Einstein Lorentzian Nilpotent Lie groups

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Seminar Algebra, Geometry, Topology and Applications

Outline

A *pseudo-Euclidean vector space* is a real vector space of finite dimension n endowed with a nondegenerate symmetric inner product of signature $(q, n-q) = (- \ldots -, + \ldots +)$.
When the signature is $(0, n)$ (resp. $(1, n-1)$) the space is called *Euclidean* (resp. *Lorentzian*).
Let $(V, \langle \ , \ \rangle)$ be a pseudo-Euclidean vector space of signature $(q, n-q)$. A vector $u \in V$ is called:

1. *spacelike* if $\langle u, u \rangle > 0$,
2. *timelike* if $\langle u, u \rangle < 0$ and
3. *isotropic* if $\langle u, u \rangle = 0$.

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A subspace F of a pseudo-Euclidean vector space $(V, \langle\ ,\ \rangle)$ is called:

1. Nondegenerate if $F \cap F^{\perp} = \{0\}$,
2. degenerate if $F \cap F^{\perp} \neq \{0\}$,
3. Isotropic if $\langle u, v\rangle = 0$ for any $u, v \in F$.

Note that we have always $\dim F + \dim F^{\perp} = \dim V$.
If F is isotropic then $\dim F \leq \min(q, n - q)$.
In particular, $\dim(F \cap F^{\perp}) \leq \min(q, n - q)$.

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A family $(u_1, \ldots, u_s)$ of vectors in V is called *orthogonal* if, for $i, j = 1, \ldots, s$ and $i \neq j$, $\langle u_i, u_j \rangle = 0$.

An orthonormal basis of V is an orthogonal basis $(e_1, \ldots, e_n)$ such that $\langle e_i, e_i \rangle = \pm 1$.

A *pseudo-Euclidean basis* of V is a basis $(e_1, \bar{e}_2, \ldots, e_q, \bar{e}_q, f_1, \ldots, f_{n-2q})$ for which the non vanishing products are $\langle \bar{e}_i, e_i \rangle = \langle f_j, f_j \rangle = 1$, $i \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, n-2q\}$. When V is Lorentzian, we call such a basis *Lorentzian*.

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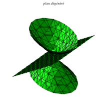
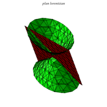
Let $(V, \langle\ ,\ \rangle)$ be Lorentzian vector space and $F \subset V$ is a vector subspace. Then either:

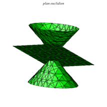
1. F is nondegenerate Euclidean and $F^{\perp}$ is nondegenerate Lorentzian,
2. F is nondegenerate Lorentzian and $F^{\perp}$ is nondegenerate Euclidean,
3. F is degenerate and $\dim(F \cap F^{\perp}) = 1$.



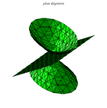
A degenerate plan.



A Lorentzian plan.



A Euclidean plan.



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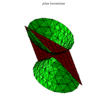

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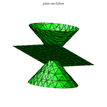

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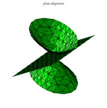

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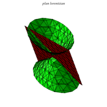

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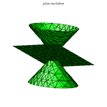

A Euclidean plan.

For any endomorphism $F : V \longrightarrow V$, we denote by $F^* : V \longrightarrow V$ its adjoint with respect to $\langle\ ,\ \rangle$. The following lemma will be useful later.

Lemma

Let $(V, \langle\ ,\ \rangle)$ be a Lorentzian vector space, e an isotropic vector and J a skew-symmetric endomorphism. Then $\langle Je, Je\rangle \geq 0$. Moreover, $\langle Je, Je\rangle = 0$ if and only if $Je = \alpha e$.

Proof.

