Geometry of symmetric cones

Khalid Koufany

Institut Elie Cartan–Nancy • Université de Lorraine

Interuniversity Geometry Seminar
09/04/2022

The main reference of this talk is :
Faraut, Jacques; Korányi, Adam. **Analysis on symmetric cones** Oxford Mathematical Monographs. Oxford Science Publications. *The Clarendon Press, Oxford University Press, New York*, 1994. xii+382 pp.

Symmetric cones

Let V be a finite dimensional real Euclidean space with the inner product $(\cdot \mid \cdot)$. A convex cone $\Omega \subset V$ (containing 0) is pointed or proper if Ω contains no affine line.
If Ω is closed, this means $\Omega \cap (-\Omega) = \{0\}$.
Define the dual cone by

$$\Omega^* = \{x \in V \mid (x|y) \geqslant 0 \text{ for all } y \in \Omega\}.$$

Then Ω^* is a closed convex cone, and has a non-trivial interior $\Omega^{*\circ}$ if and only if Ω is pointed.
The automorphism group $G(\Omega)$ of an open convex cone $\Omega \subset V$ is defined by

$$G(\Omega) = \{g \in \mathrm{GL}(V) \mid g\Omega \subset \Omega\}.$$

It is a closed subgroup of $\mathrm{GL}(V)$ and hence a Lie group. The open cone $\Omega \subset V$ is said to be symmetric if it is homogeneous ($G(\Omega)$ acts transitively on Ω) and self dual ($\Omega^* = \Omega$).

Example 1

Consider $V = \mathrm{Sym}(n, \mathbb{R})$ with the inner product $(A|B) = \mathrm{Tr}(AB)$. Then the set $\Omega = \Omega_n(\mathbb{R})$ of positive definite symmetric matrices is a convex cone in V.

Let us prove that Ω is a symmetric cone.

The group $GL(n, \mathbb{R})$ acts on Ω by

$$\rho(g)x = gxg^\top$$

and this action is transitive (if $x \in \Omega$, then $x = \alpha\alpha^\top$ where $\alpha \in \mathrm{GL}(n, \mathbb{R})$, thus $x = \alpha\alpha^\top = \rho(\alpha)I_n$.

Let $x \in \Omega^*$, then for any non zero vector $\xi \in \mathbb{R}^n$, $y := \xi\xi^\top \in \Omega$. By definition of Ω^* we have

$$(x\xi, \xi)_{\mathbb{R}^n} = (x|\xi\xi^\top) = (x|y) > 0$$

and hence $x \in \Omega$.

Conversely, any element $x \in \Omega$, can be written as

$$x = \sum_{j=1}^{k} \xi_j \xi_j^\top$$

where the ξ_j are independent vectors in $\mathbb{R}^n$. Therefore, if $y \in \Omega$, then

$$(y|x) = \sum_{j=1}^{k} (y|\xi_j \xi_j^\top) = \sum_{j=1}^{k} (y\xi_j, \xi_j)_{\mathbb{R}^n} > 0.$$

Thus $x \in \Omega^*$ and $\Omega = \Omega^*$.

Example 2

Let $\Omega = \Lambda_n$ to be the Lorentz cone in the Euclidean space $V = \mathbb{R}^n$,

$$\Omega = \left\{x \in \mathbb{R}^n \mid x_1^2 - x_2^2 - \cdots - x_n^2 > 0, x_1 > 0\right\}.$$

Let $[,]$ be the following bilinear form on $\mathbb{R}^n$

$$[x, y] = x_1 y_1 - x_2 y_2 - \cdots - x_n y_n = x^\top J y$$

where x, y are written as $n \times 1$ matrices and

$$J = \begin{pmatrix} 1 & 0 & \cdots & \cdots \\ 0 & -1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & -1 \end{pmatrix}.$$

Then

$$\Omega = \{x \in \mathbb{R}^n \mid [x, x] > 0, x_1 > 0\}.$$

To prove that Ω is homogeneous we consider the group $SO_0(1, n-1)$ (= the identity component of the group of $n \times n$ real matrices g such that $[gx, gy] = [x, y]$ for all x, y, or, equivalently, such that $g^\top J g = J$).

Each element of the group $SO_0(1, n-1)$ maps Ω onto itself, and so does $G = \mathbb{R}_+ \times SO_0(1, n-1)$, the direct product of $SO_0(1, n-1)$ with the group of positive dilations.

