On the isometry group of left invariant Riemannian metric on Lie groups

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Theorem.

If (M, g) is Riemannian and $f : M \longrightarrow M$ is a surjective smooth map then f is an isometry if and only if f preserves the Riemannian distance.

Theorem. (S. B. Myers-N. Steenrod)

Let (M, g) be a pseudo-Riemannian manifold. Then

(a) *The group $\mathrm{Isom}(M, \mathrm{g})$ is a Lie group which acts effectively and differentiably on M.*

(b) *For any $m \in M$, the isotropy subgroup*

$$\mathrm{Isom}_m(M, \mathrm{g}) = \{f \in \mathrm{Isom}(M, \mathrm{g}), f(m) = m\}$$

is a closed subgroup of $\mathrm{Isom}(M, \mathrm{g})$. Moreover, if we denote by

$$\rho : \mathrm{Isom}_m(M, \mathrm{g}) \longrightarrow \mathrm{GL}(T_m M), \quad f \mapsto \rho(f) = T_m f,$$

then ρ define an isomorphism of $\mathrm{Isom}_m(M, \mathrm{g})$ into a closed subgroup of $\mathrm{O}(T_m M, \mathrm{g}_m)$.

Corollary.

If (M, g) *is a Riemannian manifold,* $\mathrm{Isom}_m(M, \mathrm{g})$ *is compact. Moreover, if* (M, g) *is compact then* $\mathrm{Isom}(M, \mathrm{g})$ *is compact*

Remark.

1. *More generally, if* (M, g) *is Riemannian then* $\mathrm{Isom}(M, \mathrm{g})$ *act properly on* M.
2. *Note that* $\mathrm{Isom}(M, \mathrm{g})$ *may be compact (even trivial) even if* (M, g) *is not compact or non-Riemannian.*
3. *Note also that* $\dim \mathrm{Isom}(M, \mathrm{g}) \leq \frac{n(n+1)}{2}$ *with equality only if* (M, g) *has constant sectional curvature.*
4. *If* (M, g) *is Riemannian and* $n = \dim M \neq 0$ *then* $\mathrm{Isom}(M, \mathrm{g})$ *contains non closed subgroup of dimension* r *with*

$$\frac{1}{2}n(n-1) + 1 < r < \frac{1}{2}n(n+1).$$

The following theorem shows that there are deep relations between the curvature and the group of isometries.

Theorem. (S. Bochner.)

Let (M, g) be a compact manifold and ric *its Ricci curvature. Then*

1. *If for* $u \neq 0$ $\mathrm{ric}(u,u) < 0$ *then* $\mathrm{Isom}(M, \mathrm{g})$ *is finite.*
2. *If for* u $\mathrm{ric}(u,u) \leq 0$ *then* $\mathrm{Isom}(M, \mathrm{g})$ *is a torus.*
3. *If for* u $\mathrm{ric}(u,u) = 0$ *then* $\dim \mathrm{Isom}(M, \mathrm{g}) = \dim H^1(M, \mathbb{R})$.

Background on Lie algebras and Lie groups

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Theorem. (Folklore)

Let $\mathfrak{g}$ be a Lie algebra. Then the following are equivalent:

1. $\mathfrak{g}$ *is a semi-simple.*
2. $\mathfrak{g}$ *has a splitting* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$, *where* $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_s\}$ *is the set of all simple ideals of* $\mathfrak{g}$.
3. *The Killing form K of $\mathfrak{g}$ is non-degenerate.*[a]

Moreover, the integer s appearing in the splitting above is an invariant of $\mathfrak{g}$.

[a] Recall that K is given by $K(u, v) = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v)$.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Then $\mathrm{ad}(\mathfrak{g})$ is a subalgebra of $\mathrm{gl}(\mathfrak{g})$. Let $\mathrm{Int}(\mathfrak{g})$ the subgroup of $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra is $\mathrm{ad}(\mathfrak{g})$.

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The group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ and hence a Lie group. Its Lie algebra is the Lie algebra $\delta(\mathfrak{g})$ of the derivations of $\mathfrak{g}$.

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We have $\mathrm{ad}(\mathfrak{g}) \subset \delta(\mathfrak{g})$ and hence $\mathrm{Int}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$.

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Definition.

