A Laplace Operator for Poisson Manifolds

Seminar “*Geometry, Topology and Algebra*”

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Main reference

- Z. Saassai *“A Laplace operator for Poisson manifolds”*
  Differential Geometry and Its Applications
  Volume **68** (**2020**) 101576

Motivation

Riemannian Geometry

(Manifold) + (Riemannian metric)

$\Downarrow$

- Classical Levi-Civita connection
- Laplace operator

Riem. Geom. of Poisson Manifolds

(Poisson Manifold) + (Riem. metric)

$\Downarrow$

- Contravariant Levi-Civita connection
- ?

Outline

1. Poisson manifolds at a glance
2. Riemannian geometry of Poisson manifolds
3. Completing the picture
4. Two classical techniques from Riemannian geometry
5. Some classical results & their analogues

Poisson manifolds at a glance

What is a Poisson manifold?

Algebric definition

A ***Poisson manifold*** is a manifold M with a

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$

- $\mathbb{R}$-bilinear
- $\{f,g\} = -\{g,f\}$ (skew-symmetry)
- $\{f,gh\} = g\{f,h\} + h\{f,g\}$ (Leibniz)
- $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi).

Such a $\{\cdot,\cdot\}$ is called a ***Poisson bracket***.

What is a Poisson manifold?

Example

If (M, ω) is a symplectic manifold then

$$\{f, g\} := \omega(\mathcal{H}_f, \mathcal{H}_g) \quad \forall\, f, g \in \mathcal{C}^\infty(M)$$

where $\mathcal{H}_f \lrcorner\, \omega = -df$ is a Poisson bracket.

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For the Jacobi identity :

$$-d\omega(\mathcal{H}_f, \mathcal{H}_g, \mathcal{H}_h) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}.$$

Geometric definition

A ***Poisson manifold*** is a manifold M with a ***Poisson tensor*** π, i.e.

$$\pi \in \Gamma(\Lambda^2 TM) \qquad \text{s. t.} \qquad [\pi\,,\,\pi]_{SN} = 0.$$

Here $[\cdot,\cdot]_{SN}$ is the *Schouten-Nijenhuis bracket* given on bivector fields by

$$\begin{aligned}[X \wedge Y\,,\,U \wedge V]_{SN} = [X,U] \wedge Y \wedge V - [X,V] \wedge Y \wedge U \\ - [Y,U] \wedge X \wedge V + [Y,V] \wedge X \wedge U.\end{aligned} \tag{1}$$

where $[\cdot,\cdot]$ is the usual Lie bracket.

Example

Let $\mathfrak{g}$ be a finite dim. Lie algebra. For $a \in \mathfrak{g}^*$ define $\pi_a \in \Lambda^2 T_a \mathfrak{g}^* \simeq \Lambda^2 \mathfrak{g}^*$ by

$$\pi_a(u, v) := a\big([u, v]_{\mathfrak{g}}\big) \quad \forall\, u, v \in \mathfrak{g}.$$

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If (e_i) is a basis of $\mathfrak{g}$ with corresponding global linear coordinate system (x_i) on $\mathfrak{g}^*$ then

$$\pi_a = \sum_{i<j} \Big(\sum_k C_{ij}^k \, x_k(a) \Big) \, \partial x_i \wedge \partial x_j$$

where $[e_i, e_j]_{\mathfrak{g}} = \sum_k C_{ij}^k \, e_k$.

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Using (1), $[\pi,\pi]_{SN} = 0$ iff

$$\sum_m \big(C_{im}^l C_{jk}^m + C_{jm}^l C_{ki}^m + C_{km}^l C_{ij}^m\big) = 0 \quad \forall\, i,j,k,l.$$

Therefore π is a Poisson tensor on $\mathfrak{g}^*$, by the Jacobi identity of $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}}$.