The matrices

$$g_u = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}, \quad \text{with} \quad u \in SO(n-1)$$

are special elements of $SO_0(1, n-1)$, and so are the hyperbolic rotations

$$h_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-2} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

To show that G acts transitively on Ω we show that, for any x in Ω, there exists an element in G which maps $e_1 = (1, 0, \ldots, 0)^\top$ to x.

First, write $x = \lambda y$, with $\lambda = [x,x]^{\frac{1}{2}} > 0$, $[y,y] = 1$. Then there exists *a* rotation $u \in \mathrm{SO}(n-1)$ such that

$$(y_2, \ldots, y_n)^\top = u \cdot (0, \ldots, 0, r)^\top$$

with $r = \sqrt{y_2^2 + \cdots + y_n^2}$. Since $y_1^2 - r^2 = 1$ there exists $t \geqslant 0$ such that

$$y_1 = \cosh t, \quad r = \sinh t.$$

Therefore,

$$x = \lambda g_u h_t \cdot e_1.$$

Now we show that Ω is self-dual. To see that $\Omega \subset \Omega^*$, let $y \in \Omega$. Then, using Schwarz's inequality, we have ,

$$(x \mid y) \geqslant x_1 y_1 - \sqrt{x_2^2 + \cdots + x_n^2}\sqrt{y_2^2 + \cdots + y_n^2} > 0,$$

for all $x \in \bar{\Omega} \backslash \{0\}$. Hence y belongs to Ω^*.

To prove the reverse inclusion, let y be in Ω^*. One has $y_1 > 0$. If $y_2 = 0, \cdots, y_n = 0$, then y belongs to Ω. Otherwise, define x by

$$x_1 = \sqrt{y_2^2 + \ldots + y_n^2}, \quad x_2 = -y_2, \ldots, x_n = -y_n.$$

Then x belongs to $\bar{\Omega} \setminus \{0\}$, so $(x \mid y) > 0$, or

$$y_1 \sqrt{y_2^2 + \cdots + y_n^2} - \left(y_2^2 + \cdots + y_n^2\right) > 0$$

which means that y belongs to Ω.

Ω as a Riemmanian symmetric space

Theorem

Let Ω be a symmetric cone in a Euclidean vector space V. Then Ω is a Riemannian symmetric space.

Proof. For $x \in \Omega$ and $u, v \in V$, we let

$$G_x(u, v) = D_u D_v \log \varphi(x)$$

where φ is the characteristic function of Ω,

$$\varphi(x) = \int_\Omega e^{-(x|y)} dy$$

dy being the Euclidean measure on V.
We have, for $u \neq 0$,

$$G_x(u, u) = D_u^2 \log \varphi(x) = \left.\frac{d^2}{dt^2}\right|_{t=0} \log \varphi(x + tu)$$
$$= \frac{1}{\varphi^2} \left(\varphi D_u^2 \varphi - (D_u \varphi)^2 \right),$$

and

$$\frac{d}{dt}\Big|_{t=0} \varphi(x+tu) = -\int_\Omega e^{-(x|y)}(u \mid y)dy$$

$$\frac{d^2}{dt^2}\Big|_{t=0} \varphi(x+tu) = \int_\Omega e^{-(x|y)}(u \mid y)^2 dy.$$

Put

$$f(y) = e^{-\frac{1}{2}(x|y)}, \quad g(y) = e^{-\frac{1}{2}(x|y)}(u \mid y),$$

then

$$G_x(u,u) = \frac{1}{\varphi^2(x)^2}\left(\int_\Omega f(y)^2 dy \int_{\Omega*} g(y)^2 dy - \left(\int_\Omega f(y)g(y)dy\right)^2\right)$$

which is > 0 by the Schwarz inequality, since f and g are not proportional.

Therefore, the bilinear form G_x defines a Riemannian metric on Ω.

Every element g in $G(\Omega)$ is an isometry; in fact

$$G_{gx}(gu, gv) = \left(D_{gu}D_{gv}(\log\varphi)\right)(gx) = D_uD_v(\log\varphi\circ g)(x)$$
$$= D_uD_v(\log\varphi)(x) = G_x(u,v)$$

where the third equality follows form

$$\varphi(gx) = |\operatorname{Det} g|^{-1}\varphi(x).$$

Thus the metric, is invariant under $G(\Omega)$.
Further, for $x \in \Omega$, let x^* defined by

$$x^* = -\nabla\log\varphi(x)$$

Then one can prove (see Faraut-Koràny, pages 14–17) that the map $x \mapsto x^*$ is an involutive isometry on Ω with a unique fixed point, denoted by e. Now, if $y \in \Omega$, then there exists $g \in G(\Omega)$ such that $y = ge$ and the map $x \mapsto g(g^{-1}x)^*$ is an involutive isometry of Ω with the unique fixed point y.

