Subdivision schemes on manifolds

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Curve subdivision

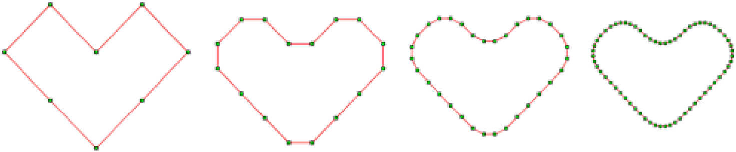


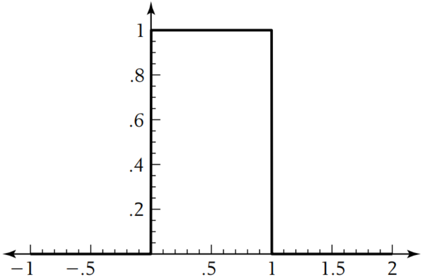
FIGURE – P^0 P^1 P^2 P^3

Let

$$n^1(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

We define the sequence of functions :

$$n^m(x) = \int_0^1 n^{m-1}(x-t)dt,$$



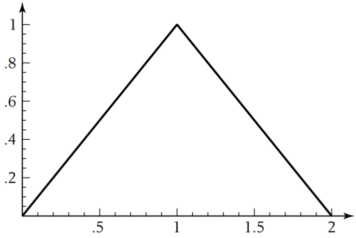


FIGURE – $n^2(x)$.

$$n^2(x) = \begin{cases} x; & \text{if } 0 < x \leqslant 1, \\ 2 - x; & \text{if } 1 < x \leqslant 2.; \\ 0; & \text{otherwise.} \end{cases}$$

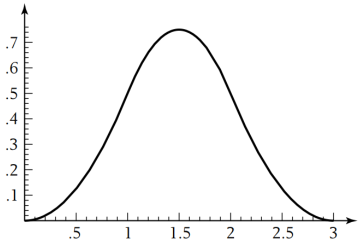


FIGURE – $n^3(x)$.

$$n^3(x) = \begin{cases} \frac{x^2}{2}; & i\text{if } 0 < x \leqslant 1 \\ -\frac{3}{2} + 3x - x^2; & \text{if } 1 < x \leqslant 2 \\ -\frac{1}{2}(-3+x)^2; & \text{if } 2 < x \leqslant 3 \\ 0; & \text{otherwise.} \end{cases}$$

1. n^m is piecewise polynomial over $[0, m]$ of nodes $i \in \{0, ..., m\}$, and over $[i, i+1]$ n^m is polynomial of degree m-1.
2. n^m is C^{m-2}
3. The family $\{n^m(. - i)\}_{i \in \mathbb{Z}}$ is free.
4. A *B-Spline* $p(x)$ of order m is of the form :

$$p(x) = \sum_{i \in \mathbb{Z}} p_i \, n^m(x - i)$$

where $\{p_i\}_{i \in \mathbb{Z}}$ is a sequence of finite support which we call *control polygon*.

Proposition

The n^m satisfies the refinement rule :

$$n^m(x) = \sum_{i\in\mathbb{Z}} a_i^m \, n^m(2x - i).$$

with

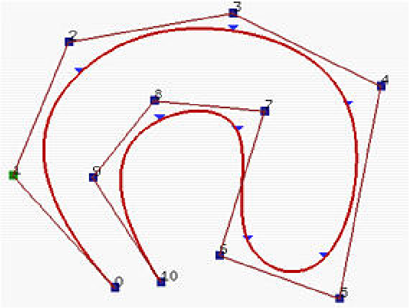
$$a_i^{[m]} = 2^{-m} \begin{pmatrix} m+1 \\ i \end{pmatrix}, \quad i \in \{0, .., m+1\},$$

For a B-Spline function $p(x)$ controlled by p_i^0 , we have :

$$p(x) = \sum_{i\in\mathbb{Z}} p_i^0 \, n^m(x-i) = \sum_{i\in\mathbb{Z}} p_i^1 \, n^m(2x-i) = \ldots = \sum_{i\in\mathbb{Z}} p_i^j \, n^m(2^j x-i).$$

Remark

For a large value of j, the points $P^j = \{p_i^j\}$ approximate p. We call such points **controle polygon**. The $P^0 = \{p_i^0\}$ is called **Initial polygon**.