1. *A Lie algebra* $\mathfrak{g}$ *is called compact if* $\mathrm{ad}(\mathfrak{g})$ *is compact.*
2. *A Lie group is called simple (resp. semi-simple) if its Lie algebra is simple (resp. semi-simple).*

Proposition.

If $\mathfrak{g}$ *is semi-simple then* $\mathrm{ad}(\mathfrak{g}) = \delta(\mathfrak{g})$, *i.e., every derivation is an inner derivation.*

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Corollary.

For a semi-simple Lie algebra over $\mathbb{R}$*, the adjoint group* $\mathrm{Int}(\mathfrak{g})$ *is the identity component of* $\mathrm{Aut}(\mathfrak{g})$*. In particular,* $\mathrm{Int}(\mathfrak{g})$ *is closed in* $\mathrm{Aut}(\mathfrak{g})$*.*

Proposition.

(a) *Let* $\mathfrak{g}$ *be a semi-simple Lie algebra over* $\mathbb{R}$. *Then* $\mathfrak{g}$ *is compact if and only if the Killing form of* $\mathfrak{g}$ *is strictly negative definite.*

(b) *Every compact Lie algebra is the direct sum* $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, *where* $Z(\mathfrak{g})$ *is the center of* $\mathfrak{g}$ *and the ideal* $[\mathfrak{g}, \mathfrak{g}]$ *is semi-simple compact.*

(b) *If* $\mathfrak{g}$ *is compact then* $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$ *where* $\mathfrak{g}_i$ *is simple compact.*

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Corollary.

A Lie algebra over $\mathbb{R}$ *is compact if and only if there exists a compact Lie group* G *with the Lie algebra isomorphic to* $\mathfrak{g}$.

Theorem.

Let G be a compact, connected semi-simple Lie group. Then the universal covering G^* of G is compact. In particular, a compact connected Lie group is semi-simple if and only if $\pi_1(G)$ is finite.

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Let G be a compact, connected semi-simple Lie group. Then $\pi_3(G) \simeq \mathbb{Z}^s$, where s is the cardinal of the set of the proper ideals in $\mathfrak{g}$.

Main result

Let G be a connected Lie group. For any $a \in G$, we denote by $\mathrm{L}_a : G \longrightarrow G$ and $\mathrm{R}_a : G \longrightarrow G$, respectively, the left translation and the right translation given by

$$\mathrm{L}_a(b) = ab \quad \text{and} \quad \mathrm{R}_a(b) = ba^{-1}.$$

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$\mathrm{L}(G)$ and $\mathrm{R}(G)$ are groups of transformations of G which are isomorphic to G. Moreover,

$$\mathrm{L}(G)\mathrm{R}(G) = \{\mathrm{L}_a \circ \mathrm{R}_b, a, b \in G\}$$

is a Lie group isomorphic to $G \times G$.

Let g is a left invariant Riemannian metric on G. Then $\mathrm{Isom}(G, \mathrm{g})$ contains $\mathrm{L}(G)$. If it contains $\mathrm{R}(G)$ then the metric is bi-invariant.

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Theorem. (Math. Ann. 223, 91-96 (1976))

Let G be a compact, connected, simple Lie group and g a left invariant Riemannian metric on G. Then $[\mathrm{Isom}(G, \mathrm{g})] \subset \mathrm{L}(G)\mathrm{R}(G)$.

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Theorem. (Math. Ann. 223, 91-96 (1976))

Let G be a compact, connected, simple Lie group and g a left invariant Riemannian metric on G. Then $[\mathrm{Isom}(G, \mathrm{g})] \subset \mathrm{L}(G)\mathrm{R}(G)$.

Remark.

We can derive from this theorem that if G is connected compact simple with a left invariant Riemannian metric of constant curvature then $G \equiv S^3$ or $\mathrm{SO}(3)$.

Preparation of the proof of the theorem

Let G be a compact, connected Lie group and g a left invariant Riemannian metric on G. Since G is compact $[\mathrm{Isom}(G, \mathrm{g})]$ is compact and connected.

