Introduction to symplectic topology

Mohamed Boucetta
boucetta@fstg-marrakech.ac.ma

Université Cadi-Ayyad
Faculté des sciences et Techniques Marrakech

Journées Géométrie, Topologie
et systèmes dynamiques
26-28 Octobre 2011
Université Hassan II Casablanca

Outline :

Outline :

1. Introduction

Outline :

1. Introduction
2. Affine Nonsqueezing theorem

Outline :

Outline :

1. Introduction
2. Affine Nonsqueezing theorem
3. Symplectic manifolds and Hamiltonian flows
4. The Hofer-Zehnder's capacity

Introduction

The standard model of a symplectic manifold is the Euclidean space $\mathbb{R}^{2n}$ endowed with its canonical symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A **symplectomorphism** of $(\mathbb{R}^{2n}, \omega_0)$ is a diffeomorphism $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ such that

$$F^* \omega_0 = \omega_0.$$

It is obvious that a symplectomorphism F is also a preserving-volume diffeomorphism since

$$F^*\Omega = \Omega,$$

where $\Omega = \wedge^n \omega_0$ is the volume form associated to ω_0.

It is obvious that a symplectomorphism F is also a preserving-volume diffeomorphism since

$$F^*\Omega = \Omega,$$

where $\Omega = \wedge^n \omega_0$ is the volume form associated to ω_0. Note that F is a preserving-volume diffeomorphism iff F preserves the Lebesgue measure.

It is obvious that a symplectomorphism F is also a preserving-volume diffeomorphism since

$$F^*\Omega = \Omega,$$

where $\Omega = \wedge^n \omega_0$ is the volume form associated to ω_0.
Note that F is a preserving-volume diffeomorphism iff F preserves the Lebesgue measure.
Question 1 : Is the group of symplectomorphisms significantly smaller than the group of preserving-volume diffeomorphisms ?

It is obvious that a symplectomorphism F is also a preserving-volume diffeomorphism since

$$F^*\Omega = \Omega,$$

where $\Omega = \wedge^n \omega_0$ is the volume form associated to ω_0.
Note that F is a preserving-volume diffeomorphism iff F preserves the Lebesgue measure.
Question 1 : Is the group of symplectomorphisms significantly smaller than the group of preserving-volume diffeomorphisms ?
Question 2 : If the answer to the first question is yes, can one find a topological characterization of a symplectomorphism ?

The symplectic cylinder of radius $R > 0$ is

$$Z^{2n}(R) = \{(x, \ldots, y) \in \mathbb{R}^{2n},\ x_1^2 + y_1^2 \leq R^2\} \simeq B^2(R) \times \mathbb{R}^{2n-2}.$$

We denote by $B^{2n}(r)$ the Euclidean closed ball of center 0 and the radius r in $\mathbb{R}^{2n}$.

The symplectic cylinder of radius $R > 0$ is

$$Z^{2n}(R) = \{(x, \ldots, y) \in \mathbb{R}^{2n},\ x_1^2 + y_1^2 \leq R^2\} \simeq B^2(R) \times \mathbb{R}^{2n-2}.$$

We denote by $B^{2n}(r)$ the Euclidean closed ball of center 0 and the radius r in $\mathbb{R}^{2n}$. Note that

$$B^{2n}(R) \subset Z^{2n}(R).$$

The symplectic cylinder of radius $R > 0$ is

$$Z^{2n}(R) = \{(x, \dots, y) \in \mathbb{R}^{2n},\ x_1^2 + y_1^2 \le R^2\} \simeq B^2(R) \times \mathbb{R}^{2n-2}.$$

We denote by $B^{2n}(r)$ the Euclidean closed ball of center 0 and the radius r in $\mathbb{R}^{2n}$. Note that

$$B^{2n}(R) \subset Z^{2n}(R).$$

Theorem.

(**Gromov 1985**) *If there is a symplectic embedding* $F : B^{2n}(r) \hookrightarrow Z^{2n}(R)$ *then* $r \le R$.

A **symplectic capacity** is a functor $\mathfrak{c}$ which assigns to every symplectic manifold (M, ω) a nonnegative (possibly infinite) number $\mathfrak{c}(M, \omega)$ and satisfies the following conditions.

A **symplectic capacity** is a functor $\mathfrak{c}$ which assigns to every symplectic manifold (M, ω) a nonnegative (possibly infinite) number $\mathfrak{c}(M, \omega)$ and satisfies the following conditions.

- (**Monotonicity**) If there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$ then $\mathfrak{c}(M_1, \omega_1) \leq \mathfrak{c}(M_2, \omega_2)$.

A **symplectic capacity** is a functor $\mathfrak{c}$ which assigns to every symplectic manifold (M, ω) a nonnegative (possibly infinite) number $\mathfrak{c}(M, \omega)$ and satisfies the following conditions.

- (**Monotonicity**) If there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$ then $\mathfrak{c}(M_1, \omega_1) \leq \mathfrak{c}(M_2, \omega_2)$.
- (**Conformality**) $\mathfrak{c}(M, \lambda\omega) = |\lambda|\mathfrak{c}(M, \omega)$.

A **symplectic capacity** is a functor $\mathfrak{c}$ which assigns to every symplectic manifold (M, ω) a nonnegative (possibly infinite) number $\mathfrak{c}(M, \omega)$ and satisfies the following conditions.

- (**Monotonicity**) If there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$ then $\mathfrak{c}(M_1, \omega_1) \leq \mathfrak{c}(M_2, \omega_2)$.
- (**Conformality**) $\mathfrak{c}(M, \lambda\omega) = |\lambda| \mathfrak{c}(M, \omega)$.
- (**Non triviality**) $\mathfrak{c}(B^{2n}(1), \omega_0) > 0$ and $\mathfrak{c}(Z^{2n}(1), \omega_0) < \infty$.

Proposition.

The existence of a symplectic capacity $\mathfrak{c}$ satisfying

$$\mathfrak{c}(B^{2n}(1), \omega_0) = \mathfrak{c}(Z^{2n}(1), \omega_0) = \pi \tag{1}$$

is equivalent to Gromov's nonsqueezing theorem.

Proposition.

The existence of a symplectic capacity $\mathfrak{c}$ *satisfying*

$$\mathfrak{c}(B^{2n}(1), \omega_0) = \mathfrak{c}(Z^{2n}(1), \omega_0) = \pi \tag{1}$$

is equivalent to Gromov's nonsqueezing theorem.

Proof : The direct sense is trivial. The converse is based on the Gromov width.

Proposition.

The existence of a symplectic capacity $\mathfrak{c}$ satisfying

$$\mathfrak{c}(B^{2n}(1), \omega_0) = \mathfrak{c}(Z^{2n}(1), \omega_0) = \pi \qquad (1)$$

is equivalent to Gromov's nonsqueezing theorem.