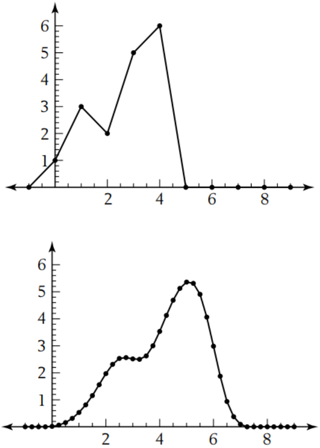


FIGURE – A B-Spline of order $m = 4$ (cubic).

Subdivision scheme

We can view a B-Spline function $p(x)$ as a limit of refined polygons. Such refinement is given by :

$$p_i^{j+1} = \sum_{k\in\mathbb{Z}} a^m_{i-2k}\, p^j_K.$$

Or shortly by : $P^{j+1} = SP^j = S^j P^0$.
Then we have $p(x) = \lim_{j\to\infty} S^j P$

Sequence of control polygons converges toward a limit curve.

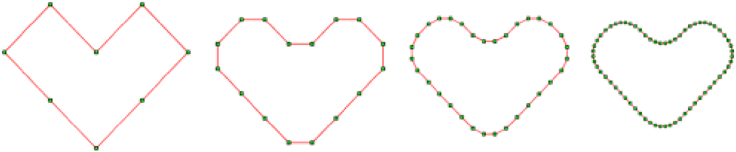


FIGURE – P^0 P^1 P^2 P^3

Quadratic B-Spline

Let the scheme of the B-Spline of order $m = 2$ (Quadratic) be :

$$(SP)_{2i} = \frac{1}{4}p_{i-1} + \frac{3}{4}p_i, \qquad (SP)_{2i+1} = \frac{3}{4}p_i + \frac{1}{4}p_{i+1}.$$

It can be expressed, equivalently, by :

$$(SP)_{2i} = av_{1/4}(p_{i-1}, p_i), \qquad (SP)_{2i+1} = av_{3/4}(p_i, p_{i+1}). \qquad (1)$$

where $av_\alpha(x, y) = \alpha x + (1 - \alpha)y$.

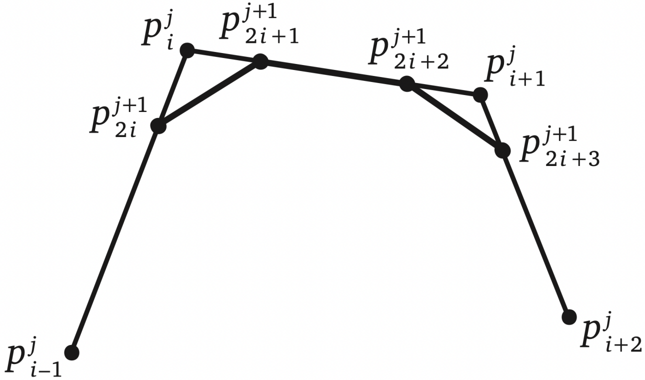


FIGURE – Construction of the new points p_{2i}^{j+1} and p_{2i+1}^{j+1} .

Cubic B-spline

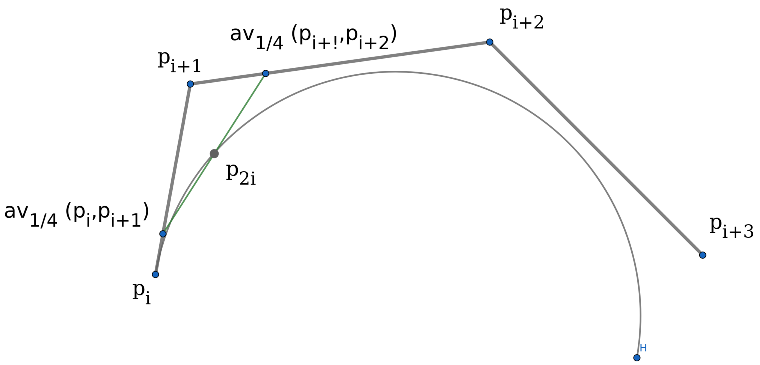
The Cubic B-spline subdivision (order $m = 3$) :