Let G be the identity component of $G(\Omega)$ and $K = G \cap O(V)$. Then G acts transitively on Ω and K is a compact subgroup of G. Furthermore (see Faraut-Korànyi, page 18) $G_e = K$, where G_e is stabilizer subgroup of e.
The map, $\theta : g \mapsto (g^{-1})^\top$ is an involution of G (Cartan involution) and $K = G^\theta$ is a maximal compact subgroup of G. We conclude that, as a symmetric space,

$$\Omega = G/K.$$

For $\Omega = \Omega_m(\mathbb{R})$, we have

$$\Omega_m(\mathbb{R}) = \mathrm{GL}(m, \mathbb{R})/\,\mathrm{SO}(m).$$

For $\Omega = \Lambda_n$, we have

$$\Lambda_n(\mathbb{R}) = \mathrm{SO}_0(1, n-1)/\,\mathrm{SO}(n-1).$$

Euclidean Jordan algebras

Let V be a finite dimensional Euclidean vector space equipped with a scalar product $(\cdot \mid \cdot)$.
V is a Euclidean Jordan algebra if,

$$xy = yx,$$
$$x^2(xy) = x(x^2y),$$
$$(xy \mid z) = (x \mid yz).$$

For $x \in V$, denote by $L(x)$ the linear operator defined by

$$y \mapsto L(x)y = xy.$$

The quadratic representation P is defined by

$$P(x) = 2L(x)^2 - L(x^2).$$

An element $x \in V$ is said to be invertible if there exists an element $y \in \mathbb{R}[x]$ such that $xy = e$. Since $\mathbb{R}[x]$ is associative, y is unique. It is called the inverse of x and is denoted by $y = x^{-1}$.

The leading example is $V = \mathrm{Sym}(n, \mathbb{R})$ with
– the product

$$x \circ y = \frac{1}{2}(xy + yx),$$

– the inner product

$$(x \mid y) = \mathrm{Tr}(x \circ y) = \mathrm{Tr}(xy),$$

– the quadratic representation

$$P(x)y = xyx.$$

Proposition

(1) *An element* $x \in V$ *is invertible if and only if* $P(x)$ *is invertible, and in this case,* $P(x^{-1}) = P(x)^{-1}$.
(2) *The differential of the map* $x \mapsto x^{-1}$ *is* $-P(x)^{-1}$, *i.e.*

$$D_u\left(x^{-1}\right) = D\left(x^{-1}\right)u = -P(x)^{-1}u.$$

(3) *If* x *and* y *are invertible, then* $P(x)y$ *is invertible and*

$$(P(x)y)^{-1} = P\left(x^{-1}\right)y^{-1}.$$

(4) *For any elements* x *and* y *of* V,

$$P(P(y)x) = P(y)P(x)P(y) \qquad \text{“the fundamental relation”}$$

(5) *If* x *and* x *are invertible, then*

$$P(x^{-1} - y^{-1}) = P(x)^{-1}P(x-y)P(y)^{-1} \qquad \text{“Hua's identity”}$$

Jordan frames

An element $c \in V$ is said to be an idempotent if $c^2 = c$. Using the identity $L\left(x^2 y\right) - L\left(x^2\right) L(y) = 2(L(xy) - L(x)L(y))L(x)$ with $x = y$, we obtain

$$L\left(x^3\right) = 3L\left(x^2\right) L(x) - 2L(x)^3,$$

and for $x = c$:

$$2L(c)^3 - 3L(c)^2 + L(c) = 0.$$

Therefore, an eigenvalue λ of $L(c)$ is a solution of

$$2\lambda^3 - 3\lambda^2 + \lambda = 0,$$

Then the only possible eigenvalues of $L(c)$ are 1, $\frac{1}{2}$, 0 and V is the direct sum of the corresponding eigenspaces $V(c,1)$, $V(c,\frac{1}{2})$ and $V(c,0)$. The decomposition

$$V = V(c,1) \oplus V(c,\frac{1}{2}) \oplus V(c,0),$$

is called the Peirce decomposition of V with respect to c.

