On the Geometry of Noncommutative Deformations

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Outline

- Origin of the problem
- Basic tools
- Hawkins's Result
- Fundamental example
- Problem
- Solution

Origin of the problem :[Hawkins, J. Diff. Geom. 77 (2007) 385-424]

Let $\mathcal{A}_0$ be an algebra. A deformation in the sense of Hawkins of $\mathcal{A}_0$ is an extension of $\mathcal{A}_0$ of the form

$$0 \longrightarrow \hbar\mathcal{A} \longrightarrow \mathcal{A} \xrightarrow{P} \mathcal{A}_0 \longrightarrow 0,$$

where $\hbar$ is central in $\mathcal{A}$ and for any $a \in \mathcal{A}$

$$\hbar a = 0 \implies a = 0.$$

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Example

Fix $a \in \mathbb{C}$. Take $\mathcal{A}_0 = \mathbb{C}$, $\mathcal{A} = \mathbb{C}[X]$, $\hbar = (X - a)$ and $P(Q) = Q(a)$.

Example

$\mathcal{A}_0 = C^\infty(M, \mathbb{C})$ and

$$\mathcal{A} := \left\{ \sum_{n \geq 0} f_n \hbar^n, f_n \in \mathcal{A}_0 \right\}$$

a $\star$-product [1] on $\mathcal{A}$ given by, $f, g \in \mathcal{A}_0 \subset \mathcal{A}$,

$$f \star g = fg + \sum_{n \geq 1} B_n(f, g) \hbar^n$$

and $P : \mathcal{A} \longrightarrow \mathcal{A}_0$ given by

$$P\left(\sum_{n \geq 0} f_n \hbar^n \right) = f_0.$$

1. By virtue of a famous theorem of Kontsevich such a $\star$-product exists.

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Let $(\Omega^*(M), \wedge, d)$ be the graded algebra of differential forms on a manifold M and

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$$\boxed{\{\mathcal{P}(\alpha), \mathcal{P}(\beta)\} = \mathcal{P}\left(\frac{1}{\hbar}[\alpha, \beta]\right), \quad \alpha, \beta \in \mathcal{A}}$$

where

$$[\alpha, \beta] = \alpha.\beta - (-1)^{\deg\alpha \deg\beta}\beta.\alpha$$

is the graded commutator in $\mathcal{A}$.

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$$\mathcal{D}_{df}\, g\alpha = \pi_\#(df)(g)\alpha + g\mathcal{D}_{df}\alpha,$$

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The triple $(M, \pi, \mathcal{D})$ constitutes the geometry of the deformation of $\Omega^*(M)$.

Deformation of the spectral triple associated to a Riemannian manifold

Let (M, g) be a Riemannian manifold. Any deformation of the **spectral triple** [2] of (M, g) induces a deformation of $\Omega^*(M)$ and hence, gives rise to a Poisson tensor π and a contravariant connexion $\mathcal{D}$ satisfying the conditions above.

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Poisson manifolds

Definition

A Poisson bracket on M is

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

- $\mathbb{R}$-bilinear
- $\{f, g\} = -\{g, f\}$ (anti-symmetric)
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi)
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Examples

- Every symplectic manifold (M, ω) : $\{f, g\} := \omega(X_f, X_g)$.
- The dual of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$:

$$\{f, g\}(a) := \prec a, \left[d_a f, d_a g\right] \succ \quad (a \in \mathfrak{g}^*, f, g \in C^\infty(\mathfrak{g}^*)).$$

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- a Lie bracket on $\Omega^1(M)$, called *de Koszul's bracket* :

$$[\alpha, \beta]_\pi := L_{\pi_\sharp(\alpha)}\beta - L_{\pi_\sharp(\beta)}\alpha - d\big(\pi(\alpha, \beta)\big).$$

Contravariant connections

Definition

A contravariant connection on (M,π) is

$$\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \text{ notée } (\alpha,\beta) \mapsto \mathcal{D}_\alpha \beta\,,$$

