Analytic linear Lie Rack Structures on Leibniz Algebras

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⇝ Lie Rack :

A Lie rack is a pointed smooth manifold $(X, 1)$ together with a smooth map $\rhd : X \times X \longrightarrow X$, $(a, b) \mapsto a \rhd b$ such that,for any $a, b, c \in X$,

- the left translation $\mathrm{L}_a : X \longrightarrow X$, $b \mapsto a \rhd b$ are diffeomorphisms,
- $a \rhd (b \rhd c) = (a \rhd b) \rhd (a \rhd c)$,
- $1 \rhd a = a$ and $a \rhd 1 = 1$.

⇝ Example :

- Any Lie group G has a Lie rack structure given by $g \rhd h := g^{-1}hg$.

$$\{\textbf{Lie groups}\} \subset \{\textbf{Lie racks}\}$$

Rack Structures And Leibniz Algebras

$\rightsquigarrow$ Leibniz algebra :

A left Leibniz algebra [1] is an algebra $(\mathfrak{h}, [\ ,\ ])$ over a field $\mathbb{K}$ such that, for every element $u \in \mathfrak{h}$, $\mathrm{ad}_u : \mathfrak{h} \longrightarrow \mathfrak{h}$, $v \mapsto [u, v]$ is a derivation of $\mathfrak{h}$, i.e.,

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]] \tag{1}$$

$\rightsquigarrow$ Example :

- If the Leibniz bracket is skew, then $(\mathfrak{h}, [\ ,\ ])$ is a Lie algebra.

$$\{\textbf{Lie algebras}\} \subset \{\textbf{Leibniz algebras}\}$$

1. J. L. Loday, *Une version non-commutative des algebres de Lie*, L'Ens. Math **39** (1993) 269-293.

$\rightsquigarrow$ Tangent Functor[2] : Given a pointed Lie rack $(X, 1)$, for any $a \in X$, we denote by $\mathrm{Ad}_a : T_1X = \mathfrak{h} \longrightarrow \mathfrak{h}$ the differential of L_a at 1. We have

$$\mathrm{L}_{a \rhd b} = \mathrm{L}_a \circ \mathrm{L}_b \circ \mathrm{L}_a^{-1} \quad \text{and} \quad \mathrm{Ad}_{a \rhd b} = \mathrm{Ad}_a \circ \mathrm{Ad}_b \circ \mathrm{Ad}_a^{-1}.$$

Thus $\mathrm{Ad} : X \longrightarrow \mathrm{GL}(\mathfrak{h})$ is an homomorphism of Lie racks. If we put

$$[u, v]_{\rhd} = \frac{d}{dt}_{|t=0} \mathrm{Ad}_{c(t)} v, \quad u, v \in \mathfrak{h}, c :]-\epsilon, \epsilon[\longrightarrow X, \; c(0) = 1, c'(0) = u.$$

Theorem

Any Lie rack $(X, \rhd, 1)$, the tangent space $(\mathfrak{h}, [\ ,\ ]_{\rhd})$ is a left Leibniz algebra.

2. M. Kinyon, *Leibniz algebras, Lie racks, and digroups*, Journal of Lie Theory, volume 17 (2007) 99–114.

Analytic Linear Lie Rack Structures On Leibniz Algebras

- Analityc Linear Lie Rack Structures on finite dimensional vector space.

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  - ↳ Rigidity of Leibniz algebras.

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- A.L.L.R.S and Rigidity of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$.

Analytic linear Lie rack structures

Definitions

- A linear Lie rack structure on a finite dimensional vector space V is a Lie rack operation $(x, y) \mapsto x \triangleright y$ pointed at 0 and such that for any x, the map $\mathrm{L}_x : y \mapsto x \triangleright y$ is linear.

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- A linear Lie rack operation $\rhd$ is called analytic if for any $x, y \in V$,

$$x \rhd y = y + [x, y] + \sum_{n=2}^{\infty} A_{n,1}(x, \ldots, x, y),$$

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where : $A_{n,1} : V \times \dots \times V \longrightarrow V$ is an $(n+1)$-multilinear map which is symmetric in the n first arguments.