It is an orthogonal decomposition with respect to any associative scalar product.

Two idempotents c and d are said to be *orthogonal* if $(c|d) = 0$, which is equivalent to $cd = 0$. An idempotent is said to be *primitive* if it is not the sum of two non-zero idempotents. An idempotent c is primitive if and only if $\dim V(c, 1) = 1$.

We say that $(c_j)_{1 \leqslant j \leqslant m}$ is a Jordan frame if each c_j is a primitive idempotent and

$$c_i c_j = 0, \quad i \neq j$$

$$c_1 + c_2 + \ldots + c_m = e.$$

All the Jordan frames have the same number of elements, denote by r and called the rank of the Jordan algebra V.

The group K acts transitively on the set of primitive idempotents, and also on the set of Jordan frames. Therefore if we fix a Jordan frame $(c_j)_{j=1}^r$, then every element $x \in V$ can be written in the form

$$x = k(\sum_{j=1}^{r} \lambda_j c_j)$$

where $k \in K$ and $\lambda_1, \ldots, \lambda_r$ real numbers. The scalars $(\lambda_j)_{1 \leqslant j \leqslant r}$ are unique and called the spectral values of x.
We define the determinant and the trace of the Jordan algebra by

$$\det(x) = \prod_{j=1}^{r} \lambda_j, \quad \mathrm{tr}(x) = \sum_{j=1}^{r} \lambda_j.$$

The trace is a linear form of V and the determinant is a homogeneous polynomial on V of degree r. Both are invariant under $\mathrm{Aut}(V)$.

Proposition

Let V be a simple Euclidean Jordan algebra on dimension n and rank r.

(i) We have

$$\operatorname{Tr} L(x) = \frac{n}{r} \operatorname{tr}(x),$$

$$\operatorname{Det} P(x) = (\det x)^{\frac{2n}{r}},$$

$$\det(P(y)x) = (\det y)^2 \det x.$$

(ii) Furthermore, if the scalar product on V is defined by

$$(x \mid y) = \operatorname{tr}(xy)$$

then

$$\nabla \log \det x = x^{-1}.$$

(ii) The set of invertible elements in V is given by

$$V^{\times} = \{x \in V \mid \det(x) \neq 0\}.$$

Proposition

In a simple Euclidean Jordan algebra V every associative scalar product is a scalar multiple of $\mathrm{tr}(xy)$.

Hence, we assume from now that the scalar product of V is given by

$$(x|y) = \mathrm{tr}(xy).$$

The example : $V = \mathrm{Sym}(r, \mathbb{R})$

In this case the determinant and the trace are the usual matrix determinant and trace.
Put $r = p + q$. An idempotent is an orthogonal projection

$$c = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$V(c,1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a : p \times p \text{ symmetric matrix} \right\},$$
$$V(c,\frac{1}{2}) = \left\{ \begin{pmatrix} 0 & d \\ d^\top & 0 \end{pmatrix} \mid d : p \times q \text{ matrix} \right\},$$
$$V(c,0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mid b : q \times p \text{ symmetric matrix} \right\}.$$

The example : Spin factor

Let B be a symmetric bilinear form on $\mathbb{R}^{n-1}$. Then $V = \mathbb{R}^{1,n-1} = \mathbb{R} \times \mathbb{R}^{n-1}$ is a Euclidean Jordan algebra of dimension n and rank 2 :

The product is

$$(\lambda, u)(\mu, v) = (\lambda\mu + B(u, v), \lambda v + \mu u).$$

We have

$$\mathrm{tr}(\lambda, u) = 2\lambda, \;\; \det(\lambda, u) = \lambda^2 - B(u, u).$$

An element is invertible if and only if $\det(\lambda, u) = \lambda^2 - B(u, u) \neq 0$. In this case

$$(\lambda, u)^{-1} = \frac{1}{\det(\lambda, u)}(\lambda, -u).$$

The associative inner product is

$$((\lambda, u) \mid (\mu, v)) = 2(\lambda\mu + B(u, v)).$$

The non-zero idempotents are

$$e = (1, 0), \quad c = \left(\frac{1}{2}, w\right)$$

with $B(w, w) = \frac{1}{4}$. For such an idempotent c :

$$V(c, 1) = \mathbb{R}c,$$
$$V(c, 0) = \mathbb{R}\left(\frac{1}{2}, -w\right),$$
$$V\left(c, \frac{1}{2}\right) = \{(0, u) \mid B(u, w) = 0\}.$$