We choose an isotropic vector $\bar{e}$ such that $\langle e, \bar{e}\rangle = 1$ and an orthonormal basis $(f_1, \ldots, f_r)$ of $\{e, \bar{e}\}^{\perp}$. Since J is skew-symmetric then

$$Je = \alpha e + \sum_{i=1}^{r} a_i f_i \quad \text{and} \quad \langle Je, Je\rangle = \sum_{i=1}^{r} a_i^2.$$

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Let (G, g) be a Lie group endowed with a left-invariant pseudo-Riemannian metric. The study of the curvature of (G, g) is equivalent to the study of $(\mathfrak{g} = T_e G, [\ ,\ ], \langle\ ,\ \rangle = g(e))$.
We refer to $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ as a pseudo-Euclidean Lie algebra.
Levi-Civita connection of (G, g) defines a product $\mathrm{L} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called *Levi-Civita product* given by Koszul's formula

$$2\langle \mathrm{L}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle. \quad (1)$$

For any $u, v \in \mathfrak{g}$, $\mathrm{L}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is skew-symmetric and $[u, v] = \mathrm{L}_u v - \mathrm{L}_v u$.
We denote by $\mathrm{R}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$, the right multiplication $\mathrm{R}_u(v) = \mathrm{L}_v u$.
We have $\mathrm{L}_u - \mathrm{R}_u = \mathrm{ad}_u$.

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The curvature of $\mathfrak{g}$ is given by

$$\begin{aligned} K(u,v)w &= \mathrm{L}_{[u,v]}w - [\mathrm{L}_u, \mathrm{L}_v]w \\ &= [\mathrm{R}_w, \mathrm{L}_u](v) - \mathrm{R}_w \circ \mathrm{R}_u(v) + \mathrm{R}_{uw}(v). \end{aligned}$$

The Ricci curvature $\mathrm{ric} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ and its Ricci operator $\mathrm{Ric} : \mathfrak{g} \longrightarrow \mathfrak{g}$ are defined by

$$\begin{aligned} \langle \mathrm{Ric}(u), v \rangle &= \mathrm{ric}(u,v) = \mathrm{tr}\left(w \longrightarrow K(u,w)v\right) \\ &= -\mathrm{tr}(\mathrm{R}_w \circ \mathrm{R}_u) + \mathrm{tr}(\mathrm{R}_{uv}). \end{aligned}$$

We have

$$\mathrm{tr}(\mathrm{R}_{uv}) = -\frac{1}{2}\left(\langle \mathrm{ad}_H u, v\rangle + \langle \mathrm{ad}_H v, u\rangle\right),$$

where H is the vector given by $\langle H, u\rangle = \mathrm{tr}(\mathrm{ad_u})$.

One can deduce easily from (1) that

$$\mathrm{R}_u = -\frac{1}{2}\left(\mathrm{ad}_u + \mathrm{ad}_u^*\right) - \frac{1}{2}J_u,$$

where $J_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the skew-symmetric the endomorphism given by $J_u(v) = \mathrm{ad}_v^* u$.

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Proposition

We have

$$\begin{aligned}
\mathrm{ric}(u,v) &= -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) - \frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(J_u \circ J_v) \\
&\quad -\frac{1}{2}\langle \mathrm{ad}_H u, v\rangle - \frac{1}{2}\langle \mathrm{ad}_H v, u\rangle, \\
\langle H, u\rangle &= \mathrm{tr}(\mathrm{ad}_u), \\
J_u(v) &= \mathrm{ad}_v^t(u). \qquad (J_u = 0 \iff u \in [\mathfrak{g},\mathfrak{g}]^\perp).
\end{aligned}$$

Proposition

If $\mathfrak{g}$ is nilpotent then

$$\begin{aligned}
\mathrm{ric}(u,v) &= -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(J_u \circ J_v) \\
&= -\frac{1}{2}\langle \mathcal{J}_1(u), v\rangle + \frac{1}{4}\langle \mathcal{J}_2(u), v\rangle, \\
\mathrm{Ric} &= -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2.
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Remark

If $\langle\ ,\ \rangle$ is Euclidean then $\langle \mathcal{J}_1(u), u\rangle \geq 0$ (resp. $\langle \mathcal{J}_2(u), u\rangle \geq 0$) and $\langle \mathcal{J}_1(u), u\rangle = 0$ (resp. $\langle \mathcal{J}_2(u), u\rangle = 0$) if $u \in Z(\mathfrak{g})$ (resp. $u \in [\mathfrak{g},\mathfrak{g}]^{\perp}$).

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Let $(e_1, \ldots, e_p)$ be a basis of $\mathfrak{g}$. Then, for any $u, v \in \mathfrak{g}$, the Lie bracket can be written

$$[u, v] = \sum_{i=1}^{p} \langle S_i u, v \rangle e_i, \tag{2}$$

where $S_i : \mathfrak{g} \longrightarrow \mathfrak{g}$ are skew-symmetric endomorphisms with respect to $\langle\ ,\ \rangle$.