One and the same

$$\pi(df, dg) = \{f, g\}$$

$$\frac{1}{2}[\pi, \pi]_{SN}(df, dg, dh) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$$

Elements attached to a Poisson manifold

- *Anchor* $\pi_\sharp : T^*M \longrightarrow TM\,, \quad a \mapsto \pi(a, \cdot\,).$

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- *Poisson differential* $d_\pi : \mathfrak{X}^\bullet(M) \longrightarrow \mathfrak{X}^{\bullet+1}(M),$

$$d_\pi P\,(\alpha_1, \dots, \alpha_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1}\, \pi_\sharp(\alpha_i) \cdot P\big(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_{p+1}\big)$$
$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j}\, P\big([\alpha_i, \alpha_j]_\pi, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \widehat{\alpha}_j, \dots, \alpha_{p+1}\big).$$

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- *Poisson codifferential* $\delta_\pi : \Omega^\bullet(M) \longrightarrow \Omega^{\bullet-1}(M),$

$$\delta_\pi := i_\pi \circ d - d \circ i_\pi\,.$$

Riemannian geometry of Poisson manifolds

Contravariant Levi-Civita connection

Analogy

Recall that given a Riemannian metric g on M,

$$\exists! \quad \nabla : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \longrightarrow \mathfrak{X}^1(M), \quad (X,Y) \mapsto \nabla_X Y$$

1. $\mathbb{R}$-bilinear
2. $\nabla_{fX} Y = f \nabla_X Y$ (tensoriality)
3. $\nabla_X (fY) = X(f)\, Y + f \nabla_X Y$ (Leibniz)
4. $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$ (torsionlessness)
5. $X \cdot \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ (compatibility with g)

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This is determined by

$$\langle \nabla_X Y \, , Z \, \rangle = \frac{1}{2} \{ X \cdot \langle Y, Z \rangle + Y \cdot \langle X, Z \rangle - Z \cdot \langle X, Y \rangle + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \} \, .$$

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$$+ \langle [\alpha, \beta]_\pi, \gamma \rangle - \langle [\beta, \gamma]_\pi, \alpha \rangle + \langle [\gamma, \alpha]_\pi, \beta \rangle \Big\}\,.$$

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$$\mathfrak{R}^{\nabla}(u \wedge v) \;:=\; \frac{1}{2} \sum_i \big(R(u,v)\,e_i\big) \wedge e_i\,.$$

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- *Weitzenböck curvature*

$$\mathfrak{W}^{\nabla}(\eta) \;:=\; -\sum_{i,j} \varepsilon_i \wedge [e_j \lrcorner R(e_i,e_j)\,\eta]\,, \quad \forall\, \eta \in \Omega^{\bullet}(M).$$

(e_i) beeing an orthonormal basis of $T_x M$ and (ε_i), the dual basis of $T_x^* M$.

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$$\mathcal{R}(\alpha,\beta)\gamma := \mathcal{D}_\alpha\mathcal{D}_\beta\gamma - \mathcal{D}_\beta\mathcal{D}_\alpha\gamma - \mathcal{D}_{[\alpha,\beta]_\pi}\gamma\,.$$

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(e_i) beeing an orthonormal basis of T_xM and (ε_i), the dual basis of T_x^*M.

Curvatures

- *Riemann curvature*

$$\mathcal{R}(\alpha,\beta)\gamma := \mathcal{D}_\alpha\mathcal{D}_\beta\gamma - \mathcal{D}_\beta\mathcal{D}_\alpha\gamma - \mathcal{D}_{[\alpha,\beta]_\pi}\gamma\,.$$

- *Ricci curvature*

$$\mathcal{R}ic\,(a) := \sum_i \mathcal{R}(a,\varepsilon_i)\varepsilon_i\,, \quad \forall\, a \in T_x^*M.$$

- *Curvature operator*

$$\mathfrak{R}^\nabla(u\wedge v) \;:=\; \frac{1}{2}\sum_i \big(R(u,v)\,e_i\big)\wedge e_i\,.$$

