# Topological methods in division algebras: Hopf's Theorem and $(1, 2, 4, 8)$ Theorem

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# Topological consequences of the existence of a division algebra structure on $\mathbb{R}^n$

**Definition.**

*A division algebra structure on* $\mathbb{R}^n$ *is a bilinear product* $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(x, y) \mapsto x.y$ *such that*

$$x.y = 0 \iff x = 0 \text{ or } y = 0.$$

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This is equivalent to: for any $x \neq 0$,

$$\mathrm{L}_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ y \mapsto \mathrm{L}_x y := x.y$$

and

$$\mathrm{R}_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ y \mapsto \mathrm{R}_x y := y.x$$

are isomorphisms.

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3. *In dimension 4:* $\mathbb{H}$ *endowed with its canonical product:non commutative associative.*
4. *In dimension 8:* $\mathbb{O}$ *endowed with its canonical product: non commutative non associative, alternative (see The Octonions, John C. Baez, arXiv:math/0105155v4 [math.RA] 23 Apr 2002 ).*

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Hop's map is odd in the sense that

$$\forall x, y \in S^{n-1}, \quad h(-x, y) = h(x, -y) = -h(x, y).$$

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Note that

$$H^*(\mathbb{P}^{n-1}, \mathbb{Z}_2) = \bigoplus_{k=0}^{n-1} H^k(\mathbb{P}^{n-1}, \mathbb{Z}_2)$$

and

$$H^*(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}, \mathbb{Z}_2) = \bigoplus_{k=0}^{2n-2} H^k(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}, \mathbb{Z}_2).$$

# Hopf's Theorem

**Theorem (Hopf 1940)**

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**Corollary.**

*If* $\mathbb{R}^n$ *has a division algebra structure then* $n = 2^p$.

# Degree modulo 2 of a continuous map $f : M \longrightarrow M$

Let $M$ be a compact connected topological space, $f : M \longrightarrow M$ be a continuous map. An element $y \in M$ is called regular value if for any $x \in f^{-1}(y)$, $f$ is an homeomorphism from a neighborhood of $x$ to a neighborhood of $y$. In this case $f^{-1}(y)$ is finite and its cardinal mod 2 doesn't depend on $y$. We call it $\deg_2(f) \in \mathbb{Z}_2$.

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We have

$$f \text{ homotopic to } g \;\Rightarrow\; \deg_2(f) = \deg_2(g).$$

Any continuous function $f : M \longrightarrow M$ is homotopic to a function $g : M \longrightarrow M$ which is $C^\infty$ and according to Sard's theorem $g$ has a regular value so we put

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Let $F : S^{n-1} \longrightarrow \mathrm{GL}(n, \mathbb{R})$ a continuous map. For any $v \in \mathbb{R}^n \setminus \{0\}$, we denote by $F_v : S^{n-1} \longrightarrow S^{n-1}$, $x \mapsto \frac{F(x)(v)}{|F(x)(v)|}$.

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We define the mod 2 invariant of $F$ by

$$\alpha(F) = \deg_2(F_v).$$

# Second topological consequence of the existence of a division algebra structure on $\mathbb{R}^n$

Suppose that $\mathbb{R}^n$ carries a subdivision algebra structure. Then the map

$$F : S^{n-1} \longrightarrow \mathrm{GL}(n, \mathbb{R}),\ x \mapsto F(x) = \mathrm{L}_x$$

is continuous and for any $v \in \mathbb{R}^n \setminus \{0\}$,

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is an homeomorphism.
Thus

$$\alpha(F) = 1.$$

**Theorem.**

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## Proof.

