Some developments and problems in symplectic topology

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Seminar Algebra, Geometry, Topology and applications

Theorem. (Darboux's theorem)

Let (M, ω) be a symplectic manifold. Then, for any $p \in M$, there exists a coordinates system $(p_i, q_i)_{i=1}^n$ such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Theorem. (Moser's argument (1965))

Let ω_t a family of symplectic forms such that

$$\frac{d}{dt}\omega_t = d\sigma_t.$$

Then there exists a family of diffeomorphisms ϕ_t such that $\phi_t^* \omega_t = \omega_0$.

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Corollary.

There is no local invariant in symplectic geometry and the group of symplectomorphisms $\mathrm{Symp}(M, \omega)$ *has an infinite dimension.*

Purpose.

The subject of symplectic topology is the global structure of a symplectic manifold and the behavior of symplectomorphisms.

Questions

1. Which manifold support a symplectic structure? What symplectic invariants are there to distinguish one from another?
2. Must a symplectomorphism always have a lot of fixed points?
3. Are there any special distinguishing features of Hamiltonian flows? For example, must they always have a periodic orbit?
4. Is there a geometric way to understand the fact that a symplectic structure makes 2-dimensional measurements?
5. Are there special symplectic connections on a symplectic manifold?

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Fact.

Gromov (1985) studied the existence of symplectic structures on open manifolds and built an invariant of compact symplectic manifolds using J-holomorphic curves.

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Theorem. (Poincaré's last geometric theorem)

Every **area-preserving** *homeomorphism of the annulus*

$$A = \{(u, v) \in \mathbb{R}^2 : a \leq u^2 + v^2 \leq b\}$$

which preserves the two boundary components and twists them in opposite directions must have at least two fixed points.

This result was proved by Birkhoff in 1925.

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Conjecture. (Arnold's conjecture (1965))

If ϕ is a 1-time of a time dependent Hamiltonian flow on a compact symplectic manifold M, then ϕ has at least $\mathrm{Crit}(M)$ *distinct fixed points.*

Theorem. (Conley-Zehnder (1983))

The Arnold's conjecture is true for the standard torus.

By using Floer homology:

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Is there a geometric way to understand the fact that a symplectic structure makes 2-dimensional measurements?

Theorem. (Nonsqueezing theorem (Gromov))

If there is a symplectomorphism

$$\phi : B^{2n}(r) \longrightarrow Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$$

then $r \leq R$.
$B^{2n}(r) = \{u \in \mathbb{R}^{2n} : ||u|| \leq r\}$.

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- *(***Conformality***)* $\mathfrak{c}(M, \lambda\omega) = |\lambda| \mathfrak{c}(M, \omega)$.
- *(***Non triviality***)* $\mathfrak{c}(B^{2n}(1), \omega_0) > 0$ *and* $\mathfrak{c}(Z^{2n}(1), \omega_0) < \infty$.

Proposition.

The existence of a symplectic capacity $\mathfrak{c}$ *satisfying*

$$\mathfrak{c}(B^{2n}(1), \omega_0) = \mathfrak{c}(Z^{2n}(1), \omega_0) = \pi \tag{1}$$

is equivalent to Gromov's nonsqueezing theorem.

Theorem. (Hofer-Zehnder (1990))

There exists a capacity $\mathfrak{c}_{HZ}$ *satisfying*

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for every $r > 0$.

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There are many capacities.

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The birth of symplectic topology

Theorem. (Rigidity (Eliasberg-Gromov))

The group of symplectomorphisms of a symplectic manifold (M, ω) is C^0-closed in the group of all diffeomorphisms of M.

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There is a notion of topological symplectic manifold.

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Unlike the pseudo-Riemannian case any symplectic structure have an infinity of symplectic connections.

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Symplectic connections have been used in the quantification by deformation in the sense of Fedosov.

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The different curvatures of a symplectic connection have been defined and studied in the spirit of the pseudo-Riemannian case.

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