$\mathbb{R}$-bilinear and satisfying

$$\mathcal{D}_{f\alpha}\beta = f\,\mathcal{D}_\alpha\beta\,, \quad \mathcal{D}_\alpha(f\beta) = f\,\mathcal{D}_\alpha\beta + \pi_\sharp(\alpha)(f)\beta \quad \bigl(f \in \mathcal{C}^\infty(M)\bigr).$$

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The torsion and *the curvature* of $\mathcal{D}$ are :

$$T(\alpha,\beta) := \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha - [\alpha,\beta]_\pi\,,$$

$$R(\alpha,\beta)\gamma := \mathcal{D}_\alpha\mathcal{D}_\beta\gamma - \mathcal{D}_\beta\mathcal{D}_\alpha\gamma - \mathcal{D}_{[\alpha,\beta]_\pi}\gamma\,.$$

When $T = 0$ (resp. $R = 0$), $\mathcal{D}$ is called *torsionless* (resp. *flat*).

Contravariant connexion

Fundamental example

Given a Riemannian metric g on (M,π), $\exists!$ contravariant connection $\mathcal{D}$ of (M,π) such that $T=0$ et $\mathcal{D}g=0$; it is given by :

$$\boxed{\begin{aligned}\langle \mathcal{D}_\alpha \beta, \gamma\rangle = \ & \frac{1}{2}\big\{\pi_\sharp(\alpha)\cdot\langle\beta,\gamma\rangle + \pi_\sharp(\beta)\cdot\langle\alpha,\gamma\rangle - \pi_\sharp(\gamma)\cdot\langle\alpha,\beta\rangle \\ & + \langle[\alpha,\beta]_\pi,\gamma\rangle - \langle[\beta,\gamma]_\pi,\alpha\rangle + \langle[\gamma,\alpha]_\pi,\beta\rangle\big\}\end{aligned}}$$

and called *the Levi-Civita contravariant connection* associated to (π, g) (in short : *CLCC*).

Metacurvature ?

Hawkins's bracket

If $\mathcal{D}$ is a contravariant connection <u>torsionless</u> on (M, π),

$$\exists! \ \{\cdot, \cdot\} : \Omega^*(M) \times \Omega^*(M) \longrightarrow \Omega^*(M)$$

- $\mathbb{R}$*-bilinear,*
- *degree* 0 *:* $\quad \deg\{\sigma, \tau\} = \deg\sigma + \deg\tau\,,$
- *graded commutative :* $\quad \{\sigma, \tau\} = -(-1)^{\deg\sigma \deg\tau}\{\tau, \sigma\}\,,$
- *Leibniz :* $\quad \{\sigma, \tau \wedge \rho\} = \{\sigma, \tau\} \wedge \rho + (-1)^{\deg\sigma \deg\tau}\, \tau \wedge \{\sigma, \rho\}\,,$
- *derivation :* $\quad d\{\sigma, \tau\} = \{d\sigma, \tau\} + (-1)^{\deg\sigma}\, \{\sigma, d\tau\}\,,$
- *For any* $f, g \in C^\infty(M)$ *and any* $\alpha \in \Omega^1(M)$*,*

$$\{f, g\} = \pi(df, dg), \quad \{f, \alpha\} = \mathcal{D}_{df}\alpha\,.$$

Metacurvature ?