For any $y \in S^{n-1}$, $(e_1.y, \ldots, e_n.y)$ are linearly independent and give by orthonormalisation a family of vectors $(X_1(y), \ldots, X_n(y))$ with

$$X_1(y) = \frac{e_1.y}{|e_1.y|} = F_{e_1}(y) \quad \text{and} \quad \langle X_i(y), F_{e_1}(y) \rangle = 0, i = 2, \ldots$$

Thus, for $i = 2, \ldots, n$, $Y_i(y) = X_i(F_{e_1}^{-1}(y))$ define a family of $n-1$ vector fields on $S^{n-1}$ which are linearly independent and hence $S^{n-1}$ is parallelizable. Moreover, $Y_i(-y) = -Y_i(y)$ and hence $\mathbb{P}^{n-1}$ is parallelizable. □

**Remark.**

*If $S^n$ is parallelizable then there exists a continuous function $F : S^n \longrightarrow \mathrm{GL}(n+1, \mathbb{R})$ such that $\alpha(F) = 1$.*

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*Indeed, If $X_i : S^n \longrightarrow \mathbb{R}^{n+1}$ parallelize $S^n$ then*

$$F : S^n \longrightarrow \mathrm{GL}(n+1, \mathbb{R}),\ x \mapsto (x, X_1(x), \ldots, X_n(x))$$

*satisfies $F_{e_1}(x) = x$ and hence $\alpha(F) = 1$.*

# Theorem $(1, 2, 4, 8)$

**Theorem.** (Kervaire-Milnor 1958)

*If there exists a continuous function* $F : S^{n-1} \longrightarrow \mathrm{GL}(n, \mathbb{R})$ *such that* $\alpha(F) = 1$ *then* $n = 1, 2, 4$ *or 8.*

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**Corollary.**

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$\mathbb{R}^n$ *has a division algebra structure if and only if* $n = 1, 2, 4$ *or 8.*

# Back to Hopf's Theorem

**Theorem (Hopf 1940)**

*If there exists a continuous odd mapping $h : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ then $n = 2^p$.*

# Homology and cohomology with coefficients in $\mathbb{Z}_2$

Denote by $\Delta^n = [0, 1, \ldots, n]$ the convex hull of the origin with the canonical basis of $\mathbb{R}^n$,

$$\Delta[0, 1, \ldots, n] = \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n, t_i \geq 0, \sum t_i \leq 1\right\}.$$

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We denote the face of $\Delta^n$ opposite to the $i$-th vertex by $\Delta < i >$, i.e.,

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$\Delta^0 = \{0\}$, $\Delta^1 = [0, 1]$ and $\Delta^2$ is the triangle $0, e_1, e_2$ etc..

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By a singular $n$-chain we mean any finite formal linear combination of singular n-simplices with coefficients from $\mathbb{Z}_2$, and write

$$C_n(X) = \left\{ \sum_{i=1}^{r} n_i \sigma_i, n_i \in \mathbb{Z}_2, \sigma_i \in \Sigma^n \right\}.$$

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$$C_0(X) = \left\{ \sum_{i=1}^{r} n_i P_i, n_i \in \mathbb{Z}_2, P_i \in X \right\}$$

Next we define the boundary operator
$\delta_n : C_n(X) \longrightarrow C_{n-1}(X)$ on each simplex $\sigma$ by setting

$$\delta_n(\sigma) = \sum_{q=0}^{n} (-1)^q \sigma_{|\Delta<q>},$$

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**Lemma. (Poincaré)**

*The boundary operator of singular chains satisfies*

$$\delta_{n-1} \circ \delta_n = 0 \quad \forall n \geq 1,$$

*so that the complex of singular chains is a differential complex.*

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Every continuous closed curve $c : [0,1] \longrightarrow X$ defines an homology class $[c] \in H_1(X;\mathbb{Z}_2)$. If $X$ is pathwise connected then $H_0(X;\mathbb{Z}_2) \simeq \mathbb{Z}_2$.