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$$x \rhd y = y + [x, y] + \sum_{n=2}^{\infty} A_{n,1}(x, \ldots, x, y),$$

where : $A_{1,1}$ is the left Leibniz bracket associated to $\rhd$.

Characterization of analytic linear Lie rack structures

"Main Theorem"

Let V be a real finite dimensional vector space and $(A_{n,1})_{n\geq 1}$ a sequence of $n+1$-multilinear maps symmetric in the n first arguments. We suppose that the operation $\rhd$ given by

$$x \rhd y = y + \sum_{n=1}^{\infty} A_{n,1}(x, \dots, x, y)$$

converges. Then $\rhd$ is a Lie rack structure on V if and only if for any $p, q \in \mathbb{N}^*$ and $x, y, z \in V$,

$$A_{p,\mathbf{1}}(x, A_{q,\mathbf{1}}(y, z)) = \sum_{s_{\mathbf{1}}+\dots+s_q+k=p} A_{q,\mathbf{1}}(A_{s_{\mathbf{1}},\mathbf{1}}(x, y), \dots, A_{s_q,\mathbf{1}}(x, y), A_{k,\mathbf{1}}(x, z)),$$

where for sake of simplicity $A_{p,1}(x, y) := A_{p,1}(x, \dots, x, y)$.

In particular, if $p = q = 1$ we get that $[\ ,\ ] := A_{1,1}$ is a left Leibniz bracket which is actually the left Leibniz bracket associated to $(V, \rhd)$.

Invariant maps

If $p = 1$ and $q \in \mathbb{N}^*$, the relation [3] becomes

$$\begin{aligned}\mathcal{L}_x A_{q,1}(y_1, \dots, y_{q+1}) &:= [x, A_{q,1}(y_1, \dots, y_{q+1})] \\ &\quad - \sum_{i=1}^{q+1} A_{q,1}(y_1, \dots, [x, y_i], \dots, y_{q+1}) \\ &= 0.\end{aligned}$$

A multilinear map A on a left Leibniz algebra satisfying $\mathcal{L}_x A = 0$ will be called invariant.

3.

$$A_{p,\mathbf{1}}(x, A_{q,\mathbf{1}}(y,z)) = \sum_{s_{\mathbf{1}}+\dots+s_q+k=p} A_{q,\mathbf{1}}(A_{s_{\mathbf{1}},\mathbf{1}}(x,y), \dots, A_{s_q,\mathbf{1}}(x,y), A_{k,\mathbf{1}}(x,z)).$$

Canonical A.L.L.R.S on Leibniz algebras

- If $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra then the operation $\overset{c}{\rhd}: \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ given by

$$u \overset{c}{\rhd} v = \exp(\mathrm{ad}_u)(v)$$

defines an (canonical) analytic linear Lie rack structure on $\mathfrak{h}$ such that the associated left Leibniz bracket on $T_0\mathfrak{h} = \mathfrak{h}$ is the initial bracket $[\ ,\ ]$. Where $A^0_{0,1}(x, y) = y$ and

$$A^0_{n,1}(x_1, \ldots, x_n, y) := \frac{1}{(n!)^2} \sum_{\sigma \in S_n} \mathrm{ad}_{x_{\sigma(1)}} \circ \ldots \circ \mathrm{ad}_{x_{\sigma(n)}}(y).$$

- Corollary

The $\left(A^0_{n,1}\right)_{n \in \mathbb{N}}$ *satisfy the sequence of equations*[4].

4.

$$A_{p,\mathbf{1}}(x, A_{q,\mathbf{1}}(y, z)) = \sum_{s_{\mathbf{1}}+\ldots+s_q+k=p} A_{q,\mathbf{1}}(A_{s_{\mathbf{1}},\mathbf{1}}(x, y), \ldots, A_{s_q,\mathbf{1}}(x, y), A_{k,\mathbf{1}}(x, z)).$$

Main Proposition

Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra, $F : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function and $P : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathbb{R}$ a symmetric multilinear p-form such that, for any $y, x_1 \ldots, x_p \in \mathfrak{h}$,

$$\sum_{i=1}^{p} P(x_1, \ldots, [y, x_i], \ldots, x_p) = 0.$$

Then the operation $\rhd$ given by

$$x \rhd y = \exp(F(P(x, \ldots, x))\mathrm{ad}_x)(y) \tag{2}$$

is a linear Lie rack structure on $\mathfrak{h}$ and its associated left Leibniz bracket is $[\ ,\ ]_\rhd = F(0)[\ ,\ ]$. Moreover, if F is analytic then $\rhd$ is analytic .