Proof : The direct sense is trivial. The converse is based on the Gromov width. For any symplectic $2n$-dimensional manifold (M, ω), put

$$\mathfrak{c}_G(M, \omega) = \sup \mathrm{E}(M, \omega),$$

where

$$\mathrm{E}(M, \omega) = \left\{\pi r^2 |\ (B^{2n}(r), \omega_0) \text{ embeds symplectically in } M\right\}.$$

□

The key to understanding symplectic capacities is the observation that the non triviality axiom makes it impossible for the volume of M to be a capacity. The requirement that $\mathfrak{c}(Z^{2n}(1), \omega_0)$ be finite means that these capacities are 2-dimensional invariants.

The existence of symplectic capacities is non trivial.

The existence of symplectic capacities is non trivial.
In this course we give another proof of Gromov's theorem using the notion of symplectic capacity, namely, the symplectic capacity introduced by Hofer-Zehnder in [4] based on the highly difficult theorem :

Theorem.

(**Hofer-Zehnder 1990**) *Assume* $H \in \mathcal{H}(Z^{2n}(1))$ *with* $\sup H > \pi$. *Then the Hamiltonian flow of* H *has a nonconstant periodic orbit of period 1.*

Affine nonsqueezing theorem
Symplectic vector spaces

Let $(e_1, \ldots, e_{2n})$ denote the canonical basis of $\mathbb{R}^{2n}$. The bilinear skew-symmetric 2-form

$$\omega_0 = \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^*$$

is non-degenerate, i.e.,

$$\omega_0(u, v) = 0 \quad \forall v \in \mathbb{R}^{2n} \implies u = 0.$$

The couple $(\mathbb{R}^{2n}, \omega_0)$ is the standard example of symplectic vector space.

More generally, a **symplectic vector space** is a couple (V, ω) where V is finite dimensional $\mathbb{R}$-vector space and ω is a bilinear skew-symmetric 2-form on V which is nondegenerate.

More generally, a **symplectic vector space** is a couple (V, ω) where V is finite dimensional $\mathbb{R}$-vector space and ω is a bilinear skew-symmetric 2-form on V which is nondegenerate. This means that ω satisfies :

1. ω is bilinear ;
2. for any $u, v \in V$, $\omega(u, v) = -\omega(v, u)$;
3. for any $u \in V$,

$$\omega(u, v) = 0 \quad \forall v \in V \implies u = 0.$$

A symplectic vector space must be even dimensional.

Let (V, ω) be a symplectic vector.

- A **linear symplectomorphism** of V is a vector space isomorphism $\Phi : V \longrightarrow V$ which preserves the symplectic form ω, i.e., for any $u, v \in V$,

$$\Phi^*\omega(u, v) := \omega(\Phi u, \Phi v) = \omega(u, v).$$

Let (V, ω) be a symplectic vector.

- A **linear symplectomorphism** of V is a vector space isomorphism $\Phi : V \longrightarrow V$ which preserves the symplectic form ω, i.e., for any $u, v \in V$,

$$\Phi^*\omega(u, v) := \omega(\Phi u, \Phi v) = \omega(u, v).$$

  The linear symplectomorphisms of (V, ω) form a group which we denote by $\mathrm{Sp}(V, \omega)$. In the case of the standard symplectic structure on $\mathbb{R}^{2n}$, we denote $\mathrm{Sp}(2n) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

Let (V, ω) be a symplectic vector.

- A **linear symplectomorphism** of V is a vector space isomorphism $\Phi : V \longrightarrow V$ which preserves the symplectic form ω, i.e., for any $u, v \in V$,

$$\Phi^*\omega(u, v) := \omega(\Phi u, \Phi v) = \omega(u, v).$$

  The linear symplectomorphisms of (V, ω) form a group which we denote by $\mathrm{Sp}(V, \omega)$. In the case of the standard symplectic structure on $\mathbb{R}^{2n}$, we denote $\mathrm{Sp}(2n) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

- Let $W \subset V$ be a vector subspace. The **symplectic orthogonal** of W is the vector space

$$W^\omega = \{u \in V, \omega(u, v) = 0 \ \forall v \in W\}.$$

Proposition.

We have

$$\dim W^{\omega} + \dim W = \dim V \quad \textit{and} \quad (W^{\omega})^{\omega} = W.$$

Proposition.

We have

$$\dim W^\omega + \dim W = \dim V \quad \textit{and} \quad (W^\omega)^\omega = W.$$

Theorem.

Let (V, ω) *be a symplectic vector space of dimension* $2n$. *Then there exists a basis* $(e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n)$ *such that*

$$\omega(e_i, e_j) = \omega(\bar{e}_i, \bar{e}_j) = 0 \quad \textit{and} \quad \omega(e_i, \bar{e}_j) = \delta_{ij}.$$

Such a basis is called a **symplectic basis**. *Moreover, there exists a vector space isomorphism* $\Phi : \mathbb{R}^{2n} \longrightarrow V$ *such that*

$$\Phi^* \omega = \omega_0.$$

The **volume form** associated to a symplectic vector space (V, ω) is the $2n$-form given by

$$\Omega = \omega^n = \overbrace{\omega \wedge \ldots \wedge \omega}^{n} .$$

Note that $\Omega \neq 0$ and, more precisely, if $(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)$ is a symplectic basis then

$$\Omega = n! \left(e_1^* \wedge \bar{e}_1^* \wedge \ldots \wedge e_n^* \wedge \bar{e}_n^* \right) .$$

Linear symplectic group

Let $\mathbb{B}_0$ be the canonical basis of $\mathbb{R}^{2n}$ and $\langle\ ,\ \rangle$ the Euclidean inner product of $\mathbb{R}^{2n}$. The matrix of ω_0 in $\mathbb{B}_0$ is the matrix

$$\mathrm{J}_0 = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

We have obviously $\mathrm{J}_0^2 = -\mathrm{I}_{2n}$,

$$\langle \mathrm{J}_0 u, \mathrm{J}_0 v \rangle = \langle u, v \rangle \quad \text{and} \quad \omega_0(u, v) = \langle \mathrm{J}_0 u, v \rangle. \tag{2}$$

An isomorphism of $\mathbb{R}^{2n}$ is a linear symplectomorphism iff its matrix Φ in $\mathbb{B}_0$ satisfies

$$\Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0. \tag{3}$$

An isomorphism of $\mathbb{R}^{2n}$ is a linear symplectomorphism iff its matrix Φ in $\mathbb{B}_0$ satisfies

$$\Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0. \tag{3}$$

$$\mathrm{Sp}(2n) \simeq \left\{ \Phi \in \mathrm{GL}(2n, \mathbb{R}), \Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0. \right\}.$$

An isomorphism of $\mathbb{R}^{2n}$ is a linear symplectomorphism iff its matrix Φ in $\mathbb{B}_0$ satisfies

$$\Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0. \tag{3}$$

$$\mathrm{Sp}(2n) \simeq \left\{\Phi \in \mathrm{GL}(2n, \mathbb{R}), \Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0.\right\}.$$

