Riemannian Lie algebroids

Mohamed Boucetta

Cadi-Ayyad University

28-12-2019

Seminar Algebra, Geometry, Topology and Applications

Outline

An *anchored vector bundle* is a triple (A, M, ρ) where :

1. $p : A \longrightarrow M$ is a vector bundle,
2. $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor.

Let (A, M, ρ) be an anchored vector bundle.
A bracket on $\Gamma(A)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.
It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \qquad (1)$$

$[\ ,\ ]_A$ is local:

$$a_{|U} = 0 \Longrightarrow (\forall b \in \Gamma(A), \quad [a, b]_A = 0 \text{ on } U.)$$

An *anchored vector bundle* is a triple (A, M, ρ) where :

1. $p : A \longrightarrow M$ is a vector bundle,
2. $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor.

Let (A, M, ρ) be an anchored vector bundle.
A bracket on $\Gamma(A)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.
It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \qquad (1)$$

$[\ ,\ ]_A$ is local:

$$a_{|U} = 0 \Longrightarrow (\forall b \in \Gamma(A), \quad [a, b]_A = 0 \text{ on } U.)$$

An *anchored vector bundle* is a triple (A, M, ρ) where :

1. $p : A \longrightarrow M$ is a vector bundle,
2. $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor.

Let (A, M, ρ) be an anchored vector bundle.

A bracket on $\Gamma(A)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.

It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \qquad (1)$$

$[\ ,\ ]_A$ is local:

$$a_{|U} = 0 \Longrightarrow (\forall b \in \Gamma(A), \quad [a, b]_A = 0 \text{ on } U.)$$

An *anchored vector bundle* is a triple (A, M, ρ) where :

1. $p : A \longrightarrow M$ is a vector bundle,
2. $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor.

Let (A, M, ρ) be an anchored vector bundle.
A bracket on $\Gamma(A)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.
It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \tag{1}$$

$[\ ,\ ]_A$ is local:

$$a_{|U} = 0 \Longrightarrow (\forall b \in \Gamma(A), \quad [a, b]_A = 0 \text{ on } U.)$$

An *anchored vector bundle* is a triple (A, M, ρ) where :

1. $p : A \longrightarrow M$ is a vector bundle,
2. $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor.

Let (A, M, ρ) be an anchored vector bundle.
A bracket on $\Gamma(A)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.
It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \tag{1}$$

$[\ ,\ ]_A$ is local:

$$a_{|U} = 0 \Longrightarrow (\forall b \in \Gamma(A), \quad [a, b]_A = 0 \text{ on } U.)$$

An *anchored vector bundle* is a triple (A, M, ρ) where :

1. $p : A \longrightarrow M$ is a vector bundle,
2. $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor.

Let (A, M, ρ) be an anchored vector bundle.
A bracket on $\Gamma(A)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.
It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \qquad (1)$$

$[\ ,\ ]_A$ is local:

$$a_{|U} = 0 \Longrightarrow (\forall b \in \Gamma(A), \quad [a, b]_A = 0 \text{ on } U.)$$

The *torsion* of $[\ ,\ ]_A$ is the map $\Theta_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \longrightarrow \mathcal{X}(M)$ given by

$$\Theta_{[\ ,\ ]_A}(a, b) = \rho([a, b]_A) - [\rho(a), \rho(b)]. \qquad (2)$$

$\Theta_{[\ ,\ ]_A}$ is $\mathbb{R}$-bilinear, skew-symmetric and, for any $f \in C^\infty(M)$,

$$\Theta_{[\ ,\ ]_A}(fa, b) = \Theta_{[\ ,\ ]_A}(a, fb) = f\Theta_{[\ ,\ ]_A}(a, b).$$

So $\Theta_{[\ ,\ ]_A} \in \Gamma(\wedge^2 A^* \otimes TM)$.

The *torsion* of $[\ ,\ ]_A$ is the map $\Theta_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \longrightarrow \mathcal{X}(M)$ given by

$$\Theta_{[\ ,\ ]_A}(a, b) = \rho([a, b]_A) - [\rho(a), \rho(b)]. \qquad (2)$$

$\Theta_{[\ ,\ ]_A}$ is $\mathbb{R}$-bilinear, skew-symmetric and, for any $f \in C^\infty(M)$,

$$\Theta_{[\ ,\ ]_A}(fa, b) = \Theta_{[\ ,\ ]_A}(a, fb) = f\Theta_{[\ ,\ ]_A}(a, b).$$

So $\Theta_{[\ ,\ ]_A} \in \Gamma(\wedge^2 A^* \otimes TM)$.

The *torsion* of $[\ ,\ ]_A$ is the map $\Theta_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \longrightarrow \mathcal{X}(M)$ given by

$$\Theta_{[\ ,\ ]_A}(a, b) = \rho([a, b]_A) - [\rho(a), \rho(b)]. \tag{2}$$

$\Theta_{[\ ,\ ]_A}$ is $\mathbb{R}$-bilinear, skew-symmetric and, for any $f \in C^\infty(M)$,

$$\Theta_{[\ ,\ ]_A}(fa, b) = \Theta_{[\ ,\ ]_A}(a, fb) = f\Theta_{[\ ,\ ]_A}(a, b).$$

So $\Theta_{[\ ,\ ]_A} \in \Gamma(\wedge^2 A^* \otimes TM)$.

The Jacobiator of $[\ ,\ ]_A$ as $J_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ given by

$$J_{[\ ,\ ]_A}(a,b,c) = [[a,b]_A, c]_A + [[b,c]_A, a]_A + [[c,a]_A, b]_A.$$

J is $\mathbb{R}$-trilinear and skew-symmetric. Thus $[\ ,\ ]_A$ is a Lie bracket if and only if $J_{[\ ,\ ]_A} = 0$.
For any $a, b, c \in \Gamma(A)$ and any $f \in C^\infty(M)$,

$$J_{[\ ,\ ]_A}(a,b,fc) = fJ_{[\ ,\ ]_A}(a,b,c) + \Theta_{[\ ,\ ]_A}(a,b)(f)c. \qquad (3)$$

The Jacobiator of $[\ ,\ ]_A$ as $J_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ given by

$$J_{[\ ,\ ]_A}(a, b, c) = [[a, b]_A, c]_A + [[b, c]_A, a]_A + [[c, a]_A, b]_A.$$

J is $\mathbb{R}$-trilinear and skew-symmetric. Thus $[\ ,\ ]_A$ is a Lie bracket if and only if $J_{[\ ,\ ]_A} = 0$.

For any $a, b, c \in \Gamma(A)$ and any $f \in C^\infty(M)$,

$$J_{[\ ,\ ]_A}(a, b, fc) = fJ_{[\ ,\ ]_A}(a, b, c) + \Theta_{[\ ,\ ]_A}(a, b)(f)c. \qquad (3)$$

The Jacobiator of $[\ ,\ ]_A$ as $J_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ given by

$$J_{[\ ,\ ]_A}(a,b,c) = [[a,b]_A,c]_A + [[b,c]_A,a]_A + [[c,a]_A,b]_A.$$

J is $\mathbb{R}$-trilinear and skew-symmetric. Thus $[\ ,\ ]_A$ is a Lie bracket if and only if $J_{[\ ,\ ]_A} = 0$.
For any $a,b,c \in \Gamma(A)$ and any $f \in C^\infty(M)$,

$$J_{[\ ,\ ]_A}(a,b,fc) = fJ_{[\ ,\ ]_A}(a,b,c) + \Theta_{[\ ,\ ]_A}(a,b)(f)c. \qquad (3)$$

This relation shows that:

1. $J_{[\ ,\ ]_A}$ is local:

$$a_{|U} = 0 \quad \Longrightarrow \quad J_A(a,.,.)_{|U} = 0,$$

2. if $\Theta_{[\ ,\ ]_A}$ vanishes then $J_{[\ ,\ ]_A}$ becomes a tensor, namely, $J_{[\ ,\ ]_A} \in \Gamma(\wedge^3 A^* \otimes TM)$,

3. $$J_{[\ ,\ ]_A} \equiv 0 \quad \Longrightarrow \quad \Theta_{[\ ,\ ]_A} \equiv 0.$$

This relation shows that:

1. $J_{[\ ,\ ]_A}$ is local:

$$a_{|U} = 0 \quad \Longrightarrow \quad J_A(a, ., .)_{|U} = 0,$$

2. if $\Theta_{[\ ,\ ]_A}$ vanishes then $J_{[\ ,\ ]_A}$ becomes a tensor, namely, $J_{[\ ,\ ]_A} \in \Gamma(\wedge^3 A^* \otimes TM)$,

3. $$J_{[\ ,\ ]_A} \equiv 0 \quad \Longrightarrow \quad \Theta_{[\ ,\ ]_A} \equiv 0.$$

This relation shows that:

1. $J_{[\ ,\ ]_A}$ is local:

$$a_{|U} = 0 \quad \Longrightarrow \quad J_A(a, ., .)_{|U} = 0,$$

2. if $\Theta_{[\ ,\ ]_A}$ vanishes then $J_{[\ ,\ ]_A}$ becomes a tensor, namely, $J_{[\ ,\ ]_A} \in \Gamma(\wedge^3 A^* \otimes TM)$,
3.

$$J_{[\ ,\ ]_A} \equiv 0 \quad \Longrightarrow \quad \Theta_{[\ ,\ ]_A} \equiv 0.$$

Proposition

Let (A, M, ρ) be an anchored bundle and $[\ ,\ ]_A$ an anchored bracket on $\Gamma(A)$. Then the following assertions are equivalent:

- (i) $(\Gamma(A), [\ ,\ ]_A)$ *is a Lie algebra, i.e., J_A vanishes identically.*
- (ii) *For any $x \in M$ there exists an open set U of M containing x and a basis of sections $(a_1, \ldots, a_r)$ over U such that*

$$J_{[\ ,\ ]_A}(a_i, a_j, a_k) = 0 \quad \textit{and} \quad \Theta(a_i, a_j) = 0, \quad 1 \le i < j < k \le r.$$

Definition

A Lie algebroid is an anchored vector bundle (A, M, ρ) together with an anchored bracket $[\ ,\ ]_A$ satisfying (i) or (ii) of Proposition 1.3.

Proposition

Let (A, M, ρ) be an anchored bundle and $[\ ,\ ]_A$ an anchored bracket on $\Gamma(A)$. Then the following assertions are equivalent:

(i) $(\Gamma(A), [\ ,\ ]_A)$ *is a Lie algebra, i.e., J_A vanishes identically.*

(ii) *For any $x \in M$ there exists an open set U of M containing x and a basis of sections $(a_1, \ldots, a_r)$ over U such that*

$$J_{[\ ,\ ]_A}(a_i, a_j, a_k) = 0 \quad \text{and} \quad \Theta(a_i, a_j) = 0, \quad 1 \leq i < j < k \leq r.$$

Definition

A Lie algebroid is an anchored vector bundle (A, M, ρ) together with an anchored bracket $[\ ,\ ]_A$ satisfying (i) or (ii) of Proposition 1.3.

There are some well-known properties of a Lie algebroid $(A, M, \rho, [\ ,\ ]_A)$.

(a) The smooth distribution $\mathrm{Im}\rho$ is integrable in the sense of Sussmann and, for any leaf L of $\mathrm{Im}\rho$, $(A_{|L}, L, \rho, [\ ,\ ]_A)$ is a transitive Lie algebroid.

(b) For any $x \in M$, there is an induced Lie bracket say $[\ ,\ ]_x$ on $\mathfrak{g}_x = \ker(\rho_x) \subset A_x$ which makes it into a finite dimensional Lie algebra.

There are some well-known properties of a Lie algebroid $(A, M, \rho, [\ ,\ ]_A)$.