Main Proposition

Let $(\mathfrak{h},[\ ,\ ])$ be a left Leibniz algebra, $F:\mathbb{R}\longrightarrow\mathbb{R}$ a smooth function and $P:\mathfrak{h}\times\ldots\times\mathfrak{h}\longrightarrow\mathbb{R}$ a symmetric multilinear p-form such that, for any $y,x_1\ldots,x_p\in\mathfrak{h}$,

$$\sum_{i=1}^{p}P(x_1,\ldots,[y,x_i],\ldots,x_p)=0.$$

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$\rightsquigarrow$ If F is the identity map, the $\rhd$ is the canonical A.L.L.R.S on $\mathfrak{h}$.

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$$\sum_{i=1}^{p} P(x_1, \ldots, [y, x_i], \ldots, x_p) = 0.$$

Then the operation $\triangleright$ given by

$$x \triangleright y = \exp(F(P(x, \ldots, x))\mathrm{ad}_x)(y) \tag{2}$$

is a linear Lie rack structure on $\mathfrak{h}$ and its associated left Leibniz bracket is $[\ ,\ ]_\triangleright = F(0)[\ ,\ ]$. Moreover, if F is analytic then $\triangleright$ is analytic .

$\rightsquigarrow$ If one takes $F(0) = 0$, the two pointed Lie rack structures

$$x \triangleright_0 y = y \quad \text{and} \quad x \triangleright_1 y = \exp(F(P(x, \ldots, x)))\mathrm{ad}_x)(y)$$

are two pointed Lie rack structures on abelian Leibniz algebra which are not equivalent (even locally near 0).

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Rigid Leibniz algebra :

A left Leibniz algebra $(\mathfrak{h},[\ ,\ ])$ is called rigid if any analytic linear Lie rack structure $\rhd$ on $\mathfrak{h}$ such that $[\ ,\ ]_\rhd=[\ ,\ ]$ is given by

$$x\rhd y=\exp(F(P(x,\ldots,x)))\mathrm{ad}_x)(y),$$

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is a linear Lie rack structure on $\mathfrak{h}$ and its associated left Leibniz bracket is $[\ ,\ ]_\rhd = F(0)[\ ,\ ]$. Moreover, if F is analytic then $\rhd$ is analytic .

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where $P : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathbb{R}$ invariant symmetric multilinear p-form on $\mathfrak{h}$.

Characterization of A.L.L.R.S and Cohomological interpretation

$$A_{p,1}(x, A_{q,1}(y,z)) = \sum_{s_1+\ldots+s_q+k=p} A_{q,1}(A_{s_1,1}(x,y),\ldots,A_{s_q,1}(x,y),A_{k,1}(x,z)),$$

where for sake of simplicity $A_{p,1}(x,y) := A_{p,1}(x,\ldots,x,y)$.
For $q = 1$, The above equation can be written for any $x, y, z \in \mathfrak{h}$,

$$\delta(i_x \ldots i_x A_{p,1})(y,z) = -\sum_{r=1}^{p-1}[A_{r,1}(x,y), A_{p-r,1}(x,z)],$$

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where

$\rightsquigarrow$ $\delta : Hom(\otimes^n \mathfrak{h}, \mathfrak{h}) \longrightarrow Hom(\otimes^{n+1}\mathfrak{h}, \mathfrak{h})$ given by

$$\begin{aligned}\delta(\omega)(x_0,\ldots,x_n) &= \sum_{i=0}^{n-1}[x_i, \omega(x_0,\ldots,\hat{x}_i,\ldots,x_n)] + (-1)^{n-1}[\omega(x_0,\ldots,x_{n-1}), x_n] \\ &+ \sum_{i<j}(-1)^{i+1}\omega(x_0,\ldots,\hat{x}_i,\ldots,x_{j-1},[x_i,x_j],x_{j+1},\ldots,x_n),\end{aligned}$$

and then defines a cohomology $H^p(\mathfrak{h})$ for $p \in \mathbb{N}$.

