AGTA Seminar Session 05-08-2021

# A Wide Family of 3-Manifolds Consolidating the L-Space Conjecture

Part 1 : Introduction by H. Abchir

I Motivation
II Some preliminaries

## I- Motivation

① **Main concern:** exploring the world of 3-dimensional manifolds with the ultimate objective their classification

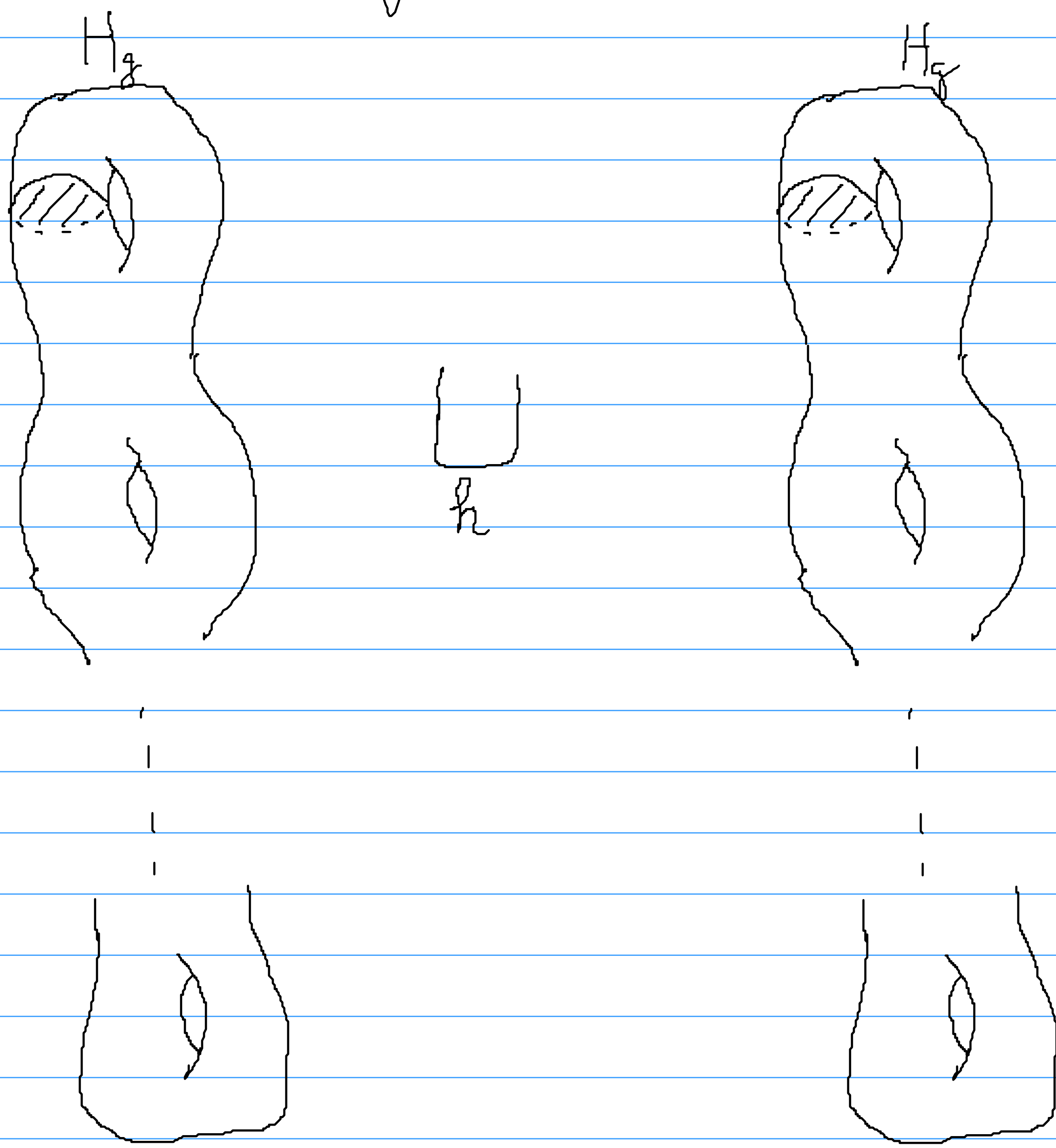
② **Main results:**

a. The P.L. (piecewise linear) and the differentiable classifications of 3-manifolds are both equivalent to the topological one: Moise 1952, 1954.
Munkres 1966.

6.