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$$H = \{f \in [\mathrm{Isom}(G, \mathrm{g})], f(e) = e\}\,.$$

H is a compact subgroup of $[\mathrm{Isom}(G, \mathrm{g})]$ and we have

$$[\mathrm{Isom}(G, \mathrm{g})] = \mathrm{L}(G)H \quad \text{and} \quad \mathrm{L}(G) \cap H = \{e\}. \qquad (1)$$

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This implies that $[\mathrm{Isom}(G, \mathrm{g})]$ is a diffeomorphic to a product manifold $\mathrm{L}(G) \times H$ and H is connected.

If $[\mathrm{Isom}(G, \mathrm{g})]$ is contained in $\mathrm{L}(G)\mathrm{R}(G)$ then $\mathrm{L}(G)$ is a normal subgroup of $[\mathrm{Isom}(G, \mathrm{g})]$. Moreover, each element of H has form $\mathrm{L}_a \circ \mathrm{R}_a$, i.e., H is contained in the inner automorphism group of G.

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Theorem.

(a) *If* $[\mathrm{Isom}(G,\mathrm{g})]$ *is contained in* $\mathrm{L}(G)\mathrm{R}(G)$ *then* $\mathrm{L}(G)$ *is a normal subgroup of* $[\mathrm{Isom}(G,\mathrm{g})]$.

(b) $[\mathrm{Isom}(G,\mathrm{g})]$ *is contained in* $\mathrm{L}(G)\mathrm{R}(G)$ *if and only if* H *is contained in the inner automorphism group of* G.

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Thus $x = h(y)$ and hence h is an automorphism of G. So $H \subset \mathrm{Aut}(G)$. But G is semi-simple and hence $\mathrm{Aut}_0(G) = \mathrm{Inner}(G)$.

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Theorem.

Let (G, g) *be a compact, connected semi-simple Lie group. Then* $[\mathrm{Isom}(G, \mathrm{g})]$ *is contained in* $\mathrm{L}(G)\mathrm{R}(G)$ *if and only if* $\mathrm{L}(G)$ *is a normal subgroup of* $[\mathrm{Isom}(G, \mathrm{g})]$.

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3. $[\mathrm{Isom}(G, \mathrm{g})] = \mathrm{L}(G)H$ and $\mathrm{L}(G) \cap H = \{e\}$ with H closed.

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4. H contains non normal subgroup of $[\mathrm{Isom}(G, \mathrm{g})]$ except $\{e\}$.
5. $[\mathrm{Isom}(G, \mathrm{g})]$ is contained in $\mathrm{L}(G)\mathrm{R}(G)$ if and only if $\mathrm{L}(G)$ is a normal subgroup of $[\mathrm{Isom}(G, \mathrm{g})]$.

To prove our main theorem it suffices to prove the following theorem.

Theorem.

Let K be a compact connected Lie group. Let G and H be closed connected subgroups of K such that

(A) *G is simple.*

(B) *$K = GH$ and $G \cap H = \{e\}$.*

(C) *H contains non normal subgroup of K except $\{e\}$.*

Then G is a normal subgroup of K.

A fundamental Lemma

Lemma.

Let K be a compact connected Lie group. Let G and H be **closed proper** *subgroups of K such that $K = GH$ and $G \cap H = \{e\}$. Then*

(a) *G and H are connected.*

(b) *K is semi-simple if and only if both G and H are semi-simple.*

(c) *If K, G and H are semi-simple then $s(K) = s(G) + s(H)$.*[a]

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[a] $s(G)$ is the number of proper ideals contained in the Lie algebra $\mathfrak{g}$ of G.

Proof.

From the assumption it follows that G and H and compact and K is diffeomorphic to $G \times H$. Hence G and H are connected and

$$\pi_i(K) = \pi_i(G) + \pi_i(H), \quad i = 1, \ldots.$$

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Then $\pi_1(K)$ is finite if and only if $\pi_1(G)$ and $\pi_1(H)$ are finite, which implies (b).

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Then $\pi_1(K)$ is finite if and only if $\pi_1(G)$ and $\pi_1(H)$ are finite, which implies (b). Now if K, G and H are semi-simple then

$$\mathbb{Z}^{s(K)} = \pi_3(K) = \pi_3(G) + \pi_3(H) = \mathbb{Z}^{s(G)} + \mathbb{Z}^{s(H)}$$

and hence $s(K) = s(G) + s(K)$ and (c) follows. □

Corollary.