Characterization of A.L.L.R.S and Cohomological interpretation

$$A_{p,1}(x, A_{q,1}(y,z)) = \sum_{s_1+\ldots+s_q+k=p} A_{q,1}(A_{s_1,1}(x,y), \ldots, A_{s_q,1}(x,y), A_{k,1}(x,z)),$$

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  and then defines a cohomology $H^p(\mathfrak{h})$ for $p \in \mathbb{N}$.
- $\rightsquigarrow$ $i_x \ldots i_x A_{p,1} : \mathfrak{h} \longrightarrow \mathfrak{h},\ y \mapsto A_{p,1}(x,\ldots,x,y)$.

A.L.L.R.S on Leibniz algebras with $H^0 = H^1 = 0$

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$\rightsquigarrow$ The sequence $(A^0_{n,1})_{n\in\mathbb{N}}$ defining the canonical linear Lie rack structure of $\mathfrak{h}$ satisfies

$$\delta(i_x \dots i_x A^0_{p,1})(y,z) = -\sum_{r=1}^{p-1}[A^0_{r,1}(x,y), A^0_{p-r,1}(x,z)],$$

A.L.L.R.S on Leibniz algebras with $H^0 = H^1 = 0$

⇝ **Main Theorem** :
Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra such that $H^0(\mathfrak{h}) = H^1(\mathfrak{h}) = 0$. Let $(A_{n,1})_{n\geq 0}$ be a sequence where $A_{0,1}(x,y) = y$ and $A_{1,1}(x,y) = [x,y]$ and, for any $n \geq 2$, $A_{n,1} : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathfrak{h}$ is multilinear invariant and symmetric in the n first arguments. We suppose that the $A_{n,1}$ satisfy [5]. Then there exists a unique sequence $(B_n)_{n\geq 2}$ of invariant symmetric multilinear maps $B_n : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathfrak{h}$ such that, for any $x, y \in \mathfrak{h}$,

$$A_{n,\mathbf{1}}(x,y) = A^{\mathbf{0}}_{n,\mathbf{1}}(x,y) + \sum_{\substack{1 \leq k \leq \left[\frac{n}{2}\right] \\ s = l_1 + \ldots + l_k \leq n}} A^{\mathbf{0}}_{k,\mathbf{1}}(B_{l_1}(x), \ldots, B_{l_k}(x), A^{\mathbf{0}}_{n-s,\mathbf{1}}(x,y)), \qquad (3)$$

where $A_{p,1}(x,y) = A_{p,1}(x,\ldots,x,y)$ and $B_l(x) = B_l(x,\ldots,x)$.

5.

$$A_{p,\mathbf{1}}(x, A_{q,\mathbf{1}}(y,z)) = \sum_{s_1+\ldots+s_q+k=p} A_{q,\mathbf{1}}(A_{s_1,\mathbf{1}}(x,y), \ldots, A_{s_q,\mathbf{1}}(x,y), A_{k,\mathbf{1}}(x,z)).$$

Let $\mathfrak{h}$ be either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}(3)$ and $\rhd$ an analytic linear Lie rack structure on $\mathfrak{h}$ such that $[\ ,\ ]_\rhd$ is the Lie algebra bracket of $\mathfrak{h}$. Then there exists an analytic function $F : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$F(u) = 1 + \sum_{k=1}^{\infty} a_k u^k$$

such that, for any $x, y \in \mathfrak{h}$,

$$x \rhd y = \exp(F(\langle x, x\rangle)\mathrm{ad}_x)(y),$$

where $\langle x, x\rangle = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_x)$. So $\mathfrak{h}$ is rigid.