Jacobi identity and metacurvature

What about the Jocobi identity,

$$\{\sigma, \{\tau, \rho\}\} - \{\{\sigma, \tau\}, \rho\} - (-1)^{\deg\sigma \deg\tau}\{\tau, \{\sigma, \rho\}\} = 0\ ?$$

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In low degrees, we have :

- $\mathcal{J}(f,g,h) = 0$ since π is Poisson.
- $\mathcal{J}(f,g,\alpha) = \mathcal{D}_{df}\mathcal{D}_{dg}\alpha - \mathcal{D}_{d\{f,g\}}\alpha - \mathcal{D}_{dg}\mathcal{D}_{df}\alpha = R(df,dg)\alpha$.
- If $\mathcal{D}$ is flat then the formula

  $$\boxed{\mathcal{M}(df,\alpha,\beta) := \mathcal{J}(f,\alpha,\beta) = \{f,\{\alpha,\beta\}\} - \{\{f,\alpha\},\beta\} - \{\{f,\beta\},\alpha\}}$$

  where $\alpha,\beta\in\Omega^1(M)$ defines a tensor field $\mathcal{M}$ of type $(2,3)$.

Proposition

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It is $C^\infty(M)$-trilinear symmetric

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One can see $\mathcal{M}$ as an element of $\Gamma(S^3TM \otimes \wedge^2 T^*M)$.

Computation of Hawkins's bracket and the metacurvature

Let $(M, \pi, \mathcal{D})$ be a manifold endowed with a Poisson tensor and a contravariant connexion torsionless and flat. The Hawkins's bracket in low degrees is given by

$$\{f, g\} = \pi(df, dg), \quad \{f, \alpha\} = \mathcal{D}_{df}\alpha, \; f, g \in C^\infty(M), \; \alpha \in \Omega^1(M).$$

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The metacurvature is given

$$\mathcal{M}(df, \alpha, \beta) = \{f, \{\alpha, \beta\}\} - \{\{f, \alpha\}, \beta\} - \{\alpha, \{f, \beta\}\}, \ f \in C^\infty(M), \ \alpha, \beta \in \Omega^1(M).$$

Important Remark

If α is parallel, i.e., $\mathcal{D}\alpha = 0$, for any $\beta \in \Omega^1(M)$,

$$\boxed{\{\alpha, \beta\} = -\mathcal{D}_\beta d\alpha.} \tag{1}$$

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If α an β are parallel, for any $\gamma \in \Omega^1(M)$,

$$\boxed{\mathcal{M}(\alpha, \beta, \gamma) = -\mathcal{D}_\gamma \mathcal{D}_\beta d\alpha.} \tag{2}$$

Back to the initial problem

If (M, g) is a Riemannian manifold,

$$\Big[\text{A deformation of the spectrale triple of } (M,g)\,\Big]$$

$$\Downarrow$$

$$\left[\begin{array}{l} \exists \ \text{ a Poisson tensor } \pi \text{ on } M \text{ such that :} \\ \quad (H_1)\ \text{The CLCC } \mathcal{D} \text{ assocatied to } (\pi, g) \text{ is flat} \\ \quad (H_2)\ \text{The metacurvature of } \mathcal{D} \text{ vanishes} \\ \quad (H_3)\ d(i_\pi \mu) = 0, \text{ where } \mu \text{ is the Riemannian volume} \end{array}\right]$$

Main result of Hawkins

Theorem 1 [Hawkins, J. Diff. Geom. 77 (2007) 385-424]

Let (M, π) be a Poisson manifold endowed with a Riemannian metric g. Assume that M is compact satisfying (H_1), (H_2) and (H_3). Then, near any $x \in M^{reg}$,

$$\pi = \frac{1}{2} \sum_{i,j} a_{ij} \, X_i \wedge X_j$$

where (a_{ij}) is constant and invertible and $X_1, \ldots, X_{2r}$ are linearly independent commuting Killing vector fields. Moreover, $\mathcal{D}\pi = 0$.[3].

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- of Poisson manifolds (M, π) endowed with a contravariant connection $\mathcal{D}$ such the torsion, the curvature and the metacurvature of $\mathcal{D}$ vanish.
- of Poisson manifold (M, g, π) endowed with a Riemannian metric such that the Levi-Civita contravariant connection associated à (π, g) is flat and metaflat.