If a real Lie algebra $\mathfrak{K} = \mathfrak{g} \oplus \mathfrak{h}$ *where* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are compact semi-simple then* $\mathfrak{K}$ *is compact semi-simple and* $s(\mathfrak{K}) = s(\mathfrak{g}) + s(\mathfrak{h})$*.*

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Proof.

Let K be the simply connected Lie group of Lie algebra $\mathfrak{K}$, G and H the connected subgroups of K whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$. Then G and H are compact and semi-simple. We have also $G \cap H = \{e\}$ and $K = GH$ and the corollary follows from the lemma. □

Proof of Theorem 18

Denote by $\mathfrak{K}$, $\mathfrak{g}$ and $\mathfrak{h}$, respectively, the Lie algebras of K, G and H. We have

1. $\mathfrak{g}$ is simple,
2. $\mathfrak{K} = \mathfrak{g} \oplus \mathfrak{h}$,
3. $\mathfrak{h}$ contains no proper ideal of $\mathfrak{K}$ except $\{0\}$.
4. $\mathfrak{K} = Z(\mathfrak{K}) \oplus \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_s$, where $\mathfrak{a}_i$ is simple compact.

To show the theorem, it suffices to show that $\mathfrak{g}$ is an ideal of $\mathfrak{K}$.

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To show the theorem, it suffices to show that $\mathfrak{g}$ is an ideal of $\mathfrak{K}$.

To do so, we first prove the following: **If there exists i such that $\mathfrak{g} \subset \mathfrak{a}_i$ then $\mathfrak{g}$ is an ideal of $\mathfrak{K}$.**

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Suppose that $\mathfrak{g} \subset \mathfrak{a}$ and $\mathfrak{a}$ is a simple ideal of $\mathfrak{K}$. The connected Lie subgroup A of K corresponding to $\mathfrak{a}$ is **compact** and simple. G is closed subgroup of A and $H' = A \cap H$ is compact. From the assumption, we have $A = GH'$, $G \cap H' = \{e\}$. Suppose that G is proper subgroup of A. Then H' is also proper subgroup of A. Then from the fundamental lemma we have $s(A) = s(G) + s(H')$. Or $s(G) = S(A) = 1$ and hence $s(H') = 0$, which is a contradiction, so $A = G$.

To complete the proof, we suppose that for any $i = 1, \ldots, s$ $\mathfrak{g}$ is not contained in $\mathfrak{a}_i$ and deduce a contradiction.

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$$\mathfrak{K} = Z(\mathfrak{K}) \oplus \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_s, \quad Z(\mathfrak{K}) = \mathfrak{a}_0.$$

To complete the proof, we suppose that for any $i = 1, \ldots, s$ $\mathfrak{g}$ is not contained in $\mathfrak{a}_i$ and deduce a contradiction. Denote by $p_i : \mathfrak{K} \longrightarrow \mathfrak{a}_i$ the projection with respect to the splitting

$$\mathfrak{K} = Z(\mathfrak{K}) \oplus \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_s, \quad Z(\mathfrak{K}) = \mathfrak{a}_0.$$

For any $i = 0, \ldots, s$, p_i is a Lie algebra homomorphism and hence $\mathfrak{g} \cap \ker p_i$ is an ideal in $\mathfrak{g}$ and hence $\mathfrak{g} \cap \ker p_i = \{0\}$ or $\mathfrak{g} \cap \ker p_i = \mathfrak{g}$. Since $\mathfrak{g} \neq \{0\}$ then there exist i_0 such that $\mathfrak{g} \cap \ker p_{i_0} \neq \mathfrak{g}$ and hence $\mathfrak{g} \cap \ker p_{i_0} = \{0\}$. Suppose that $i_0 = 0$. Then $p_0(\mathfrak{g})$ is abelian and isomorphic to $\mathfrak{g}$ which is impossible. So $i_0 \neq 0$. Put $i_0 = 1$.

We have $\mathfrak{g} \cap \ker p_1 = \{0\}$. Moreover, $\mathfrak{g} \cap \mathfrak{a}_1$ is an ideal in $\mathfrak{g}$ and since $\mathfrak{g}$ is not contained in $\mathfrak{a}_1$ then $\mathfrak{g} \cap \mathfrak{a}_1 = \{0\}$. This implies

$$\mathfrak{g} \cap p_1(\mathfrak{g}) = \{0\}. \tag{2}$$