- Consider $\mathfrak{h}$ be either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}(3)$. Since $H^0(\mathfrak{h}) = H^1(\mathfrak{h}) = 0$, we will use characterization [6] of A.L.L.R.S with $H^0 = H^1 = 0$. Therefore, we will need to determine the space of Invariant multilinear symmetric forms on $\mathfrak{h}$.
- Explicit A.L.L.R.S on $\mathfrak{h}$.
- Proof that there exists a unique sequence $(a_n)_{n\geq 1}$ such that the function $F(t) = 1 + \sum_{t=1}^{\infty} a_n t^n$ converge and

$$x \rhd y = \exp(F(\langle x, x\rangle)\mathrm{ad}_x)(y)$$
$$= y + \sum_{n=0}^{\infty} F(\langle x, x\rangle)^{2n+1} A^0_{2n+1,1}(x, y) + \sum_{n=1}^{\infty} F(\langle x, x\rangle)^{2n} A^0_{2n,1}(x, y).$$

6.
$$A_{n,\mathbf{1}}(x, y) = A^{\mathbf{0}}_{n,\mathbf{1}}(x, y) + \sum_{\substack{\mathbf{1} \leq k \leq \left[\frac{n}{\mathbf{2}}\right] \\ s = l_{\mathbf{1}} + \ldots + l_k \leq n}} A^{\mathbf{0}}_{k,\mathbf{1}}(B_{l_{\mathbf{1}}}(x), \ldots, B_{l_k}(x), A^{\mathbf{0}}_{n-s,\mathbf{1}}(x, y)), \qquad \mathbf{(4)}$$

A.L.L.R.S and Rigidity of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(3)$.

- Since $H^0(\mathfrak{g})) = H^1(\mathfrak{g}) = 0$, then the $A_{n,1}$ which define the A.L.L.R.S on $\mathfrak{g}$ can be written as follows :

$$A_{n,1}(x,y) = A^0_{n,1}(x,y) + \sum_{\substack{1 \leq k \leq \left[\frac{n}{2}\right] \\ s = l_1 + \ldots + l_k \leq n}} A^0_{k,1}(B_{l_1}(x), \ldots, B_{l_k}(x), A^0_{n-s,1}(x,y)).$$

- For any $n \in \mathbb{N}^*$, we define $P : \mathfrak{g}^{2n} \longrightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) by

$$P_n(x_1, \ldots, x_{2n}) = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \langle x_{\sigma(1)}, x_{\sigma(2)} \rangle \ldots \langle x_{\sigma(2n-1)}, x_{\sigma(2n)} \rangle \quad \text{and} \quad P_0 = 1,$$

where $\langle x, x \rangle = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x^2)$. This defines a symmetric invariant form on $\mathfrak{g}$ and the map $B_n^{\mathfrak{g}} : \mathfrak{g}^{2n+1} \longrightarrow \mathfrak{g}$ given by

$$B_n^{\mathfrak{g}}(x_1, \ldots, x_{2n+1}) = \sum_{k=1}^{2n+1} P_n(x_1, \ldots, \hat{x}_k, \ldots, x_{2n+1}) x_k$$

is symmetric and invariant. We denote by $S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ the vector space of $\mathfrak{g}$-invariant n-multilinear symmetric forms on $\mathfrak{g}$ with values in $\mathfrak{g}$.

Theorem :

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then [7], for any $n \in \mathbb{N}^*$, we have

$$S^{\mathfrak{g}}_{2n}(\mathfrak{g}, \mathfrak{g}) = 0 \quad \text{and} \quad S^{\mathfrak{g}}_{2n+1}(\mathfrak{g}, \mathfrak{g}) = \mathbb{C}B^{\mathfrak{g}}_n.$$

Corollary :

If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(3)$ then, for any $n \in \mathbb{N}^*$, we have

$$S^{\mathfrak{g}}_{2n}(\mathfrak{g}, \mathfrak{g}) = 0 \quad \text{and} \quad S^{\mathfrak{g}}_{2n+1}(\mathfrak{g}, \mathfrak{g}) = \mathbb{R}B^{\mathfrak{g}}_n.$$

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A.L.L.R.S and Rigidity of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$

Main Proposition :
Let $\mathfrak{h}$ be either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}(3)$ and $\rhd$ an analytic linear Lie rack product on $\mathfrak{h}$ such that $[\ ,\ ]_\rhd$ is the Lie algebra bracket of $\mathfrak{h}$. Then there exists a sequence $(U_n)_{n\in\mathbb{N}^*}$ with $U_1 = 1$, $U_2 = \frac{1}{2}$, for any $x, y \in \mathfrak{h}$,

$$x \rhd y = y + \left(\sum_{n=0}^{\infty} U_{2n+1}\langle x, x\rangle^n\right)[x, y] + \left(\sum_{n=1}^{\infty} U_{2n}\langle x, x\rangle^{n-1}\right)\mathrm{ad}_x^2(y)$$

and for any $n \in \mathbb{N}^*$,

$$U_{2n} = \frac{1}{2}\left[\sum_{r=0}^{n-1} U_{2r+1}U_{2(n-r)-1} - \sum_{r=1}^{n-1} U_{2r}U_{2(n-r)}\right]. \qquad (5)$$