The family $(S_1, \ldots, S_p)$ will be called *structure endomorphisms* associated to $(e_1, \ldots, e_p)$.

We have $Z(\mathfrak{g}) = \bigcap_{i=1}^{p} \ker S_i$ and for any $u \in \mathfrak{g}$,

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Recall that if $\mathfrak{g}$ is nilpotent

$$\begin{aligned} \mathrm{ric}(u,v) &= -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(J_u \circ J_v) \\ &= -\frac{1}{2}\langle \mathcal{J}_1(u), v\rangle + \frac{1}{4}\langle \mathcal{J}_2(u), v\rangle. \end{aligned}$$

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra, $(e_1, \ldots, e_p)$ a basis of $\mathfrak{g}$ and $(S_1, \ldots, S_p)$ the corresponding structure endomorphisms. Then

$$\mathcal{J}_1 = -\sum_{i,j=1}^{p} \langle e_i, e_j\rangle S_i \circ S_j \quad \text{and} \quad \mathcal{J}_2 u = -\sum_{i,j=1}^{p} \langle e_i, u\rangle \mathrm{tr}(S_i \circ S_j) e_j. \tag{4}$$

In particular, $\mathrm{tr}\mathcal{J}_1 = \mathrm{tr}\mathcal{J}_2$.

A pseudo-Euclidean Lie algebra $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is called Einstein if there exists a $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{g}}.$$

If $\lambda = 0$ then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is called Ricci flat.

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A nilpotent pseudo-Euclidean Lie algebra $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is Einstein if there exists a $\lambda \in \mathbb{R}$ such that

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}.$$

This is a very complicated quadratic equation and we will solve it for $\dim \mathfrak{g} \leq 5$.

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The following proposition appeared first in the Euclidean context.

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra and let Q denote the symmetric endomorphism $Q = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2$. Then for any orthonormal basis $(e_1, \ldots, e_p)$ of $\mathfrak{g}$ and any endomorphism E of $\mathfrak{g}$, we have

$$\mathrm{tr}(QE) = \frac{1}{4}\sum_{i,j}\varepsilon_i\varepsilon_j\langle E([e_i, e_j]) - [E(e_i), e_j] - [e_i, E(e_j)], [e_i, e_j]\rangle, \tag{5}$$

where $\langle e_i, e_i\rangle = \varepsilon_i$.

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a pseudo-Euclidean nilpotent Lie algebra having a derivation with non null trace. Then* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *is Einstein if and only if it is Ricci flat.*

Remark

The Lie algebra of derivations of nilpotent Lie algebras has been widely studied and computed. It turns out that nilpotent Lie algebras having a derivation with non null trace are the most common. For instance, any nilpotent Lie algebra up to dimension 6 has this property.

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra. We have

$$Z(\mathfrak{g}) \subset M = \ker \mathcal{J}_1 \quad \text{and} \quad [\mathfrak{g}, \mathfrak{g}]^\perp \subset N = \ker \mathcal{J}_2.$$

Since $\mathcal{J}_1$ and $\mathcal{J}_2$ are symmetric

$$\mathrm{Im}\mathcal{J}_1 = M^\perp \subset Z(\mathfrak{g})^\perp \quad \text{and} \quad \mathrm{Im}\mathcal{J}_2 = N^\perp \subset [\mathfrak{g}, \mathfrak{g}].$$

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Suppose that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is nilpotent and Einstein with $\lambda \neq 0$, i.e.,

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}.$$

This implies

$$Z(\mathfrak{g}) \subset \mathrm{Im}\mathcal{J}_2 \subset [\mathfrak{g}, \mathfrak{g}].$$

Moreover, $M \cap N = \{0\}$. If $\dim Z(\mathfrak{g}) \geq \dim[\mathfrak{g}, \mathfrak{g}]$ then

$$\dim M + \dim N \geq \dim Z(\mathfrak{g}) + \dim[\mathfrak{g}, \mathfrak{g}]^{\perp} \geq \dim \mathfrak{g}$$

and hence $\mathfrak{g} = M \oplus N$. This contradicts $\mathrm{tr}(\mathcal{J}_1) = tr(\mathcal{J}_2)$.

