Symplectic Lie algebras

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Seminar Algebra, Geometry, Topology and Applications

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Basic concepts

- Isotropic ideals and symplectic rank
- Symplectic reduction
- Induced left symmetric product on normal quotients
- Totally geodesic subalgebras
- Symplectic oxidation

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A symplectic Lie algebra is a Lie algebra \mathfrak{g} endowed with $\omega \in \wedge^2 \mathfrak{g}^*$ such that:

- ω is a nondegenerate, i.e., $(\omega(u, v) = 0, \forall v \in \mathfrak{g}) \Longrightarrow u = 0.$
- 2 ω is closed, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\omega([u,v],w) + \omega([v,w],u) + \omega([w,u],v) = 0.$$

An ideal ℑ of g is called an isotropic ideal of (g,ω) if ℑ is an isotropic subspace for ω, i.e., ω(u, v) = 0 for any u, v ∈ ℑ. In this case ℑ ⊂ ℑ[⊥] and dim ℑ ≤ ½ dim g. If the orthogonal ℑ[⊥] is an ideal in g we call ℑ a normal isotropic ideal. If ℑ is a maximal isotropic subspace ℑ is called a Lagrangian ideal.

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An ideal \mathfrak{J} of \mathfrak{g} is called an isotropic ideal of (\mathfrak{g}, ω) if \mathfrak{J} is an isotropic subspace for ω , i.e., $\omega(u, v) = 0$ for any $u, v \in \mathfrak{J}$. In this case $\mathfrak{J} \subset \mathfrak{J}^{\perp}$ and dim $\mathfrak{J} \leq \frac{1}{2}$ dim \mathfrak{g} . If the orthogonal \mathfrak{J}^{\perp} is an ideal in \mathfrak{g} we call \mathfrak{J} a normal isotropic ideal. If \mathfrak{J} is a maximal isotropic subspace \mathfrak{J} is called a Lagrangiar ideal.

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Lemma

Let \mathfrak{J} be an ideal of (\mathfrak{g}, ω) . Then

- **1** If \mathfrak{J} is isotropic then \mathfrak{J} is abelian.
- **2** \mathfrak{J}^{\perp} is a subalgebra of \mathfrak{g} .
- $\ \ \, \mathfrak{J}^{\perp} \ \, \text{is an ideal in } \mathfrak{g} \ \, \text{if and only if } [\mathfrak{J}^{\perp},\mathfrak{J}]=\{0\}.$

Proof.

It is a consequence of the relation:

 $\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0.$

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Let (\mathfrak{g}, ω) be a symplectic Lie algebra and $\mathfrak{J} \subset \mathfrak{g}$ an isotropic ideal.

The orthogonal \mathfrak{J}^{\perp} is a subalgebra of \mathfrak{g} which contains \mathfrak{J} , and therefore ω descends to a symplectic form $\overline{\omega}$ on the quotient Lie algebra

$$\overline{\mathfrak{g}}=\mathfrak{J}^{\perp}/\mathfrak{J}.$$

Definition The symplectic Lie algebra $(\overline{\mathfrak{g}}, \overline{\omega})$ is called the symplectic reduction of (\mathfrak{g}, ω) with respect to the isotropic ideal \mathfrak{J} . If \mathfrak{J}^{\perp} is an ideal we call the reduction normal.



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The reduction of (\mathfrak{g}, ω) with respect to \mathfrak{J} is called:

- Lagrangian reduction if \mathfrak{J} is a Lagrangian ideal. In this case, $\mathfrak{J} = \mathfrak{J}^{\perp}$ and $\overline{\mathfrak{g}} = \{0\}$.
- ② Central reduction if J is central. In this case, J[⊥] is an ideal in g, which contains [g, g].
- (a) Codimension one normal reduction if \mathfrak{J} is one-dimensional and $[\mathfrak{J},\mathfrak{J}^{\perp}] = \{0\}.$

Remark

Any one dimensional ideal in (\mathfrak{g},ω) is isotropic.

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Let (\mathfrak{g}, ω) be symplectic Lie algebra, \mathfrak{J} an ideal such that \mathfrak{J}^{\perp} is an ideal, i.e., $[\mathfrak{J}, \mathfrak{J}^{\perp}] = 0$.