$$(SP)_{2i} = \frac{1}{2}p_i + \frac{1}{2}p_{i+1}, \qquad (SP)_{2i+1} = \frac{3}{8}p_i + \frac{1}{4}p_{i+1} + \frac{3}{8}p_{i+2}.$$

It reads

$$(SP)_{2i} = av_{1/2}(p_i, p_{i+1}),$$
$$(SP)_{2i+1} = av_{1/2}\Big(av_{1/4}(p_i, p_{i+1}), av_{1/4}(p_{i+1}, p_{i+2})\Big).$$

Cubic B-Spline construction



$$p_{2i} = av_{1/2}\Big(av_{1/4}(p_i, p_{i+1}), av_{1/4}(p_{i+1}, p_{i+2})\Big).$$

Definition

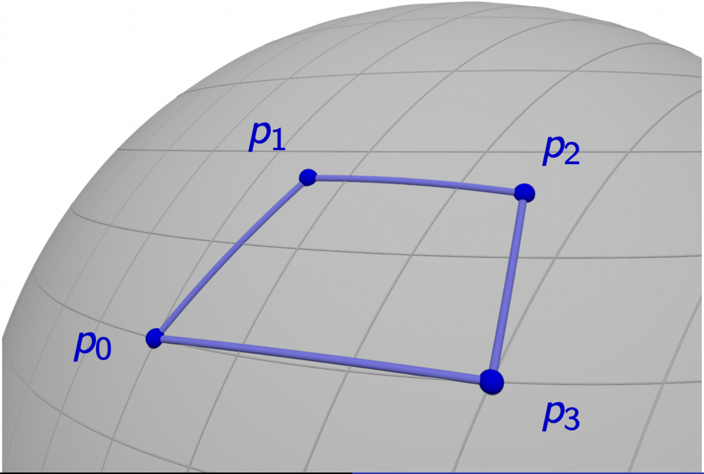
A linear scheme $p_i^{j+1} = \sum_{k\in\mathbb{Z}} a_{i-2k}\, p_k^j$ is affinely invariant if

$$\sum_{k\in\mathbb{Z}} a_{2k} = \sum_{k\in\mathbb{Z}} a_{2k+1} = 1.$$

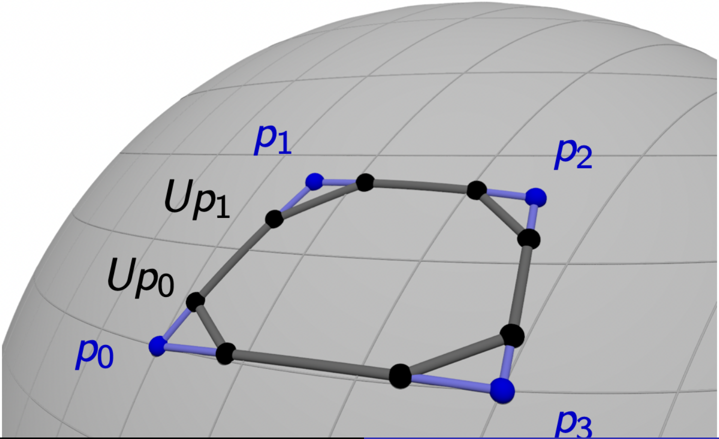
Theorem

Any affinely invariant linear subdivision is expressible via the *av* operator.

