An Introduction to the Geometry of Homogeneous Manifolds

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What is a Geometry ?

- First direction The theory of Riemannian manifolds, developed by B. Riemann, and is a generalization of Gauss' theory of surfaces. . .
- Second direction Cartan's theory of connections on a fiber bundle (espaces généralisés), which was used as an appropriate mathematical framework in recent physical theories (Yang-Mills theory, quantum gravity). . .
- Third direction F. Klein's Erlangen Program: The study of invariant geometric objets on a homogeneous space : Riemannian metric, affine connection, symplectic structure . . .

$$\mathrm{Diff}(M) \supset \mathrm{Aut}(M, \mathcal{T}) \supset G$$

The main questions

- What are the invariant tensors on a homogeneous manifold?
- What is the tangent bundle of a homogeneous manifold?
- Can we give an algebraic description of the invariant connections on a homogeneous manifold? The same question remains true on flat connections?

Thoughout answers to these questions, we will emphasize some basic tools and ideas that everyone should know in dealing with homogeneous apaces.

Outline

Homogeneous spaces

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- The quotient map:

$$p : G \to G/H, \quad p(b) = \bar{b} = bH$$

- The canonical (the homogeneous) action of G on G/H:

$$G \times G/H \to G/H, \quad (g, \bar{b}) \mapsto g \cdot \bar{b} = \overline{gb}.$$

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Remark. The isotopy subgroup in $\bar{b}$ is $G_{\bar{b}} = bHb^{-1}$.

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1. *G/H has a unique smooth manifold structure such that the quotient map p is a submersion. The left action of G on G/H turns G/H into a homogeneous G-space.*
2. *If M be a homogeneous G-space, and $o \in M$ be any point of M. Then the isotropy subgroup G_o is a closed subgroup of G, and the map $F : G/G_o \to M$ defined by $F(gG_o) = g \cdot o$ is a G-equivariant diffeomorphism.*

Proposition

Let X be a set and $\rho : G \to \mathcal{B}(X)$ a group morphism from a Lie group G to the group of bijections of X such that he action is transitive and the istropy subgroup in some point $o \in X$ is a closed in G.

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Proof. Let H be the isotropy subgroup in o, that is the set of element of G which map o to itself. Since $H \subset G$ is closed the left coset space G/H becomes a smooth G-homogeneous space. The map

$$\varphi : G/H \to X, \quad gH \mapsto \rho(g)(o)$$

is a well defined bijection G-equivariant.

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Examples

For all $k, n \in \mathbb{N}$, $1 \leq k \leq n - 1$, let
$G(n, k)$: the Grassmannian manifold, which is the set of k-dimensional real subspaces of dimension of $\mathbb{R}^n$.

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These are homogeneous $O(n)$-spaces

$$V(n,k) \stackrel{\cong}{\to} O(n)/O(n-k),$$

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$$V(n,k) \overset{\cong}{\to} O(n)/O(n-k),$$

$$G(n,k) \cong \; O(n)/O(k) \times O(n-k) \cong GL(n,\mathbb{R})/H,$$

where

$$H = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \; / \quad \text{with } A \in GL(k,\mathbb{R}),\; D \in GL(n-k,\mathbb{R}) \right\}.$$

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which is identified to $GL(n, \mathbb{C})$. Hence

$$M \cong GL(2n, \mathbb{R})/GL(n, \mathbb{C}).$$

Example $T(G/H)$ as a homogeneous manifold

We recall that for any Lie group G, the tangent manifold TG is also a Lie group, where the product $\bullet$ is induced from the differential of the multiplication $\mu : G \times G \to G$

$$TG \times TG \to TG, \quad u_a \bullet v_b := T\mu(u_a, u_b)$$

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Moreover, the differential of the canonical homogeneous action $\lambda : G \times G/H \to G/H$ induces a map

$$T\lambda : TG \times T(G/H) \to T(G/H), \quad u_a.v_{\bar{b}} := T_{(a,\bar{b})}\lambda(u_a, v_{\bar{b}}),$$

where $a, b \in G, \bar{b} = bH, u_a \in T_aG$ and $v_{\bar{b}} \in T_{\bar{b}}(G/H)$.

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where $a, b \in G, \bar{b} = bH, u_a \in T_aG$ and $v_{\bar{b}} \in T_{\bar{b}}(G/H)$. We can show that it is a homogeneous action of TG on $T(G/H)$, and hence we have a TG-equivariant diffeomorphism :

$$\boxed{TG/TH \stackrel{\cong}{\to} T(G/H), \quad u_a.TH \mapsto Tp(u_a)}$$

Homogeneous G-vector bundles

Definition

A homogeneous G-vector bundle over $M = G/H$ is a vector bundle $\pi : E \to M$, together with an action of G on E, such that

1. π is a G-map, i.e. $\pi(g \cdot u) = g \cdot \pi(u)$
2. If $g \in G$ then $g : \pi^{-1}(x) \to \pi^{-1}(gx)$ is linear map.