We have

$$\mathrm{Sp}(2n) \subset \mathrm{SL}(2n, \mathbb{R}) := \{\Phi \in \mathrm{GL}(2n, \mathbb{R}),\ \det \Phi = 1\}.$$

An isomorphism of $\mathbb{R}^{2n}$ is a linear symplectomorphism iff its matrix Φ in $\mathbb{B}_0$ satisfies

$$\Phi^{\mathrm{T}}\mathrm{J}_0\Phi = \mathrm{J}_0. \tag{3}$$

$$\mathrm{Sp}(2n) \simeq \left\{\Phi \in \mathrm{GL}(2n,\mathbb{R}), \Phi^{\mathrm{T}}\mathrm{J}_0\Phi = \mathrm{J}_0.\right\}.$$

We have

$$\mathrm{Sp}(2n) \subset \mathrm{SL}(2n,\mathbb{R}) := \{\Phi \in \mathrm{GL}(2n,\mathbb{R}),\ \det\Phi = 1\}.$$

Since $J_0^{-1} = -J_0$ we get that $\Phi \in \mathrm{Sp}(2n)$ iff $\Phi^{\mathrm{T}} \in \mathrm{Sp}(2n)$.

We identify $\mathrm{GL}(n, \mathbb{C})$ as

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C}) &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, X, Y \in \mathrm{GL}(n, \mathbb{R}) \right\} \\ &= \left\{ \Phi \in \mathrm{GL}(2n, \mathbb{R}), \Phi \mathrm{J}_0 = \mathrm{J}_0 \Phi \right\}. \end{aligned}$$

We identify $\mathrm{GL}(n,\mathbb{C})$ as

$$\begin{aligned} \mathrm{GL}(n,\mathbb{C}) &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, X, Y \in \mathrm{GL}(n,\mathbb{R}) \right\} \\ &= \left\{ \Phi \in \mathrm{GL}(2n,\mathbb{R}), \Phi \mathrm{J}_0 = \mathrm{J}_0 \Phi \right\}. \end{aligned}$$

The unitary group is identified to

$$\mathrm{U}(n) = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(n,\mathbb{C}), (X + \imath Y)(X - \imath Y)^{\mathrm{T}} = \mathrm{I}_n \right\}.$$

Lemma.

We have

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n).$$

Lemma.

We have

$$\mathrm{Sp}(2n)\cap\mathrm{O}(2n) = \mathrm{Sp}(2n)\cap\mathrm{GL}(n,\mathbb{C}) = \mathrm{O}(2n)\cap\mathrm{GL}(n,\mathbb{C}) = \mathrm{U}(n).$$

Let Φ be a $2n \times 2n$ real matrix. We have the following equivalence :

$$\begin{aligned}
\Phi \in \mathrm{GL}(n,\mathbb{C}) \quad &\Longleftrightarrow \quad \Phi \mathrm{J}_0 = \mathrm{J}_0 \Phi, \\
\Phi \in \mathrm{Sp}(2n) \quad &\Longleftrightarrow \quad \Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0, \\
\Phi \in \mathrm{O}(2n) \quad &\Longleftrightarrow \quad \Phi^{\mathrm{T}} \Phi = \mathrm{I}_{2n}.
\end{aligned}$$

It is obvious that any of these conditions imply the third.

Now, according to (**??**), the subgroup $\mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C})$ consists of this matrix

$$\Phi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$$

which satisfy

$$XY^{\mathrm{T}} = YX^{\mathrm{T}} \quad \text{and} \quad XX^{\mathrm{T}} + YY^{\mathrm{T}} = \mathrm{I}_n.$$

This is precisely the condition

$$(X + \imath Y)(X - \imath Y)^{\mathrm{T}} = \mathrm{I}_n.$$

□

Lemma.

If $P = P^{\mathrm{T}} \in \mathrm{Sp}(2n)$ *is symmetric, positive definite symplectic matrix then* $P^{\alpha} \in \mathrm{Sp}(2n)$ *for any real number* $\alpha > 0$.

Lemma.

If $P = P^{\mathrm{T}} \in \mathrm{Sp}(2n)$ is symmetric, positive definite symplectic matrix then $P^\alpha \in \mathrm{Sp}(2n)$ for any real number $\alpha > 0$.

Proof. We will show that, for any $z, z' \in \mathbb{R}^{2n}$,

$$\omega_0(P^\alpha z, P^\alpha z') = \omega_0(z, z'). \qquad (*)$$

First, denote by $0 < \lambda_1 < \ldots < \lambda_r$ the different eigenvalues of P and $V_{\lambda_1}, \ldots, V_{\lambda_r}$ the corresponding eigenspaces. We have

$$\mathbb{R}^{2n} = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}.$$

We distinguish two cases :

- $z \in V_{\lambda_i}$, $z' \in V_{\lambda_j}$ and $\lambda_i \lambda_j \neq 1$. Then $P^\alpha z = \lambda_i^\alpha z$ and $P^\alpha z' = \lambda_j^\alpha z'$ and according to Lemma **??** $\omega_0(z, z') = 0$ and $(*)$ holds.
- $z \in V_{\lambda_i}$, $z' \in V_{\lambda_j}$ and $\lambda_i \lambda_j = 1$. Then $P^\alpha z = \lambda_i^\alpha z$, $P^\alpha z' = \lambda_j^\alpha z'$ and $(*)$ holds. $\square$

Proposition.

The unitary group $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(2n)$ *and the quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible.*

Proposition.

The unitary group $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(2n)$ *and the quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible.*

Proof : Thus the map

$$\mathrm{Sp}(2n) \times [0,1] \longrightarrow \mathrm{Sp}(2n) \ : \ (\Phi, t) \mapsto (\Phi\Phi^{\mathrm{T}})^{-\frac{t}{2}}\Phi$$

is a retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$.

Proposition.

The unitary group $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(2n)$ *and the quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible.*

Proof : Thus the map

$$\mathrm{Sp}(2n) \times [0,1] \longrightarrow \mathrm{Sp}(2n) \; : \; (\Phi, t) \mapsto (\Phi\Phi^{\mathrm{T}})^{-\frac{t}{2}}\Phi$$

is a retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$.
Let $\mathrm{G} \subset \mathrm{Sp}(2n)$ be any compact subgroup. Put $P = \int_G g^{\mathrm{T}} g dg$. We have, for any $\Phi \in \mathrm{G}$,

$$\Phi^{\mathrm{T}} P \Phi = P$$

and hence

$$P^{\frac{1}{2}} \mathrm{G} P^{-\frac{1}{2}} \subset \mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n).$$

□

An **affine symplectomorphism** of $\mathbb{R}^{2n}$ is a map $\phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ of the form

$$\phi(z) = \Phi z + z_0,$$

where $\Phi \in \mathrm{Sp}(2n)$ and $z_0 \in \mathbb{R}^{2n}$. We denote by $\mathrm{ASp}(2n)$ the group of affine symplectomorphisms. The affine nonsqueezing theorem asserts that a ball in $\mathbb{R}^{2n}$ can only be embedded into a symplectic cylinder by an affine symplectomorphism if it has a smaller radius. The symplectic cylinder of radius $R > 0$ is

$$Z^{2n}(R) = \left\{(x, y) \in \mathbb{R}^{2n},\ x_1^2 + y_1^2 \leq R^2\right\} \simeq B^2(R) \times \mathbb{R}^{2n-2}.$$

We denote the Euclidean closed ball of center 0 and the radius r in $\mathbb{R}^{2n}$ by $B^{2n}(r)$.