The cone of squares in V

Lat V be a Euclidean Jordan algebra with unit element e. Let C be the set of all squares

$$C = \left\{x^2 \mid x \in V\right\}.$$

The set C is a cone and therefore its closed dual

$$C^* = \left\{y \in V \mid \forall x \in V, (y \mid x^2) \geqslant 0\right\}$$

is a closed convex cone.
Since

$$(y \mid x^2) = (yx \mid x) = (L(y)x \mid x)$$

we have

$$C^* = \{y \mid L(y) \text{ is positive semi-definite } \}.$$

Let

$$\Omega = C^\circ = \mathrm{Int}(C).$$

Theorem

Ω is a symmetric cone. Furthermore, Ω is the connected component of e in $V^{\times}$, and also is the set of elements $x \in V$ for which $L(x)$ is positive definite.

Proof. (1) To prove that Ω is self-dual, we show that $C^* = C$. If $x = \sum_{j=1}^{k} \lambda_j c_j$ is the spectral decomposition of an element $x \in V$, then $x^2 = \sum_{j=1}^{k} \lambda_j^2 c_j$ and

$$L\left(x^2\right) = \sum_{j=1}^{k} \lambda_j^2 L\left(c_j\right).$$

Since the operators $L\left(c_j\right)$ are positive, $L\left(x^2\right)$ is positive and $C \subset C^*$.

Conversely, let $x \in C^*$. Since the idempotents c_j are orthogonal, we have

$$\begin{aligned}\lambda_j &= \frac{1}{\|c_j\|^2}(x \mid c_j) \\ &= \frac{1}{\|c_j\|^2}\left(x \mid c_j^2\right) \\ &= \frac{1}{\|c_j\|^2}(xc_j \mid c_j) \\ &= \frac{1}{\|c_j\|^2}(L(x)c_j \mid c_j) \geqslant 0.\end{aligned}$$

Therefore, $x = y^2$ with

$$y = \sum_{j=1}^{k} \sqrt{\lambda_j} c_j.$$

So we have shown that $C^* \subset C$, and finally that $C^* = C$.

(2) Let us consider the set

$$B = \{y \in V \mid L(y) \text{ is positive definite }\}$$

then B is open, therefore B is contained in Ω. For a non-zero element y in V let us consider the linear form on V

$$\ell(x) = \left(x \mid y^2\right) = (L(y)x \mid y).$$

Then ℓ is not identically zero and for x in C we have $\ell(x) \geqslant 0$, and for x in Ω we have $\ell(x) > 0$ since Ω is open, which means that Ω is contained in B.

(3) If $x = \sum_{j=1}^{k} \lambda_j c_j \in V$, then it is clear that

$$\exp x = \sum_{j=1}^{k} e^{\lambda_j} c_j \in \Omega$$

Thus $\exp(V) \subset \Omega$. The converse is also clear. Hence we can also define Ω as

$$\Omega = \exp V = \{\exp x \mid x \in V\}.$$

(4) The set $V^\times = \{x \in V \mid \det x \neq 0\}$ is open. Let $x \in \bar{\Omega} \cap V^\times$. Since x belongs to $\bar{\Omega}$ the eigenvalues of x are non-negative, and since x belongs to $V^\times$ they are non-zero, therefore x belongs to Ω, and $\bar{\Omega} \cap V^\times = \Omega$. This means that Ω is closed in $V^\times$, hence Ω is the connected component of $V^\times$ containing the identity element e.
(5) It remains to show that Ω is homogeneous. If x is invertible, then $P(x)$ is invertible, and $P(x)\Omega$ is a connected open subset of $V^\times$ (if x and y are invertible then $P(x)y$ is invertible). Since $x^2 = P(x)e$ belongs to Ω, we have $P(x)\Omega \subset \Omega$. On the other hand, $P(x)^{-1}\Omega = P\left(x^{-1}\right)\Omega \subset \Omega$. Hence, $P(x)$ belongs to $G(\Omega)$ and

$$\Omega = \left\{x^2 \mid x \in V^\times\right\} = \{P(x)e \mid x \in V^\times\} \subset G(\Omega)e = \Omega.$$