For every continuous map $f : X \longrightarrow Y$ there is a natural homomorphism

$$f_* : H_n(X;\mathbb{Z}_2) \longrightarrow H_n(Y;\mathbb{Z}_2), \quad n \in \mathbb{N}.$$

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There is a product on $H^*(M;\mathbb{Z}_2)$ which makes it into a graded algebra

$$\cup : H^n(X;\mathbb{Z}_2) \otimes H^m(X;\mathbb{Z}_2) \longrightarrow H^{n+m}(X;\mathbb{Z}_2), (\alpha,\beta) \mapsto \alpha \cup \beta.$$

# Poincaré Duality

**Theorem. (Poincaré)**

*Let $M$ be a closed manifold of dimension $n$. For any $p \in \mathbb{N}$ there exists a natural isomorphism*

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**Corollary.**

*Let $M$ be a closed manifold of dimension $n$. Then for any $p \geq n+1$,*

$$H_p(M;\mathbb{Z}_2) = 0 \quad \textit{and} \quad H_n(M,\mathbb{Z}_2) \simeq (\mathbb{Z}_2)^m,$$

*where $m$ is the number of connected components of $M$.*

# Fundamental class

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Any closed connected $q$-dimensional submanifold $Y \subset M$ defines an element $|Y| \in H_q(M;\mathbb{Z}_2)$ via the map
$i_* : H_q(Y;\mathbb{Z}_2) \longrightarrow H_q(M;\mathbb{Z}_2)$

# Cohomology and homology of $\mathbb{P}^n$

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3. Denote by $X = \pi(|\mathbb{P}^{n-1}|) \in H^1(\mathbb{P}^n; \mathbb{Z}_2)$. Then for any $q$, $X^q$ is the generator of $H^q(\mathbb{P}^n; \mathbb{Z}_2)$ and hence

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$$H^*(\mathbb{P}^n; \mathbb{Z}_2) = \left\{ \sum_{q=0}^{n} t_i X^i, t_i \in \mathbb{Z}_2, X^{n+1} = 0 \right\}.$$
4. $\langle X, |\mathbb{P}^1| \rangle = 1$.

# Cohomology and homology of $\mathbb{P}^n \times \mathbb{P}^n$

1. For $0 \leq q \leq 2n$,

   $H_q(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}_2) = \text{Vect}\left\{|\mathbb{P}^r \times \mathbb{P}^s|, r + s = q, 0 \leq r, s \leq n\right\}.$

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2. Put $Y = \pi(|\mathbb{P}^{n-1} \times \mathbb{P}^n|) \in H^1(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}_2)$ and $Z = \pi(|\mathbb{P}^n \times \mathbb{P}^{n-1}|) \in H^1(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}_2)$. Then

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3. $\langle Y, |\mathbb{P}^1 \times \{*\}|\rangle = 1$ and $\langle Y, |\{*\} \times \mathbb{P}^1|\rangle = 0$

# Cohomology and homology of $\mathbb{P}^n \times \mathbb{P}^n$

1. For $0 \leq q \leq 2n$,

$$H_q(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}_2) = \mathrm{Vect}\left\{|\mathbb{P}^r \times \mathbb{P}^s|, r+s=q, 0 \leq r, s \leq n\right\}.$$

2. Put $Y = \pi(|\mathbb{P}^{n-1} \times \mathbb{P}^n|) \in H^1(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}_2)$ and $Z = \pi(|\mathbb{P}^n \times \mathbb{P}^{n-1}|) \in H^1(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}_2)$. Then

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4. $\langle Z, |\mathbb{P}^1 \times \{*\}| \rangle = 0.$ and $\langle Z, |\{*\} \times \mathbb{P}^1| \rangle = 1.$

# Proof of Hopf's Theorem

Suppose that there exist a continuous odd map
$g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ and denote by
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Now

$$G^*(X) = Y + Z.$$

Now since $X^n = 0$ then

$$(Y+Z)^n = \sum_{q=0}^{n} \frac{n!}{q!(n-q)!} Y^q Z^{n-q} = \sum_{q=1}^{n-1} \frac{n!}{q!(n-q)!} Y^q Z^{n-q} = 0$$

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This implies that for any $1 \leq q \leq n-1$, $\frac{n!}{q!(n-q)!}$ is even and this implies that $n = 2^p$. $\square$

# Kervaire-Milnor Theorem

We give the needed material for a sketch of a proof of the following theorem.