Fundamental Example

Let

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and $\mathcal{D}^r : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$,

$$(\alpha, \beta) \mapsto \mathcal{D}^r_\alpha\beta := \sum_{i,j} a_{ij}\alpha(\zeta(u_i))\mathcal{L}_{\zeta(u_j)}\beta$$

where $r = \sum a_{ij}u_i \wedge u_j$ and $\{u_1, \ldots, u_n\}$ is a basis of $\mathfrak{g}$.

Fundamental Example

Theorem 2 [Boucetta, Lett. Math. Phys. 83 (2008) 69-81]

(a) π^r and $\mathcal{D}^r$ depend only on r and ζ and define, respectively, a Poisson tensor and a contravariant connection torsionless and flat on M.

(b) If g is a Riemannian metric on M and ζ preserves g, i.e., for any $u \in \mathfrak{g}$, $\zeta(u)$ is a Killing vector field, then $\mathcal{D}^r$ is the contravariant Levi-Civita connection of (π^r, g).

(c) If ζ is **free**, i.e., for any $x \in M$, the map $v \mapsto \zeta(v)(x)$ is injective then the metacurvature of $\mathcal{D}^r$ vanishes.

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Remark. In (c), we cannot drop the hypothesis ζ free.

The problem ?

Given a Poisson manifold (M, π) endowed with a contravariant connection without torsion, flat and metaflat, for any regular point x there exists a neighborhood U of x, a free action of a finite dimensional Lie algebra $\zeta : \mathfrak{g} \to \mathfrak{X}(U)$, and a solution $r \in \wedge^2\mathfrak{g}$ of CYBE such that $\pi_{|U} = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$?

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- In the fundamental example, if ζ is free the it is $\mathbf{T}$ which vanishes implying the vanishing of the metacurvature.

The problem reformulated :

Given a Poisson manifold (M, π) endowed with a $\mathcal{F}^{reg}$-contravariant connection without torsion, flat and $\mathbf{T} = 0$, for any regular point x there exists a neighborhood U of x, a free action of a finite dimensional Lie algebra $\zeta : \mathfrak{g} \to \mathfrak{X}(U)$, and a solution $r \in \wedge^2 \mathfrak{g}$ of CYBE such that $\pi_{|U} = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$?

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Moreover, if $\mathcal{D}$ is the Levi-Civita contravariant connection of (M,π,g), the action ζ preserves g ?

Solution : Main result

Théorème 3 [Boucetta & Saassai, J. Geom. Phys. 82 (2014) 64-74]

Let $(M, \pi, \mathcal{D})$ be a Poisson manifold endowed with a contravariant connection torsionless and flat.

- If $\mathcal{D}$ is a $\mathcal{F}^{reg}$-connection and $\mathbf{T} = 0$, then for any $x_0 \in M^{reg}$ there exists a neighborhood U of x, a free action of a finite dimensional Lie algebra $\zeta : \mathfrak{g} \to \mathfrak{X}(U)$, and an invertible solution $r \in \wedge^2\mathfrak{g}$ of CYBE such that $\pi_{|U} = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$.
- Moreover, if $\mathcal{D}$ is the Levi-Civita contravariant connection of (M, π, g) then the action ζ preserves g.

Solution : Building a flat co-frame

Let $(M, \pi, \mathcal{D})$ such that $\mathcal{D}$ is a $\mathcal{F}^{reg}$-connection, torsionless and flat. Let $x_0 \in M^{reg}$ and $(a_1, \ldots, a_{2r})$ a family of covectors in $T^*_{x_0}M$ such that $(\pi_\#(a_1), \ldots, \pi_\#(a_{2r}))$ is a basis of $\mathrm{Im}\pi_\#(x_0)$.

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- For any $a \in T^*_{x_0}M$, there exists an open set $U \ni x_0$ and $\beta^a \in \Omega^1(U)$ such that $\beta^a(x_0) = a$ and $\mathcal{D}\beta^a = 0$.