Theorem.

Let $\phi \in \mathrm{ASp}(2n)$ *and assume that* $\phi(B^{2n}(r)) \subset Z^{2n}(R)$. *Then* $r \leq R$.

Theorem.

Let $\phi \in \mathrm{ASp}(2n)$ and assume that $\phi(B^{2n}(r)) \subset Z^{2n}(R)$. Then $r \leq R$.

Proof. Write $\phi(z) = \Phi(z) + z_0$ with $\Phi \in \mathrm{Sp}(2n)$ and $z_0 \in \mathbb{R}^{2n}$ and denote by $(e_1, \ldots, e_{2n})$ the canonical basis of $\mathbb{R}^{2n}$. The condition $\phi(B^{2n}(r)) \subset Z^{2n}(R)$ is equivalent to

$$\forall u \in B^{2n}(r), \quad ((\Phi(u))_1 + z_0^1)^2 + ((\Phi(u))_{n+1} + z_0^{n+1})^2 \leq R^2. \tag{$*$}$$

Now it is easy to see that

$$(\Phi(u))_1 = \langle \Phi^{\mathrm{T}} e_1, u \rangle \quad \text{and} \quad (\Phi(u))_{n+1} = \langle \Phi^{\mathrm{T}} e_{n+1}, u \rangle.$$

The crucial point is that since $\Phi^{\mathrm{T}} \in \mathrm{Sp}(2n)$,

$$\omega_0(\Phi^{\mathrm{T}} e_1, \Phi^{\mathrm{T}} e_{n+1}) = \omega_0(e_1, e_{n+1}) = 1.$$

So, by using (2) and the Cauchy-Schwarz inequality, we get

$$1 = \omega_0(\Phi^{\mathrm{T}} e_1, \Phi^{\mathrm{T}} e_{n+1}) \leq |\Phi^{\mathrm{T}} e_1||\Phi^{\mathrm{T}} e_{n+1}|.$$

This inequality implies that either $|\Phi^{\mathrm{T}} e_1|$ or $|\Phi^{\mathrm{T}} e_{n+1}|$ is greater than or equal to one. Assume without loss of generality that $|\Phi^{\mathrm{T}} e_1| \geq 1$ and choose in $(*)$ $u = \epsilon r \frac{\Phi^{\mathrm{T}} e_1}{|\Phi^{\mathrm{T}} e_1|}$ where ϵ is the sign of z_0^1. We get

$$r^2 \leq (r|\Phi^{\mathrm{T}} e_1| + |z_0^1|)^2 + ((\Phi(u))_{n+1} + z_0^{n+1})^2 \leq R^2,$$

and the theorem follows. □

We call a subset $A \subset \mathbb{R}^{2n}$ a **linear symplectic ball** of **radius** r if there exists $\Phi \in \mathrm{Sp}(2n)$ such that $A = \Phi(B^{2n}(r))$. It results that A and $B^{2n}(r)$ must have the same volume and hence r does not depend on Φ.

We call a subset $A \subset \mathbb{R}^{2n}$ a **linear symplectic ball** of **radius** r if there exists $\Phi \in \mathrm{Sp}(2n)$ such that $A = \Phi(B^{2n}(r))$. It results that A and $B^{2n}(r)$ must have the same volume and hence r does not depend on Φ.
In a similar way, a subset $Z \in \mathbb{R}^{2n}$ is called **linear symplectic cylinder** if there exists $\Phi \in \mathrm{Sp}(2n)$ and $r > 0$ such that $Z = \Phi(Z^{2n}(r))$. It follows from Theorem 9 that for any linear symplectic cylinder Z the number $r > 0$ is a linear symplectic invariant.

A nonsingular $2n \times 2n$ matrix Φ is said to have the **linear nonsqueezing property** if for every linear symplectic ball B of radius r and every linear symplectic cylinder Z of radius R we have

$$\Phi(B) \subset Z \quad \Longrightarrow \quad r \leq R.$$

A nonsingular $2n \times 2n$ matrix Φ is said to have the **linear nonsqueezing property** if for every linear symplectic ball B of radius r and every linear symplectic cylinder Z of radius R we have

$$\Phi(B) \subset Z \quad \Longrightarrow \quad r \leq R.$$

Theorem.

Let Φ be a non singular $2n \times 2n$ matrix such that Φ and Φ^{-1} have the linear nonsqueezing property. Then Φ is either symplectic or anti-symplectic.

Proof.

Suppose that Φ is neither symplectic or anti-symplectic. Then there exist $u, v \in \mathbb{R}^{2n}$ such that

$$0 < \lambda^2 = \omega_0(\Phi^{\mathrm{T}} u, \Phi^{\mathrm{T}} v) < \omega_0(u, v) = 1.$$

Proof.

Suppose that Φ is neither symplectic or anti-symplectic. Then there exist $u, v \in \mathbb{R}^{2n}$ such that

$$0 < \lambda^2 = \omega_0(\Phi^{\mathrm{T}} u, \Phi^{\mathrm{T}} v) < \omega_0(u, v) = 1.$$

Hence there exist symplectic bases $\mathbb{B}_1 = (u_1, v_1, \ldots, u_n, v_n)$ and $\mathbb{B}_2 = (u'_1, v'_1, \ldots, u'_n, v'_n)$ of $\mathbb{R}^{2n}$ such that

$$u_1 = u, \quad v_1 = v, \quad u'_1 = \lambda^{-1}\Phi^{\mathrm{T}} u, \quad v'_1 = \lambda^{-1}\Phi^{\mathrm{T}} v.$$

Denote by $\Psi \in \mathrm{Sp}(2n)$ (resp. $\Psi' \in \mathrm{Sp}(2n)$) the matrix which maps $\mathbb{B}_0$ to $\mathbb{B}_1$ (resp. $\mathbb{B}_2$).