To summary, we have

$$\begin{aligned}\Omega &= \exp V \\ &= \operatorname{int}\left\{x^2 \mid x \in V\right\} \\ &= \text{ the identity component of } V^\times \\ &= \left\{x^2 \mid x \in V^\times\right\} \\ &= \left\{P(x)e \mid x \in V^\times\right\} \\ &= \{x \in V \mid L(x) \text{ positive definite }\}\end{aligned}$$

The structure group of V

The structure group of V is the subgroup of $\mathrm{GL}(V)$ given by

$$\mathrm{Str}(V) = \{g \in \mathrm{GL}(V) \mid P(gx) = gP(x)g^\top, \quad \text{for any } x \in V\}.$$

It is known that if $x \in V^\times$, then $P(x) \in \mathrm{Str}(V)$ and $P(x)^\top = P(x)$. Furthermore, the automorphism group $\mathrm{Aut}(V)$ is a subgroup of $\mathrm{Str}(V)$ and an element g in $\mathrm{Str}(V)$ belongs to $\mathrm{Aut}(V)$ if and only if $ge = e$. In particular, $g^\top = g^{-1}$ for $g \in \mathrm{Aut}(V)$.

Proposition

If V is a simple Euclidean Jordan algebra, then

$$\mathrm{Str}(V) = \{\pm I\}G(\Omega).$$

In particular, if $g \in \mathrm{Str}(V)$, then $g(\Omega) = \Omega$ or $g(\Omega) = -\Omega$.

Example

If $V = \mathrm{Sym}(m, \mathbb{R})$, then $\Omega = \Omega_m(\mathbb{R})$ symmetric positive definite matrices.
In this case, $G(\Omega) = \mathrm{GL}(m, \mathbb{R})/\{\pm I_m\}$ and $\mathrm{Str}(V) = \mathrm{GL}(m, \mathbb{R})$.

The Jordan algebra associated with a symmetric cone

Let Ω be a symmetric cone in a Euclidean space V.
As above, we denote by $G(\Omega)$ the automorphism group of Ω, G its identity component and $K = G \cap O(V)$.
We write $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{k}$ for the Lie algebra of K.
We choose a point e in Ω whose stabilizer is K.
An element $X \in \mathfrak{g}$ belongs to $\mathfrak{k}$ if and only if $X \cdot e = 0$.
Since $G \cdot e = \Omega$, we have $\mathfrak{g} \cdot e = V$.
Therefore, the mapping from $\mathfrak{p}$ into V defined by $X \mapsto X \cdot e$ is a bijection.
We denote by L its inverse: for x in $V, L(x)$ is the unique element in $\mathfrak{p}$ such that $L(x)e = x$.

Theorem (Vinberg-Kœcher)

Let Ω be a symmetric cone in a Euclidean vector space V. Defining on V the product

$$xy = L(x)y,$$

V is a Euclidean Jordan algebra with identity element e and

$$\bar{\Omega} = \left\{x^2 \mid x \in V\right\}.$$

Proof (Folowing Satake proof).
It is clear that the product we have defined is bilinear. It is also commutative, since

$$xy - yx = [L(x), L(y)] \cdot e = 0$$

by $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $\mathfrak{k} \cdot e = 0$.
The inner product of V is an associative bilinear form, since each $L(x)$ belongs to $\mathfrak{p}$ and is therefore symmetric.

In order to prove (J2 : Jordan identity), we define the associator of three elements x, y, z in V by

$$[x, z, y] = x(zy) - (xz)y = [L(x), L(y)]z.$$

For any $x, y \in V$ we must show that $x^2(xy) = x(x^2y)$, i.e.
$\left[x^2, y, x\right] = 0$.
Using $[L(x), L(y)] \in [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we have

$$\begin{aligned}[][L(x), L(y)], L(z)]e &= [L(x), L(y)](ze) \\ &= [x, z, y] = L([x, z, y])e.\end{aligned}$$

Since $X \mapsto X \cdot e$ is bijective from $\mathfrak{p}$ onto V, it follows that

$$[[L(x), L(y)], L(z)] = L([x, z, y]).$$

Applying this identity to z, we immediately find

$$\left[x, z^2, y\right] = 2[x, z, y]z. \tag{1}$$

Now, for any x, y, z in V, the associativity of the scalar product gives

$$\begin{aligned} \left(\left[x^2, y, x\right] \mid z\right) &= \left(x^2(xy) \mid z\right) - \left(x\left(x^2 y\right) \mid z\right) \\ &= \left(x^2 \mid (xy)z - y(xz)\right) = \left(x^2 \mid [z, x, y]\right). \end{aligned} \tag{2}$$