(a) The smooth distribution $\mathrm{Im}\rho$ is integrable in the sense of Sussmann and, for any leaf L of $\mathrm{Im}\rho$, $(A_{|L}, L, \rho, [\ ,\ ]_A)$ is a transitive Lie algebroid.

(b) For any $x \in M$, there is an induced Lie bracket say $[\ ,\ ]_x$ on $\mathfrak{g}_x = \ker(\rho_x) \subset A_x$ which makes it into a finite dimensional Lie algebra.

(c) The map $d_A : \Gamma(\wedge A^*) \longrightarrow \Gamma(\wedge A^*)$ by

$$d_A Q(a_1, \ldots, a_p) = \sum_{i=1}^{p} (-1)^{i+1} \rho(a_i).Q(a_1, \ldots, \hat{a}_i, \ldots, a_p)$$
$$- \sum_{1 \leq i < j \leq p} (-1)^{i+j+1} Q([a_i, a_j]_A, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_p),$$

is a differential, i.e., $d_A^2 = 0$. In particular, for any $a, b \in \Gamma(A)$, $f \in C^\infty(M)$ and $Q \in \Gamma(\wedge A^*)$,

$$d_A f(a) = \rho(a)(f)$$
$$d_A Q(a, b) = \rho(a).Q(b) - \rho(b).Q(a) - Q([a, b]_A).$$

These two relations show that there is a correspondence between Lie algebroids structure on (A, M) and differentials on $\Gamma(\wedge A^*)$.

Theorem (Local splitting))

Let $x_0 \in M$ be a point where $\#_{x_0}$ has rank q. There exists a system of coordinates $(x_1, \ldots, x_q, y_1, \ldots, y_{n-q})$ valid in a neighborhood U of x_0 and a basis of sections $\{a_1, \ldots, a_r\}$ of A over U, such that

$$\begin{aligned} \#(a_i) &= \partial_{x_i} \qquad (i = 1, \ldots, q), \\ \#(a_i) &= \sum_j b^{ij} \partial_{y_j} \qquad (i = q+1, \ldots, r), \end{aligned}$$

where $b^{ij} \in C^\infty(U)$ are smooth functions depending only on the $y's$ and vanishing at x_0: $b^{ij} = b^{ij}(y^s)$, $b^{ij}(x_0) = 0$. Moreover, for any $i, j = 1, \ldots, r$,

$$[a_i, a_j] = \sum_u C_{ij}^u a_u,$$

where $C_{ij}^u \in C^\infty(U)$ vanish if $u < q$ and satisfy $\sum \frac{\partial C_{ij}^u}{\partial \ldots} b^{ut} = 0.$

Some examples of Lie algebroids

(a) The basic example of a Lie algebroid over M is the tangent bundle itself, with the identity mapping as anchor.

(b) Every finite dimensional Lie algebra is a Lie algebroid over a one point space.

(c) Let (M, π) be a Poisson manifold. The bivector field π defines a bundle homomorphism $\pi_\# : T^*M \longrightarrow TM$ and a bracket on $\Omega^1(M)$ by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\#(\alpha)}\beta - \mathcal{L}_{\pi_\#(\beta)}\alpha - d\pi(\alpha, \beta)$$

such that $(T^*M, M, \pi_\#, [\ ,\ ]_\pi)$ is a Lie algebroid.

Some examples of Lie algebroids

(a) The basic example of a Lie algebroid over M is the tangent bundle itself, with the identity mapping as anchor.

(b) Every finite dimensional Lie algebra is a Lie algebroid over a one point space.

(c) Let (M, π) be a Poisson manifold. The bivector field π defines a bundle homomorphism $\pi_\# : T^*M \longrightarrow TM$ and a bracket on $\Omega^1(M)$ by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\#(\alpha)}\beta - \mathcal{L}_{\pi_\#(\beta)}\alpha - d\pi(\alpha, \beta)$$

such that $(T^*M, M, \pi_\#, [\ ,\ ]_\pi)$ is a Lie algebroid.

Some examples of Lie algebroids

(*a*) The basic example of a Lie algebroid over M is the tangent bundle itself, with the identity mapping as anchor.

(*b*) Every finite dimensional Lie algebra is a Lie algebroid over a one point space.

(*c*) Let (M, π) be a Poisson manifold. The bivector field π defines a bundle homomorphism $\pi_\# : T^*M \longrightarrow TM$ and a bracket on $\Omega^1(M)$ by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\#(\alpha)}\beta - \mathcal{L}_{\pi_\#(\beta)}\alpha - d\pi(\alpha, \beta)$$

such that $(T^*M, M, \pi_\#, [\ ,\ ]_\pi)$ is a Lie algebroid.

(*d*) Let $\mathfrak{g} \stackrel{\tau}{\longrightarrow} \mathcal{X}(M)$ be an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M, i.e., a morphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of vector fields on M.
Consider $(A, M, \rho, [\ ,\ ]_A)$, where $A = M \times \mathfrak{g}$ as a trivial bundle and

$$\rho((m, \xi)) = \tau(\xi)(m) \quad \text{and} \quad [\xi, \eta]_A = \mathcal{L}_{\rho(\xi)}\eta - \mathcal{L}_{\rho(\eta)}\xi + [\xi, \eta]_{\mathfrak{g}},$$

$\eta, \xi \in \Gamma(A) = C^\infty(M, \mathfrak{g})$.
By using (*ii*) of Proposition 1.3, it is easy to check that $(A, M, \rho, [\ ,\ ]_A)$ is a Lie algebroid.

(d) Let $\mathfrak{g} \xrightarrow{\tau} \mathcal{X}(M)$ be an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M, i.e., a morphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Consider $(A, M, \rho, [\ ,\ ]_A)$, where $A = M \times \mathfrak{g}$ as a trivial bundle and

$$\rho((m,\xi)) = \tau(\xi)(m) \quad \text{and} \quad [\xi,\eta]_A = \mathcal{L}_{\rho(\xi)}\eta - \mathcal{L}_{\rho(\eta)}\xi + [\xi,\eta]_{\mathfrak{g}},$$

$\eta, \xi \in \Gamma(A) = C^\infty(M, \mathfrak{g})$.

By using (ii) of Proposition 1.3, it is easy to check that $(A, M, \rho, [\ ,\ ]_A)$ is a Lie algebroid.

(d) Let $\mathfrak{g} \xrightarrow{\tau} \mathcal{X}(M)$ be an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M, i.e., a morphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Consider $(A, M, \rho, [\ ,\ ]_A)$, where $A = M \times \mathfrak{g}$ as a trivial bundle and

$$\rho((m,\xi)) = \tau(\xi)(m) \quad \text{and} \quad [\xi,\eta]_A = \mathcal{L}_{\rho(\xi)}\eta - \mathcal{L}_{\rho(\eta)}\xi + [\xi,\eta]_{\mathfrak{g}},$$

$\eta, \xi \in \Gamma(A) = C^\infty(M, \mathfrak{g})$.
By using (ii) of Proposition 1.3, it is easy to check that $(A, M, \rho, [\ ,\ ]_A)$ is a Lie algebroid.

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor map $\#$. An A-connection on a vector bundle $E \longrightarrow M$ is an operator $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ satisfying:

1. $\nabla_{a+b}s = \nabla_a s + \nabla_b s$ for any $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$;
2. $\nabla_a(s_1 + s_2) = \nabla_a s_1 + \nabla_a s_2$ for any $a \in \Gamma(A)$ and $s_1, s_2 \in \Gamma(E)$;
3. $\nabla_{fa}s = f\nabla_a s$ for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$;
4. $\nabla_a(fs) = f\nabla_a s + \#(a)(f)s$ for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

Definition

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor $\#$.

1. An A-path is a smooth path $\alpha : [t_0, t_1] \longrightarrow A$ such that

$$\#(\alpha(t)) = \frac{d}{dt} p(\alpha(t)), \qquad t \in [t_0, t_1].$$

   We call the curve $\gamma : [t_0, t_1] \longrightarrow M$ given by $\gamma(t) = p(\alpha(t))$ the *base path of* α.

2. An A-path α is called *vertical* if $\#(\alpha(t)) = 0$ for any $t \in [t_0, t_1]$.

Remark

When $A = TM$, a A-path is just the derivative $\dot{c} : [t_0, t_1] \longrightarrow TM$ of a curve $c : [t_0, t_1] \longrightarrow M$.

Definition

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor $\#$.

1. An A-path is a smooth path $\alpha : [t_0, t_1] \longrightarrow A$ such that

$$\#(\alpha(t)) = \frac{d}{dt}p(\alpha(t)), \qquad t \in [t_0, t_1].$$

   We call the curve $\gamma : [t_0, t_1] \longrightarrow M$ given by $\gamma(t) = p(\alpha(t))$ the *base path of* α.

2. An A-path α is called *vertical* if $\#(\alpha(t)) = 0$ for any $t \in [t_0, t_1]$.

Remark

When $A = TM$, a A-path is just the derivative $\dot{c} : [t_0, t_1] \longrightarrow TM$ of a curve $c : [t_0, t_1] \longrightarrow M$.

Let $p : A \longrightarrow M$ be a Lie algebroid, $\pi : E \longrightarrow M$ a vector bundle and ∇ an A-connection on E. Fix an A-path $\alpha : [t_0, t_1] \longrightarrow A$.

An α-section of E is a smooth map $s : [t_0, t_1] \longrightarrow E$ such that the projections on M of α and s define the same base path, i.e.,

$$p(\alpha) = \pi(s).$$

We denote by $\Gamma(E)_\alpha$ the space of α-sections of E.

Let $p : A \longrightarrow M$ be a Lie algebroid, $\pi : E \longrightarrow M$ a vector bundle and ∇ an A-connection on E. Fix an A-path $\alpha : [t_0, t_1] \longrightarrow A$.

An α-section of E is a smooth map $s : [t_0, t_1] \longrightarrow E$ such that the projections on M of α and s define the same base path, i.e.,

$$p(\alpha) = \pi(s).$$

We denote by $\Gamma(E)_\alpha$ the space of α-sections of E.

Let $p : A \longrightarrow M$ be a Lie algebroid, $\pi : E \longrightarrow M$ a vector bundle and ∇ an A-connection on E. Fix an A-path $\alpha : [t_0, t_1] \longrightarrow A$.

An α-section of E is a smooth map $s : [t_0, t_1] \longrightarrow E$ such that the projections on M of α and s define the same base path, i.e.,

$$p(\alpha) = \pi(s).$$

We denote by $\Gamma(E)_\alpha$ the space of α-sections of E.