We have the exact sequences of Lie algebras

$$0 \longrightarrow \mathfrak{J}^{\perp} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{J}^{\perp} = \mathfrak{h} \longrightarrow 0.$$

We define $\omega_{\mathfrak{h}}:\mathfrak{h}\longrightarrow\mathfrak{J}^*$, $\rho:\mathfrak{h}\longrightarrow\mathrm{End}(\mathfrak{J}^*)$ and $\bullet:\mathfrak{h}\times\mathfrak{h}\longrightarrow\mathfrak{h}$ by

 $\omega_{\mathfrak{h}}(\overline{u})(a) = \omega(u, a) \quad \text{and} \quad \prec \rho(\overline{u})(\alpha), a \succ = - \prec \alpha, [u, a] \succ,$

$$\omega_{\mathfrak{h}}(\nabla^{\mathfrak{h}}_{\overline{u}}\overline{v})(a) = -\omega(v, [u, a])$$

for $u, v \in \mathfrak{g}$, $a \in \mathfrak{J}$ and $\alpha \in \mathfrak{J}^*$.

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Proposition

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 $\omega_{\mathfrak{h}}([\overline{u},\overline{v}]) = \rho(\overline{u})(\omega_{\mathfrak{h}}(\overline{v})) - \rho(\overline{v})(\omega_{\mathfrak{h}}(\overline{u})).$

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If we take $\mathfrak{J} = \mathfrak{g}$, $\mathfrak{g} = \mathfrak{h}$ and we recover the well-known fact that \mathfrak{g} carries a torsion free flat product given by

$$\omega(\nabla_u v, w) = -\omega(v, [u, w]), \quad u, v, w \in \mathfrak{g}.$$

Moreover, if \mathfrak{J} is an ideal such that \mathfrak{J}^{\perp} is an ideal then $\pi: \mathfrak{g} \longrightarrow \mathfrak{h} = \mathfrak{g}/\mathfrak{J}^{\perp}$ satisfies

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In the context when \mathfrak{J} is isotropic and normal, the flat Lie algebra $(\mathfrak{h}, \nabla^{\mathfrak{h}})$ is called the *quotient flat Lie algebra* associated to the reduction with respect to the normal ideal \mathfrak{J} .

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Proposition

Let (\mathfrak{g}, ω) be a symplectic Lie algebra and ∇ its associated torsion-free flat connection. Then:

Basic concepts

- A subalgebra L of g is a totally geodesic subalgebra with respect to ∇ if and only if [L, L[⊥]] ⊂ L[⊥].
- ② For every ideal ℑ of g, the orthogonal subalgebra ℑ[⊥] is totally geodesic.
- Severy isotropic ideal ℑ of (g,ω) is a totally geodesic subalgebra and the induced connection ∇³ on ℑ is trivial.
- Lagrangian subalgebras of (g, ω) are totally geodesic, in particular, they carry a flat connection.

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- Solution For every ideal ℑ of 𝔅, the orthogonal subalgebra ℑ[⊥] is totally geodesic.
- Every isotropic ideal ℑ of (𝔅, ω) is a totally geodesic subalgebra and the induced connection ∇^ℑ on ℑ is trivial.
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Purpose

Given an appropriate set of additional data on a symplectic Lie algebra $(\bar{\mathfrak{g}}, \bar{\omega})$ the process of reduction can be reversed to construct a symplectic Lie algebra (\mathfrak{g}, ω) which reduces to $(\bar{\mathfrak{g}}, \bar{\omega})$.

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Oxidation of Lie algebras. Let $\overline{\mathfrak{g}}$ be a Lie algebra and $[,]_0$ its Lie bracket. Assume the following set of additional data is given:

- **1** a derivation $\phi \in \text{Der}(\overline{\mathfrak{g}})$,
- ② a two-cocycle $lpha\in Z^2(\overline{\mathfrak{g}})$, i.e.,

$$\oint_{u,v,w} \alpha([u,v]_0,w) = 0,$$

In a linear form λ ∈ g^{*}.
Define on g = ⟨ξ⟩ ⊕ ḡ ⊕ ⟨H⟩ the Lie bracket

$$\begin{cases} [u, v] = [u, v]_0 + \alpha(u, v)H, \quad u, v \in \overline{\mathfrak{g}}, \\ [\xi, u] = \phi(u) + \lambda(u)H, \quad u \in \overline{\mathfrak{g}}. \end{cases}$$



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Proposition $(\mathfrak{g}, [,])$ is a Lie algebra if and only if

$$\alpha_{\phi} = -d\lambda \in B^2(\overline{\mathfrak{g}}),$$

where

$$lpha_\phi(u,v) = lpha(\phi(u),v) + lpha(u,\phi(v))$$
 and $dlpha(u,v) = -\lambda([u,v]_0).$

Proof. It is an immediate consequence of the Jacobi identity.