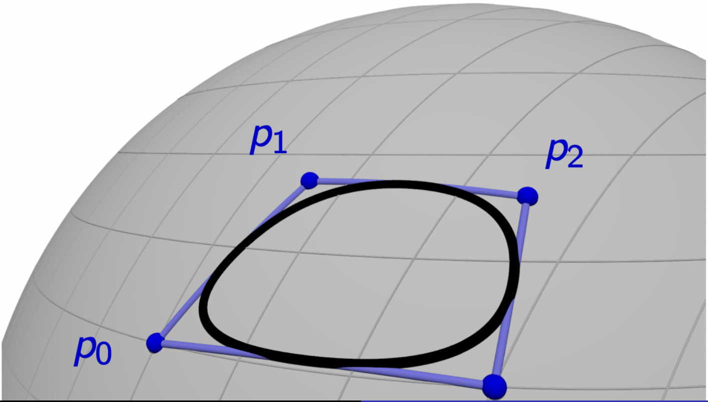
Cubic Spline on manifold



Cubic Spline on manifold



Cubic Spline on manifold



Geodesic average

Definition

If c is the geodesic curve which joints x and y such that $c(0) = x$ and $c(t) = y$, then we let

$$\text{g-av}_{\alpha}(x, y) = c(\alpha t).$$

Definition

The geodesic analogue T of an affinely invariant linear scheme S, which is expressed in terms of averages, is defined by replacing each occurrence of the av operator by the g-av operator.

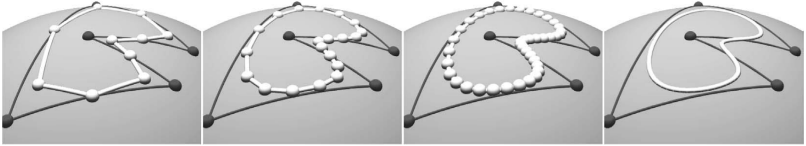


FIGURE – Geodesic cubic B-spline subdivision on the sphere. Black : Control geodesic polygon P. White (from left to right) : TP, T^2P, T^3P, $T^\infty P$.

The geodesic analogous of the linear cubic B-spline is :

$$(TP)_{2i} = \text{g-av}_{1/2}\left(p_i, p_{i+1}\right),$$

$$(TP)_{2i+1} = \text{g-av}_{1/2}\Big(\text{g-av}_{1/4}\left(p_{i+1}, p_i\right), \text{g-av}_{1/4}\left(p_{i+1}, p_{i+2}\right)\Big).$$

Hyperbolic

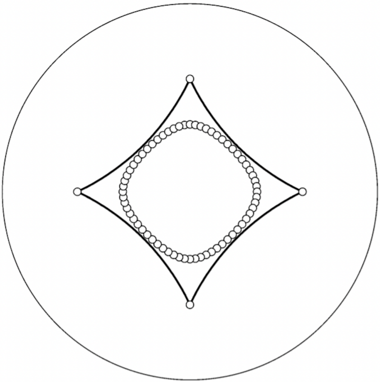


FIGURE – Geodesic cubic B-spline subdivision in the Hyperbolic plane. Polygons P and T^4P.

The displacements group

FIGURE – Geodesic cubic B-spline subdivision in the Euclidean group $S0_3 \times \mathbb{R}^3$.

Convergence

If P is a sequence of points, we use the symbol ΔP for the sequence of differences $(\Delta P)_i = p_{i+1} - p_i$. Further we define

$$d(P) := \sup_i \|p_{i+1} - p_i\| = \|\Delta P\|_\infty, \qquad \text{where} \quad \|p\|_\infty = \sup_i \|p\|.$$

Convergence

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Defintion

A subdivision scheme S is said to satisfy a convergence condition with factor $\mu_0 < 1$, if

$$d(S^l P) \leqslant \mu_0^l \, d(P) \qquad \text{for all} \quad l, P; \tag{2}$$

Remark

A convergent scheme means that $\lim_{l \to \infty} \|\Delta(S^l P)\|_\infty = 0$ (the points of $S^l P$ shrink).

The proximity condition

Most of our statements consider polygons whose points are contained in some subset M of $\mathbb{R}^d$, and fulfil $d(P) < \varepsilon$. Such a class of polygons is denoted by $\mathcal{P}_{M,\varepsilon}$.