Then there is exists an unique map

$$\nabla^\alpha : \Gamma(E)_\alpha \longrightarrow \Gamma(E)_\alpha$$

satisfying:

1. $\nabla^\alpha(c_1 s_1 + c_2 s_2) = c_1 \nabla^\alpha s_1 + c_2 \nabla^\alpha s_2$, $c_1, c_2 \in \mathbb{R}$;
2. $\nabla^\alpha f s = f' s + f \nabla^\alpha s$ where $f : [t_0, t_1] \longrightarrow \mathbb{R}$ is a smooth function;
3. if $\widetilde{s}$ is a local section of E which extends s then

$$\nabla^\alpha s(t) = \nabla_{\alpha(t)} \widetilde{s};$$

Let $(x_1, \ldots, x_n)$ be a local system of coordinates on an open set U, $(a_1, \ldots, a_r)$ is local frame of A and $(e_1, \ldots, e_q)$ is a local frame of E over U and

$$\#a_k = \sum_{i=1}^{n} b^{ki} \partial_{x_i} \qquad (k = 1, \ldots, r).$$

Then

$$\alpha(t) = \sum_{i=1}^{r} \alpha_i(t) a_i \quad \text{and} \quad s(t) = \sum_{i=1}^{q} s_i(t) e_i.$$

We have

$$p(\alpha(t)) = \pi(s(t)) = (x_1(t), \ldots, x_n(t)) \quad \text{and} \quad \#(\alpha(t)) = \sum_{i=1}^{n} x_i'(t) \partial_{x_i}.$$

Let $(x_1, \ldots, x_n)$ be a local system of coordinates on an open set U, $(a_1, \ldots, a_r)$ is local frame of A and $(e_1, \ldots, e_q)$ is a local frame of E over U and

$$\#a_k = \sum_{i=1}^{n} b^{ki}\partial_{x_i} \qquad (k = 1, \ldots, r).$$

Then

$$\alpha(t) = \sum_{i=1}^{r} \alpha_i(t)a_i \quad \text{and} \quad s(t) = \sum_{i=1}^{q} s_i(t)e_i.$$

We have

$$p(\alpha(t)) = \pi(s(t)) = (x_1(t), \ldots, x_n(t)) \quad \text{and} \quad \#(\alpha(t)) = \sum_{i=1}^{n} x_i'(t)\partial_{x_i}.$$

Let $(x_1, \ldots, x_n)$ be a local system of coordinates on an open set U, $(a_1, \ldots, a_r)$ is local frame of A and $(e_1, \ldots, e_q)$ is a local frame of E over U and

$$\#a_k = \sum_{i=1}^{n} b^{ki} \partial_{x_i} \qquad (k = 1, \ldots, r).$$

Then

$$\alpha(t) = \sum_{i=1}^{r} \alpha_i(t) a_i \quad \text{and} \quad s(t) = \sum_{i=1}^{q} s_i(t) e_i.$$

We have

$$p(\alpha(t)) = \pi(s(t)) = (x_1(t), \ldots, x_n(t)) \quad \text{and} \quad \#(\alpha(t)) = \sum_{i=1}^{n} x_i'(t) \partial_{x_i}.$$

$$\begin{aligned}
\nabla^{\alpha} s &= \sum_{i=1}^{q} s_i'(t) e_i + \sum_{i=1}^{q} s_i(t) \nabla^{\alpha} e_i \\
&= \sum_{i=1}^{q} s_i'(t) e_i + \sum_{i=1}^{q} s_i(t) \nabla_{\alpha(t)} e_i \\
&= \sum_{i=1}^{q} s_i'(t) e_i + \sum_{i=1}^{q} \sum_{j=1}^{r} s_i(t) \alpha_j(t) \nabla_{a_j} e_i \\
&= \sum_{k=1}^{q} \left(s_k'(t) + \sum_{i=1}^{q} \sum_{j=1}^{r} s_i(t) \alpha_j(t) \Gamma_{ji}^{k}(x(t)) \right) e_k .
\end{aligned}$$

$$\nabla_{a_j} e_i = \sum_{k=1}^{q} \Gamma_{ji}^{k} e_k .$$

An α-section s is called parallel along α if $\nabla^\alpha s = 0$,i.e.,

$$s_k'(t) + \sum_{i=1}^{q}\sum_{j=1}^{r} s_i(t)\alpha_j(t)\Gamma_{ji}^k(x(t)) = 0, \quad k = 1, \ldots, q.$$

One has then the notion of parallel transport along α, denoted by

$$\tau_\alpha^t : E_{\gamma(t_0)} \longrightarrow E_{\gamma(t)},$$

and $\tau_\alpha^t(s_0) = s(t)$ where s is the unique parallel α-section satisfying $s(0) = s_0$.

An α-section s is called parallel along α if $\nabla^\alpha s = 0$,i.e.,

$$s_k'(t) + \sum_{i=1}^{q}\sum_{j=1}^{r} s_i(t)\alpha_j(t)\Gamma_{ji}^k(x(t)) = 0, \quad k = 1, \ldots, q.$$

One has then the notion of parallel transport along α, denoted by

$$\tau_\alpha^t : E_{\gamma(t_0)} \longrightarrow E_{\gamma(t)},$$

and $\tau_\alpha^t(s_0) = s(t)$ where s is the unique parallel α-section satisfying $s(0) = s_0$.

If $\alpha_0 \in A_x$ and s is a section of E in a neighborhood of x, one can check easily that

$$\nabla_{\alpha_0} s = \frac{d}{dt}_{|t=0} (\tau_\alpha^t)^{-1}(s(\gamma(t))), \qquad (4)$$

where α is any A-path satisfying $\alpha(0) = \alpha_0$.

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor $\#$. We shall call A-connections on the vector bundle $A \longrightarrow M$ *linear A-connections*.
Let $\mathcal{D}$ be a linear A-connection. An A-path $\alpha : [t_0, t_1] \longrightarrow A$ is a *geodesic* of $\mathcal{D}$ if

$$\mathcal{D}^{\alpha}\alpha = 0.$$

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor $\#$. We shall call A-connections on the vector bundle $A \longrightarrow M$ *linear A-connections*. Let $\mathcal{D}$ be a linear A-connection. An A-path $\alpha : [t_0, t_1] \longrightarrow A$ is a *geodesic* of $\mathcal{D}$ if

$$\mathcal{D}^{\alpha}\alpha = 0.$$

Let $(x_1, \ldots, x_n)$ be a local system of coordinates on an open set U and $(a_1, \ldots, a_r)$ a basis of local sections over U. The structure functions $b^{si}, C^u_{st} \in C^\infty(U)$ are given by

$$\#a_s = \sum_{i=1}^{n} b^{si}\partial_{x_i} \qquad (s = 1, \ldots, r),$$

We define the Christoffel symbols of $\mathcal{D}$ according to $(a_1, \ldots, a_r)$ as usually by

$$\mathcal{D}_{a_s} a_t = \sum_{u=1}^{r} \Gamma^u_{st} a_u.$$

The A-path α is a geodesic if, for $i = 1, \ldots, n$ and $j = 1, \ldots, r$,

$$\left\{ \begin{array}{rcl} \dot{x}_i(t) & = & \displaystyle\sum_{j=1}^{r} \alpha_j(t) b^{ji}(x_1(t), \ldots, x_n(t)), \\ \dot{\alpha}_j(t) & = & -\displaystyle\sum_{s,u=1}^{r} \alpha_s(t)\alpha_u(t)\Gamma_{su}^{j}(x_1(t), \ldots, x_n(t)), \end{array} \right. \tag{5}$$

where $\alpha(t) = \sum_{i=1}^{r} \alpha_i(t) a_i$ is the local expression of α and $p(\alpha(t)) = (x_1(t), \ldots, x_n(t))$ is the local expression of its base path.

Exactly as in the classical case, one has existence and uniqueness of geodesics with given initial base point $x \in M$ and "initial speed" $a_0 \in A_x$. Actually, there exists a vector field G on A such that the geodesics of $\mathcal{D}$ are the integral curves of G. We call G the *geodesic vector field* associated to $\mathcal{D}$ and $\mathcal{D}$ is called complete if G is complete.

Remark

The notions of connection, parallel transport and geodesic can be defined in any anchored vector bundle.

We introduce now two natural notions of compatibility between linear A-connections and the structures of Lie algebroids.

Definition

1. *A linear A-connection $\mathcal{D}$ is strongly compatible with the Lie algebroid structure if, for any A-path α , the parallel transport τ_α preserves* $\ker\#$.
2. *A linear A-connection $\mathcal{D}$ is weakly compatible with the Lie algebroid structure if, for any vertical A-path α, the parallel transport τ_α preserves* $\ker\#$.

We introduce now two natural notions of compatibility between linear A-connections and the structures of Lie algebroids.

Definition

1. *A linear A-connection $\mathcal{D}$ is strongly compatible with the Lie algebroid structure if, for any A-path α , the parallel transport τ_α preserves* $\ker\#$.
2. *A linear A-connection $\mathcal{D}$ is weakly compatible with the Lie algebroid structure if, for any vertical A-path α, the parallel transport τ_α preserves* $\ker\#$.

The following proposition gives an useful characterization of the these notions of compatibility.

Proposition

1. *A linear A-connection $\mathcal{D}$ is strongly compatible with the Lie algebroid structure if and only if, for any leaf L and for any sections $\alpha \in \Gamma(A_L)$ and $\beta \in \Gamma(\mathfrak{g}_L)$, $\mathcal{D}_\alpha\beta \in \Gamma(\mathfrak{g}_L)$.*
2. *A linear A-connection $\mathcal{D}$ is weakly compatible with the Lie algebroid structure if and only if, for any leaf L and for any sections $\alpha \in \Gamma(\mathfrak{g}_L)$ and $\beta \in \Gamma(\mathfrak{g}_L)$, $\mathcal{D}_\alpha\beta \in \Gamma(\mathfrak{g}_L)$.*

Example

Let $p : A \longrightarrow M$ be a Lie algebroid and ∇ be a TM-connection on A. Associated with ∇ there is an obvious linear A-connection

$$\mathcal{D}^0_a b = \nabla_{\#(a)} b$$

which is clearly weakly compatible with the Lie algebroid structure.

A bit more subtle is the following linear A-connection

$$\mathcal{D}^1_a b = \nabla_{\#(b)} a + [a, b]_A$$

which is strongly compatible with the Lie algebroid structure. These connections play a fundamental role in the theory of characteristic classes.

Example

Let $p : A \longrightarrow M$ be a Lie algebroid and ∇ be a TM-connection on A. Associated with ∇ there is an obvious linear A-connection

$$\mathcal{D}^0_a b = \nabla_{\#(a)} b$$

which is clearly weakly compatible with the Lie algebroid structure.

A bit more subtle is the following linear A-connection

$$\mathcal{D}^1_a b = \nabla_{\#(b)} a + [a, b]_A$$

which is strongly compatible with the Lie algebroid structure. These connections play a fundamental role in the theory of characteristic classes.

We give an interpretation of the torsion and the curvature of an A-connection which leads naturally to the notion of homotopy of A-paths. This notion plays a crucial role in the integrability of Lie algebroids.

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor $\#$ and $E \longrightarrow M$ a vector bundle. The curvature of an A-connection ∇ on E is formally identical to the usual definition

$$R(a,b)s = \nabla_a \nabla_b s - \nabla_b \nabla_a s - \nabla_{[a,b]_A} s,$$

where $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$. The connection ∇ is called flat if R vanishes identically.

If $\mathcal{D}$ is a linear A-connection the torsion of $\mathcal{D}$ is given by

$$T_{\mathcal{D}}(a,b) = \mathcal{D}_a b - \mathcal{D}_b a - [a,b]_A.$$

Let $p : A \longrightarrow M$ be a Lie algebroid with anchor $\#$ and $E \longrightarrow M$ a vector bundle. The curvature of an A-connection ∇ on E is formally identical to the usual definition

$$R(a, b)s = \nabla_a \nabla_b s - \nabla_b \nabla_a s - \nabla_{[a,b]_A} s,$$

where $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$. The connection ∇ is called flat if R vanishes identically.
If $\mathcal{D}$ is a linear A-connection the torsion of $\mathcal{D}$ is given by

$$T_{\mathcal{D}}(a, b) = \mathcal{D}_a b - \mathcal{D}_b a - [a, b]_A.$$

In the classical case ($A = TM$), the curvature and the torsion can be interpreted by using variations of paths. We will show now that we have a similar interpretation in the general case. First, let us recall the notion of variation of paths in the classical case in order to find the appropriate generalization.

Let M be a manifold and ∇ a connection on TM.

A variation of curves is a smooth map $\Gamma = (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$. Any variation of curves defines two collections of curves: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting $s = constant$ and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $(-\varepsilon, \varepsilon)$ by setting $t = constant$.