**Theorem: (Heegaard splitting).** P. Heegaard 1898.

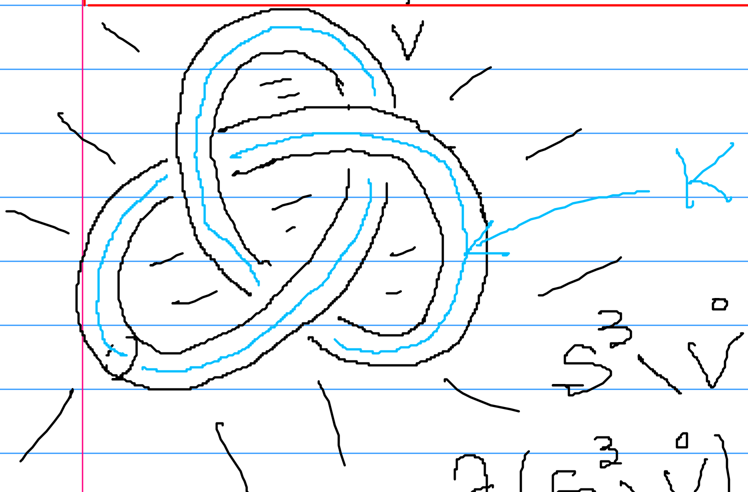
Any orientable 3-manifold has a Heegaard splitting




$$h : \partial H_g \xrightarrow{\simeq} \partial H_g$$

Theorem: Lickorish 1962

Any orientable 3-manifold may be obtained by cutting out some solid tori from $S^3$ and then pasting them back in, but along different homeomorphisms of their boundaries.




$$S^3 \setminus \mathring{V}$$

$$\partial(S^3 \setminus \mathring{V}) \simeq S^1 \times S^1$$

$$(S^3 \setminus \mathring{V}) \sqcup_f D^2 \times S^1$$

$$f : \partial(D^2 \times_K S^1) \longrightarrow \partial(S^3 \setminus \mathring{V})$$

$$S^1 \times S^1 \longrightarrow S^1 \times S^1$$

meridian $\partial D^2 \times \{*\} \longmapsto \alpha$

$[\alpha] = pm + ql$ . $r = \frac{p}{q}$ = slope

That is an $r$-surgery in $S^3$ along $K$

**Theorem** Kirby 1978

Two links in $S^3$ with integer slopes produce the same 3-manifold if and only if they can be obtained from each other by a finite sequence of the two "Kirby moves"

$$\text{Closed connected oriented 3-manifolds} \Big/ \text{up to diffeomorphism} \xleftrightarrow{\text{bijection}} \text{Isotopy classes of links} \Big/ \text{up to Kirby moves 1 and 2}$$