**Theorem.**

$\mathbb{R}^n$ *has a structure of division algebra if and only if* $n = 1, 2, 4$ *or* $8$.

# Vector bundles over a topological space

Let $X$ be a topological space. A vector bundle of rank $n$ over $X$ is a topological space $E$ together with a surjective continuous map $\pi : E \longrightarrow X$ such that:

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1. for any $x \in X$, $E_x = \pi^{-1}(x)$ has a structure of $n$-dimensional real vector space,
2. for any $x \in X$, there exists $n$ sections $s_1, \ldots, s_n : U \longrightarrow E$ such that, for any $y \in U$, $(s_1(y), \ldots, s_n(y))$ is a basis of $E_y$.

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**Example**

1\. $\pi : X \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector bundle called the trivial vector bundle of rank $n$ over $X$ .
2\. The tangent space to a manifold is a vector bundle.

A $n$-vector bundle $\pi : E \longrightarrow X$ is called trivializable if it admits $n$ linearly independent global sections.

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**Theorem.**

*Every vector bundle over a contractile topological space is trivializable.*

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We define a continuous $f_E : S^{n-1} \longrightarrow \mathrm{GL}(m)$ by

$$f_E = P((s_i^+(y))_{i=1}^m, (s_i^-(y))_{i=1}^m).$$

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$$f_E = P((s_i^+(y))_{i=1}^m, (s_i^-(y))_{i=1}^m).$$

Conversely, any continuous map $f : S^{n-1} \longrightarrow \mathrm{GL}(m)$ defines a $m$-vector bundle $E_f$ over $S^n$.

# Hopf vector bundles

Suppose that $\mathbb{R}^n$ carries a subdivision algebra structure. Then the map

$$F : S^{n-1} \longrightarrow \mathrm{GL}(n, \mathbb{R}),\ x \mapsto F(x) = \mathrm{L}_x$$

is continuous and for any $v \in \mathbb{R}^n \setminus \{0\}$,

$$F_v : S^{n-1} \longrightarrow S^{n-1},\ x \mapsto \frac{x.v}{||x.v||}$$

is an homeomorphism.

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is an homeomorphism.
Thus

$$\alpha(F) = 1.$$

We denote by $E_F \longrightarrow S^n$ the associated vector bundle.

From the canonical division algebra structures on $\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ we get four vector bundles

$$H_{\mathbb{R}} \longrightarrow S^1,\ H_{\mathbb{C}} \longrightarrow S^2,\ H_{\mathbb{H}} \longrightarrow S^4 \quad \text{and} \quad H_{\mathbb{O}} \longrightarrow S^8$$

known as Hopf's vector bundles.
Note that $H_{\mathbb{R}} \longrightarrow S^1$ is the Möbius strip.

# Operations on vector bundles

Let $X$ be a topological space and $E \longrightarrow X$ and $F \longrightarrow X$ two vector bundles of rank $n$ and $m$.

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2. $E \otimes F := \bigcup_{x \in X} (E_x \otimes F_x) \longrightarrow X$ is a $mn$-vector bundle.
3. Let $f : Y \longrightarrow X$ be continuous map. Then $f^*E \longrightarrow Y$ is a $n$-vector bundle (pull-back) where

$$f^*E = \{(y, v) \in Y \times E, f(y) = \pi(v)\}.$$

# The ring of vector bundles $\mathrm{KO}(X)$

Let $X$ is a topological space. We denote by $\mathrm{Vect}(X)$ the set of classes of isomorphism of vector bundles over $X$.

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**Proposition.**

1. *The direct sum* $\oplus$ *induces an operation on* $\mathrm{Vect}(X)$ *which is commutative, associative and has a neutral element.*
2. *The tensor product* $\otimes$ *induces an operation on* $\mathrm{Vect}(X)$ *which is commutative, associative and has a neutral element.*
3. *The operation* $\otimes$ *is distributive with respect to* $\oplus$.