The tangent vectors to these two families of curves are examples of vector fields along Γ, we denote them by

$$T(s,t) = \partial_t \Gamma(s,t) = \frac{d}{dt}\Gamma_s(t), \quad S(s,t) = \partial_s \Gamma(s,t) = \frac{d}{ds}\Gamma^{(t)}(s).$$

Let M be a manifold and ∇ a connection on TM.

A variation of curves is a smooth map $\Gamma = (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$.
Any variation of curves defines two collections of curves: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting $s = constant$ and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $(-\varepsilon, \varepsilon)$ by setting $t = constant$.
The tangent vectors to these two families of curves are examples of vector fields along Γ, we denote them by

$$T(s,t) = \partial_t \Gamma(s,t) = \frac{d}{dt}\Gamma_s(t), \quad S(s,t) = \partial_s \Gamma(s,t) = \frac{d}{ds}\Gamma^{(t)}(s).$$

Let M be a manifold and ∇ a connection on TM.

A variation of curves is a smooth map $\Gamma = (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$. Any variation of curves defines two collections of curves: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting $s = constant$ and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $(-\varepsilon, \varepsilon)$ by setting $t = constant$.

The tangent vectors to these two families of curves are examples of vector fields along Γ, we denote them by

$$T(s,t) = \partial_t \Gamma(s,t) = \frac{d}{dt}\Gamma_s(t), \quad S(s,t) = \partial_s \Gamma(s,t) = \frac{d}{ds}\Gamma^{(t)}(s).$$

Let M be a manifold and ∇ a connection on TM.

A variation of curves is a smooth map $\Gamma = (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$. Any variation of curves defines two collections of curves: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting $s = constant$ and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $(-\varepsilon, \varepsilon)$ by setting $t = constant$.
The tangent vectors to these two families of curves are examples of vector fields along Γ, we denote them by

$$T(s,t) = \partial_t \Gamma(s,t) = \frac{d}{dt}\Gamma_s(t), \quad S(s,t) = \partial_s \Gamma(s,t) = \frac{d}{ds}\Gamma^{(t)}(s).$$

If V is a vector field along Γ, we can compute the covariant derivative of V either along the main curves or along the transverse curves, the resulting vector fields along Γ are denoted by $D_t V$ and $D_s V$. The following lemma is classical.

Lemma

With the notation above, we have

$$D_s T - D_t S = T^{\nabla}(S, T). \tag{6}$$

and

$$D_s D_t Y - D_t D_s Y = -R^{\nabla}(S, T)Y. \tag{7}$$

Let (A, M, ρ) a Lie algebroid. Let $\alpha : [0,1] \times [0,1] \longrightarrow A$ and $\Gamma : [0,1] \times [0,1] \longrightarrow M$ it projection.

We call α *variation* of A-paths if

1. for any $s \in [0,1]$, the map $t \mapsto \alpha(s,t)$ is an A-path, i.e.,

$$\#(\alpha(s,t)) = \frac{\partial \Gamma}{\partial t}(s,t),$$

2. the base variation $\Gamma(s,t) = p(\alpha(s,t))$ lies entirely in a fixed leaf L of the characteristic foliation.

Let (A, M, ρ) a Lie algebroid. Let $\alpha : [0,1] \times [0,1] \longrightarrow A$ and $\Gamma : [0,1] \times [0,1] \longrightarrow M$ it projection.
We call α *variation* of A-paths if

1. for any $s \in [0,1]$, the map $t \mapsto \alpha(s,t)$ is an A-path, i.e.,

$$\#(\alpha(s,t)) = \frac{\partial \Gamma}{\partial t}(s,t),$$

2. the base variation $\Gamma(s,t) = p(\alpha(s,t))$ lies entirely in a fixed leaf L of the characteristic foliation.

Let α be a variation of A-paths and Γ its projection.

A *transverse variation* to α is a smooth map $\beta : [0,1] \times [0,1] \longrightarrow A$ such that $p(\beta) = \Gamma$ and

$$\#(\beta(s,t)) = \frac{\partial \Gamma}{\partial s}(s,t).$$

It is clear that if $\#$ is injective, there is an unique transverse variation to a given variation of A-paths. However, if $\#$ is not injective, a given variation of A-paths admits many transverse variations to it. There is a way which permit the control of transverse variations to a fixed variation of A-path. Let us explain this important fact which is at the origin of the notion of homotopy of A-paths.

Let α be a variation of A-paths and Γ its projection. A *transverse variation* to α is a smooth map $\beta : [0,1] \times [0,1] \longrightarrow A$ such that $p(\beta) = \Gamma$ and

$$\#(\beta(s,t)) = \frac{\partial \Gamma}{\partial s}(s,t).$$

It is clear that if $\#$ is injective, there is an unique transverse variation to a given variation of A-paths. However, if $\#$ is not injective, a given variation of A-paths admits many transverse variations to it. There is a way which permit the control of transverse variations to a fixed variation of A-path. Let us explain this important fact which is at the origin of the notion of homotopy of A-paths.

Let α be a variation of A-paths and Γ its projection.
A *transverse variation* to α is a smooth map $\beta : [0,1] \times [0,1] \longrightarrow A$ such that $p(\beta) = \Gamma$ and

$$\#(\beta(s,t)) = \frac{\partial \Gamma}{\partial s}(s,t).$$

It is clear that if $\#$ is injective, there is an unique transverse variation to a given variation of A-paths. However, if $\#$ is not injective, a given variation of A-paths admits many transverse variations to it. There is a way which permit the control of transverse variations to a fixed variation of A-path. Let us explain this important fact which is at the origin of the notion of homotopy of A-paths.

First, let us fix some notations. Let α and β be, respectively, a variation of A-paths and a transverse variation and let $\Gamma = p(\alpha) = p(\beta)$ denote the commune base path.

Let ∇ be an A-connection on a vector bundle $\pi : E \longrightarrow M$ and let $s : [0,1] \times [0,1] \longrightarrow E$ be a section over Γ.

For any $\varepsilon \in [0,1]$, $t \mapsto \alpha(\varepsilon, t)$ is an A-path and $\nabla_t s$ denotes the derivative of $t \mapsto s(\varepsilon, t)$ along this A-path.

On the other hand, for any $t \in [0,1]$, $\varepsilon \mapsto \beta(\varepsilon, t)$ is an A-path and $\nabla_\varepsilon s$ denotes the derivative of $\varepsilon \mapsto s(\varepsilon, t)$ along this A-path.

First, let us fix some notations. Let α and β be, respectively, a variation of A-paths and a transverse variation and let $\Gamma = p(\alpha) = p(\beta)$ denote the commune base path.

Let ∇ be an A-connection on a vector bundle $\pi : E \longrightarrow M$ and let $s : [0,1] \times [0,1] \longrightarrow E$ be a section over Γ.

For any $\varepsilon \in [0,1]$, $t \mapsto \alpha(\varepsilon, t)$ is an A-path and $\nabla_t s$ denotes the derivative of $t \mapsto s(\varepsilon, t)$ along this A-path.

On the other hand, for any $t \in [0,1]$, $\varepsilon \mapsto \beta(\varepsilon, t)$ is an A-path and $\nabla_\varepsilon s$ denotes the derivative of $\varepsilon \mapsto s(\varepsilon, t)$ along this A-path.

First, let us fix some notations. Let α and β be, respectively, a variation of A-paths and a transverse variation and let $\Gamma = p(\alpha) = p(\beta)$ denote the commune base path.

Let ∇ be an A-connection on a vector bundle $\pi : E \longrightarrow M$ and let $s : [0,1] \times [0,1] \longrightarrow E$ be a section over Γ.

For any $\varepsilon \in [0,1]$, $t \mapsto \alpha(\varepsilon, t)$ is an A-path and $\nabla_t s$ denotes the derivative of $t \mapsto s(\varepsilon, t)$ along this A-path.

On the other hand, for any $t \in [0,1]$, $\varepsilon \mapsto \beta(\varepsilon, t)$ is an A-path and $\nabla_\varepsilon s$ denotes the derivative of $\varepsilon \mapsto s(\varepsilon, t)$ along this A-path.

First, let us fix some notations. Let α and β be, respectively, a variation of A-paths and a transverse variation and let $\Gamma = p(\alpha) = p(\beta)$ denote the commune base path.

Let ∇ be an A-connection on a vector bundle $\pi : E \longrightarrow M$ and let $s : [0,1] \times [0,1] \longrightarrow E$ be a section over Γ.

For any $\varepsilon \in [0,1]$, $t \mapsto \alpha(\varepsilon, t)$ is an A-path and $\nabla_t s$ denotes the derivative of $t \mapsto s(\varepsilon, t)$ along this A-path.

On the other hand, for any $t \in [0,1]$, $\varepsilon \mapsto \beta(\varepsilon, t)$ is an A-path and $\nabla_\varepsilon s$ denotes the derivative of $\varepsilon \mapsto s(\varepsilon, t)$ along this A-path.

Proposition

With the notation above the following assertions hold.

1. *For any linear A-connection $\mathcal{D}$, the variation*

$$\Delta(\alpha,\beta) = \mathcal{D}_t\beta - \mathcal{D}_\varepsilon\alpha - T_{\mathcal{D}}(\alpha,\beta)$$

does not depend on $\mathcal{D}$ and satisfies $\#(\Delta(\alpha,\beta)) = 0$.

2. *for any A-connection ∇ on E and for any section s of E over Γ*

$$\nabla_t\nabla_\varepsilon s - \nabla_\varepsilon\nabla_t s = R(\alpha,\beta)s + \nabla_{\Delta(\alpha,\beta)}s.$$

Proof.

Fix $(\varepsilon_0, t_0) \in [0,1] \times [0,1]$ and choose a local coordinates $(x_1, \dots, x_q, y_1, \dots, y_{n-q})$ near $x_0 = \Gamma(\varepsilon_0, t_0)$ and a basis of sections $(a_1, \dots, a_r)$, $(q = \mathrm{r}ank\#_{x_0})$, such that

$$\begin{aligned} \#(a_i) &= \partial_{x_i} \qquad (i = 1, \dots, q), \\ \#(a_i) &= \sum_j b^{ij} \partial_{y_j} \qquad (i = q+1, \dots, r), \end{aligned}$$

where $b^{ij} \in C^\infty(U)$ are smooth functions depending only on the y's and vanishing at x_0: $b^{ij} = b^{ij}(y^s)$, $b^{ij}(x_0) = 0$. Moreover, for any $i, j = 1, \dots, r$,

$$[a_i, a_j] = \sum_u C_{ij}^u a_u,$$

where $C_{ij}^u \in C^\infty(U)$ vanish if $u \leq q$ and satisfy $\sum \frac{\partial C_{ij}^u}{\partial \dots} b^{ut} = 0.$

In these coordinates, we have

$$\begin{cases} \alpha(\varepsilon,t) = \sum_{i=1}^{r} \alpha^i(\varepsilon,t) a_i, \\ \beta(\varepsilon,t) = \sum_{i=1}^{r} \beta^i(\varepsilon,t) a_i, \\ \Gamma(\varepsilon,t) = (x_1(\varepsilon,t),\ldots,x_q(\varepsilon,t),c_1,\ldots,c_{n-q}), \\ \frac{\partial\Gamma}{\partial t} = \sum_{j=1}^{q} \frac{\partial x_j}{\partial t}\partial_{x_j} = \sum_{i=1}^{q} \alpha^j(\varepsilon,t)\partial_{x_j}, \\ \frac{\partial\Gamma}{\partial\varepsilon} = \sum_{j=1}^{q} \frac{\partial x_j}{\partial\varepsilon}\partial_{x_j} = \sum_{i=1}^{q} \beta^j(\varepsilon,t)\partial_{x_j}, \end{cases} \tag{8}$$

where $c_1,\ldots,c_{n-q}$ are constant.