- *Weitzenböck curvature*

$$\mathfrak{W}^\nabla(\eta) \;:=\; -\sum_{i,j}\varepsilon_i\wedge[e_j \lrcorner R(e_i,e_j)\,\eta]\,, \quad \forall \eta\in\Omega^\bullet(M).$$

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$$\mathfrak{R}^{\mathcal{D}}(a \wedge b) \;:=\; \frac{1}{2} \sum_i \big(\mathcal{R}(a,b)\,\varepsilon_i\big) \wedge \varepsilon_i\,.$$

- *Weitzenböck curvature*

$$\mathfrak{W}^{\nabla}(\eta) \;:=\; -\sum_{i,j} \varepsilon_i \wedge [e_j \lrcorner\, R(e_i,e_j)\,\eta]\,, \quad \forall \eta \in \Omega^\bullet(M).$$

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$$\mathfrak{W}^{\mathcal{D}}(P) \;:=\; -\sum_{i,j} e_i \wedge [\varepsilon_j \lrcorner\, \mathcal{R}(\varepsilon_i,\varepsilon_j)P], \quad \forall\, P \in \mathfrak{X}^\bullet(M).$$

(e_i) beeing an orthonormal basis of $T_x M$ and (ε_i), the dual basis of $T_x^* M$.

A formula on the way

For any $\eta \in \Omega^p(M)$ and any $X_1, \ldots, X_p \in \mathfrak{X}^1(M)$,

$$\mathfrak{W}^{\nabla}(\eta)(X_1, \ldots, X_p) = \sum_{i=1}^{p} \eta\big(X_1, \ldots, X_{i-1}, \mathrm{Ric}(X_i), X_{i+1}, \ldots, X_p\big)$$
$$+ 2 \sum_{1 \leq i < j \leq p} (-1)^{i+j} \big(\mathfrak{R}^{\nabla}(X_i \wedge X_j) \lrcorner \, \eta\big)(X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_p)$$

A formula on the way

For any $P \in \mathfrak{X}^p(M)$ and any $\alpha_1, \ldots, \alpha_p \in \Omega^1(M)$,

$$\mathfrak{W}^{\mathcal{D}}(P)(\alpha_1, \ldots, \alpha_p) = \sum_{i=1}^{p} P\big(\alpha_1, \ldots, \alpha_{i-1}, \mathcal{R}ic\,(\alpha_i), \alpha_{i+1}, \ldots, \alpha_p\big)$$
$$+ 2 \sum_{1 \leq i < j \leq p} (-1)^{i+j} \big(\mathfrak{R}^{\mathcal{D}}(\alpha_i \wedge \alpha_j) \lrcorner P\big)(\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \widehat{\alpha}_j, \ldots, \alpha_p).$$

Completing the picture

Introducing $\Delta^{\pi,g}$

Riemann

Connection Laplacian

$\Delta^{\nabla} = \sum_i (\nabla_{\nabla_{E_i} E_i} - \nabla_{E_i} \circ \nabla_{E_i})$

(E_i) is a local orthonormal frame.

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$\Longrightarrow$

Poisson

(Contravariant) connection Laplacian

$\Delta^{\mathcal{D}} := \sum_i (\mathcal{D}_{\mathcal{D}_{\theta_i} \theta_i} - \mathcal{D}_{\theta_i} \circ \mathcal{D}_{\theta_i})$

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Hodge-de Rham Laplacian

$\Delta = d \circ \delta + \delta \circ d$

where $\delta := \sum_i i_{E_i} \circ \nabla_{E_i}$ is the co-differential.

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The operator $\Delta^{\pi,g}$

$$\Delta^{\pi,g} := d_\pi \circ \delta_\pi^g + \delta_\pi^g \circ d_\pi$$

where $\delta_\pi^g := \sharp \circ \delta_\pi \circ \flat$ and $\sharp, \flat$ are the musical isomorphisms.