Suppose that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is nilpotent and Einstein with $\lambda \neq 0$, i.e.,

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}.$$

This implies

$$Z(\mathfrak{g}) \subset \mathrm{Im}\mathcal{J}_2 \subset [\mathfrak{g}, \mathfrak{g}].$$

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Proposition

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *be a nilpotent pseudo-Euclidean Lie algebra. Then*

1. *If* $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *is Einstein with* $\lambda \neq 0$ *then* $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$.
2. *If* $\dim Z(\mathfrak{g}) \geq \dim[\mathfrak{g}, \mathfrak{g}]$ *then* $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *is Einstein if and only if it is Ricci flat.*

Corollary

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *be a 2-nilpotent pseudo-Euclidean Lie algebra. Then* $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *is Einstein if and only if it is Ricci flat.*

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Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent non abelian Lie algebra. If $[\mathfrak{g}, \mathfrak{g}]$ is non degenerate then it is Lorentzian.

Proof.

Suppose that $[\mathfrak{g}, \mathfrak{g}]$ is nondegenerate Euclidean, choose an orthonormal basis $(e_1, \ldots, e_d)$ of $[\mathfrak{g}, \mathfrak{g}]$ and denote by $(S_1, \ldots, S_d)$ the associated structure endomorphisms. We have

$$\begin{aligned} \lambda \mathrm{Id}_{\mathfrak{g}} &= -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2, \\ \mathcal{J}_1 &= -\sum_{i=1}^{d} S_i^2, \\ \mathcal{J}_2 u &= -\sum_{i,j=1}^{d} \langle u, e_i \rangle \mathrm{tr}(S_i \circ S_j) e_j. \end{aligned}$$

Proof.

Since $\mathfrak{g}$ is nilpotent then $\dim[\mathfrak{g},\mathfrak{g}]^\perp \geq 2$ and we can choose a couple $(e,\bar{e})$ of isotropic vectors in $[\mathfrak{g},\mathfrak{g}]^\perp$ such that $\langle e,\bar{e}\rangle = 1$.

So

$$\frac{1}{2}\mathcal{J}_1 e = -\lambda e,\ \frac{1}{2}\mathcal{J}_1 \bar{e} = -\lambda \bar{e} \quad \text{and} \quad \sum_{i=1}^{d}\langle S_i e, S_i e\rangle = \sum_{i=1}^{d}\langle S_i \bar{e}, S_i \bar{e}\rangle = 0.$$

So, according to Lemma 1.1, for any $i \in \{1,\ldots,d\}$, $S_i e = \alpha_i e$ and $S_i \bar{e} = -\alpha_i \bar{e}$. Thus

$$\lambda = \frac{1}{2}\sum_{i=1}^{d}\alpha_i^2 \geq 0.$$

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Proof.

Let $K_i = (S_i)_{|\{e,\bar{e}\}^\perp}$. So $\mathrm{tr}(S_i^2) = 2\alpha_i^2 + \mathrm{tr}(K_i^2)$ and $\mathrm{tr}(K_i^2) \leq 0$. Now, since $\mathrm{tr}(\mathcal{J}_1) = \mathrm{tr}(\mathcal{J}_2)$, we get

$$(\dim \mathfrak{g})\lambda = -\frac{1}{4}\mathrm{tr}(\mathcal{J}_1) = \frac{1}{4}\sum_{i=1}^{d}(2\alpha_i^2 + \mathrm{tr}(K_i^2)) = \lambda + \frac{1}{4}\sum_{i=1}^{d}\mathrm{tr}(K_i^2).$$

This shows that $\lambda = 0$ and $\mathrm{tr}(K_i^2) = 0$ for any i thus $S_i = 0$ which implies that $\mathfrak{g}$ is abelian. This is a contradiction which completes the proof. □

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein pseudo-Riemannian non abelian nilpotent Lie algebra of signature $(p, q) = (-, \ldots, -, +, \ldots, +)$. If $Z(\mathfrak{g})$ is nondegenerate then $Z(\mathfrak{g})^\perp$ is not Euclidean.

Proof.