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Definition

Subdivision schemes S, T satisfy a proximity condition for a class $\mathcal{P}_{M,\varepsilon}$ of polygons P, if there is a constante C such that for all $P \in \mathcal{P}_{M,\varepsilon}$,

$$\|SP - TP\|_\infty \leqslant C d(P)^2. \tag{3}$$

Convergence Theorem 1 (Dyn and Walner 2005)

Suppose that S, T satisfy a proximity condition for all $P \in \mathcal{P}_{M,\varepsilon}$, and S satisfies a convergence condition with factor $\mu_0 < 1$. Then there is $\delta > 0$ and $\bar{\mu} < 1$ such that T satisfies a convergence condition with factor $\bar{\mu}_0$ for all $P \in \mathcal{P}_{M,\delta}$.

Corollary

If S is convergent and is in proximity with T, then T is also convergent.

Theorem 2 (Dyn and Walner 2005)

We use the requirements and notation of Theorem 1, and we assume that S has the property that $\|S^l\| \leqslant A$. Then for any polygon $P \in \mathcal{P}_{M,\delta}$,

$$\|S^\infty P - T^\infty P\|_\infty \leqslant \frac{AC}{1-\bar{\mu}^2}\, d(P)^2.$$

Theorem 2 allows to transfer stability properties of S to T. If e.g. $\|S^\infty(P+\varepsilon) - S^\infty P\|_\infty \leqslant D\|\varepsilon\|_\infty$, then

$$\begin{aligned}\|T^\infty(P+\varepsilon) - T^\infty P\|_\infty &\leqslant \frac{AC}{1-\bar{\mu}^2}\Big(d(P+\varepsilon)^2 + d(P)^2\Big) + D\|\varepsilon\|_\infty,\\ &\leqslant \frac{AC}{1-\bar{\mu}^2}\Big(2d(P)^2 + 4d(P)\|\varepsilon\|_\infty + 4\|\varepsilon\|_\infty^2\Big)\\ &+ D\|\varepsilon\|_\infty.\end{aligned}$$

We want to prove that the geodesic analogous T of a linear scheme S fulfils a proximity condition .
Let $T_x M$ be the tangent plane of the surface M, and $\mathbb{I}_x$ be the second fundamental form at the point x. We consider such open subsets V of M where there exists a constant $D > 0$ with the property that

$$x \in V, \quad w \in T_x M, \quad \|w\| \leqslant 1 \Longrightarrow \|\mathbb{I}_x(w)\| \leqslant D \qquad (4)$$

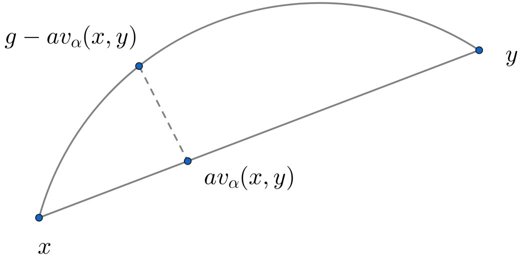
Clearly all points in M have a neighbourhood V where there exists $D > 0$ such that (4) holds true.

Proposition 1

Assume that (4) holds true with $D > 0$ and an open set V, and that the points x, y are joined by the geodesic of length $\leqslant 1/D$. If the geodesic used in g-av$_\alpha(x, y)$ is contained in V, then

$$\|av_\alpha(x,y) - \text{g-av}_\alpha(x,y)\| \leqslant 2Dmin(|\alpha| + \alpha^2, |\beta| + \beta^2)\|x - y\|^2,$$

with $\beta = 1 - \alpha$.



Lemma 1

Assume that c is a curve with $\|\dot{c}\| = 1$ and $\|\ddot{c}\| \leqslant C$. Then

$$\|c(0) + t\dot{c}(0) - c(t)\| \leqslant \frac{Ct^2}{2}, \qquad |t| - Ct^2/2 \leqslant \|c(t) - c(0)\|, \quad (5)$$

$$t < 1/C \Longrightarrow |t| \leqslant 2\|c(t) - c(0)\|. \quad (6)$$

Taylor's formula $c(t) = c(0) + t\dot{c}(0) + \frac{t^2}{2}\ddot{c}(\theta t)$ with $\theta \in [0, 1]$ implies that

$$\|c(t) - c(0) - t\dot{c}(0)\| = \left\| \frac{t^2}{2}\ddot{c}(\theta t) \right\| \leqslant \frac{Ct^2}{2}, \tag{7}$$

$$\|c(t) - c(0)\| = \|t\dot{c}(0) + \frac{Ct^2}{2}\ddot{c}(\theta t)\| \geqslant \|t\dot{c}(0)\| - \|\frac{t^2}{2}\ddot{c}(\theta t)\|. \tag{8}$$

Eqs. (7) and (8) immediately imply (5).