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Beltrami Laplacian

$\Delta f = \Delta^{\nabla} f = -\mathrm{div}_g(\mathrm{grad}\, f)$

for any function f.

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The zero degree case

$\Delta^{\pi,g} f = \Delta^{\mathcal{D}} f$

for any function f ?

The compatibility condition $d(\pi \lrcorner \mu_g) = 0$

DOES $\Delta^{\pi,g} = \Delta^{\mathcal{D}}$ ON $\mathcal{C}^\infty(M)$?

Proposition

Assume (M, π, g) to be oriented, with Riemannian volume element μ_g. Then

$$\Delta^{\pi,g} = \Delta^{\mathcal{D}} - \pi_\sharp(\phi_g^\flat) \quad \text{on} \quad \mathcal{C}^\infty(M)$$

where ϕ_g is the unique vector field on M s. t. $\phi_g \lrcorner \mu_g = d(\pi \lrcorner \mu_g)$.

Consequently, $\Delta^{\pi,g} = \Delta^{\mathcal{D}}$ on $\mathcal{C}^\infty(M)$ iff $d(\pi \lrcorner \mu_g) = 0$. In which case,

$$\Delta^{\pi,g}(f) = \mathrm{div}_{\mathcal{D}}(\mathcal{H}_f^\flat) \quad \forall f \in \mathcal{C}^\infty(M).$$

The compatibility condition $d(\pi \lrcorner \mu_g) = 0$

Does $\Delta^{\pi,g} = \Delta^{\mathcal{D}}$ on $\mathcal{C}^\infty(M)$?

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Consequently, $\Delta^{\pi,g} = \Delta^{\mathcal{D}}$ *on* $\mathcal{C}^\infty(M)$ *iff* $d(\pi \lrcorner \mu_g) = 0$. *In which case,*

$$\Delta^{\pi,g}(f) = \operatorname{div}_{\mathcal{D}}(\mathcal{H}_f^\flat) \qquad \forall f \in \mathcal{C}^\infty(M).$$

Here, $\operatorname{div}_{\mathcal{D}} : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ is the ***contravariant divergence*** defined by

$$\operatorname{div}_{\mathcal{D}}(\eta)\big|_x := \sum_i e_i \lrcorner \mathcal{D}_{\varepsilon_i}\eta$$

where (e_i) is any basis of T_xM with dual basis (ε_i). And, $\mathcal{H}_f := \pi_\sharp(df)$ is the ***Hamiltonian vector field*** of f.

Divergences

If (M, π, g) is oriented, then for any $\eta \in \Omega^\bullet(M)$

$$\pi_\sharp\,(\mathrm{div}_{\mathcal{D}}\,\eta) = \mathrm{div}_g(\pi_\sharp\,\eta) - 2\,\pi_\sharp\,(\phi_g \lrcorner\,\eta).$$

In particular, for any 1-form α

$$\mathrm{div}_{\mathcal{D}}(\alpha) = \mathrm{div}_g(\pi_\sharp\,\alpha)$$

provided that $d(\pi \lrcorner \mu_g) = 0$.

The compatibility condition $d(\pi \lrcorner \mu_g) = 0$

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In particular, for any 1-form α

$$\mathrm{div}_\mathcal{D}(\alpha) = \mathrm{div}_g(\pi_\sharp\,\alpha)$$

provided that $d(\pi \lrcorner \mu_g) = 0$.

Theorem

If (M, π, g) *is closed and s. t.* $d(\pi \lrcorner \mu_g) = 0$ *then*

$$\int_M \delta_\pi(\alpha)\,\mu_g = \int_M \mathrm{div}_\mathcal{D}(\alpha)\,\mu_g = 0 \quad \forall \alpha \in \Omega^1(M).$$

Self-adjointness and non-negativity

Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denote the *global inner product* defined on $\mathfrak{X}^\bullet(M)$ by

$$\langle\!\langle\, P,\, Q \,\rangle\!\rangle := \int_M \langle P,\, Q \rangle\, \mu_g \quad \forall\, P,\, Q \in \mathfrak{X}^p(M).$$