Suppose that $Z(\mathfrak{g})$ is nondegenerate and $Z(\mathfrak{g})^\perp$ is Euclidean and choose a family of vector $(e_1, \ldots, e_p) \in Z(\mathfrak{g})$ such that $\langle e_i, e_i\rangle = -1$. We have $\mathfrak{g} = \mathrm{span}\{e_1, \ldots, e_p\} \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = \{e_1, \ldots, e_p\}^\perp$.
For any $u, v \in \mathfrak{g}_0$, put

$$[u, v] = \sum_{i=1}^{p} \langle K_i u, v\rangle e_i + [u, v]_0,$$

where $K_i : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$ are skew-symmetric and $[u, v]_0 \in \mathfrak{g}_0$. □

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Proof.

It is obvious that $(\mathfrak{g}_0, [\ ,\ ]_0, \langle\ ,\ \rangle)$ is an Euclidean nilpotent Lie algebra.

We claim that if $(\mathfrak{g}, \langle\ ,\ \rangle)$ is Einstein then $\lambda = \frac{1}{4}\mathrm{tr}(K_i^2)$, for $i = 1, \ldots, p$

$$\mathrm{Ric}_{\langle\ ,\ \rangle_0} = \lambda \mathrm{Id}_{\mathfrak{g}_0} + \frac{1}{2}\sum_{i=1}^{p} K_i^2.$$

This implies that the Ricci curvature of $(\mathfrak{g}_0, \langle\ ,\ \rangle)$ is nonpositive. But a non abelian nilpotent Euclidean Lie algebra has always a Ricci negative direction and Ricci positive direction. So the only possibility is the $K_i = 0$ and $\mathfrak{g}_0$ is abelian. We get a contradiction which completes the proof. □

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Corollary

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be an Einstein Lorentzian non abelian nilpotent Lie algebra. If* $Z(\mathfrak{g})$ *is nondegenerate then it is Euclidean.*

Corollary

Let $\mathfrak{g}$ *be an Einstein Lorentzian non abelian 2-step nilpotent Lie algebra. Then* $Z(\mathfrak{g})$ *is degenerate.*

Proof.

Suppose that $Z(\mathfrak{g})$ is nondegenerate. According to Proposition 3.5, $Z(\mathfrak{g})$ is nondegenerate Euclidean. But $\mathfrak{g}$ is 2-step nilpotent and hence $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})$. Thus $[\mathfrak{g}, \mathfrak{g}]$ is nondegenerate Euclidean which contradicts Proposition 3.4. □

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Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent Lie algebra such that $[\mathfrak{g}, \mathfrak{g}]$ is degenerate then $[\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^{\perp} \subset Z(\mathfrak{g})$ and $\lambda = 0$.

Proof.

Let $(e, \bar{e}, f_1, \ldots, f_d, g_1, \ldots, g_s)$ be a Lorentzian basis of $\mathfrak{g}$ such that $(e, f_1, \ldots, f_d)$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$, $(e, g_1, \ldots, g_s)$ is a basis of $[\mathfrak{g}, \mathfrak{g}]^{\perp}$. Denote by $(A, S_1, \ldots, S_d)$ the associated structure endomorphisms, i.e., for any $u, v \in \mathfrak{g}$,

$$[u, v] = \langle Au, v\rangle e + \sum_{i=1}^{d} \langle S_i u, v\rangle f_i.$$

We have

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}} \quad \text{and} \quad \mathcal{J}_1 = -\sum_{i=1}^{d} S_i^2.$$

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein Lorentzian nilpotent Lie algebra such that $[\mathfrak{g}, \mathfrak{g}]$ is degenerate then $[\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp \subset Z(\mathfrak{g})$ and $\lambda = 0$.

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Proof.

Since $e \in [\mathfrak{g}, \mathfrak{g}]^\perp$ and isotropic, we have $\mathcal{J}_2 e = 0$, $-\frac{1}{2}\mathcal{J}_1 e = \lambda e$, and hence $\sum_{j=1}^d \langle S_j e, S_j e \rangle = 0$. We get $S_j e = a_j e$ for any $j = 1, \ldots, d$ and hence

$$\lambda = \frac{1}{2}\sum_{i=1}^{d} a_i^2 \geq 0.$$

On the other hand, since $\mathrm{tr}(\mathcal{J}_1) = \mathrm{tr}(\mathcal{J}_2)$, we get

$$(\dim \mathfrak{g})\lambda = -\frac{1}{4}\mathrm{tr}(\mathcal{J}_1) = \frac{1}{4}\sum_{j=1}^{d} \mathrm{tr}(S_j^2).$$

□

Proof.