Now

$$\mathcal{D}_t\beta = \sum_{i=1}^{r}\frac{\partial\beta^i}{\partial t}a_i + \sum_{i,j=1}^{r}\alpha^j\beta^i\mathcal{D}_{a_j}a_i$$

$$\mathcal{D}_\varepsilon\alpha = \sum_{i=1}^{r}\frac{\partial\alpha^i}{\partial\varepsilon}a_i + \sum_{i,j=1}^{r}\alpha^i\beta^j\mathcal{D}_{a_j}a_i.$$

Hence

$$\mathcal{D}_t\beta - \mathcal{D}_\varepsilon\alpha = \sum_{i=1}^{r}\left(\frac{\partial\beta^i}{\partial t} - \frac{\partial\alpha^i}{\partial\varepsilon}\right)a_i + T_{\mathcal{D}}(\alpha,\beta) + \sum_{i,j=1}^{r}\alpha^i\beta^j[a_i,a_j].$$

Now, form (8), we have $\frac{\partial \beta^i}{\partial t} = \frac{\partial \alpha^i}{\partial \varepsilon}$ for any $i = 1, \ldots, q$, so

$$\boxed{\mathcal{D}_t \beta - \mathcal{D}_\varepsilon \alpha - T_{\mathcal{D}}(\alpha, \beta) = \sum_{i=q+1}^{r} \left(\frac{\partial \beta^i}{\partial t} - \frac{\partial \alpha^i}{\partial \varepsilon} \right) a_i + \sum_{i,j=1}^{r} \alpha^i \beta^j [a_i, a_j].} \tag{9}$$

One can see that the right hand of this equality lies in $\ker \#$ and does not depend on $\mathcal{D}$.

Now, form (8), we have $\frac{\partial \beta^i}{\partial t} = \frac{\partial \alpha^i}{\partial \varepsilon}$ for any $i = 1, \ldots, q$, so

$$\boxed{\mathcal{D}_t\beta - \mathcal{D}_\varepsilon\alpha - T_{\mathcal{D}}(\alpha, \beta) = \sum_{i=q+1}^{r} \left(\frac{\partial \beta^i}{\partial t} - \frac{\partial \alpha^i}{\partial \varepsilon} \right) a_i + \sum_{i,j=1}^{r} \alpha^i \beta^j [a_i, a_j].} \tag{9}$$

One can see that the right hand of this equality lies in ker$\#$ and does not depend on $\mathcal{D}$.

We choose a local trivialization $(x_1, \ldots, x_q, y_1, \ldots, y_{n-q}, a_1, \ldots, a_r)$ as above, we trivialize E near x_0 by a local basis of sections $(e_1, \ldots, e_\mu)$ and put

$$s(\varepsilon, t) = \sum_{j=1}^{\mu} s^j(\varepsilon, t) e_j.$$

$$\nabla_t s = \sum_{j=1}^{\mu} \frac{\partial s^j}{\partial t} e_j + \sum_{i,j} \alpha^i s^j \nabla_{a_i} e_j.$$

We choose a local trivialization $(x_1, \ldots, x_q, y_1, \ldots, y_{n-q}, a_1, \ldots, a_r)$ as above, we trivialize E near x_0 by a local basis of sections $(e_1, \ldots, e_\mu)$ and put

$$s(\varepsilon, t) = \sum_{j=1}^{\mu} s^j(\varepsilon, t) e_j.$$

$$\nabla_t s = \sum_{j=1}^{\mu} \frac{\partial s^j}{\partial t} e_j + \sum_{i,j} \alpha^i s^j \nabla_{a_i} e_j.$$

$$\nabla_\varepsilon\nabla_t s = \sum_{j=1}^{\mu}\frac{\partial^2 s^j}{\partial\varepsilon\partial t}e_j + \sum_{i,j}\left(\beta^i\frac{\partial s^j}{\partial t} + \frac{\partial\alpha^i}{\partial\varepsilon}s^j + \alpha^i\frac{\partial s^j}{\partial\varepsilon}\right)\nabla_{a_i}e_j + \sum_{i,j,k}\beta^k\alpha^i s^j\nabla_{a_k}\nabla_{a_i}e_j.$$

$$\nabla_t\nabla_\varepsilon s = \sum_{j=1}^{\mu}\frac{\partial^2 s^j}{\partial t\partial\varepsilon}e_j + \sum_{i,j}\left(\alpha^i\frac{\partial s^j}{\partial\varepsilon} + \frac{\partial\beta^i}{\partial t}s^j + \beta^i\frac{\partial s^j}{\partial t}\right)\nabla_{a_i}e_j + \sum_{i,j,k}\alpha^k\beta^i s^j\nabla_{a_k}\nabla_{a_i}e_j.$$

$$\nabla_t\nabla_\varepsilon s - \nabla_\varepsilon\nabla_t s - R(\alpha,\beta)s = \sum_{i,j}\left(\frac{\partial\beta^i}{\partial t} - \frac{\partial\alpha^i}{\partial\varepsilon}\right)s^j\nabla_{a_i}e_j + \sum_{i,j,k}\alpha^k\beta^i s^j\nabla_{[a_k,a_i]}e_j.$$

From the expression of $\Delta(\alpha,\beta)$ given by (9) and from (8), we have

$$\Delta(\alpha,\beta)=0 \Leftrightarrow \begin{cases} \dfrac{\partial\alpha^i}{\partial\varepsilon}-\dfrac{\partial\beta^i}{\partial t}=\displaystyle\sum_{l,k=1}^{r}\alpha^l\beta^k C_{lk}^i & i=q+1,\ldots,r,\\ \alpha^j=\dfrac{\partial x_j}{\partial t},\ \beta^j=\dfrac{\partial x_j}{\partial\varepsilon} & j=1,\ldots,q. \end{cases} \tag{10}$$

Now by using the standard results about linear differential systems one can deduce easily the following useful proposition.

Proposition

Let $p : A \longrightarrow M$ be a Lie algebroid. Then, for a given variation of A-paths α and for given $\beta_0 : [0,1] \longrightarrow A$ such that

$$\#(\beta_0)(\varepsilon) = \frac{\partial p \circ \alpha}{\partial \varepsilon}(\varepsilon, 0)$$

there exists an unique transverse variation β to α such that

$$\Delta(\alpha, \beta) = 0 \quad \textit{and} \quad \beta(\varepsilon, 0) = \beta_0(\varepsilon) \quad \textit{for any } \varepsilon \in [0,1].$$

We can now define the homotopy of A-paths with fixed end-points.

Definition

Let α_0 and α_1 be two A-paths on a Lie algebroid $p : A \longrightarrow M$ such that $p(\alpha_0(0)) = p(\alpha_1(0))$ and $p(\alpha_0(1)) = p(\alpha_1(1))$.

An A-homotopy with fixed end-points from α_0 to α_1 is a variation of A-paths α such that:

1. $p(\alpha(\varepsilon, 0)) = p(\alpha(0, 0))$ *and* $p(\alpha(\varepsilon, 1)) = p(\alpha(0, 1))$ *for any* $\varepsilon \in [0, 1]$, $\alpha(0, .) = \alpha_0$ *and* $\alpha(1, .) = \alpha_1$,
2. *the unique transverse variation β to α satisfying $\Delta(\alpha, \beta) = 0$ and $\beta(\varepsilon, 0) = 0$ satisfies also $\beta(\varepsilon, 1) = 0$.*

We can now define the homotopy of A-paths with fixed end-points.

Definition

Let α_0 and α_1 be two A-paths on a Lie algebroid $p : A \longrightarrow M$ such that $p(\alpha_0(0)) = p(\alpha_1(0))$ and $p(\alpha_0(1)) = p(\alpha_1(1))$.

An A-homotopy with fixed end-points from α_0 to α_1 is a variation of A-paths α such that:

1. $p(\alpha(\varepsilon, 0)) = p(\alpha(0, 0))$ *and* $p(\alpha(\varepsilon, 1)) = p(\alpha(0, 1))$ *for any* $\varepsilon \in [0, 1]$, $\alpha(0, .) = \alpha_0$ *and* $\alpha(1, .) = \alpha_1$,
2. *the unique transverse variation β to α satisfying $\Delta(\alpha, \beta) = 0$ and $\beta(\varepsilon, 0) = 0$ satisfies also $\beta(\varepsilon, 1) = 0$.*

We can now define the homotopy of A-paths with fixed end-points.

Definition

Let α_0 and α_1 be two A-paths on a Lie algebroid $p : A \longrightarrow M$ such that $p(\alpha_0(0)) = p(\alpha_1(0))$ and $p(\alpha_0(1)) = p(\alpha_1(1))$.

An A-homotopy with fixed end-points from α_0 to α_1 is a variation of A-paths α such that:

1. $p(\alpha(\varepsilon, 0)) = p(\alpha(0, 0))$ *and* $p(\alpha(\varepsilon, 1)) = p(\alpha(0, 1))$ *for any* $\varepsilon \in [0, 1]$, $\alpha(0, .) = \alpha_0$ *and* $\alpha(1, .) = \alpha_1$,
2. *the unique transverse variation β to α satisfying $\Delta(\alpha, \beta) = 0$ and $\beta(\varepsilon, 0) = 0$ satisfies also $\beta(\varepsilon, 1) = 0$.*

The following Lemma will be useful latter.

Lemma

Let $\alpha_0 : [0,1] \longrightarrow A$ be an A-path and $\beta_0 : [0,1] \longrightarrow A$ an α_0-section such that $\beta_0(0) = \beta_0(1) = 0$. Then there exists an A-homotopy α with fixed end-points such that $\alpha(0,.) = \alpha_0$ and the corresponding transverse variation β satisfies $\beta(0,.) = \beta_0$.

Proof.

Consider the base path $\gamma_0 : [0,1] \longrightarrow M$ of α_0 and choose an homotopy $\gamma : [0,1] \times [0,1] \longrightarrow M$ with fixed end points such that γ lies in the same leaf as γ_0, $\gamma(0,.) = \gamma_0$ and $\frac{\partial \gamma}{\partial \varepsilon}(0,t) = \#(\beta_0(t))$. We choose also $\beta : [0,1] \times [0,1] \longrightarrow A$ such that $\beta(0,t) = \beta_0(t)$ for any $t \in [0,1]$, $\beta(\varepsilon, 0) = \beta(\varepsilon, 1) = 0$ for any $\varepsilon \in [0,1]$ and $\frac{\partial \gamma}{\partial \varepsilon}(\varepsilon, t) = \#(\beta(\varepsilon, t))$ for any (ε, t). From (10), one can deduce that there exists an unique variation $\alpha : [0,1] \times [0,1] \longrightarrow A$ such that the base path of α is γ, $\frac{\partial \gamma}{\partial t}(\varepsilon, t) = \#(\alpha(\varepsilon, t))$, $\alpha(0,.) = \alpha_0$ and $\Delta(\alpha, \beta) = 0$. This variation is clearly an A-homotopy with fixed end-points and satisfies the required properties.

A Riemannian metric on a Lie algebroid $p : A \longrightarrow M$ is the data, for any $x \in M$, of a scalar product $\langle\ ,\ \rangle_x$ on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $\langle a, b\rangle$ is smooth.
the formula

$$\begin{aligned} 2\langle \mathcal{D}_a b, c\rangle &= \#(a).\langle b, c\rangle + \#(b).\langle a, c\rangle - \#(c).\langle a, b\rangle \\ &+\langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle \end{aligned}$$

defines a linear A-connection which is characterized by the two following properties:
(i) $\mathcal{D}$ is metric, i.e., $\#(a).\langle b, c\rangle = \langle \mathcal{D}_a b, c\rangle + \langle b, \mathcal{D}_a c\rangle$,
(ii) $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]$.
We call $\mathcal{D}$ the *Levi-Civita A-connection* associated to the Riemannian metric $\langle\ ,\ \rangle$.