We call the Lie algebra $\mathfrak{g} = \overline{\mathfrak{g}}_{\phi,\alpha,\lambda}$ the central oxidation of $\overline{\mathfrak{g}}$ (with respect to the data ϕ, α, λ).

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We call the Lie algebra $\mathfrak{g} = \overline{\mathfrak{g}}_{\phi,\alpha,\lambda}$ the central oxidation of $\overline{\mathfrak{g}}$ (with respect to the data ϕ, α, λ).

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Proposition

Let $\overline{\mathfrak{g}}$ be a nilpotent Lie algebra and assume that the derivation ϕ above is nilpotent. Then the Lie algebra $\mathfrak{g} = \overline{\mathfrak{g}}_{\phi,\alpha,\lambda}$ is nilpotent, and H is contained in the center of \mathfrak{g} .

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Now let $(\overline{\mathfrak{g}}, \overline{\omega})$ be a symplectic Lie algebra. We define a non-degenerate two-form ω on the vector space $\mathfrak{g} = \langle \xi \rangle \oplus \overline{\mathfrak{g}} \oplus \langle H \rangle$ by requiring that

$$\begin{array}{l} \bullet \quad \omega_{\mid \overline{\mathfrak{g}}} = \overline{\omega}, \\ \hline \bullet \quad \overline{\mathfrak{g}}^{\perp} = \langle \xi, H \rangle \text{ and } \omega(\xi, H) = 1 \end{array}$$

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Proposition

Let $\phi \in \text{Der}(\overline{\mathfrak{g}})$, $\alpha \in Z^2(\overline{\mathfrak{g}})$ such that $\alpha_{\phi} = -d\lambda$ for some $\lambda \in \overline{\mathfrak{g}}$. Then $(\overline{\mathfrak{g}}_{\phi,\alpha,\lambda}, \omega)$ is a symplectic Lie algebra if and only if

$$lpha(u,v)=\overline{\omega}_{\phi}(u,v):=\overline{\omega}(\phi(u),v)+\overline{\omega}(u,\phi(v)).$$

Proof.

$$0 = \omega([\xi, u], v) + \omega([v, \xi], u) + \omega([u, v], \xi)$$

= $\overline{\omega}(\phi(u), v) + \overline{\omega}(u, \phi(v)) - \alpha(u, v).$

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The symplectic Lie algebra $(\overline{\mathfrak{g}}_{\phi,\alpha,\lambda},\omega)$ is called symplectic oxidation of $(\overline{\mathfrak{g}},\overline{\omega})$ with respect to the data ϕ and λ . Observe that the symplectic oxidation reduces to $(\overline{\mathfrak{g}},\overline{\omega})$ with respect to the one-dimensional central ideal $\mathfrak{J} = \langle H \rangle$.

Remark

Let \mathfrak{g} be a Lie algebra, $\alpha \in Z^2(\mathfrak{g})$ and ϕ a derivation of \mathfrak{g} . Then $\alpha_{\phi} \in Z^2(\mathfrak{g})$.

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The symplectic Lie algebra $(\bar{\mathfrak{g}}_{\phi,\alpha,\lambda},\omega)$ is called symplectic oxidation of $(\bar{\mathfrak{g}},\bar{\omega})$ with respect to the data ϕ and λ . Observe that the symplectic oxidation reduces to $(\bar{\mathfrak{g}},\bar{\omega})$ with respect to the one-dimensional central ideal $\mathfrak{J} = \langle H \rangle$.

Remark

Let \mathfrak{g} be a Lie algebra, $\alpha \in Z^2(\mathfrak{g})$ and ϕ a derivation of \mathfrak{g} . Then $\alpha_{\phi} \in Z^2(\mathfrak{g})$.

Corollary

Let $(\overline{\mathfrak{g}}, \overline{\omega})$ be a symplectic Lie algebra and $\phi \in \text{Der}(\overline{\mathfrak{g}})$ a derivation. Then there exists a symplectic oxidation $(\overline{\mathfrak{g}}_{\phi,\overline{\omega}_{\phi},\lambda},\omega)$ if and only if the cohomology class

$$[\overline{\omega}_{\phi,\phi}] \in H^2(\overline{\mathfrak{g}}).$$

vanishes.