Or

$$\begin{aligned} \mathrm{tr}(S_j^2) &= \langle S_j^2 e, \bar{e}\rangle + \langle S_j^2 \bar{e}, e\rangle + \sum_I \langle S_j^2 f_I, f_I\rangle + \sum_I \langle S_j^2 g_I, g_I\rangle \\ &= 2a_j^2 - \sum_I \langle S_j f_I, S_j f_I\rangle - \sum_I \langle S_j g_I, S_j g_I\rangle. \end{aligned}$$

Thus

$$(\dim \mathfrak{g} - 1)\lambda = -\sum_{I,j} \langle S_j f_I, S_j f_I\rangle - \sum_{I,j} \langle S_j g_I, S_j g_I\rangle.$$

□

Proof.

Or

$$\begin{aligned}\mathrm{tr}(S_j^2) &= \langle S_j^2 e, \bar{e}\rangle + \langle S_j^2 \bar{e}, e\rangle + \sum_l \langle S_j^2 f_l, f_l\rangle + \sum_l \langle S_j^2 g_l, g_l\rangle \\ &= 2a_j^2 - \sum_l \langle S_j f_l, S_j f_l\rangle - \sum_l \langle S_j g_l, S_j g_l\rangle.\end{aligned}$$

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$$(\dim \mathfrak{g} - 1)\lambda = -\sum_{l,j} \langle S_j f_l, S_j f_l\rangle - \sum_{l,j} \langle S_j g_l, S_j g_l\rangle.$$

□

Proof.

Since S_j leaves invariant e, it leaves invariant its orthogonal $\mathrm{span}\{e, f_l, g_k\}$ and hence $\langle S_j f_l, S_j f_l\rangle \geq 0$ and $\langle S_j g_l, S_j g_l\rangle \geq 0$. So $\lambda = 0$, $S_j(e) = 0$. Thus, for any $u \in \mathfrak{g}$, $[e, u] = \langle A(e), u\rangle e$. But ad_u is nilpotent an hence $[e, u] = 0$ which completes the proof. □

Consider $(V, \langle\ ,\ \rangle_0)$ an Euclidean vector space, $b \in V$, $K, D : V \longrightarrow V$ two endomorphisms of V such that K is skew-symmetric. We endow the vector space $\mathfrak{g} = \mathbb{R}e \oplus V \oplus \mathbb{R}\bar{e}$ with the inner product $\langle\ ,\ \rangle$ which extends $\langle\ ,\ \rangle_0$, for which $\mathrm{span}\{e, \bar{e}\}$ and V are orthogonal, $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$ and $\langle e, \bar{e}\rangle = 1$. We define also on $\mathfrak{g}$ the bracket

$$\begin{cases} [\bar{e}, e] = \mu e, \\ [\bar{e}, u] = D(u) + \langle b, u\rangle_0 e, \\ [u, v] = \langle K(u), v\rangle_0 e, \quad u, v \in V. \end{cases} \tag{6}$$

Theorem

(*i*) $(\mathfrak{g}, [\ ,\ ])$ *is a Lie algebra if and only if* $KD + D^*K = \mu K$.

(*i*) *If the condition in* (i) *is satisfied then* $(\mathfrak{g}, \langle\ ,\ \rangle, [\ ,\ ])$ *is an Einstein Lorentzian Lie algebra if and only if*

$$4\mu(\mu + \mathrm{tr}(D)) = \mathrm{tr}(K^2) + 2\mathrm{tr}(D^2) + 2\mathrm{tr}(DD^*).$$

In this case it is Ricci flat.

A data (K, D, μ, b) satisfying the conditions in Theorem 4.1 are called admissible.