The idea of the proof is showing that there exists a unique sequence $(a_n)_{n\geq 1}$ such that the function $F(t) = 1 + \sum_{t=1}^{\infty} a_n t^n$ converge and

$$x \triangleright y = \exp(F(\langle x, x\rangle)\mathrm{ad}_x)(y)$$
$$= y + \sum_{n=0}^{\infty} F(\langle x, x\rangle)^{2n+1} A^0_{2n+1,1}(x, y) + \sum_{n=1}^{\infty} F(\langle x, x\rangle)^{2n} A^0_{2n,1}(x, y).$$

Using the following formulas

$$\begin{cases} A^0_{2n}(x, y) = \frac{\langle x, x\rangle^{n-1}}{(2n)!}\mathrm{ad}_x^2(y) = \frac{\langle x, x\rangle^{n}}{(2n)!}y - \frac{\langle x, x\rangle^{n-1}\langle x, y\rangle}{(2n)!}x, \ \ n \geq 1, \\ A^0_{2n+1}(x, y) = \frac{\langle x,x\rangle^{n}}{(2n+1)!}[x, y], \quad n \geq 0. \end{cases} \tag{6}$$

one can see that

$$\begin{aligned} \exp(F(\langle x, x\rangle)\mathrm{ad}_x)(y) &= y + \left(\sum_{n=0}^{\infty} \frac{[F(\langle x, x\rangle)]^{2n+1}\langle x, x\rangle^{n}}{(2n+1)!}\right)[x, y] \\ &\quad + \left(\sum_{n=1}^{\infty} \frac{[F(\langle x, x\rangle)]^{2n}\langle x, x\rangle^{n-1}}{(2n)!}\right)\mathrm{ad}_x^2(y). \end{aligned}$$

Put $[F(\langle x, x\rangle)]^n = \sum_{m=0}^{\infty} B_{n,m}\langle x, x\rangle^m$ and compute the coefficients $B_{n,m}$.

$$\begin{aligned} [F(\langle x, x\rangle)]^n &= \left(1 + a_1\langle x, x\rangle + a_2\langle x, x\rangle^2 + \ldots + a_m\langle x, x\rangle^m + R\right)^n \\ &= \left(1 + a_1\langle x, x\rangle + a_2\langle x, x\rangle^2 + \ldots + a_m\langle x, x\rangle^m\right)^n + P, \end{aligned}$$

where P contains terms of degree $\geq m+1$.

The multinomial theorem gives

$$\left(1 + a_1 \langle x, x\rangle + a_2 \langle x, x\rangle^2 + \ldots + a_m \langle x, x\rangle^m\right)^n = \sum_{k_0+\ldots+k_m=n} \frac{n!}{k_0!k_1!\ldots k_m!} a_1^{k_1} \ldots a_m^{k_m} \langle x, x\rangle^{k_1+2k_2+\ldots+mk_m}.$$

Thus

$$B_{n,m} = \sum_{k_1+2k_2+\ldots+mk_m=m, k_0+k_1+\ldots+k_m=n} \frac{n!}{k_0!k_1!\ldots k_m!} a_1^{k_1} \ldots a_m^{k_m},$$

for $m \geq 1$ and $B_{n,0} = 1$.