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- Note $\phi_i = \beta^{a_i}$. The vector fields $(\pi_\#(\phi_1), \ldots, \pi_\#(\phi_{2r}))$ are commuting linearly independent so there exists a coordinates system $\left((x^i)_{i=1}^{2r}, (y^j)_{j=1}^{d-2r}\right)$ such that $\pi_\#(\phi_i) = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, 2r$.

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- For any $x \in U$, put $\mathcal{H}_x = \mathrm{vect}\{\phi_1(x), \ldots, \phi_{2r}(x)\}$. We have $T^*_U M = \ker \pi_\# \oplus \mathcal{H}$ and $\mathcal{D}\mathcal{H} \subset \mathcal{H}$.

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- $\mathbf{F}^* = \left\{\phi_1, \ldots, \phi_{2r}, dy^1, \ldots, dy^{d-2r}\right\}$ is a flat co-frame.

Solution : The frame dual of $\mathbf{F}^*$

For any $i = 1, \ldots, 2r$, there exists a unique family of functions $A_i^1, \ldots, A_i^s$ such that $dx^i + \sum_u A_i^u \, dy_u \in \mathcal{H}$. Consider

$$X^i := -X_{x^i} = -\pi_\sharp(dx^i)\,, \qquad Y_u := \frac{\partial}{\partial y_u} - \sum_{i=1}^{2r} A_i^u \frac{\partial}{\partial x^i}\,.$$

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Lemma 3

$\{X_i, Y_u\}$ is the dual frame of $\mathbf{F}^*$. Moreover, the vector fields X_i and Y_u are, respectively, Hamiltonian and Poisson, and satisfy :

$$[X_i, X_j] = -\sum_{k=1}^{2r} \frac{\partial \pi_{ij}}{\partial x_k} X_k\,; \quad [X_i, Y_u] = \sum_{j=1}^{2r} \frac{\partial A_i^u}{\partial x_j} X_j\,;$$

$$[Y_u, Y_v] = \sum_{i,j=1}^{2r} \pi^{ij} \left(\frac{\partial A_j^u}{\partial y_v} - \frac{\partial A_j^v}{\partial y_u} + \sum_{k=1}^{2r} A_k^u \frac{\partial A_j^v}{\partial x_k} - A_k^v \frac{\partial A_j^u}{\partial x_k} \right) X_i\,.$$

with $(\pi_{ij}) = (\pi(dx^i, dx^j))$ and (π^{ij}) is the inverse of the matrix (π_{ij}).

Solution : The tensor fields $\mathcal{M}$ et T

Theorem 4

- For any u, $\mathcal{M}(dy_u, \cdot, \cdot) = 0$.
- For any i, j, k,

$$\mathcal{M}(\phi_i, \phi_j, \phi_k) = -\sum_{l<m} \frac{\partial^3 \pi_{lm}}{\partial x_i \partial x_j \partial x_k} \phi_l \wedge \phi_m + \sum_{l,u} \frac{\partial^3 A_l^u}{\partial x_i \partial x_j \partial x_k} \phi_l \wedge dy_u$$

$$+ \sum_{u<v,\, l} \frac{\partial^2}{\partial x_i \partial x_j} \left(\pi^{kl} \left(\frac{\partial A_l^u}{\partial y_v} - \frac{\partial A_l^v}{\partial y_u} + \sum_m A_m^u \frac{\partial A_l^v}{\partial x_m} - A_m^v \frac{\partial A_l^u}{\partial x_m} \right) \right) dy_u \wedge dy_v .$$

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Theorem 5

- For any u, $\mathbf{T}(dy_u, \cdot) = 0$.
- For any i, j,

$$\mathbf{T}(\phi_i, \phi_j) = -\sum_{k<l} \frac{\partial^2 \pi_{kl}}{\partial x_i \partial x_j} \phi_k \wedge \phi_l + \sum_{k,u} \frac{\partial^2 A_k^u}{\partial x_i \partial x_j} \phi_k \wedge dy_u$$

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Solution : Proof of Theorem 3