Theorem

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein nilpotent non abelian Lorentzian Lie algebra and suppose that there exists $e \in Z(\mathfrak{g})$ a central isotropic vector and denote $\mathcal{I} = \mathbb{R}e$. Then:

1. *$Z(\mathfrak{g})$ is degenerate and $\lambda = 0$.*
2. *$\mathcal{I}^\perp$ is an ideal and $\mathfrak{g}_0 = \mathcal{I}^\perp/\mathcal{I}$ is an Euclidean abelian Lie algebra.*
3. *$\mathfrak{g}$ is obtained from $\mathfrak{g}_0$ by the double extension process with admissible data $(K, D, 0, b)$ and D is nilpotent.*

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Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be an Einstein nilpotent non abelian Lorentzian Lie algebra and suppose that there exists $e \in Z(\mathfrak{g})$ a central isotropic vector and denote $\mathcal{I} = \mathbb{R}e$. Then:

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If $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a Lorentzian Einstein nilpotent Lie algebra with generate center then $\mathfrak{g}$ is Ricci flat and it is isomorphic to

$$\mathbb{R}\bar{e} \oplus V \oplus \mathbb{R}e$$

with the non vanishing Lie brackets are given by

$$\begin{cases} [\bar{e}, u] = D(u) + \langle b, u\rangle_0 e, \\ [u, v] = \langle K(u), v\rangle_0 e, \quad u, v \in V. \end{cases} \tag{7}$$

K is skew-symmetric and

$$KD + D^*K = 0,\ D^{\dim V} = 0 \quad \text{and} \quad \mathrm{tr}(K^2) = -2\mathrm{tr}(D^*D).$$

Theorem

Let (G, g) be an Einstein Lorentzian nilpotent Lie group of dimension ≤ 5. Then the center of $\mathfrak{g}$ is degenerate.

Lie Algebra	Lie brackets	Non Trace-free Derivation
$L_{3,2}$	$[e_1, e_2] = e_3$	$e^1 \otimes e_1 + e^3 \otimes e_3$
$L_{4,2}$	$[e_1, e_2] = e_3$	$e^1 \otimes e_1 + e^3 \otimes e_3$
$L_{4,3}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$	$2e^2 \otimes e_2 - e^1 \otimes e_1 + e^3 \otimes e_3$
$L_{5,2}$	$[e_1, e_2] = e_3$	$e^1 \otimes e_1 + e^3 \otimes e_3$
$L_{5,3}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$	$2e^2 \otimes e_2 - e^1 \otimes e_1 + e^3 \otimes e_3$
$L_{5,4}$	$[e_1, e_2] = e_5$, $[e_3, e_4] = e_5$	$e^1 \otimes e_1 + e^3 \otimes e_3 + e^5 \otimes e_5$
$L_{5,5}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_2, e_4] = e_5$	$e^3 \otimes e_3 + 2e^2 \otimes e_2 + 2e^5 \otimes e_5 - e^1 \otimes e_1$
$L_{5,6}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_5$	$e^1 \otimes e_1 + 2e^2 \otimes e_2 + 3e^3 \otimes e_3 + 4e^4 \otimes e_4 + 5e^5 \otimes e_5$
$L_{5,7}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$	$e^1 \otimes e_1 - 2e^2 \otimes e_2 - e^3 \otimes e_3 + e^5 \otimes e_5$
$L_{5,8}$	$[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$	$e^1 \otimes e_1 - e^2 \otimes e_2 + e^5 \otimes e_5$
$L_{5,9}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$	$2e^1 \otimes e_1 - e^2 \otimes e_2 + e^3 \otimes e_3 + 3e^4 \otimes e_4$

Table of nilpotent Lie algebras of dimension ≤ 5 with non null trace derivation

Theorem

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a Ricci-flat nilpotent Lie algebra of dimension ≤ 5. Then

1. *If* $\dim \mathfrak{g} = 3$ *then* $\mathfrak{g}$ *is isomorphic to* $L_{3,2}$ *with the metric* $\langle\ ,\ \rangle_{3,2} = \alpha e_1 \odot e_3 + e_2 \otimes e_2$ *with* $\alpha > 0$*. This metric is actually flat.*
2. *If* $\dim \mathfrak{g} = 4$ *then* $\mathfrak{g}$ *is isomorphic to* $L_{4,2}$ *with the metric*

$$\langle\ ,\ \rangle_{4,2} = \alpha e_1 \odot e_3 + e_2 \otimes e_2 + e_4 \otimes e_4 + b e_2 \odot e_4, \quad \alpha \neq 0, |b| < 1,$$

or to $L_{4,3}$ *with the metric*

$$\langle\ ,\ \rangle_{4,3} = e_1 \otimes e_1 + a e_1 \odot e_2 + (a^2 + b^2) e_2 \otimes e_2 + c e_2 \odot e_3 + \varepsilon e_2 \odot e_4 + e_3 \otimes e_3,\ a, b \in \mathbb{R},$$