Theorem

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner\, \mu_g) = 0$. *Then*

1. δ^g_π *is the formal adjoint of* d_π *:*

$$\langle\!\langle\, d_\pi P,\, Q \,\rangle\!\rangle = \langle\!\langle\, P,\, \delta^g_\pi\, Q \,\rangle\!\rangle \quad \forall\, P \in \mathfrak{X}^p(M),\, Q \in \mathfrak{X}^{p+1}(M).$$

2. $\Delta^{\pi,g}$ *is formally self-adjoint :*

$$\langle\!\langle\, \Delta^{\pi,g}(P),\, Q \,\rangle\!\rangle = \langle\!\langle\, P,\, \Delta^{\pi,g}(Q) \,\rangle\!\rangle \quad \forall\, P, Q \in \mathfrak{X}^p(M).$$

3. $\Delta^{\pi,g}$ *is non-negative :* $\quad \langle\!\langle\, \Delta^{\pi,g}(P),\, P \,\rangle\!\rangle \geq 0 \quad \forall\, P \in \mathfrak{X}^p(M).$

Two classical techniques from Riemannian geometry

Normal (co-)frames at a point

Lemma

Around any $x \in (M, g)$ *there exists a local orthonormal frame* (E_k) *s. t.* $(\nabla E_k)|_x = 0$ *for all* k.

Normal (co-)frames at a point

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Let M^{reg} denote the open dense set of (M, π, g) where the map

$$M \ni x \mapsto \operatorname{rank}(\pi_\sharp|_x : T_x^* M \to T_x M)$$

is locally constant.

Proposition

The following are equivalent.

1. *Around any* $x \in M^{reg}$ *there exists a local orthonormal co-frame* (θ_k) *s. t.* $(\mathcal{D}\,\theta_k)|_x = 0$ *for all* k.
2. $\mathcal{D}$ *is an* $\mathcal{F}^{reg}$*-connection :* $\mathcal{D}_a = 0$ *whenever* $\pi_\sharp(a) = 0$*, for all* $a \in T_x^* M$ *with* $x \in M^{reg}$.

1^{st} INGREDIENT

Lemma (E. Hopf, 1927)

Assume (M, g) *to be closed. If* f *is a function on* M *s. t.* $\Delta f \geq 0$ *then* f *is constant and* $\Delta f = 0$.

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Assume (M, g) *to be closed. If* f *is a function on* M *s. t.* $\Delta f \geq 0$ *then* f *is constant and* $\Delta f = 0$.

Lemma

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner \mu_g) = 0$. *If* f *is a function on* M *s. t.* $\Delta^{\pi,g}(f) \geq 0$ *then* f *is a Casimir function* (*i.e.* $\mathcal{H}_f = 0$) *and* $\Delta^{\pi,g}(f) = 0$.

2^{d} INGREDIENT

Theorem (R. Weitzenböck, 1923)

On (M, g) *the following formula holds good*

$$\Delta = \Delta^{\nabla} + \mathfrak{W}^{\nabla}.$$

2^d INGREDIENT

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Theorem

If (M, π, g) *is s. t.* $d(\pi \lrcorner \mu_g) = 0$ *then*

$$\Delta^{\pi,g} = \Delta^{\mathcal{D}} + \mathfrak{W}^{\mathcal{D}}.$$

The recipe

- Start with following general formula :

$$\Delta\left(-\frac{1}{2}|\omega|^2\right) = |\nabla\omega|^2 - \langle\Delta\omega,\omega\rangle + \langle\mathfrak{W}^{\nabla}\omega,\omega\rangle \qquad (*)$$

for ω a differential form.

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- Try to find some assumptions on ω and on the curvature so that

$$-\langle\Delta\omega,\omega\rangle + \langle\mathfrak{W}^\nabla\omega,\omega\rangle \geq 0\,.$$

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  for ω a differential form.