By a similar computation we also have

$$\left(\left[x^2, y, x\right] \mid z\right) = \left(x \mid y\left(x^2 z\right) - \left(x^2 y\right) z\right) = \left(x \mid \left[y, x^2, z\right]\right)$$

and using (1) we see that this is further equal to

$$2(x \mid [y, x, z]x) = 2\left(x^2 \mid [y, x, z]\right) = -2\left(x^2 \mid [z, x, y]\right).$$

Comparing with (2) we see that

$$\left(\left[x^2, y, x\right] \mid z\right) = 0$$

This holds for every z in V, therefore $\left[x^2, y, x\right] = 0$, that is,

$$x^2(xy) = x\left(x^2 y\right).$$

Let Ω_1 be the symmetric cone associated with the Jordan algebra V (the interior of the set of squares). We have

$$\begin{aligned}\Omega_1 = \exp V &= \{\exp x \mid x \in V\} \\ &= \{\exp L(x) \cdot e \mid x \in V\} \\ &= \{\exp X \cdot e \mid X \in \mathfrak{p}\} \\ &\subset G \cdot e = \Omega\end{aligned}$$

Since Ω and Ω_1 are self-dual, then $\Omega_1 = \Omega$. □

If V is a Euclidean Jordan algebra, then the cone $\Omega = \exp V$ is called the associated symmetric cone.

Let V be a Euclidean Jordan algebra and Ω the associated symmetric cone.

Theorem

For x in Ω and u, v in V we set

$$\gamma_x(u, v) = \left(P(x)^{-1}u \mid v\right).$$

The family of bilinear forms γ_x defines a G-invariant Riemannian metric on Ω. The map $x \mapsto x^{-1}$ is an involutive isometry with unique fixed point e.

Proof. If x belongs to Ω then $P(x)$ is positive definite. The invariance of the metric:

$$\left(P(gx)^{-1}gu \mid gv\right) = \left(P(x)^{-1}u \mid v\right), \quad \forall g \in G(\Omega)$$

follows from the fact $P(gx) = gP(x)g^\top$, since $g \in G(\Omega) \subset \mathrm{Str}(V)$.

Let us now consider the map $x \mapsto x^{-1}$. We know that its differential is $-P(x)^{-1}$. To show that it is an isometry we have to prove that

$$\left(P\left(x^{-1}\right)^{-1} P(x)^{-1}u \mid P(x)^{-1}v\right) = \left(P(x)^{-1}u \mid v\right).$$

But this follows from the fact that $P(x^{-1}) = P(x)^{-1}$. $\square$

For any symmetric cone Ω we constructed above a G-invariant Riemannian metric G_x and an isometric involution $x \mapsto x^*$. Further, the G-invariant metric γ_x and the isometric involution $x \mapsto x^{-1}$, provide another, independent, proof that Ω is a symmetric Riemannian space.

The metrics G_x and γ_x are not exactly the same; a G-invariant metric on Ω is not unique unless V is simple.

In this case, we have

$$G_x = \frac{n}{r}\gamma_x,$$

and

$$x \mapsto x^* = \frac{n}{r}x^{-1}.$$

Theorem

(1) *We have*

$$\mathrm{Aut}(V)_0 = K,$$
$$\mathrm{Der}(V) = \mathfrak{k},$$

where $\mathrm{Aut}(V)_0$ *denotes the identity component of* $\mathrm{Aut}(V)$*.*

(2) *Every* g *in* G *can be uniquely written as*

$$g = P(x)k, \quad \text{with } x \in \Omega,\ k \in K$$

i.e.

$$G = P(\Omega)K \quad \text{"polar decomposition"}$$

(3) *Let* $\mathfrak{g}$ *be the Lie algebra of* G (*or* $G(\Omega)$) *and* $\mathfrak{p} = \{L(x) \mid x \in V\}$. *Then we have*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

(*Cartan decomposition of* $\mathfrak{g}$ *w.r.t.* $X \mapsto \theta(X) = -X^\top$).