$\varepsilon = \pm 1$*. The metric* $\langle\ ,\ \rangle_{4,2}$ *is flat and* $\langle\ ,\ \rangle_{4,3}$ *is flat if and only if* $\varepsilon = -1$*.*

If $\dim \mathfrak{g} = 5$ then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:

(a) $L_{5,2}$ with the metric

$$\alpha e_1 \odot e_3 + e_2 \otimes e_2 + e_4 \otimes e_4 + e_5 \otimes e_5 + be_2 \odot e_4 + ce_2 \odot e_5 + bce_4 \odot e_5,$$

$\alpha \neq 0, |b| < 1, |c| < 1.$ This metric is flat.

(b) $L_{5,8}$ with the metric

$$e_1 \otimes e_1 + ae_1 \odot e_2 - yx^{-1} e_1 \odot e_3 + (b - ayx^{-1}) e_2 \odot e_3 + (a^2 + b^2) e_2 \odot e_2$$
$$+ \sqrt{x^2 + y^2} e_2 \otimes e_5 + (1 + (yx^{-1})^2) e_3 \otimes e_3 + x^2 e_4 \otimes e_4,$$
$$(x \neq 0, a, b, y \in \mathbb{R}).$$

(c) $L_{5,9}$ with the the metric

$$(a^2 + b^2) e_1 \otimes e_1 + (b - ayx^{-1}) e_1 \odot e_2 + ae_1 \odot e_3 + \varepsilon \sqrt{x^2 + y^2 + 1} e_1 \odot e_5$$
$$(1 + (yx^{-1})^2) e_2 \otimes e_2 - yx^{-1} e_2 \odot e_3 + e_3 \otimes e_3 + x^2 e_4 \otimes e_4,$$
$$(x \neq 0, a, b, y \in \mathbb{R}).$$

(d) $L_{5,3}$ with the metric

$$e_1 \otimes e_1 + ae_1 \odot e_2 + (a^2 + b^2)e_2 \otimes e_2 + be_2 \odot e_3 + \varepsilon\sqrt{x^2+1}e_2 \odot e_4$$
$$+(1+x^2)e_3 \otimes e_3 - xe_3 \odot e_5 + e_5 \otimes e_5, \ (x, a, b \in \mathbb{R}).$$

(e) $L_{5,5}$ with the metric

$$(a^2+b^2)e_1 \otimes e_1 + a\gamma^{-1}e_1 \odot e_2 + \gamma(b - ax^{-1}y)e_1 \odot e_4 + \sqrt{x^2+y^2}e_1 \odot e_5$$
$$+\gamma^{-2}e_2 \otimes e_2 - x^{-1}ye_2 \odot e_4 + x^2\gamma^{-2}e_3 \otimes e_3 + \gamma^2(1 + (x^{-1}y)^2)e_4 \otimes e_4,$$
$$(x \neq 0, \gamma \neq 0, a, b, y \in \mathbb{R})$$

or

$$e_1 \otimes e_1 + be_1 \odot e_2 + (a^2+b^2)e_2 \otimes e_2 + ae_2 \odot e_3 + \varepsilon\sqrt{x^2+1}e_2 \odot e_5$$
$$(1+x^2)e_3 \otimes e_3 + x\gamma e_3 \odot e_4 + \gamma^2 e_4 \otimes e_4, \quad (\gamma \neq 0, x, a, b \in \mathbb{R}).$$

(f) $L_{5,6}$ with the metric

$$(a^2+b^2)e_1 \otimes e_1 + (b + ax^{-1}y)e_1 \odot e_2 + \mu ae_1 \odot e_3 + \varepsilon\mu^2\sqrt{x^2+y^2+1}e_1 \odot$$
$$+(1+x^{-2}y^2)e_2 \otimes e_2 + \mu x^{-1}ye_2 \odot e_3 + \mu^2 e_3 \otimes e_3 + \mu^4x^2 e_4 \otimes e_4,$$
$$\mu \neq 0, \gamma \neq 0, x \neq 0, a, b, y \in \mathbb{R}.$$