Then the matrix

$$A = \Psi'^{-1}\Phi^{\mathrm{T}}\Psi$$

satisfies

$$Ae_1 = \lambda e_1 \quad \text{and} \quad Ae_{n+1} = \pm\lambda e_{n+1}.$$

This implies that the transposed matrix A^{T} maps the unit ball $B^{2n}(1)$ to cylinder $Z^{2n}(\lambda)$. But since $\lambda < 1$ this means that Φ does not have the nonsqueezing property in contradiction to our assumption. This proves the theorem. □

The affine nonsqueezing theorem gives rise to the notion of the **linear symplectic width** of an arbitrary subset $A \subset \mathbb{R}^{2n}$, defined by

$$\mathfrak{W}_L(A) = \sup \left\{ \pi r^2 \,|\, \phi(B^{2n}(r)) \subset A \text{ for some } \phi \in \mathrm{ASp}(\mathbb{R}^{2n}) \right\}.$$

The affine nonsqueezing theorem gives rise to the notion of the **linear symplectic width** of an arbitrary subset $A \subset \mathbb{R}^{2n}$, defined by

$$\mathfrak{W}_L(A) = \sup\left\{\pi r^2 |\ \phi(B^{2n}(r)) \subset A \text{ for some } \phi \in \mathrm{ASp}(\mathbb{R}^{2n})\right\}.$$

It follows from Theorem 9 that the linear symplectic width has the following properties :

- (**Monotonicity**) If $\phi(A) \subset B$ for some $\phi \in \mathrm{ASp}(\mathbb{R}^{2n})$ then $\mathfrak{W}_L(A) \leq \mathfrak{W}_L(B)$.
- (**Conformality**) $\mathfrak{W}_L(\lambda A) = \lambda^2 \mathfrak{W}_L(A)$.
- (**Nontriviality**) $\mathfrak{W}_L(B^{2n}(r)) = \mathfrak{W}_L(Z^{2n}(r)) = \pi r^2$.

The nontriviality axiom implies that $\mathfrak{W}_L$ is a two-dimensional invariant. It is obvious from the monotonicity property that affine symplectomorphisms preserve the linear symplectic width. We shall prove that this property in fact characterizes symplectic and anti-symplectic linear maps.

Recall that an ellipsoid centered at 0 in the Euclidean space $\mathbb{R}^{2n}$ is given by

$$E = \left\{ x \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} x_i x_j = \langle Ax, x \rangle \leq 1 \right\}$$

where the $2n \times 2n$ matrix $A = (a_{ij})$ is symmetric positive definite.

Recall that an ellipsoid centered at 0 in the Euclidean space $\mathbb{R}^{2n}$ is given by

$$E = \left\{ x \in \mathbb{R}^{2n} \middle| \sum_{i,j=1}^{2n} a_{ij} x_i x_j = \langle Ax, x \rangle \leq 1 \right\}$$

where the $2n \times 2n$ matrix $A = (a_{ij})$ is symmetric positive definite.

Proposition.

1. *For any $r > 0$ and for any Φ an isomorphism, $\Phi(B^{2n}(r))$ is an ellipsoid centered at 0.*
2. *If E is an ellipsoid centered at 0 then for any $r > 0$ there exists Φ an isomorphism such that $E = \Phi(B^{2n}(r))$.*

There exists $\Phi \in \mathrm{O}(2n)$ such that

$$\Phi^{-1}(E) = \left\{ (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n} | \sum_{i=1}^{2n} \frac{x_i^2}{\rho_i^2} \leq 1 \right\},$$

where $\rho_i = \sqrt{\lambda_i^{-1}}$ and $0 < \lambda_1 \leq \dots \leq \lambda_{2n}$ are the eigenvalues of the matrix (a_{ij}).

Symplectically an ellipsoid can be characterized as follows.

Lemma.

Given any ellipsoid

$$E = \left\{ (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \middle| \sum_{i,j=1}^{2n} a_{ij} x_i x_j \leq 1 \right\}$$

there is a linear symplectomorphism $\Phi \in \mathrm{Sp}(2n)$ *such that*

$$\Phi(E) = E(r) := \left\{ (x, y) \in \mathbb{R}^{2n} \middle| \sum_{j=1}^{n} \frac{x_j^2 + y_j^2}{r_j^2} \leq 1 \right\},$$

for some n-uple $r = (r_1, \ldots, r_n)$ *with* $0 < r_1 \leq \ldots \leq r_n$.
Moreover, r *is entirely determined by* E.

Proof.

Since ω_0 is nondegenerate there exists a skew-symmetric (with respect to $\langle\ ,\ \rangle_A$) nonsingular endomorphism J such that

$$\omega_0(u, v) = \langle Ju, v\rangle_A.$$

According to a classical result in linear algebra there exists an orthonormal basis of $\langle\ ,\ \rangle_A$ say $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ and a family of real number $0 < a_1 \leq \ldots \leq a_n$ such that, for $i = 1, \ldots, n$,

$$Ju_i = a_i v_i \quad \text{and} \quad Jv_i = -a_i u_i.$$

For $i = 1, \ldots, n$, put $u_i' = \sqrt{a_i^{-1}}u_i$ and $v_i' = \sqrt{a_i^{-1}}v_i$. It is easy to check that $(u_1', \ldots, u_n', v_1', \ldots, v_n')$ is a symplectic basis of $\mathbb{R}^{2n}$. Denote by Φ the element of $\mathrm{Sp}(2n)$ which maps the canonical basis to this basis.

Now, we have

$$\begin{aligned}
\langle u, u\rangle_A &= \omega_0(J^{-1}u, u) \\
&= \sum_{i=1}^{n} \left(\omega_0(J^{-1}u, v_i')\omega_0(u_i', u) - \omega_0(J^{-1}u, u_i')\omega_0(v_i', u)\right) \\
&= \sum_{i=1}^{n} \left(\omega_0(J^{-1}v_i', u)\omega_0(\Phi e_i, u) - \omega_0(J^{-1}u_i', u)\omega_0(\Phi e_{n+1}, u)\right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{a_i}(\omega_0(u_i', u)\omega_0(\Phi e_i, u) + \omega_0(v_i', u)\omega_0(\Phi e_{n+1}, u))\right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{a_i}(\omega_0(\Phi e_i, u)\omega_0(\Phi e_i, u) + \omega_0(\Phi e_{n+1}, u)\omega_0(\Phi e_{n+1}, u))\right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{a_i}(\omega_0(e_i, \Phi^{-1}u)^2 + \omega_0(e_{n+1}, \Phi^{-1}u)^2)\right),
\end{aligned}$$

and the first statement of the lemma follows.