We define on $\mathrm{Vect}(X) \times \mathrm{Vect}(X)$ the equivalence relation $\simeq$ by

$$(E, F) \simeq (E', F') \iff \exists G, E \oplus F' \oplus G = F \oplus E' \oplus G.$$

Put

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**Proposition.**

$(\mathrm{KO}(X), \oplus, \otimes)$ *is a ring and*

$$\epsilon : \mathrm{KO}(X) \longrightarrow \mathbb{Z}, \ [E, F] \longrightarrow \mathrm{rank}(F) - \mathrm{rank}(E)$$

*is an homomorphism. We denote by* $\widetilde{\mathrm{KO}}(X) = \ker \epsilon$.

For any continuous function $f : X \longrightarrow Y$ the pull-back defines an homomorphism of ring

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For any two vector bundles $E$ and $F$, we have

$$-[E, F] = [F, E]$$

and

$$[E, F] = [E, 0] + [0, F] := F - E.$$

# Stiefel-Whitney classes

**Axiom 1.** To each vector bundle $\xi$ corresponds a sequence of cohomology classes

$$w_i(\xi) \in H^i(B(\xi); \mathbb{Z}_2), \quad i = 0, 1 \ldots,$$

called the Stiefel-Whitney classes of $\xi$. The class $w_0(\xi)$ corresponds to the element $1 \in H^i(B(\xi); \mathbb{Z}_2)$ and $w_i(\xi) = 0$ for $i > \operatorname{rank}(\xi)$.

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**Axiom 2. Naturality.** If $f : B(\xi) \longrightarrow B(\eta)$ is covered by a bundle map from $\xi$ to $\eta$ then

$$w_i(\xi) = f^* w_i(\eta).$$

**Axiom 3. The Whitney product theorem.** If $\xi$ and $\eta$ are two vector bundles over the same basis then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi).w_{k-i}(\eta).$$

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The total Stiefel-Whitney class of $\xi$ is given by

$$w(\xi) = 1 + w_1(\xi) + \ldots + w_n(\xi), \quad n = \mathrm{rank}(\xi).$$

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The Whitney product theorem can be written

$$w(\xi \oplus \eta) = w(\xi).w(\eta).$$

# Consequences of the four axioms.

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**Proposition.**

*If $\xi$ is an $n$-vector bundle with an Euclidean product and $k$ linearly independent sections then*

$$w_n(\xi) = w_{n-1}(\xi) = \ldots = w_{n-k+1}(\xi) = 0.$$

**Proposition.**

*Let* $f : S^{n-1} \longrightarrow \mathrm{GL}(n)$ *and* $\xi_f$ *the associated vector bundle over* $S^n$ *then*

$$\mathbb{Z}_2 = H^n(S^n; \mathbb{Z}_2) \ni w_n(\xi_f) = \alpha(f) \in \mathbb{Z}_2.$$

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*If there exists a vector bundle* $\xi$ *over* $S^n$ *with* $w_n(\xi) \neq 0$ *then* $n = 1, 2, 4$ *or* $8$*.*

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**Theorem.**

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**Corollary.**

*If there exists a division algebra structure on* $\mathbb{R}^n$ *then* $n = 1, 2, 4$ *or* $8$.

Let $X$ be a topological space. Put

$$G(X) = \{1 + a_1 + \ldots + a_i + \ldots ; a_i \in H^i(X; \mathbb{Z}_2)\}.$$

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**Proposition.**

*$G(X)$ endowed with the multiplication is an abelian group and $\Phi : \mathrm{KO}(X) \longrightarrow G(X)$, $[E, F] \mapsto w(E).w(F)^{-1}$ is an homomorphism of groups from the additive group $\mathrm{KO}(X)$ to the multiplicative group $G(X)$.*