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{F(\langle x,x\rangle)^{2n+1}\langle x,x\rangle^{n}}{(2n+1)!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{B_{2n+1,m}\langle x,x\rangle^{m+n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{p=0}^{n} \frac{B_{2p+1,n-p}}{(2p+1)!}\right)\langle x,x\rangle^{n}, \\
\sum_{n=1}^{\infty} \frac{F(\langle x,x\rangle)^{2n}\langle x,x\rangle^{n-1}}{(2n)!} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{B_{2n,m}\langle x,x\rangle^{m+n-1}}{(2n)!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{p=1}^{n} \frac{B_{2p,n-p}}{(2p)!}\right)\langle x,x\rangle^{n-1}.
\end{aligned}
$$

For sake of simplicity and clarity, put

$$\begin{aligned} V_{n,m}(a_1,\ldots,a_m) &= \frac{B_{n,m}}{n!} \\ &= \sum_{k_1+2k_2+\ldots+mk_m=m, k_0+k_1+\ldots+k_m=n} \frac{a_1^{k_1}\ldots a_m^{k_m}}{k_0!k_1!\ldots k_m!}. \end{aligned}$$

To prove the theorem we need to show that there exists a unique sequences $(a_n)_{n\geq 1}$ such that

$$U_{2n+1} = \sum_{p=0}^{n} V_{2p+1,n-p}(a_1,\ldots,a_{n-p}), \quad n \geq 1, \tag{7}$$

$$U_{2n} = \sum_{p=1}^{n} V_{2p,n-p}(a_1,\ldots,a_{n-p}), \quad n \geq 1. \tag{8}$$

Note first that the relation(5) and the fact that $U_2 = \frac{1}{2}$ defines the sequence $(U_{2n})_{n\geq 1}$ entirely in function of the sequence $(U_{2n+1})_{n\geq 0}$. On the other hand, since $V_{1,n}(a_1, \dots, a_n) = a_n$ and $U_1 = 1$ then

$$U_3 = a_1 + \frac{1}{3!} \quad \text{and} \quad U_{2n+1} = a_n + \sum_{p=1}^{n} V_{2p+1,n-p}(a_1, \dots, a_{n-p}), \quad n \geq 2.$$

Since the quantity $\sum_{p=1}^{n} V_{2p+1,n-p}(a_1, \dots, a_{n-p})$ depends only on $(a_1, \dots, a_{n-1})$, these relations define inductively and uniquely the sequence $(a_n)_{n\geq 1}$ in function of $(U_{2n+1})_{n\geq 0}$. To achieve the proof we need to prove (8). We will proceed by induction and we will use the following relation

$$\frac{\partial V_{n,m}}{\partial a_l}(a_1, \dots, a_m) = V_{n-1,m-l}(a_1, \dots, a_{m-l}), \quad l = 1, \dots, m. \tag{9}$$

Indeed,

$$\frac{\partial V_{n,m}}{\partial a_l}(a_1,\ldots,a_m) = \sum_{\substack{k_1+2k_2+\ldots+mk_m=m,\\ k_0+k_1+\ldots+k_m=n, k_l\geq 1}} \frac{a_1^{k_1}\ldots a_l^{k_l-1}\ldots a_m^{k_m}}{k_0!k_1!\ldots(k_l-1)!\ldots k_m!}$$

$$\overset{k_l'=k_l-1}{=} \sum_{\substack{k_1+2k_2+\ldots+lk_l'+\ldots+mk_m=m-l,\\ k_0+k_1+\ldots+k_l'+\ldots+k_m=n-1}} \frac{a_1^{k_1}\ldots a_l^{k_l'}\ldots a_m^{k_m}}{k_0!k_1!\ldots(k_l')!\ldots k_m!}.$$

To conclude, one needs to remark that in the relation

$$k_1+2k_2+\ldots+lk_l'+\ldots+mk_m = m-l$$

the left side is a sum of nonnegative number and the right side is nonnegative so $(m-l+1)k_{m-l+1} = \ldots = mk_m = 0$ and hence the relation is equivalent to

$$k_1+2k_2+\ldots+(m-l)k_{m-l} = m-l.$$

We can now prove (8). We proceed by induction. For $n = 1$, we have $U_2 = \frac{1}{2}$ and $V_{2,0} = \frac{1}{2}$. Suppose that the relation holds from 1 to $n-1$. By virtue of (5), we have

$$U_{2n} = \frac{1}{2}\left[\sum_{r=0}^{n-1} U_{2r+1}U_{2(n-r)-1} - \sum_{r=1}^{n-1} U_{2r}U_{2(n-r)}\right]$$

and all the U_r appearing in this formula are given by (7) and (8) this implies that $U_{2n} = H(a_1, \ldots, a_{n-1})$. On the other hand, we have

$$\sum_{p=1}^{n} V_{2p,n-p}(a_1, \ldots, a_{n-p}) = G(a_1, \ldots, a_{n-1}).$$

To show that U_{2n} satisfies (8) is equivalent to showing

$$H(0) = G(0) \quad \text{and} \quad \frac{\partial H}{\partial a_l} = \frac{\partial G}{\partial a_l}, \qquad l = 1, \ldots n-1.$$