Sketch of the proof

The idea is to build near x_0 a family of linearly independent vector fields $Z_1, \dots, Z_{2r} \in \Gamma(\mathcal{C})$ which commute with X_i and Y_u. In this case

- $[Z_i, Z_j] = \sum_k c_{ij}^k Z_k$ with $c_{ij}^k = cst$ hence $Z_1, \dots, Z_{2r}$ generate a Lie algebra of dimension $2r$ which acts freely near x_0.
- $\pi = \frac{1}{2} \sum_{i,j} a_{ij} \, Z_i \wedge Z_j$ where (a_{ij}) is constant and invertible.
- $\mathcal{D}_\alpha \beta = \sum_{i,j} a_{ij} \alpha(Z_i) \, \mathcal{L}_{Z_j} \beta$; indeed, this is true for $\beta = \phi_i$ or dy_u since $\mathcal{L}_{Z_i} \phi_j = \mathcal{L}_{Z_i} dy_u = 0$. And $\mathcal{D}_\alpha \beta - \sum_{i,j} a_{ij} \alpha(Z_i) \, \mathcal{L}_{Z_j} \beta$ is tensorial in β.

Solution : Proof of Theorem 3

We proceed on two steps :

First step

We build a family of vector fields $T_1, \dots, T_{2r} \in \Gamma(\mathcal{C})$ which commute with the X_i. Indeed, the vanishing of $\mathbf{T}$ and Lemma 3, imply :

$$[X_i, X_j] = \sum_{k=1}^{2r} \lambda_{ij}^k X_k \,, \quad [X_i, Y_u] = \sum_{j=1}^{2r} \mu_{iu}^j X_j \,, \quad [Y_u, Y_v] = \sum_{i=1}^{2r} \nu_{uv}^i X_i$$

where λ_{ij}^k, μ_{iu}^j, ν_{uv}^i are Casimir, i.e., depend only on the y^i.

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We choose a transversal $\mathcal{T}$ to the symplectic foliation $\mathcal{S}$ passing through x_0. For $y \in \mathcal{T}$ fix, $X_{1|\mathcal{S}_y}, \ldots, X_{2r|\mathcal{S}_y}$ span a Lie algebra $\mathfrak{g}_y$ which act freely and transitively on $\mathcal{S}_y$, so $\exists$ an anti-homomorphism of Lie algebras $\Gamma_y : \mathfrak{g}_y \to \mathfrak{X}^1(\mathcal{S}_y)$, such that

$$\Gamma_y(X_{i|\mathcal{S}_y})(y) = X_i(y) \,, \quad [\Gamma_y(X_{i|\mathcal{S}_y}), X_{j|\mathcal{S}_y}] = 0 \;\; \forall i, j.$$

We take $T_i(z) := \Gamma_y(X_i^y)(z), \; z \in \mathcal{S}_y$ and we variate y to obtain T_i.

Solution : Proof of Theorem 3

Now, the μ_{iu}^j are Casimir

$$[T_i, Y_u] = \sum_{j=1}^{2r} \gamma_{iu}^j T_j$$

with γ_{iu}^j Casimir and satisfy

$$\frac{\partial \gamma_{ju}^i}{\partial y_v} - \frac{\partial \gamma_{jv}^i}{\partial y_u} + \sum_{k=1}^{2r} \gamma_{ku}^i \gamma_{jv}^k - \gamma_{kv}^i \gamma_{ju}^k = 0 \qquad (*)$$

since the ν_{uv}^i are Casimir and $[T_i, [Y_u, Y_v]] = 0$ for any i, u, v.

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since the ν_{uv}^{i} are Casimir and $[T_i, [Y_u, Y_v]] = 0$ for any i, u, v.

Second step

We look for the Z_i in the form :

$$Z_i := \sum_{j=1}^{2r} \xi_{ji} \, T_j$$

where ξ_{ij} are Casimir and $\xi = (\xi_{ij})$ is invertible. They exist by virtue of $(*)$.

Thank you