The function $\varphi(t) := t - Ct^2/2$ is monotonically increasing for $t \in [0; 1/C]$ with

$$\varphi(1/C) = 1/2C =: L.$$

So we have $\varphi(t) \geqslant t/2$ if $t \in [0; 1/C]$.
As $\psi(t) := \|c(t) - c(0)\|$ has the property that $\psi(t) > \varphi(t)$ (By (5)), then $|t| < 2\psi(t)$.

Lemma 2

Assume that c is a curve with $\|\ddot{c}\| \leqslant C$. Then

$$\|av_\alpha(c(0), c(t)) - c(\alpha t)\| \leqslant \frac{|\alpha| + \alpha^2}{2} Ct^2. \qquad (9)$$

We use Taylor's formula and find that the left hand side of (9) expands to

$$\frac{1}{2}\|\alpha t^2 \ddot{c}(\theta t) - \alpha^2 t^2 \ddot{c}(\theta' \alpha t)\|$$

with $\theta, \theta' \in [0; 1]$, which implies the upper bound given by (9).

Proof of Proposition 1

We assume that c is the geodesic connecting $c(0) = x$ to $c(t) = y$. By (9) , an upper bound is given by

$$D\frac{|\alpha| + \alpha^2}{2}t^2.$$

Because of the symmetry of the geodesic average ; namely

$$\text{g-av}_{1-\alpha}(y, x) = \text{g-av}_{\alpha}(x, y),$$

this relation remains true if we replace α by $1 - \alpha$. Thanks to (6), $t \leqslant 2\|x - y\|$, which completes the proof.

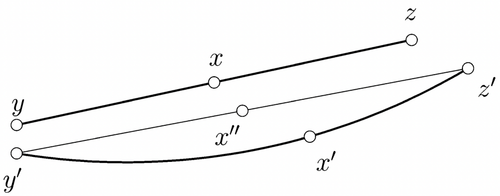
Theorem

Let V and D as in (4). Consider an affinely invariant subdivision scheme S and its analogous geodesic scheme T. Let the class $\mathcal{P}'_{V,\delta}$ consists of all polygons P in V with $d(P) < \delta$ and which have the property that all geodesics used in subdividing according to T are contained in V.
Then S and T fulfil a proximity condition for all polygons $P \in \mathcal{P}'_{V,\delta}$.

Proof of the Theorem

As to two or more steps of averaging, we perform an induction step. We assume that points x and x' are defined in a linear and a nonlinear way, respectively, by $x = av_\alpha(y,z)$ and $x' = \text{g-av}_\alpha(y',z')$. We also assume that

$$\|y - z\| \leqslant Cd(P), \qquad \|y' - y\|, \|z' - z\| \leqslant C'd(P)^2. \qquad (10)$$



Our aim is to show that x and x' meet a proximity condition. By induction, this would show that T and S are in proximity. We introduce $x" = av_\alpha(y', z')$ and use Proposition 1 again :

$$\begin{aligned} \|x - x'\| &\leqslant \|x' - x''\| + \|x - x''\| \\ &\leqslant C''\|y' - z'\|^2 + \|av_\alpha(y - y', z - z')\| \\ &\leqslant C''\Big(\|y - y'\| + \|y - z\| + \|z - z'\|\Big)^2 + \\ & C''max(\|y - y'\|, \|z - z'\|). \end{aligned}$$

Thus by (10) and $d(P) < \delta$, we have $\|x - x'\| \leqslant C'''d(P)^2$.