Classification

Let V be a simple Euclidean Jordan algebra of dimension n and rank r. Recall that if $c \in V$ is idempotent ($c^2 = c$), then the only possible eigenvalues of $L(c)$ are $1, \frac{1}{2}$, and 0 , and V is the direct sum of the corresponding subspaces $V(c,1), V\left(c, \frac{1}{2}\right)$ and $V(c,0)$.

$$V = V\left(c_{,},1\right) \oplus V\left(c_{,},\frac{1}{2}\right) \oplus V\left(c_1,0\right)$$

(Peirce decomposition of V with respect to the idempotent c). This decomposition is orthogonal with respect to any associative symmetric bilinear form, since the transformations $L(x)$ are symmetric with respect to any such form.

Proposition

The subspaces $V(c,1)$ and $V(c,0)$ are Jordan subalgebras of V. They are orthogonal in the sense that

$$V(c,1)\cdot V(c,0)=\{0\}$$

Furthermore,

$$\big(V(c,1)+V(c,0)\big)\cdot V\left(c,\frac{1}{2}\right)\subset V\left(c,\frac{1}{2}\right),$$

$$V\left(c,\frac{1}{2}\right)\cdot V\left(c_1,\frac{1}{2}\right)\subset V(c,1)+V(c,0).$$

Let $\{c_1, \ldots, c_r\}$ be a *Jordan frame*,

$$c_i c_j = 0 \text{ if } i \neq j, \quad c_i^2 = c_i$$
$$c_1 + \cdots + c_r = e$$

Since the operators $L(c_i)$ commute, they admit a simultaneous diagonalization. We consider the following subspaces of V

$$V_{ii} = V(c_{i,}, 1) = \mathbb{R} c_i,$$
$$V_{ij} = V\left(c_{i,} \frac{1}{2}\right) \cap V\left(c_j, \frac{1}{2}\right).$$

Theorem

(1) *The space V decomposes in the following orthogonal direct sum:*

$$V = \bigoplus_{1\leqslant i\leqslant j\leqslant r} V_{ij} = \sum_{1\leqslant i\leqslant r}^{\oplus} \mathbb{R}c_i \oplus \sum_{1\leqslant i<j\leqslant r}^{\oplus} V_{ij}$$

(3) *Furthermore,*

$$V_{ij}\cdot V_{ij} \subset V_{ij} + V_{jj},$$
$$V_{ij}\cdot V_{jk} \subset V_{ik},\ \ \textit{if } i \neq k,$$
$$V_{ij} + V_{ki} = \{0\},\ \ \textit{if } \{i,j\}\cap\{k,\ell\} = 0.$$

(4) *For $i \mp j$, all the spaces V_{ij} have de same dimension, denoted by d.*

(5) *We have*

$$n = r + \frac{d}{2}r(r-1).$$

Classification of simple Euclidean Jordan algebras

V	$\dim V$	$\operatorname{rank} V$	d
$\mathrm{Sym}(m,\mathbb{R})$	$\frac{1}{2}m(m+1)$	m	1
$\mathrm{Herm}(m,\mathbb{C})$	m^2	m	2
$\mathrm{Herm}(m,\mathbb{H})$	$m(2m-1)$	m	4
$\mathbb{R}\times\mathbb{R}^{n-1}$	n	2	$n-2$
$\mathrm{Herm}(3,\mathbb{O})$	27	3	8

V	Ω	$\mathfrak{g}$	$\mathfrak{k}$
$\mathrm{Sym}(m,\mathbb{R})$	$\Omega_m(\mathbb{R})$	$\mathfrak{sl}(m,\mathbb{R})\oplus\mathbb{R}$	$\mathfrak{o}(m)$
$\mathrm{Herm}(m,\mathbb{C})$	$\Omega_m(\mathbb{C})$	$\mathfrak{sl}(m,\mathbb{C})\oplus\mathbb{R}$	$\mathfrak{su}(m)$
$\mathrm{Herm}(m,\mathbb{H})$	$\Omega_m(\mathbb{H})$	$\mathfrak{sl}(m,\mathbb{H})\oplus\mathbb{R}$	$\mathfrak{su}(m,\mathbb{H})4$
$\mathbb{R}\times\mathbb{R}^{n-1}$	Λ_n	$\mathfrak{o}(1,n-1)\oplus\mathbb{R}$	$o(n-1)$
$\mathrm{Herm}(3,\mathbb{O})$	$\Omega_3(\mathbb{O})$	$\mathfrak{e}_{6(-26)}\oplus\mathbb{R}$	$\mathfrak{f}_4$