A Riemannian metric on a Lie algebroid $p : A \longrightarrow M$ is the data, for any $x \in M$, of a scalar product $\langle\ ,\ \rangle_x$ on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $\langle a, b\rangle$ is smooth. the formula

$$\begin{aligned} 2\langle \mathcal{D}_a b, c\rangle &= \#(a).\langle b, c\rangle + \#(b).\langle a, c\rangle - \#(c).\langle a, b\rangle \\ &\quad + \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle \end{aligned}$$

defines a linear A-connection which is characterized by the two following properties:

(i) $\mathcal{D}$ is metric, i.e., $\#(a).\langle b, c\rangle = \langle \mathcal{D}_a b, c\rangle + \langle b, \mathcal{D}_a c\rangle$,

(ii) $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]$.

We call $\mathcal{D}$ the *Levi-Civita A-connection* associated to the Riemannian metric $\langle\ ,\ \rangle$.

A Riemannian metric on a Lie algebroid $p : A \longrightarrow M$ is the data, for any $x \in M$, of a scalar product $\langle\ ,\ \rangle_x$ on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $\langle a, b\rangle$ is smooth. the formula

$$\begin{aligned} 2\langle \mathcal{D}_a b, c\rangle &= \#(a).\langle b, c\rangle + \#(b).\langle a, c\rangle - \#(c).\langle a, b\rangle \\ &\quad + \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle \end{aligned}$$

defines a linear A-connection which is characterized by the two following properties:

(*i*) $\mathcal{D}$ is metric, i.e., $\#(a).\langle b, c\rangle = \langle \mathcal{D}_a b, c\rangle + \langle b, \mathcal{D}_a c\rangle$,

(*ii*) $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]$.

We call $\mathcal{D}$ the *Levi-Civita A-connection* associated to the Riemannian metric $\langle\ ,\ \rangle$.

A Riemannian metric on a Lie algebroid $p : A \longrightarrow M$ is the data, for any $x \in M$, of a scalar product $\langle\ ,\ \rangle_x$ on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $\langle a, b\rangle$ is smooth. the formula

$$\begin{aligned} 2\langle \mathcal{D}_a b, c\rangle &= \#(a).\langle b, c\rangle + \#(b).\langle a, c\rangle - \#(c).\langle a, b\rangle \\ &\quad + \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle \end{aligned}$$

defines a linear A-connection which is characterized by the two following properties:
(i) $\mathcal{D}$ is metric, i.e., $\#(a).\langle b, c\rangle = \langle \mathcal{D}_a b, c\rangle + \langle b, \mathcal{D}_a c\rangle$,
(ii) $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]$.
We call $\mathcal{D}$ the *Levi-Civita A-connection* associated to the Riemannian metric $\langle\ ,\ \rangle$.

A Riemannian metric on a Lie algebroid $p : A \longrightarrow M$ is the data, for any $x \in M$, of a scalar product $\langle\ ,\ \rangle_x$ on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $\langle a, b\rangle$ is smooth. the formula

$$\begin{aligned} 2\langle \mathcal{D}_a b, c\rangle &= \#(a).\langle b, c\rangle + \#(b).\langle a, c\rangle - \#(c).\langle a, b\rangle \\ &\quad + \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle \end{aligned}$$

defines a linear A-connection which is characterized by the two following properties:
(i) $\mathcal{D}$ is metric, i.e., $\#(a).\langle b, c\rangle = \langle \mathcal{D}_a b, c\rangle + \langle b, \mathcal{D}_a c\rangle$,
(ii) $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]$.
We call $\mathcal{D}$ the *Levi-Civita A-connection* associated to the Riemannian metric $\langle\ ,\ \rangle$.

In a system of coordinates $(x_1, \ldots, x_n)$ over a trivializing neighborhood U of M where A admits a basis of local sections $(a_1, \ldots, a_r)$ the Levi-Civita A-connection is determined by the Christoffel's symbols defined by $\mathcal{D}_{a_i} a_j = \sum_k \Gamma_{ij}^k a_k$. A direct computation gives

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{n} g^{kl}\left(b^{iu}\partial_{x_u}(g_{jl}) + b^{ju}\partial_{x_u}(g_{il}) - b^{lu}\partial_{x_u}(g_{ij})\right)\\ &+\frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{r} g^{kl}\left(C_{ij}^u g_{ul} + C_{li}^u g_{uj} + C_{lj}^u g_{ui}\right), \qquad (11)\end{aligned}$$

where the structure functions $b^{si}, C_{st}^u \in C^\infty(U)$ are given by

$$\#a_s = \sum_{i=1}^{n} b^{si}\partial_{x_i} \qquad (s = 1, \ldots, r),$$

Remark

There are two extremal cases:

1. *The Lie algebroid A is the tangent bundle TM of a manifold and we recover the classical notion of Riemannian manifold.*
2. *The Lie algebroid A is a Lie algebra $\mathfrak{g}$ considered as a Lie algebroid over a point. In this case a Riemannian metric on $\mathfrak{g}$ is a scalar product $\langle\ ,\ \rangle$ and the Levi-Civita $\mathfrak{g}$-connection is the product $\mathcal{D} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by*

$$2\langle \mathcal{D}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle.$$

Actually $\mathcal{D}$ is the infinitesimal data associated to the Levi-Civita connection of the left invariant metric associated to $\langle\ ,\ \rangle$ on any Lie group with $\mathfrak{g}$ as a Lie algebra.

Remark

There are two extremal cases:

1. *The Lie algebroid A is the tangent bundle TM of a manifold and we recover the classical notion of Riemannian manifold.*
2. *The Lie algebroid A is a Lie algebra $\mathfrak{g}$ considered as a Lie algebroid over a point. In this case a Riemannian metric on $\mathfrak{g}$ is a scalar product $\langle\ ,\ \rangle$ and the Levi-Civita $\mathfrak{g}$-connection is the product $\mathcal{D} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by*

$$2\langle \mathcal{D}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle.$$

Actually $\mathcal{D}$ is the infinitesimal data associated to the Levi-Civita connection of the left invariant metric associated to $\langle\ ,\ \rangle$ on any Lie group with $\mathfrak{g}$ as a Lie algebra.

Let $\langle\ ,\ \rangle$ be a Riemannian metric on a Lie algebroid $p : A \longrightarrow M$ with anchor $\#$. For any leaf L of the characteristic foliation and for any $x \in L$,

$$A_x = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp,$$

where $\mathfrak{g}_x^\perp$ is the orthogonal to $\mathfrak{g}_x$ with respect $\langle\ ,\ \rangle_x$. The restriction of the anchor $\#$ to $\mathfrak{g}_x^\perp$ is an isomorphism into T_xL and hence induces a scalar product on T_xL

$$\langle u, v\rangle_L = \langle a, b\rangle,$$

where $a, b \in \mathfrak{g}_x^\perp$ and $\#(a) = u$ and $\#(b) = v$. Thus $\langle\ ,\ \rangle$ induces a Riemannian metric $\langle\ ,\ \rangle_L$ on L. We call it the *induced Riemannian metric* on L. On the other hand, the scalar product $\langle\ ,\ \rangle_x$ induces a scalar product on $\mathfrak{g}_x$ and we denote by $\widehat{\mathcal{D}}$ the Levi-Civita $\mathfrak{g}_x$-connection associated with $(\mathfrak{g}_x, \langle\ ,\ \rangle_x)$.

Let us precise more this situation. Fix a leaf L and consider $p_L : A_L \longrightarrow L$. We have

$$A_L = \mathfrak{g}_L \oplus \mathfrak{g}_L^{\perp}.$$

We call the elements of $\Gamma(\mathfrak{g}_L)$ *vertical sections* and the elements of $\Gamma(\mathfrak{g}_L^{\perp})$ *horizontal sections*. For any section a, we denote by a^v and a^h, respectively, its horizontal and vertical component. Note that the bracket of a vertical section with every section is a vertical section. Thus, in the Riemannian point of view, the short exact sequence

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \longrightarrow TL$$

is formally identical to a Riemannian submersion. So we can introduce the O'Neill tensors.

We denote by T and H the elements of $\Gamma(A^* \otimes A^* \otimes A)$ whose values on sections a, b are given by

$$T_a b = (\mathcal{D}_{a^v} b^v)^h + (\mathcal{D}_{a^v} b^h)^v \quad \text{and} \quad H_a b = (\mathcal{D}_{a^h} b^v)^h + (\mathcal{D}_{a^h} b^h)^v.$$

The following properties of T and H follow immediately from the definition: for any $a, b \in \Gamma(A)$,

$$H_{a^h} b^h = \frac{1}{2}[a^h, b^h]^v, \tag{12}$$

$$\mathcal{D}_{a^v} b^h = T_{a^v} b^h + (\mathcal{D}_{a^v} b^h)^h, \tag{13}$$

$$\mathcal{D}_{a^h} b^v = (\mathcal{D}_{a^h} b^v)^v + H_{a^h} b^v, \tag{14}$$

$$\mathcal{D}_{a^h} b^h = H_{a^h} b^h + (\mathcal{D}_{a^h} b^h)^h. \tag{15}$$

Moreover, for any $u, v \in \mathfrak{g}_x$,

$$\mathcal{D}_u v = \widehat{\mathcal{D}}_u v + T_u v. \tag{16}$$

The following proposition is an immediate consequence of (15).

Proposition

Let $\gamma : [t_0, t_1] \longrightarrow L$ be a smooth path and let $\gamma^h : [t_0, t_1] \longrightarrow \mathfrak{g}_L^\perp$ be the unique A-path with the base path γ. Then γ is a geodesic with respect to the induced Riemannian metric on L if and only if γ^h is a geodesic of the Levi-Civita A-connexion.

The following proposition gives an interpretation of the tensors T and H.

Proposition

1. *The Levi-Civita A-connection is strongly compatible with the Lie algebroid structure if and only if* $T = H = 0$.
2. *The Levi-Civita A-connection is weakly compatible with the Lie algebroid structure if and only if* $T = 0$.

Let $p : A \longrightarrow M$ be a Lie algebroid and $\langle\ ,\ \rangle$ a Riemannian metric on A. The Riemannian metric defines a bundle isomorphism between A and A^* which transport the Lie-Poisson structure on A^* into a Poisson structure say $\pi_{\langle\ ,\ \rangle}$ in A. Let $E : A \longrightarrow \mathbb{R}$ be the energy function given by $E(a) = \frac{1}{2}\langle a, a\rangle$ and let X_E denote the Hamiltonian vector field associated to E with respect to $\pi_{\langle\ ,\ \rangle}$. The following result is a generalization of a well-known result in Riemannian geometry.

Theorem

The geodesics of the Levi-Civita A-connection associated to $\langle\ ,\ \rangle$ are the integral curves of the Hamiltonian vector field X_E.

Let $p : A \longrightarrow M$ be a Lie algebroid and $\langle\ ,\ \rangle$ a Riemannian metric on A. The Riemannian metric defines a bundle isomorphism between A and A^* which transport the Lie-Poisson structure on A^* into a Poisson structure say $\pi_{\langle\ ,\ \rangle}$ in A. Let $E : A \longrightarrow \mathbb{R}$ be the energy function given by $E(a) = \frac{1}{2}\langle a, a\rangle$ and let X_E denote the Hamiltonian vector field associated to E with respect to $\pi_{\langle\ ,\ \rangle}$. The following result is a generalization of a well-known result in Riemannian geometry.