- Try to find some assumptions on ω and on the curvature so that

$$-\langle\Delta\omega,\omega\rangle + \langle\mathfrak{W}^\nabla\omega,\omega\rangle \geq 0\,.$$

- Once succeeded, the R.H.S. of $(*)$ vanishes, implying in particular that $\nabla\omega = 0$.

Some classical results & their analogues

1^{st} Bochner type theorem

Recall that a *Killing vector field* on (M, g) is a vector field $X \in \mathfrak{X}^1(M)$ verifying

$$\langle \nabla_Y X, Z \rangle = -\langle Y, \nabla_Z X \rangle \quad \forall\, Y, Z \in \mathfrak{X}^1(M).$$

Theorem (S. Bochner, 1946)

Assume (M, g) *to be closed. If* $\mathrm{Ric} \leq 0$ $($*i.e.* $\langle \mathrm{Ric}\, v, v \rangle \leq 0 \quad \forall v \in TM)$ *then every Killing vector field* X *is parallel, i.e.* $\nabla X = 0$*. Furthermore, if* $\mathrm{Ric} < 0$ *then there are no non-zero Killing vector field on* M*.*

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Theorem (S. Bochner, 1946)

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If f is a Casimir function on (M, π, g) then

$$\langle \mathcal{D}_\alpha df, \beta \rangle = -\langle \alpha, \mathcal{D}_\beta df \rangle \quad \forall\, \alpha, \beta \in \Omega^1(M).$$

Theorem

Assume (M, π, g) to be closed and s. t. $d(\pi \lrcorner \mu_g) = 0$. If $\mathcal{R}ic \leq 0$ then, for any Casimir function $f \in \mathcal{C}^\infty(M)$, $\mathcal{D}\, df = 0$. Furthermore, if $\mathcal{R}ic < 0$ then there are no non-constant Casimir functions on M.

2d Bochner type theorem

Theorem (S. Bochner, 1946)

Assume (M, g) *to be closed. If* $\mathrm{Ric} \geq 0$ *then a* 1*-form* α *on* M *is harmonic, i.e.* $\Delta\alpha = 0$*, iff* $\nabla\alpha = 0$*. Moreover, if* $\mathrm{Ric} > 0$ *then there are no non-zero harmonic* 1*-forms on* M*.*

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Theorem

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner\, \mu_g) = 0$. *If* $\mathcal{R}ic \geq 0$ *then a vector field* X *on* M *is harmonic, i.e.* $\Delta^{\pi,g} X = 0$, *iff* $\mathcal{D}X = 0$. *Moreover, if* $\mathcal{R}ic > 0$ *then there are no non-zero harmonic vector fields on* M.

Another form of it

Theorem

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner \mu_g) = 0$. *If* $\mathcal{R}ic \geq 0$ *then for any* 1*-form* α

$$\mathcal{D}\alpha = 0 \quad \textit{iff} \quad \begin{cases} \alpha^\sharp \text{ preserves } \pi, \text{ i.e. } \mathcal{L}_{\alpha^\sharp}\pi = 0\,; \text{ and} \\ \pi_\sharp(\alpha) \text{ preserves } \mu_g\,, \text{ i.e. } \mathcal{L}_{\pi_\sharp(\alpha)}\mu_g = 0\,. \end{cases}$$

Moreover, if $\mathcal{R}ic > 0$ *then every* $\mathcal{D}$*-parallel* 1*-form vanishes.*

Meyer-Gallot type theorem

Theorem (D. Meyer & S. Gallot, 1975)

Assume (M, g) *to be closed. If* $\mathfrak{R}^{\nabla} \geq 0$ (*i.e. if all the eigenvalues of* $\mathfrak{R}^{\nabla}$ *are* ≥ 0) *then a* p*-form* ω *on* M *is harmonic, i.e.* $\Delta\omega = 0$, *iff* $\nabla\omega = 0$. *Moreover, if* $\mathfrak{R}^{\nabla} > 0$ *then every harmonic* p*-form vanishes for* $p = 1, \ldots, \dim M - 1$.