To prove uniqueness of the n-uple $r_1 \leq \ldots \leq r_n$ consider the diagonal matrix

$$D(r) = \mathrm{diag}(1/r_1^2, \ldots, 1/r_n^2, 1/r_1^2, \ldots, 1/r_n^2).$$

We must show that if there is a symplectic matrix Φ such that

$$\Phi^{\mathrm{T}} D(r) \Phi = D(r')$$

then $r = r'$. Since $\mathrm{J}_0 \Phi^{\mathrm{T}} = \Phi^{-1} \mathrm{J}_0$ the above identity is equivalent to

$$\Phi^{-1} \mathrm{J}_0 D(r) \Phi = \mathrm{J}_0 D(r').$$

Hence $\mathrm{J}_0 D(r)$ and $\mathrm{J}_0 D(r')$ have the same eigenvalues. But it is easy the check that the eigenvalues of $\mathrm{J}_0 D(r)$ are $\pm \imath / r_1^2, \ldots, \pm \imath / r_n^2$. This proves the lemma. $\square$

In view of Lemma 12 we define the **symplectic spectrum** of an ellipsoid E to be the unique n-uple $r = (r_1, \ldots, r_n)$ with $0 < r_1 \leq \ldots \leq r_n$ such that E is linearly symplectomorphic to $E(r)$. The spectrum is invariant under linear symplectomorphisms and, in fact, two ellipsoids in $\mathbb{R}^n$, which are centered at 0, are linearly symplectomorphic if and only if they have the same spectrum. Moreover, the volume of an ellipsoid $E \subset \mathbb{R}^{2n}$ is given by

$$\mathrm{Vol}(E) = \int_E \frac{\omega_0^n}{n!} = \pi^n \prod_{i=1}^{n} r_i^2.$$

In view of Lemma 12 we define the **symplectic spectrum** of an ellipsoid E to be the unique n-uple $r = (r_1, \ldots, r_n)$ with $0 < r_1 \leq \ldots \leq r_n$ such that E is linearly symplectomorphic to $E(r)$. The spectrum is invariant under linear symplectomorphisms and, in fact, two ellipsoids in $\mathbb{R}^n$, which are centered at 0, are linearly symplectomorphic if and only if they have the same spectrum. Moreover, the volume of an ellipsoid $E \subset \mathbb{R}^{2n}$ is given by

$$\mathrm{Vol}(E) = \int_E \frac{\omega_0^n}{n!} = \pi^n \prod_{i=1}^{n} r_i^2.$$

Note that

$$B^{2n}(r_1) \subset E(r_1, \ldots, r_n) \subset Z^{2n}(r_1).$$

Thus

$$\mathfrak{W}_L(E(r_1, \ldots, r_n)) = \pi r_1^2.$$

The following theorem characterizes the linear symplectic width of an ellipsoid in terms of the spectrum.

Proposition.

Let $E \subset \mathbb{R}^{2n}$ an ellipsoid centered at 0. Then

$$\mathfrak{W}_L(E) = \pi r_1^2,$$

where $r = (r_1, \ldots, r_n)$ is the symplectic spectrum associated to E.

We finish this section by the following characterization of linear symplectic or anti-symplectic maps.

Theorem.

Let $\Phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a linear map. Then the following are equivalent.
(i) Φ *preserves the linear width of ellipsoids centered at 0.*
(ii) *The matrix* Φ *is either symplectic or anti-symplectic, i.e.,* $\Phi^*\omega_0 = \pm\omega_0$.

Proof.

(ii) implies (i) is obvious. Now assume (i). Note first that Φ is invertible and Φ^{-1} preserves the linear width of ellipsoids centered at 0. Indeed,

$$\mathfrak{W}_L(\Phi^{-1}E) = \mathfrak{W}_L(\Phi\Phi^{-1}E) = \mathfrak{W}_L(E)$$

for every ellipsoid E which is centered at zero.
We shall prove that Φ has the nonsqueezing property. To see this let B be a linear symplectic ball or radius r and Z be a linear symplectic cylinder of radius R such that

$$\Phi B \subset Z.$$

Then it follows that

$$\pi r^2 = \mathfrak{W}_L(B) = \mathfrak{W}_L(\Phi B) \leq \mathfrak{W}_L(Z) = \pi R^2$$

and hence $r \leq R$. □

Symplectic manifolds and Hamiltonian flows

A **symplectic structure** on a manifold M is non-degenerate closed 2-form $\omega \in \Omega^2(M)$, i.e., ω is a differential 2-form such that :

1. for any $x \in M$, $(T_x M, \omega_x)$ is a symplectic vector space,
2. $d\omega = 0$.

The couple (M, ω) is called **symplectic manifold**.

Symplectic manifolds and Hamiltonian flows

A **symplectic structure** on a manifold M is non-degenerate closed 2-form $\omega \in \Omega^2(M)$, i.e., ω is a differential 2-form such that :

1. for any $x \in M$, (T_xM, ω_x) is a symplectic vector space,
2. $d\omega = 0$.

The couple (M, ω) is called **symplectic manifold**.
Let (M, ω) be symplectic manifold. The nondegeneracy implies to the existence of a canonical isomorphism between the tangent and the cotangent bundle, namely,

$$\omega^\flat : TM \longrightarrow T^*M : \quad u \longrightarrow i_u\omega = \omega(u, .).$$

In particular, for any function $H \in C^\infty(M)$, there exists a unique vector field denoted by X_H such that

$$i_{X_H}\omega = dH. \tag{4}$$

The vector field X_H is called **Hamiltonian vector field** associated to H.

The vector field X_H is called **Hamiltonian vector field** associated to H.
On the other hand, the nondegeneracy is equivalent to the fact that the maximal form $\Omega = \wedge^n \omega$ is a volume form and hence any symplectic manifold is orientable.

The vector field X_H is called **Hamiltonian vector field** associated to H.
On the other hand, the nondegeneracy is equivalent to the fact that the maximal form $\Omega = \wedge^n \omega$ is a volume form and hence any symplectic manifold is orientable.
A **symplectomorphism** of (M, ω) is a diffeomorphism $\phi : M \longrightarrow M$ such that $\phi^* \omega = \omega$. We denote the group of symplectomorphisms by $\mathrm{Symp}(M, \omega)$.

A vector field X is called **symplectic** if its flow preserves ω, i.e., the Lie derivative of ω is the direction of X.

A vector field X is called **symplectic** if its flow preserves ω, i.e., the Lie derivative of ω is the direction of X.
Note that according to the Cartan's formula

$$\mathcal{L}_X\omega = di_X\omega + i_X d\omega$$

and since $d\omega = 0$, X is symplectic if and only if $i_X\omega$ is closed.
We denote by $\mathcal{X}(M,\omega)$ the space of symplectic vector fields.
It is obvious that any Hamiltonian vector field is symplectic.

Example.

1. $(\mathbb{R}^{2n}, \omega_0)$ *is the standard model of symplectic manifold.*

Example.

1. $(\mathbb{R}^{2n}, \omega_0)$ *is the standard model of symplectic manifold.*
2. *Any oriented surface* S *is a symplectic manifold.*

Example.

