Symplectic Lie algebras

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Seminar Algebra, Geometry, Topology and Applications

Outline

A symplectic Lie algebra is a Lie algebra $\mathfrak{g}$ endowed with $\omega \in \wedge^2\mathfrak{g}^*$ such that:

1. ω is a nondegenerate, i.e., $(\omega(u,v) = 0,\ \forall v \in \mathfrak{g}) \Longrightarrow u = 0.$
2. ω is closed, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\omega([u,v],w) + \omega([v,w],u) + \omega([w,u],v) = 0.$$

An ideal $\mathfrak{J}$ of $\mathfrak{g}$ is called an isotropic ideal of $(\mathfrak{g},\omega)$ if $\mathfrak{J}$ is an isotropic subspace for ω, i.e., $\omega(u,v) = 0$ for any $u, v \in \mathfrak{J}$.
In this case $\mathfrak{J} \subset \mathfrak{J}^\perp$ and $\dim \mathfrak{J} \leq \frac{1}{2}\dim \mathfrak{g}$.
If the orthogonal $\mathfrak{J}^\perp$ is an ideal in $\mathfrak{g}$ we call $\mathfrak{J}$ a normal isotropic ideal. If $\mathfrak{J}$ is a maximal isotropic subspace $\mathfrak{J}$ is called a Lagrangian ideal.

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Lemma

Let $\mathfrak{J}$ be an ideal of $(\mathfrak{g}, \omega)$. Then

1. *If $\mathfrak{J}$ is isotropic then $\mathfrak{J}$ is abelian.*
2. *$\mathfrak{J}^{\perp}$ is a subalgebra of $\mathfrak{g}$.*
3. *$\mathfrak{J}^{\perp}$ is an ideal in $\mathfrak{g}$ if and only if $[\mathfrak{J}^{\perp}, \mathfrak{J}] = \{0\}$.*

Proof.

It is a consequence of the relation:

$$\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0.$$

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Definition

Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra. The symplectic rank of $(\mathfrak{g}, \omega)$ is the maximal dimension of any isotropic ideal in $(\mathfrak{g}, \omega)$.

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Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and $\mathfrak{J} \subset \mathfrak{g}$ an isotropic ideal. The orthogonal $\mathfrak{J}^{\perp}$ is a subalgebra of $\mathfrak{g}$ which contains $\mathfrak{J}$, and therefore ω descends to a symplectic form $\overline{\omega}$ on the quotient Lie algebra

$$\overline{\mathfrak{g}} = \mathfrak{J}^{\perp}/\mathfrak{J}.$$

Definition

The symplectic Lie algebra $(\overline{\mathfrak{g}}, \overline{\omega})$ is called the symplectic reduction of $(\mathfrak{g}, \omega)$ with respect to the isotropic ideal $\mathfrak{J}$. If $\mathfrak{J}^{\perp}$ is an ideal we call the reduction normal.

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1. *Lagrangian reduction* if $\mathfrak{J}$ is a Lagrangian ideal. In this case, $\mathfrak{J} = \mathfrak{J}^{\perp}$ and $\overline{\mathfrak{g}} = \{0\}$.
2. *Central reduction* if $\mathfrak{J}$ is central. In this case, $\mathfrak{J}^{\perp}$ is an ideal in $\mathfrak{g}$, which contains $[\mathfrak{g}, \mathfrak{g}]$.
3. *Codimension one normal reduction* if $\mathfrak{J}$ is one-dimensional and $[\mathfrak{J}, \mathfrak{J}^{\perp}] = \{0\}$.

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Any one dimensional ideal in $(\mathfrak{g}, \omega)$ is isotropic.

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Any one dimensional ideal in $(\mathfrak{g}, \omega)$ *is isotropic.*

Let $(\mathfrak{g}, \omega)$ be symplectic Lie algebra, $\mathfrak{J}$ an ideal such that $\mathfrak{J}^\perp$ is an ideal, i.e., $[\mathfrak{J}, \mathfrak{J}^\perp] = 0$.

We have the exact sequences of Lie algebras

$$0 \longrightarrow \mathfrak{J}^\perp \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{J}^\perp = \mathfrak{h} \longrightarrow 0.$$

We define $\omega_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathfrak{J}^*$, $\rho : \mathfrak{h} \longrightarrow \mathrm{End}(\mathfrak{J}^*)$ and $\bullet : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ by

$$\boxed{\omega_{\mathfrak{h}}(\overline{u})(a) = \omega(u, a) \quad \text{and} \quad \prec \rho(\overline{u})(\alpha), a \succ = - \prec \alpha, [u, a] \succ,}$$

$$\boxed{\omega_{\mathfrak{h}}(\nabla^{\mathfrak{h}}_{\overline{u}} \overline{v})(a) = -\omega(v, [u, a])}$$

for $u, v \in \mathfrak{g}$, $a \in \mathfrak{J}$ and $\alpha \in \mathfrak{J}^*$.

