassociative left symmetric algebra. However, if $\mathcal{A}^-$ is nilpotent, we have the following result:

Corollary (Kim). Let $(\mathcal{A}, .)$ be a left-symmetric algebra over $\mathbb{K}$ such that $\mathcal{A}^-$ is nilpotent, then $N(\mathcal{A}) = K(\mathcal{A})$.

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Merci pour votre attention!