**Theorem. ( Bott's periodicity Theorem)**

*We have the following identifications as additive groups:*

$$\widetilde{\mathrm{KO}}(S^1) = \widetilde{\mathrm{KO}}(S^2) = \mathbb{Z}_2,\ \widetilde{\mathrm{KO}}(S^3) = 0,\ \widetilde{\mathrm{KO}}(S^4) = \mathbb{Z}$$
$$\widetilde{\mathrm{KO}}(S^5) = \widetilde{\mathrm{KO}}(S^6) = \widetilde{\mathrm{KO}}(S^7) = 0,\ \widetilde{\mathrm{KO}}(S^8) = \mathbb{Z},$$
$$\widetilde{\mathrm{KO}}(S^{n+8}) = \widetilde{\mathrm{KO}}(S^n).$$

# The isomorphism between $\widetilde{\mathrm{KO}}(S^n)$ and $\widetilde{\mathrm{KO}}(S^{n+8})$

We consider the cartezian product $S^n \times S^m$, the projections $\pi_1 : S^n \times S^m \longrightarrow S^n$, $\pi_2 : S^n \times S^m \longrightarrow S^m$ and the axial cross $S^n \vee S^m = \{x_0\} \times S^m \cup S^n \times \{y_0\} \subset S^n \times S^m$. We collapse it to a point and $S^n \times S^m$ becomes $S^{n+m}$.

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From

$$S^n \vee S^m \xrightarrow{i} S^n \times S^m \xrightarrow{p} S^{n+m}$$

we get an exact sequence

$$0 \longrightarrow \widetilde{\mathrm{KO}}(S^{n+m}) \xrightarrow{p^*} \widetilde{\mathrm{KO}}(S^n \times S^m) \xrightarrow{i^*} \widetilde{\mathrm{KO}}(S^n \vee S^m) \longrightarrow 0.$$

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We have also

$$\widetilde{\mathrm{KO}}(S^n) \xrightarrow{\pi_1^*} \widetilde{\mathrm{KO}}(S^n \times S^m) \quad \text{and} \quad \widetilde{\mathrm{KO}}(S^m) \xrightarrow{\pi_2^*} \widetilde{\mathrm{KO}}(S^n \times S^m).$$

Given $a \in \widetilde{\mathrm{KO}}(S^n)$ and $b \in \widetilde{\mathrm{KO}}(S^m)$ we form

$$a.b = \pi_1^*(a).\pi_2^*(b) \in \widetilde{\mathrm{KO}}(S^n \times S^m).$$

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Since $i^*(a.b) = 0$ there exists an unique element $a \circ b \in \widetilde{\mathrm{KO}}(S^{n+m})$ such that $p^*(a \circ b) = a.b$. So we have defined a bilinear map

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So the isomorphism

$$\widetilde{\mathrm{KO}}(S^n) \longrightarrow \widetilde{\mathrm{KO}}(S^{n+8})$$

is given by

$$a \mapsto a \circ (I_8, H_{\mathbb{O}}) = a \circ (H_{\mathbb{O}} - I_8).$$

# End of the proof of Kervaire-Milnor

**Theorem.**

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The end of the proof is based on the following

**Proposition.**

*If* $n \neq 1, 2, 4, 8$ *then* $w : \widetilde{\mathrm{KO}}(S^n) \longrightarrow G(S^n)$ *satisfies* $w(a) = 1$ *for any* $a \in \widetilde{\mathrm{KO}}(S^n)$.

# Proof of the proposition

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If $n = 3, 5, 6$ or $7$ it is a consequence of Bott's periodicity theorem.
If $n = m + 8$. For $a \in \widetilde{\mathrm{KO}}(S^n)$, we have

$$\begin{aligned} a &= (E - F) \circ (H_{\mathbb{O}} - I_8) \\ &= E \circ H_{\mathbb{O}} - E \circ I_8 - F \circ H_{\mathbb{O}} + F \circ I_8, \end{aligned}$$

and the result follows from a formula of $w$ applied to a tensor product.