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Proposition

1. *ρ is a representation of $\mathfrak{h}$.*
2. *$\omega_{\mathfrak{h}}$ is bijective and a 1-cocycle with respect to ρ, i.e,*

$$\omega_{\mathfrak{h}}([\overline{u},\overline{v}]) = \rho(\overline{u})(\omega_{\mathfrak{h}}(\overline{v})) - \rho(\overline{v})(\omega_{\mathfrak{h}}(\overline{u})).$$

3. *$\nabla^{\mathfrak{h}}$ is a torsion free and flat product:*

$$[\overline{u},\overline{v}] = \nabla^{\mathfrak{h}}_{\overline{u}}\overline{v} - \nabla^{\mathfrak{h}}_{\overline{v}}\overline{u} \quad \text{and} \quad \nabla^{\mathfrak{h}}_{[\overline{u},\overline{v}]} = \nabla^{\mathfrak{h}}_{\overline{u}} \circ \nabla^{\mathfrak{h}}_{\overline{v}} - \nabla^{\mathfrak{h}}_{\overline{v}} \circ \nabla^{\mathfrak{h}}_{\overline{u}}.$$

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If we take $\mathfrak{J} = \mathfrak{g}$, $\mathfrak{g} = \mathfrak{h}$ and we recover the well-known fact that $\mathfrak{g}$ carries a torsion free flat product given by

$$\boxed{\omega(\nabla_u v, w) = -\omega(v, [u, w]), \quad u, v, w \in \mathfrak{g}.}$$

Moreover, if $\mathfrak{J}$ is an ideal such that $\mathfrak{J}^\perp$ is an ideal then $\pi : \mathfrak{g} \longrightarrow \mathfrak{h} = \mathfrak{g}/\mathfrak{J}^\perp$ satisfies

$$\boxed{\pi(\nabla_u v) = \nabla^{\mathfrak{h}}_{\pi(u)} \pi(v), \quad u, v \in \mathfrak{g}.}$$

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If $\mathfrak{J}$ is central then $\bullet = 0$ and $\mathfrak{h}$ is abelian.

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In the context when $\mathfrak{J}$ is isotropic and normal, the flat Lie algebra $(\mathfrak{h}, \nabla^{\mathfrak{h}})$ is called the *quotient flat Lie algebra* associated to the reduction with respect to the normal ideal $\mathfrak{J}$.

Proposition

Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and ∇ its associated torsion-free flat connection. Then:

1. *A subalgebra L of $\mathfrak{g}$ is a totally geodesic subalgebra with respect to ∇ if and only if $[L, L^{\perp}] \subset L^{\perp}$.*
2. *For every ideal $\mathfrak{J}$ of $\mathfrak{g}$, the orthogonal subalgebra $\mathfrak{J}^{\perp}$ is totally geodesic.*
3. *Every isotropic ideal $\mathfrak{J}$ of $(\mathfrak{g}, \omega)$ is a totally geodesic subalgebra and the induced connection $\nabla^{\mathfrak{J}}$ on $\mathfrak{J}$ is trivial.*
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Purpose

Given an appropriate set of additional data on a symplectic Lie algebra $(\overline{\mathfrak{g}}, \overline{\omega})$ *the process of reduction can be reversed to construct a symplectic Lie algebra* $(\mathfrak{g}, \omega)$ *which reduces to* $(\overline{\mathfrak{g}}, \overline{\omega})$*.*

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Oxidation of Lie algebras. Let $\overline{\mathfrak{g}}$ be a Lie algebra and $[\ ,\ ]_0$ its Lie bracket. Assume the following set of additional data is given:

1. a derivation $\phi \in \mathrm{Der}(\overline{\mathfrak{g}})$,
2. a two-cocycle $\alpha \in Z^2(\overline{\mathfrak{g}})$, i.e.,

$$\oint_{u,v,w} \alpha([u,v]_0, w) = 0,$$

3. a linear form $\lambda \in \overline{\mathfrak{g}}^*$.