1. $(\mathbb{R}^{2n}, \omega_0)$ *is the standard model of symplectic manifold.*
2. *Any oriented surface S is a symplectic manifold.*
3. **The canonical symplectic structure of the cotangent bundle.** *Let L be a smooth manifold, consider T^*L the total space of its cotangent bundle and denote by $\pi : T^*L \longrightarrow L$ the canonical projection. The* **Liouville form** *in T^*L is the differential 1-form λ in T^*L given by*

$$\lambda(Z_\alpha) = \alpha(T_\alpha \pi(Z_\alpha)),$$

where $\alpha \in T^*L$ and $Z_\alpha \in T_\alpha(T^*L)$. Let $(q_1, \ldots, q_n)$ be a coordinates system on L and $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ the associated coordinates system on T^*L. Then

$$\lambda = \sum_{i=1}^{n} p_i dq_i.$$

Darboux's Theorem asserts that there is no local invariant in symplectic geometry, more precisely, in a given dimension all symplectic forms are locally diffeomorphic.

Theorem.

Let (M, ω) be a symplectic manifold and $m \in M$. Then there exists a coordinates system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Such coordinates are called **Darboux's coordinates**.

Proof.

According to Theorem 5 there is a coordinates system $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ defined on an open set U containing m such that if $\omega_1 = \sum_{i=1}^{n} dq_i \wedge dp_i$ then

$$\omega(m) = \omega_1(m).$$

Moreover, since $\omega_1 - \omega_0$ is closed there exists $\sigma \in \Omega^1(U)$ such that

$$d\sigma = \omega_1 - \omega_0.$$

For $t \in [0, 1]$ put $\omega_t = \omega + t d\sigma$. Since $\omega_t(m)$ is nondegenerate and $[0, 1]$ is compact, we can choose U such that ω_t is nondegenerate on U for every $t \in [0, 1]$.

We consider now the family of vector fields (X_t) defined by

$$i_{X_t}\omega_t = -\sigma$$

and Φ_t the family of diffeomorphisms defined by

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t \quad \text{and} \quad \Phi_0 = \mathrm{id}.$$

Since $X_t(m) = 0$ for every $t \in [0, 1]$ we can shrink U if necessary to get Φ_t defined for every $t \in [0, 1]$ and $\Phi_t(U) \subset U$. Now

$$\begin{aligned} \frac{d}{dt}\Phi_t^*\omega_t &= \Phi_t^*\left(\frac{d}{dt}\omega_t + i_{X_t}d\omega_t + di_{X_t}\omega_t\right) \\ &= \Phi_t^*\left(d\sigma - d\sigma\right) = 0, \end{aligned}$$

and hence $\Phi_1^*\omega_1 = \omega$ and the theorem follows. □

A **Hamiltonian system** is a triple (M, ω, H) where (M, ω) is a symplectic manifold and H a function on M. The Hamiltonian vector field X_H associated to H has a flow called **Hamiltonian flow** and its integral curves are solution of

$$\dot{x}(t) = X_H(x(t)).$$

If $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are Darboux's coordinates then this differential system is equivalent to

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \quad \text{and} \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n. \tag{5}$$

Example.

The harmonic oscillator is the Hamiltonian system $(\mathbb{R}^2, \omega_0, H)$ *with*

$$H(x, y) = \frac{1}{2}(x^2 + y^2).$$

The differential system (5) *is written*

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -x$$

which equivalent to

$$\dot{x} = y \quad \text{and} \quad \ddot{x} = -x.$$

The corresponding Hamiltonian flow is given by

$$\Phi_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

The Hofer-Zehnder Capacity

In this final section we establish the existence of the Hofer-Zehnder capacity and hence prove the Gromov's nonsqueezing theorem. This capacity is based on properties of the periodic orbits of Hamiltonian flows on a symplectic manifold (M, ω) and was introduced in [4].

Let (M, ω) be a symplectic manifold. Denote the set of all nonnegative Hamiltonian functions which are compactly supported on the interior of M and which attain their maximum on some open set by

$$\mathcal{H}(M) = \left\{ H \in C_0^\infty(\text{int}M) | H \geq 0, H_{|U} = \sup H \text{ form some open set } U \right\}.$$

For every function H consider the time-independent Hamiltonian flow $\phi_H^t \in \mathrm{Symp}^c(M, \omega)$ generated by the Hamiltonian vector field X_H.

For every function H consider the time-independent Hamiltonian flow $\phi_H^t \in \mathrm{Symp}^c(M, \omega)$ generated by the Hamiltonian vector field X_H.

An orbit $x(t) = \phi_H^t(x_0)$ is called T-**periodic** if $x(t + T) = x(t)$ for every $t \in \mathbb{R}$.

For every function H consider the time-independent Hamiltonian flow $\phi_H^t \in \mathrm{Symp}^c(M,\omega)$ generated by the Hamiltonian vector field X_H.

An orbit $x(t) = \phi_H^t(x_0)$ is called T-**periodic** if $x(t+T) = x(t)$ for every $t \in \mathbb{R}$.

Call a function $H \in \mathcal{H}(M)$ **admissible** if the corresponding Hamiltonian flow has no nonconstant T-periodic orbit with period $T \leq 1$. In other word, every nonconstant periodic orbit has period > 1.

For every function H consider the time-independent Hamiltonian flow $\phi_H^t \in \mathrm{Symp}^c(M, \omega)$ generated by the Hamiltonian vector field X_H.

An orbit $x(t) = \phi_H^t(x_0)$ is called T-**periodic** if $x(t+T) = x(t)$ for every $t \in \mathbb{R}$.

Call a function $H \in \mathcal{H}(M)$ **admissible** if the corresponding Hamiltonian flow has no nonconstant T-periodic orbit with period $T \leq 1$. In other word, every nonconstant periodic orbit has period > 1.

Denote the set of admissible Hamiltonian functions by

$$\mathcal{H}_{\mathrm{ad}}(M, \omega) = \{H \in \mathcal{H}(M) \mid H \text{ admissible}\}.$$

The following lemma shows that for every Hamiltonian function $H \in \mathcal{H}(M)$ the function ϵH is admissible for $\epsilon > 0$ sufficiently small. Roughly speaking, if a vector field is small then its orbits are slow and hence the period is long.

Lemma.

Let $x(t) = x(t+T) \in \mathbb{R}^m$ be a periodic solution of the differential equation

$$\dot{x}(t) = f(x),$$

where $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is continuously differentiable. If

$$T . \sup_x \|df(x)\| < 1$$

then $x(t)$ is constant.

Proof.