But $V_{n,m}(0) = 0$ if $m \geq 1$ and $V_{n,0}(0) = \frac{1}{n!}$.

$$\begin{aligned}
H(0) &= \frac{1}{2}\left(\sum_{r=0}^{n-1} \frac{1}{(2r+1)!(2(n-r)-1)!} - \sum_{r=1}^{n-1} \frac{1}{(2r)!(2(n-r))!}\right) \\
&= \frac{1}{2}\left(\sum_{r=0}^{n-1} \frac{1}{(2r+1)!(2(n-r)-1)!} - \sum_{r=0}^{n} \frac{1}{(2r)!(2(n-r))!}\right) \\
&\quad + \frac{1}{(2n)!} \\
&= -\frac{1}{2}(1-1)^{2n} + \frac{1}{(2n)!} = \frac{1}{(2n)!},
\end{aligned}$$

$G(0) = V_{2n,0}(0) = \frac{1}{(2n)!} = H(0)$. For $r = 0, \dots, n-1$, by induction hypothesis U_{2r+1} is given by (7) and by using (9) on can see easily that $\dfrac{\partial U_{2r+1}}{\partial a_l} = U_{2(r-l)}$ if $l = 1, \dots, r$ and 0 if $l \geq r+1$. Similarly, we have $\dfrac{\partial U_{2r}}{\partial a_l} = U_{2(r-l)-1}$ if $l = 1, \dots, r-1$ and 0 if $l \geq r$. For sake of simplicity, we put

$$\frac{\partial U_{2r+1}}{\partial a_l} = U_{2(r-l)} \quad \text{and} \quad \frac{\partial U_{2r}}{\partial a_l} = U_{2(r-l)-1}$$

with the convention $U_0 = 1$ and $U_s = 0$ if s is negative. Then, for $l = 1, \dots, n-1$, we have

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Proof of the theorem

$$
\begin{aligned}
\frac{\partial H}{\partial a_l} &= \frac{1}{2}\left[\sum_{r=0}^{n-1}\left(\frac{\partial U_{2r+1}}{\partial a_l}U_{2(n-r)-1} + \frac{\partial U_{2(n-r)-1}}{\partial a_l}U_{2r+1}\right) - \sum_{r=1}^{n-1}\left(\frac{\partial U_{2r}}{\partial a_l}U_{2(n-r)} + \frac{\partial U_{2(n-r)}}{\partial a_l}U_{2r}\right)\right] \\
&= \frac{1}{2}\left[\sum_{r=0}^{n-1}\left(U_{2(r-l)}U_{2(n-r)-1} + U_{2(n-r-l-1)}U_{2r+1}\right) - \sum_{r=1}^{n-1}\left(U_{2(r-l)-1}U_{2(n-r)} + U_{2(n-r-l)-1}U_{2r}\right)\right] \\
&= \frac{1}{2}\sum_{r=0}^{n-1-l}U_{2r}U_{2(n-r-l)-1} + \frac{1}{2}\sum_{r=0}^{n-1}U_{2(n-r-l-1)}U_{2r+1} \\
&\quad -\frac{1}{2}\sum_{r=0}^{n-l-2}U_{2r+1}U_{2(n-r-l-1)} - \frac{1}{2}\sum_{r=1}^{n-1}U_{2(n-r-l)-1}U_{2r} \\
&= \frac{1}{2}U_{2(n-l)-1} + \frac{1}{2}\sum_{r=n-l-1}^{n-1}U_{2(n-r-l-1)}U_{2r+1} - \frac{1}{2}\sum_{r=n-l}^{n-1}U_{2(n-r-l)-1}U_{2r} \\
&= U_{2(n-l)-1}.
\end{aligned}
$$

This completes the proof.