Then

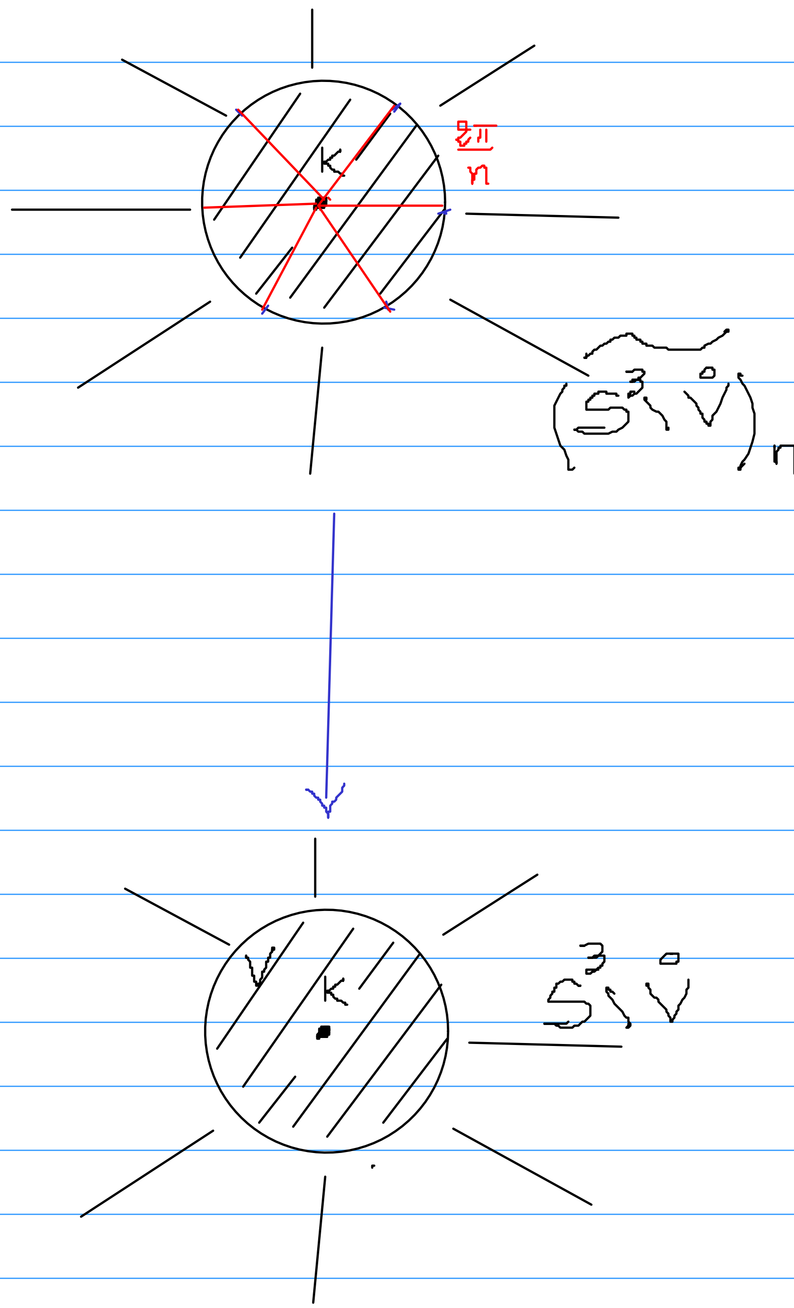
Classification of 3-manifolds

is equivalent to

Classification of links in $S^3$ up to isotopy and Kirby moves.

**Theorem** Hilden - Montesinos 1976 → 1987

For any closed oriented 3-manifold $M^3$ there exists a 3-fold covering $p: M^3 \longrightarrow S^3$ by this manifold branching along a knot



**Theorem** Poincaré's conjecture (Perelman 2003)
Each simply connected closed 3-manifold is homeomorphic to $S^3$.

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One among most important tools useful in the study of 3-manifolds is to find "powerful" invariants, allowing to distinguish them.

3 Main theories generating invariants:

a. Combinatorics: Diagrams, Graphs, Braids, Tangles, Surfaces.

b. Topological Quantum Field Theory
TQFT

Initiated by E. Witten 1988
and formalised by M. Atiyah 1988

c. Khovanov Homology. M. Khovanov 1999
= A categorification of the Jones polynomial

d. Heegaard - Floer Homology

= A categorification of the Alexander polynomial P. Ozsváth & Szabó 2003

II Preliminaries.

The Heegaard Floer homology provides a parti-cular family of 3-manifolds: L-spaces.

M a closed oriented 3-manifold s.t. $b_1(M) = 0$

We have that $|H_1(M)| \leq rk\, \widehat{HF}(M)$

① L-spaces:

Definition: Let M be a connected closed oriented 3-manifold.

M is an L-space if it is a rational homology sphere with the property that $rk\, \widehat{HF}(M) = |H_1(M; \mathbb{Z})|$

Examples:

1. Lens spaces
2. All Seifert fibered spaces with finite fundamental group
3. $L$ a link in $S^3$

$$\begin{array}{c} \Sigma(L) \\ \downarrow \\ S^3 \end{array}$$

$\Sigma(L)$ the branched double cover of $S^3$ branched along $L$

We have that:

(i) $\Sigma(L)$ is a rational homology sphere

(ii) $|H_1(M)|$ is an invariant of $L$ $= \det(L)$

(iii) If $L$ is alternating non-split then

$$\widehat{HF}(\Sigma(L)) \simeq \mathbb{Z}^{\det(L)}$$

and then $\Sigma(L)$ is an L-space.

Remark: In fact there is a more wide family of links, containing the alternating links, called quasi-alternating links s.t

$$\widehat{HF}(\Sigma(L)) \simeq \mathbb{Z}^{\det L}$$

i.e. for which $\Sigma(L)$ is an L-space.

② Left orderable group:

Def: A group $G$ is said to be left-orderable if there exists a total order $<$ on $G$ such that given $a, b \in G$, if $a < b$ then $ca < cb$ of any $c \in G$.

By convention the trivial-group is not left-orderable.

Interesting problem: relation between geometry of a 3-manifold and the left-orderability of its fundamental group.

**Theorem:** If $M$ is a closed oriented 3-manifold such that $\pi_1(M)$ is left-orderable then it is a rational homology 3-sphere.

Boyer, Rolfsen & Wiest 2005.

It seems that the only rational homology 3-spheres satisfying the converse of the theorem are the L-spaces.

**Conjecture:** Boyer, Gordon, Watson 2013

L-space conjecture

The fundamental group of a rational homology 3-sphere is non-left-orderable if and only if $M$ is an L-space

We provide infinite families of non-left-orderable L-spaces and then give further support to the conjecture