Since $x(0) = x(T)$ an easy calculation shows that

$$\dot{x}(t) = \int_0^t \frac{s}{T}\ddot{x}(s)ds + \int_t^T \frac{s-T}{T}\ddot{x}(s)ds.$$

This implies

$$|\dot{x}(t)| \leq \int_0^T |\ddot{x}(s)|ds \leq \sqrt{T}\|\ddot{x}\|_{L^2[0,T]}$$

and hence

$$\|\dot{x}\|_{L^2} \leq T\|\ddot{x}\|_{L^2}.$$

Note denote $\epsilon = \sup \|df(x)\|$ and note that

$$|\ddot{x}| \le \|df(x)\|.|\dot{x}| \le \epsilon|\dot{x}|.$$

Hence

$$\|\ddot{x}\|_{L^2} \le \epsilon\|\dot{x}\|_{L^2} \le \epsilon T\|\ddot{x}\|_{L^2}.$$

Since $\epsilon T < 1$ it follows $\ddot{x}(t) \equiv 0$. Hence $\dot{x}(t)$ is constant and periodic and hence $x(t)$ is constant. $\square$

The **Hofer-Zehnder capacity** of (M, ω) is defined by

$$\mathfrak{c}_{HZ}(M, \omega) = \sup_{H \in \mathcal{H}_{\mathrm{ad}}(M,\omega)} \|H\|$$

where $\|H\|$ is the **Hofer norm** given by

$$\|H\| = \sup_{x \in M} H(x) - \inf_{x \in M} H(x).$$

One can deduce easily from Lemma 18 that for every nonempty symplectic manifold (M, ω), $\mathfrak{c}_{HZ}(M, \omega) > 0$.

The following theorem is due to Hofer and Zehnder [4].

Theorem.

The map $(M, \omega) \mapsto \mathfrak{c}_{HZ}(M, \omega)$ *satisfies the monotonicity, conformality and normalization axioms of symplectic capacity. Moreover,*

$$\mathfrak{c}_{HZ}(B^{2n}(r), \omega_0) = \mathfrak{c}_{HZ}(Z^{2n}(r), \omega_0) = \pi r^2$$

for every $r > 0$*.*

The proof of this theorem rests on the following existence result for periodic orbits of Hamiltonian differential equation in $\mathbb{R}^{2n}$ which a proof will be given in the last section.

Theorem.

Assume $H \in \mathcal{H}(Z^{2n}(1))$ *with* $\sup H > \pi$. *Then the Hamiltonian flow of H has a nonconstant periodic orbit of period 1.*

Proof of Theorem 19.

Monotonicity.

Let $\phi : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$ be a symplectic embedding with $\dim M_1 = \dim M_2$.

Proof of Theorem 19.

Monotonicity.

Let $\phi : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$ be a symplectic embedding with $\dim M_1 = \dim M_2$.

If $H_1 : M_1 \longrightarrow \mathbb{R}$ is a compactly supported function then there is a unique compactly supported function $H_2 : M_2 \longrightarrow \mathbb{R}$ such that H_2 vanishes on $M_2 - \phi(M_1)$ and $H_1 = H_2 \circ \phi$. Since H_1 is compactly supported the function H_2 is smooth.

Since ϕ intertwine the Hamiltonian flows of H_1 et H_2 there is a one-to-one correspondence of nonconstant periodic orbits of these flow. Hence

$$\begin{aligned}
\mathfrak{c}_{HZ}(M_1,\omega_1) &= \sup_{H_1\in\mathcal{H}_{\mathrm{ad}}(M_1,\omega_1)} \|H_1\| \\
&= \sup_{\substack{H_2\in\mathcal{H}_{\mathrm{ad}}(M_2,\omega_2)\\ \mathrm{supp}(H_2)\subset\phi(M_1)}} \|H_2\| \\
&\leq \mathfrak{c}_{HZ}(M_2,\omega_2).
\end{aligned}$$

This proves monotonicity.

Conformality.

Since the Hamiltonian vector field of H with respect to ω agree with the Hamiltonian field of λH with respect to $\lambda\omega$ and hence

$$\mathcal{H}_{\mathrm{ad}}(M, \lambda\omega) = \{\lambda H | H \in \mathcal{H}_{\mathrm{ad}}(M, \omega)\}$$

and conformality follows.

Non triviality.

We shall now prove the inequality $\mathfrak{c}_{HZ}(B^{2n}(1), \omega_0) \geq \pi$.

Non triviality.

We shall now prove the inequality $\mathfrak{c}_{HZ}(B^{2n}(1),\omega_0) \geq \pi$.
Let $\epsilon > 0$ and choose a smooth function $f : [0,1] \longrightarrow \mathbb{R}$ such that

$$\forall r,\ -\pi < f'(r) \leq 0,\ f(r) = \pi - \epsilon,\ \text{for } r \text{ near } 0,$$
$$f(r) = 0 \text{ for } r \text{ near } 1.$$

Define $H(z) = f(|z|^2)$ for $z \in B^{2n}(1)$. Then

$$H \in \mathcal{H}(B^{2n}(1)) \quad \text{and} \quad \|H\| = \pi - \epsilon.$$

We must prove now that H is admissible.

But the orbits of the Hamiltonian flow are easy to calculate explicitly. According to (5), the Hamiltonian differential equation of H is of the form

$$\dot{x} = 2f'(|z|^2)y \quad \text{and} \quad \dot{y} = -2f'(|z|^2)x$$

and it follows that $r = |z(t)|^2$ is constant along the solutions. In complex notation $z = x + \imath y$ the solutions are

$$z(t) = \exp(-2\imath f'(r)t)z_0$$

and are all periodic.

They are nonconstant whenever $f'(r) \neq 0$ and in this case the period is $T = \frac{\pi}{f'(r)} > 1$. Hence for every $\epsilon > 0$ there is an admissible Hamiltonian function $H \in \mathcal{H}(B^{2n})$ with $\|H\| = \pi - \epsilon$ and this proves the inequality

$$\mathfrak{c}_{HZ}(B^{2n}(1), \omega_0) \geq \pi.$$

Now Theorem 20 asserts that for every $H \in \mathcal{H}(Z^{2n}(1))$ with $\|H\| > \pi$ the corresponding Hamiltonian flow has nonconstant periodic orbit of period 1. Hence any such function is not admissible and this implies

$$\mathfrak{c}_{HZ}(Z^{2n}(1), \omega_0) \leq \pi.$$

By the monotonicity axiom we have

$$\mathfrak{c}_{HZ}(B^{2n}(1), \omega_0) = \mathfrak{c}_{HZ}(Z^{2n}(1), \omega_0) = \pi$$

and this proves the theorem. □

Ekeland I. and Hofer H., *Symplectic topology and Hamiltonian dynamics,* Mathematische Zeitschrift, **200**, (1989) 355-378.

Eliasberg Y., *Rigidity of symplectic and contact structures,* Abstract on reports to the 7th Leningrad International Topology Conference (1982).

Eliasberg Y., *A theorem on the structure of wave front and its applications in symplectic topology,* Functional Analysis and Applications, **20**, (1987) 65-72.

Hofer H.and Zehnder E., *A new capacity for symplectic manifolds,* In Analysis et cetera (ed. P.H. Rabinowitz and E. Zehnder) (1990) 405-429, Academic Press, New York.

Gromov M., *Pseudo holomorphic curves in symplectic manifolds*, Inventiones Mathematicae **82**,(1985) 307-347.

McDuff D. and Salamon D., *Introduction to symplectic topology*, Oxford Mathematical Monographs, Calendron Press Oxford (1998).