Define on $\mathfrak{g} = \langle \xi \rangle \oplus \overline{\mathfrak{g}} \oplus \langle H \rangle$ the Lie bracket

$$\begin{cases} [u,v] = [u,v]_0 + \alpha(u,v)H, & u,v \in \overline{\mathfrak{g}}, \\ [\xi,u] = \phi(u) + \lambda(u)H, & u \in \overline{\mathfrak{g}}. \end{cases}$$

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$$\boxed{\begin{cases} [u,v] = [u,v]_0 + \alpha(u,v)H, & u,v \in \bar{\mathfrak{g}}, \\ [\xi, u] = \phi(u) + \lambda(u)H, & u \in \bar{\mathfrak{g}}. \end{cases}}$$

Proposition

$(\mathfrak{g}, [\ ,\ ])$ *is a Lie algebra if and only if*

$$\alpha_\phi = -d\lambda \in B^2(\overline{\mathfrak{g}}),$$

where

$$\alpha_\phi(u, v) = \alpha(\phi(u), v) + \alpha(u, \phi(v)) \quad \text{and} \quad d\alpha(u, v) = -\lambda([u, v]_0).$$

Proof.

It is an immediate consequence of the Jacobi identity. □

We call the Lie algebra $\mathfrak{g} = \overline{\mathfrak{g}}_{\phi,\alpha,\lambda}$ the central oxidation of $\overline{\mathfrak{g}}$ (with respect to the data ϕ, α, λ).

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Proposition

Let $\overline{\mathfrak{g}}$ be a nilpotent Lie algebra and assume that the derivation ϕ above is nilpotent. Then the Lie algebra $\mathfrak{g} = \overline{\mathfrak{g}}_{\phi,\alpha,\lambda}$ is nilpotent, and H is contained in the center of $\mathfrak{g}$.

Now let $(\overline{\mathfrak{g}}, \overline{\omega})$ be a symplectic Lie algebra. We define a non-degenerate two-form ω on the vector space $\mathfrak{g} = \langle \xi \rangle \oplus \overline{\mathfrak{g}} \oplus \langle H \rangle$ by requiring that

1. $\omega_{|\overline{\mathfrak{g}}} = \overline{\omega}$,
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Now let $(\overline{\mathfrak{g}}, \overline{\omega})$ be a symplectic Lie algebra. We define a non-degenerate two-form ω on the vector space $\mathfrak{g} = \langle \xi \rangle \oplus \overline{\mathfrak{g}} \oplus \langle H \rangle$ by requiring that

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2. $\overline{\mathfrak{g}}^{\perp} = \langle \xi, H \rangle$ and $\omega(\xi, H) = 1$.

Proposition

Let $\phi \in \mathrm{Der}(\overline{\mathfrak{g}})$, $\alpha \in Z^2(\overline{\mathfrak{g}})$ such that $\alpha_\phi = -d\lambda$ for some $\lambda \in \overline{\mathfrak{g}}$. Then $(\overline{\mathfrak{g}}_{\phi,\alpha,\lambda}, \omega)$ is a symplectic Lie algebra if and only if

$$\alpha(u, v) = \overline{\omega}_\phi(u, v) := \overline{\omega}(\phi(u), v) + \overline{\omega}(u, \phi(v)).$$

Proof.

$$\begin{aligned} 0 &= \omega([\xi, u], v) + \omega([v, \xi], u) + \omega([u, v], \xi) \\ &= \overline{\omega}(\phi(u), v) + \overline{\omega}(u, \phi(v)) - \alpha(u, v). \end{aligned}$$

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The symplectic Lie algebra $(\overline{\mathfrak{g}}_{\phi,\alpha,\lambda}, \omega)$ is called symplectic oxidation of $(\overline{\mathfrak{g}}, \overline{\omega})$ with respect to the data ϕ and λ. Observe that the symplectic oxidation reduces to $(\overline{\mathfrak{g}}, \overline{\omega})$ with respect to the one-dimensional central ideal $\mathfrak{J} = \langle H \rangle$.

Remark

Let $\mathfrak{g}$ be a Lie algebra, $\alpha \in Z^2(\mathfrak{g})$ and ϕ a derivation of $\mathfrak{g}$. Then $\alpha_\phi \in Z^2(\mathfrak{g})$.

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Corollary

Let $(\overline{\mathfrak{g}}, \overline{\omega})$ *be a symplectic Lie algebra and* $\phi \in \mathrm{Der}(\overline{\mathfrak{g}})$ *a derivation. Then there exists a symplectic oxidation* $(\overline{\mathfrak{g}}_{\phi,\overline{\omega}_\phi,\lambda}, \omega)$ *if and only if the cohomology class*

$$[\overline{\omega}_{\phi,\phi}] \in H^2(\overline{\mathfrak{g}}).$$

vanishes.