Theorem

The geodesics of the Levi-Civita A-connection associated to $\langle\ ,\ \rangle$ are the integral curves of the Hamiltonian vector field X_E.

The flow of the Hamiltonian vector field X_E is called the *geodesic flow* of $\langle\ ,\ \rangle$.

Remark

Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid. Then:

1. *For any leaf L, the geodesic vector field X_E is tangent to A_L and to $\mathfrak{g}_x$ for any $x \in L$.*
2. *From Proposition 2.1, one can deduce that, for any leaf L, the geodesic vector field X_E is tangent to $\mathfrak{g}_L^\perp$.*

Corollary

Let $p : A \longrightarrow M$ be Riemannian Lie algebroid. Then

1. *If L is a compact leaf then the geodesic flow is complete in restriction to A_L.*
2. *If M is compact then the geodesic flow is complete and for any leaf L the induced Riemannian metric $\langle\ ,\ \rangle$ is complete*

The flow of the Hamiltonian vector field X_E is called the *geodesic flow* of $\langle\ ,\ \rangle$.

Remark

Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid. Then:

1. *For any leaf L, the geodesic vector field X_E is tangent to A_L and to $\mathfrak{g}_x$ for any $x \in L$.*
2. *From Proposition 2.1, one can deduce that, for any leaf L, the geodesic vector field X_E is tangent to $\mathfrak{g}_L^{\perp}$.*

Corollary

Let $p : A \longrightarrow M$ be Riemannian Lie algebroid. Then

1. *If L is a compact leaf then the geodesic flow is complete in restriction to A_L.*
2. *If M is compact then the geodesic flow is complete and for any leaf L the induced Riemannian metric $\langle\ ,\ \rangle_L$ is complete*

Sasaki Metric of a Riemannian Lie algebroid

Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid with anchor $\#$. Fix a leaf L, consider $p_L : A_L \longrightarrow L$ and put $\mathcal{V}A_L = \mathrm{K}er dp_L$.
For any $a \in A_L$, we consider the subspace $\mathfrak{h}^\perp A_L$ of $T_a A_L$ consisting of the tangent vectors V_a such that there exists an horizontal A-path $\alpha : [0,1] \longrightarrow \mathfrak{g}_L^\perp$ satisfying $p(\alpha(0)) = p(a)$ and $V_a = \frac{d}{dt}_{|t=0} \tau_\alpha^t(a)$, where τ_α is the parallel transport along α. We have

$$TA_L = \mathcal{V}A_L \oplus \mathfrak{h}^\perp A_L. \tag{17}$$

Indeed, we define $K : TA_L \longrightarrow A_L$ as follows. Fix $a \in A_L$ and $Z \in T_aA_L$ and choose $\beta : [0,1] \longrightarrow A_L$ such that $\beta(0) = a$ and $\dot\beta(0) = Z$. There exists an unique horizontal A-path $\alpha : [0,1] \longrightarrow \mathfrak{g}_L^\perp$ with the base path $p \circ \beta(t)$. Put

$$K(Z) = (\mathcal{D}^\alpha \beta)(0).$$

It is easy to check that K is well-defined, $\mathrm{Ker} K = \mathfrak{h}^\perp A_L$ and, for any $Z \in \mathcal{V}A_L$, $K(Z) = Z$. Then the relation (17) follows.

Let $(x_1, \ldots, x_l)$ be a system of local coordinates on an open set U in L and $(a_1, \ldots, a_r)$ is a basis of local sections (over U) of A_L. This defines a system of coordinates $(x_1, \ldots, x_l, \mu_1, \ldots, \mu_r)$ on A_L and if $Z = \sum_j b_j \partial_{x_j} + \sum_j Z^j \partial_{\mu_j}$ then

$$K(Z) = \sum_l \left(Z^l + \sum_{i,j} \alpha_i \mu_j \Gamma_{ij}^l \right) a_l, \tag{18}$$

where $dp_L(Z) = \#(\sum_i \alpha_i a_i)$ and $\sum_i \alpha_i a_i \in \mathfrak{g}_L^{\perp}$.

Remark

In general, the geodesic vector field does not lies in $\mathrm{Ker} K$. *Indeed, one can check easily that for any* $a \in A_L$

$$K(X_E(a)) = -\mathcal{D}_{a^v} a.$$

Let $(x_1, \ldots, x_l)$ be a system of local coordinates on an open set U in L and $(a_1, \ldots, a_r)$ is a basis of local sections (over U) of A_L. This defines a system of coordinates $(x_1, \ldots, x_l, \mu_1, \ldots, \mu_r)$ on A_L and if $Z = \sum_j b_j \partial_{x_j} + \sum_j Z^j \partial_{\mu_j}$ then

$$K(Z) = \sum_l \left(Z^l + \sum_{i,j} \alpha_i \mu_j \Gamma_{ij}^l \right) a_l, \qquad (18)$$

where $dp_L(Z) = \#(\sum_i \alpha_i a_i)$ and $\sum_i \alpha_i a_i \in \mathfrak{g}_L^{\perp}$.

Remark

In general, the geodesic vector field does not lies in $\mathrm{Ker} K$. *Indeed, one can check easily that for any* $a \in A_L$

$$K(X_E(a)) = -\mathcal{D}_{a^v} a.$$

We define the Sasaki metric on A_L by

$$g_L(Z_a, Z_a) = \langle d_a p(Z_a), d_a p(Z_a) \rangle_L + \langle K(Z_a), K(Z_a) \rangle.$$

The projection $p_L : A_L \longrightarrow L$ becomes a Riemannian submersion. We consider now the Liouville vector field $\overrightarrow{r}$ on A_L which is the vector field generating the flow $\phi_t(a) = e^t a$. By direct computation one can get

$$[\overrightarrow{r}, X_E] = X_E. \tag{19}$$

From this relation, one deduce that X_E preserves the Riemannian volume on A_L associated to g_L if and only if X_E preserves the Riemannian volume of the restriction of g_L to the spheres bundle $UA_L = \{a \in A_L; \langle a, a \rangle = 1\}$. Let us compute the divergence of the geodesic vector field with respect to g_L.

Theorem

The divergence the geodesic vector field X_E with respect to the Sasaki metric g_L is given by

$$\mathrm{div}(X_E)(a) = \mathrm{T}rad_{a^v} + \langle a^h, N\rangle, \tag{20}$$

where $ad_{a^v} : \mathfrak{g}_{p(a)} \longrightarrow \mathfrak{g}_{p(a)}$, $b \longrightarrow [a^v, b]$ and $N = \sum_i T_{b_i} b_i$ where $(b_1, \ldots, b_s)$ is any orthonormal basis of $\mathfrak{g}_{p(a)}$ and T is the O'Neill tensor.

The following proposition gives an interesting interpretation of $\mathrm{div}X_E$, namely $\mathrm{div}X_E$ is a modular cocycle.

Proposition

Let $p : A \longrightarrow M$ be a transitive Riemannian Lie algebroid such that both A and TM are orientable. Denote by $\lambda \in \Gamma(\wedge^{top} A)$ and $\nu \in \Gamma(\wedge^{top} T^* M)$, respectively, the Riemannian volume associated to $\langle\ ,\ \rangle$ and the Riemannian volume associated to $\langle\ ,\ \rangle_M$ then

$$\mathcal{D}^A(\lambda \otimes \nu) = \mathrm{div}(X_E)(\lambda \otimes \nu),$$

where $\mathcal{D}^A$ is the canonical representation of A. Thus $\mathrm{div}(X_E)$ is a modular cocycle.

Remark

1. *If $A = TM$ then $div(X_E) = 0$ and one recover the classical Liouville Theorem.*
2. *If A is a Lie algebra then $div(X_E) = 0$ if and only if A is unimodular.*
3. *If A is a transitive unimodular Lie algebroid then there exists a Riemannian metric on A such that $div(X_E) = 0$.*

Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid with anchor $\#$. For any A-path $\alpha : [0,1] \longrightarrow A$, the energy and the length of α are given, respectively, by

$$\mathbf{E}(\alpha) = \frac{1}{2}\int_0^1 \langle \alpha(t), \alpha(t) \rangle dt \quad \text{and} \quad \mathcal{L}(\alpha) = \int_0^1 \sqrt{\langle \alpha(t), \alpha(t) \rangle} dt.$$

For any m, q lying in the same leaf of the characteristic foliation, we denote by Ω_{mq} the set of A-path α such that $p(\alpha(0)) = m$ and $p(\alpha(1)) = q$.

Proposition

(First variation formulas) *Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid. Then:*

1. *For any variation of A-paths $\alpha : [0,1] \times [0,1] \longrightarrow A$ and for any β a transverse variation to α, one has*

$$\begin{aligned}\frac{d}{d\varepsilon}\mathbf{E}(\alpha) &= \langle \beta(\varepsilon,1), \alpha(\varepsilon,1)\rangle - \langle \beta(\varepsilon,0), \alpha(\varepsilon,0)\rangle - \int_0^1 \langle \beta, \mathcal{D}_t \alpha\rangle dt \\ &\quad - \int_0^1 \langle \Delta(\alpha,\beta), \alpha\rangle dt.\end{aligned}$$

2. *The h-critical points of $\mathbf{E} : \Omega_{mq} \longrightarrow \mathbb{R}$, namely the A-paths α_0 such that*

$$\frac{d}{d\varepsilon}\mathbf{E}(\alpha)_{|\varepsilon=0} = 0$$

Proposition

(Second variation formulas) *Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid. Then the following assertions hold. For any variation of A-paths α such that α_0 is a geodesic and for any β a transverse variation to α such that $\Delta(\alpha,\beta)=0$, one has*

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathbf{E}(\alpha)_{|\varepsilon=0} &= \langle \mathcal{D}_\varepsilon\beta(0,1),\alpha(0,1)\rangle - \langle \mathcal{D}_\varepsilon\beta(0,0),\alpha(0,0)\rangle \\ &+ \int_0^1 \langle \mathcal{D}_t\beta_0, \mathcal{D}_t\beta_0\rangle dt + \int_0^1 \langle \beta_0, R(\alpha_0,\beta_0)\alpha_0\rangle dt.\end{aligned}$$

Proposition

Let α be an A-homotopy of A-paths such that α_0 is a geodesic and let β be the corresponding transverse variation. One has

$$\frac{d^2}{d\varepsilon^2}\mathbf{E}(\alpha)_{|\varepsilon=0} \quad = \quad \int_0^1 \langle \mathcal{D}_t\beta_0, \mathcal{D}_t\beta_0\rangle dt + \int_0^1 \langle \beta_0, R(\alpha_0,\beta_0)\alpha_0\rangle dt.$$

Proposition

Let α be a variation of A-paths such that α_0 is a geodesic parameterized by arc length and let β a transverse variation to α such that $\Delta(\alpha,\beta)=0$. One has

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathcal{L}(\alpha)_{|\varepsilon=0} &= \langle \mathcal{D}_\varepsilon\beta(0,1), a(0,1)\rangle - \langle \mathcal{D}_\varepsilon\beta(0,0), \alpha(0,0)\rangle \\ &+ \int_0^1 \langle \mathcal{D}_t\beta_0, \mathcal{D}_t\beta_0\rangle dt + \int_0^1 \langle \beta_0, R(\alpha_0,\beta_0)\alpha_0\rangle dt \\ &- \int_0^1 \langle \alpha_0, \mathcal{D}_t\beta_0\rangle dt.\end{aligned}$$

Proposition

Let α be an A-homotopy of A-paths such that α_0 is a geodesic parameterized by arc length and let β be the corresponding transverse variation. One has

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathcal{L}(\alpha)_{|\varepsilon=0} &= \int_0^1 \langle \mathcal{D}_t\beta_0, \mathcal{D}_t\beta_0\rangle dt + \int_0^1 \langle \beta_0, R(\alpha_0,\beta_0)\alpha_0\rangle dt \\ &\quad - \int_0^1 \langle \alpha_0, \mathcal{D}_t\beta_0\rangle dt.\end{aligned}$$

As an application of Proposition 4.1 2., we give now a description of the geodesics of a left invariant Riemannian metric on a Lie group using the geodesics of its Lie algebra considered as a Riemannian Lie algebroid.
Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. For any $u \in \mathfrak{g}$, we denote by u^+ the associated left invariant vector field on G. Suppose that G is endowed with a left invariant Riemannian metric g and put $\langle\ ,\ \rangle = g_e$. If we think $\mathfrak{g}$ as a Lie algebroid, $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a Riemannian Lie algebroid and we will explain how one can construct the geodesics of (G, g) from the geodesics of $(\mathfrak{g}, \langle\ ,\ \rangle)$.