Vanishing type theorems

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Theorem

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner \mu_g) = 0$*. If* $\mathfrak{R}^{\mathcal{D}} \geq 0$ *then a* p*-vector field* P *on* M *is harmonic, i.e.* $\Delta^{\pi,g} P = 0$*, iff* $\mathcal{D}P = 0$*. Moreover, if* $\mathfrak{R}^{\mathcal{D}} > 0$ *then every harmonic* p*-vector field vanishes for* $p = 1, \ldots, \dim M - 1$.

Case of the Poisson tensor π

Theorem

Assume (M, π, g) to be closed and s. t. $\mathfrak{R}^{\mathcal{D}} \geq 0$. The following are then equivalent.

1. *$\mathcal{D}$ is a Poisson connection, i.e. $\mathcal{D}\pi = 0$.*
2. *$d(\pi \lrcorner \mu_g) = 0$ and π is harmonic.*
3. *$\mathcal{D}$ is an $\mathcal{F}^{reg}$-connection and $d(\pi \lrcorner \mu_g) = d(\pi' \lrcorner \mu_g) = 0$ where $\pi' := \pi_\sharp(\pi^\flat)$.*

Furthermore, if any of these conditions holds, then $\mathfrak{R}^{\mathcal{D}}$ has (at least) a vanishing eigenvalue.

Lichnerowicz type theorem

Theorem (A. Lichnerowicz, 1952)

Assume (M, g) *to be closed. For any tensor field* T *on* M, *if* $\nabla^k T = 0$ *for some integer* $k \geq 2$ *then* $\nabla T = 0$.

Lichnerowicz type theorem

Theorem (A. Lichnerowicz, 1952)

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Theorem

Assume (M, π, g) to be closed and s. t. $d(\pi \lrcorner \mu_g) = 0$. For any tensor field T on M, if $\mathcal{D}^k T = 0$ for some $k \geq 2$ then $\mathcal{D} T = 0$.

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Theorem

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner \mu_g) = 0$*. For any tensor field* T *on* M*, if* $\mathcal{D}^k T = 0$ *for some* $k \geq 2$ *then* $\mathcal{D} T = 0$*.*

Corollary

Assume (M, π, g) *to be closed. Then* $\mathcal{D}\pi = 0$ *iff* $d(\pi \lrcorner \mu_g) = 0$ *and* $\mathcal{D}^k \pi = 0$ *for some* $k \geq 2$*.*

Lichnerowicz type estimate

Theorem (A. Lichnerowicz, 1958)

Assume (M, g) *to be closed. If* $\mathrm{Ric} \geq c\, g$ *for some* $c > 0$ (*i.e.* $\langle \mathrm{Ric}\, v, v\rangle \geq c\, \langle v, v\rangle$ *for all* $v \in TM$) *then*

$$\lambda \geq c \cdot (\dim M / \dim M - 1)$$

for any non-zero eigenvalue λ *of* Δ (*i.e. for any* $\lambda \in \mathbb{R}^*$ *s. t.* $\Delta f = \lambda f$ *for some non-zero function* f.

À la Lichnerowicz

Lichnerowicz type estimate

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for any non-zero eigenvalue λ *of* Δ *(i.e. for any* $\lambda \in \mathbb{R}^*$ *s. t.* $\Delta f = \lambda f$ *for some non-zero function* f*.*

Theorem

Assume (M, π, g) *to be closed and s. t.* $d(\pi \lrcorner\, \mu_g) = 0$*. If* $\mathcal{R}ic \geq c\, g$ *for* $c > 0$ *then*

$$\lambda \geq c \cdot (\dim M / \dim M - 1)$$

for any non-zero eigenvalue λ *of* $\Delta^{\pi, g}$ *(restricted to functions).*

Thank you for your attention

Any questions?