Choose a basis $(e_1, \dots, e_n)$ of $\mathfrak{g}$ and put $g_{ij} = \langle\, e_i, e_j \rangle$. Recall that the geodesics of $(\mathfrak{g}, \langle\ ,\ \rangle)$ are the integral curves of the geodesic vector field X_E given in the linear coordinates $(x_1, \dots, x_n)$ associated to $(e_1, \dots, e_n)$ by

$$X_E = -\sum_{s,t,j} x_s x_t \Gamma_{st}^j \partial_{x_j},$$

where Γ_{st}^j are given by

$$\Gamma_{st}^j = \frac{1}{2} \sum_{l,u} g^{lj} \left(g_{ul} C_{st}^u + g_{ut} C_{ls}^u + g_{us} C_{lt}^u \right).$$

Here (g^{ij}) is the inverse matrix of (g_{ij}) and C_{ij}^k are given by $[e_i, e_j] = \sum_u C_{ij}^u e_u$.

Proposition

Let $h \in G$ and $v \in T_hG$. Then the geodesic $\gamma : \mathbb{R} \longrightarrow G$ of (G, g) satisfying $\gamma(0) = h$ and $\dot{\gamma}(0) = v$ is the integral curve passing through h of the time-depending family of left invariant vector fields $(\alpha^+(t))_{t\in\mathbb{R}}$ where $\alpha : \mathbb{R} \longrightarrow \mathfrak{g}$ is the geodesic of $(\mathfrak{g}, \langle\ ,\ \rangle)$ satisfying $\alpha(0) = (L_{h^{-1}})_*(v)$.

Remark

If the Riemannian metric g is bi-invariant then $\Gamma_{ij}^k = \frac{1}{2}C_{ij}^k$ and hence X_E vanishes identically. We deduce from Proposition 3.5 that the geodesic of (G, g) passing through $h \in G$ and with initial velocity $v \in T_hG$ is the integral curve (passing through h) of the left invariant vector field $(L_{h^{-1}*}(v))^+$.

Proposition

Let $h \in G$ and $v \in T_hG$. Then the geodesic $\gamma : \mathbb{R} \longrightarrow G$ of (G, g) satisfying $\gamma(0) = h$ and $\dot{\gamma}(0) = v$ is the integral curve passing through h of the time-depending family of left invariant vector fields $(\alpha^+(t))_{t\in\mathbb{R}}$ where $\alpha : \mathbb{R} \longrightarrow \mathfrak{g}$ is the geodesic of $(\mathfrak{g}, \langle\ ,\ \rangle)$ satisfying $\alpha(0) = (L_{h^{-1}})_*(v)$.

Remark

If the Riemannian metric g is bi-invariant then $\Gamma_{ij}^k = \frac{1}{2}C_{ij}^k$ and hence X_E vanishes identically. We deduce from Proposition 3.5 that the geodesic of (G, g) passing through $h \in G$ and with initial velocity $v \in T_hG$ is the integral curve (passing through h) of the left invariant vector field $(L_{h^{-1}*}(v))^+$.

Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid. The different curvatures (sectional curvature, Ricci curvature and scalar curvature) can be defined as the classical case (when $A = TM$). For any leaf L, the short exact sequence

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \longrightarrow TL$$

is formally identical to a Riemannian submersion and hence all formulas on curvature given by O'Neill are valid in this context. We denote by K, $\hat{K}$ and $\widetilde{K}$ respectively, the sectional curvature of the Riemannian metrics $\langle\ ,\ \rangle$, the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{g}_L$ and the induced metric on L. The following proposition is a reformulation of Corollary 9.29 pp. 241 in [**?**].

Proposition

Let $\alpha, \beta, s_1, s_2 \in \Gamma(A_L)$ such that α, β are vertical, s_1, s_2 are horizontal and $|\alpha \wedge \beta| = 1$, $|s_1| = |\alpha| = 1$, $|s_1 \wedge s_2| = 1$. Then

$$
\begin{aligned}
K(\alpha, \beta) &= \hat{K}(\alpha, \beta) + |T_\alpha \beta|^2 - \langle\, T_\alpha \alpha, T_\beta \beta \rangle\ , \\
K(s_1, \alpha) &= \langle\, (\mathcal{D}_{s_1} T)_\alpha \alpha, s_1 \rangle\ - |T_\alpha s_1| + |H_{s_1} \alpha|^2, \\
K(s_1, s_2) &= \widetilde{K}(s_1, s_2) - 3|H_{s_1} s_2|^2.
\end{aligned}
$$

The last formula says that the leaves carry "more curvature" than the Lie algebroid and by applying Mayer theorem (see for instance [?]) we get:

Proposition

Let $A \longrightarrow M$ be a complete Riemannian algebroid and let L be a leaf of the characteristic foliation such that for any linearly independent horizontal sections s_1, s_2 over L, $K(s_1, s_2) \geq k$. Then $diam L \leq \frac{\pi}{\sqrt{k}}$ and hence L is compact.

There is another case when one can apply Mayer theorem. Consider a Riemannian Lie algebroid $p : A \longrightarrow M$ such that the O'Neill tensor T vanishes and fix a leaf L and denote by r and $\widetilde{r}$ respectively the Ricci curvature of the Riemannian metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_L$. The formula 9.36c pp.244 in [?] applies in our context and gives

$$r(s_1, s_2) = \widetilde{r}(\#(s_1), \#(s_2)) - 2\sum_{i=1}^{l}\langle\, H_{s_1}a_i, H_{s_2}a_i\rangle$$

where $(a_1, \ldots, a_l)$ is any orthonormal basis of $\mathfrak{g}_L^{\perp}$.

By applying Mayer theorem (see for instance [?]) we get:

Proposition

Let $A \longrightarrow M$ be a complete Riemannian algebroid such that $T = 0$ and let L be a leaf of the characteristic foliation such that there exists a constant k such that the restriction of r to $\mathfrak{g}_L^\perp$ satisfies

$$r \geq (n-1)k^{-2}\langle\ ,\ \rangle.$$

Then $diamL \leq \frac{\pi}{\sqrt{k}}$ and hence L is compact.

A groupoid is a small category $\mathcal{C}$ in which all the arrows are invertible. We shall write M for the set of objects of $\mathcal{C}$, while the set of arrows of $\mathcal{C}$ will be denoted by $\mathcal{C}$. We shall often identify M with the subset of units of $\mathcal{C}$. The structure maps of $\mathcal{C}$ will be denoted as follows: $\mathbf{s}, \mathbf{t} : \mathcal{C} \longrightarrow M$ will stand for the source map, respectively the target map, $m : \mathcal{C}^2 = \{(g, h); \mathbf{s}(g) = \mathbf{t}(h)\} \longrightarrow \mathcal{C}$ the multiplication map $(m(g, h) = gh)$, $i : \mathcal{C} \longrightarrow \mathcal{C}_1$ $(i(g) = g^{-1})$ for the inverse map and $u : M \longrightarrow \mathcal{C}$ $(u(x) = 1_x)$ for the unit map. Given $g \in \mathcal{C}$, the right multiplication by g is only defined on the $\mathbf{s}$-fiber at $\mathbf{t}(g)$, and induces a bijection

$$R_g : \mathbf{s}^{-1}(\mathbf{t}(g)) \longrightarrow \mathbf{s}^{-1}(\mathbf{s}(g)).$$

A Lie groupoid is a groupoid $\mathcal{C}$, equipped with the structure of smooth manifold both on the $\mathcal{C}$ and on the M such that all the structure maps are smooth and **s** and **t** are submersions.
The construction of a Lie algebra of a given Lie group extends to Lie groupoids. Explicitly, if $\mathcal{C}$ is a Lie groupoid, the vector bundle $T^{\mathbf{s}}\mathcal{C} = \mathrm{Ker}(d\mathbf{s})$ over $\mathcal{C}$ of **s**-vertical tangent vectors pulls back along $i : M \longrightarrow \mathcal{C}$ to a vector bundle A over M. This vector bundle has the structure of a Lie algebroid. Its anchor $\# : A \longrightarrow TM$ is induced by the differential of the target map, $d\mathbf{t} : T\mathcal{C} \longrightarrow TM$.
The sections of A over M can be identified by the space of right invariant **s**-vertical vector fields which induce a Lie bracket on the space of sections of A.

With this construction in mind, one can see that a Riemannian structure on A is equivalent to the data of a Riemannian metric on any $\mathbf{s}$-fiber such that, for any $g \in \mathcal{C}$, $R_g : \mathbf{s}^{-1}(\mathbf{t}(g)) \longrightarrow \mathbf{s}^{-1}(\mathbf{s}(g))$ is an isometry. In this case, for any $x \in M$, $\mathbf{t} : \mathbf{s}^{-1}(x) \longrightarrow L_x$ is a Riemannian submersion where the leaf L_x is endowed with the metric defined in 3.1.

A Lie algebroid A is called integrable if it is isomorphic to the Lie algebroid associated to a Lie groupoid. In [?], Crainic and Fernandes give a final solution to the problem of integrability of Lie algebroids. They show that the obstruction to integrability can be controlled by two computable quantities.
The following proposition is a direct application of Crainic-Fernandes results on integrability.

Proposition

Let $p : A \longrightarrow M$ be a Riemannian Lie algebroid such that $H = 0$. Then A is integrable.

There is a large class of Lie algebroids for which one can apply this result. Let (M, π) be a Poisson manifold. The cotangent bundle T^*M carries a structure of a Lie algebroid where the anchor is the contraction by π, $\pi_\# : T^*M \longrightarrow TM$ and the Lie bracket is given by the Koszul bracket

$$[\alpha, \beta] = \mathcal{L}_{\pi_\#(\alpha)}\beta - \mathcal{L}_{\pi_\#(\beta)}\alpha - d\pi(\alpha, \beta)$$

where $\alpha, \beta \in \Omega^1(M)$. Let $\langle \, , \, \rangle$ be a Riemannian structure in T^*M. In [**?**], the author studied the triple $(M, \pi, \langle \, , \, \rangle)$ such that π is parallel with respect the Levi-Civita T^*M-connection $\mathcal{D}$. A triple $(M, \pi, \langle \, , \, \rangle)$ satisfying $\mathcal{D}\pi = 0$ is called Riemann-Poisson manifold. The condition $\mathcal{D}\pi = 0$ implies that $\mathrm{Ker}\pi_\#$ is invariant by parallel transport and hence $\mathcal{D}$ is strongly compatible with the Lie algebroid structure of T^*M. By using Proposition 2.2, we deduce that $H = 0$.

So we get the following result.

Corollary

Let $(M, \pi, \langle\ ,\ \rangle)$ be a Riemann-Poisson manifold. Then the Lie algebroid structure of T^*M associated to π is integrable.