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- The tensor bundle $\otimes^p TM \otimes \otimes^q T^*M \to M$.
- If we start with a representation of H on a finite dimensional vector space V, then the associated bundle

$$\pi : G \times_H V \to G/H, \quad [a, v] \mapsto aH$$

where $[a, v] = [ah, h^{-1}v]$, is a homogeneous G-vector bundle (the action of G is given by $g \cdot [a, v] = [ga, v]$).

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Two G-vector bundles $E^1 \to M$ and $E^2 \to M$ are G-isomorphic, if there exists an isomorphism $\Phi : E^1 \to E^2$ of vector bundles such that f is also a G-map.

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Then we can built the associated bundle $G \times_H E_o \to G/H$.

Classification

Proposition

The map $G \times E_o \to E$ *defined by* $(g, u) \mapsto g \cdot u$, *factorizes to a* G*-isomorphism of* G*-vector bundles*

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Corollary

The mapping $E \mapsto E_o$ and $\Phi \mapsto \Phi_{|E_o}$ induces equivalences between the category $\mathrm{Vect_G(G/H)}$ of G-vector bundles and the category $\mathcal{R}(H)$ of linear representations $H \to GL(E_o)$.

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Remark. This equivalence of categories is compatible with various contructions : The Whitney sum of homogeneous G-vector bundles corresponds to the direct sum of representaions ...

The tangent bundle $T(G/H) \to G/H$

In this case, the isotropy representation is given by

$$\mathrm{Ad}^{G/H} : H \to \mathsf{GL}(T_{\overline{e}}(G/H)), \quad a \mapsto T_{\overline{e}}(\lambda_a)$$

where $\lambda_a : G/H \to G/H$ is the diffeomorphism defined by $\lambda_a : xH \mapsto axH$.

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We then have a bundle isomorphism

$$G \times_H T_{\overline{e}}(G/H) \xrightarrow{\cong} T(G/H), \quad (g, X_{\overline{e}}) \mapsto g \cdot X_{\overline{e}} = T_{\overline{e}}\lambda_g(X_{\overline{e}}).$$

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The tangent linear map $T_e p : \mathfrak{g} \to T_{\overline{e}}(G/H)$ is surjective and then induces a linear isomorphism

$$\Phi_e : \mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T_{\overline{e}}(G/H), \quad \Phi_e(u + \mathfrak{h}) = T_e p(u).$$

Lemma

For any $a \in H$, we have a commutative diagram :

$$\begin{array}{ccc}
\mathfrak{g}/\mathfrak{h} & \xrightarrow{\overline{\mathrm{Ad}}_a} & \mathfrak{g}/\mathfrak{h} \\
\Big\downarrow{\scriptstyle \Phi_e} & & \Big\downarrow{\scriptstyle \Phi_e} \\
T_{\overline{e}}(G/H) & \xrightarrow{T_{\overline{e}}(\lambda_a)} & T_{\overline{e}}(G/H)
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This means that the isotropy representation $\mathrm{Ad}^{G/H}$ is equivalente to the representation $\overline{Ad} : H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Hence, we get the following bundle isomorphism

$$\Phi : G \times_H \mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T(G/H), \quad (g, u + \mathfrak{h}) \mapsto T_{\overline{e}}\lambda_a \circ T_e p(u).$$

- By naturality, the tensor bundle

$$\otimes^p T(G/H) \otimes \otimes^q T^*(G/H) \to G/H$$

corresponds to the representation $\otimes^p(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^q(\mathfrak{g}/\mathfrak{h})^*$.

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In this case, we obtain $G \times_H \mathfrak{m} \xrightarrow{\cong} T(G/H)$, where $G \times_H \mathfrak{m} \to G/H$ is the vector bundle associated to the induced representation $Ad : H \to GL(\mathfrak{m})$.

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Example

- If H is compact, we can take $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to and $Ad(H)$-invariant scalar product on $\mathfrak{g}$.
- If H is discrete subgroup of G.

Sections of homogeneous G-vector bundles

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Explicitely, the correspondance is given by $s(gH) = g \cdot F(g)$.

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Proof. Starting from an equivariant smooth function F, equivariancy implies that $g \cdot F(g)$ depends only on gH, so we can use this expression to define $s : G/H \to E$. Choosing a local smooth section σ of the principal bundle $G \to G/H$, we get $s(x) = \sigma(x) \cdot F(\sigma(x)$, which immediately implies smoothness of s. Conversely, given $s : G/H \to E$ a smooth section ; for any $g \in G$ we have $\pi(g^{-1} \cdot s(gH)) = o$, hence we

Example

Consider $M = G/H$, $u \in \mathfrak{g}$ and $u^* \in \Gamma(TM)$ the fundamental vector field induced by $\exp(-tu)$.

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Example

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Question. What-is the H-equivariant map $F^u : G \to \mathfrak{g}/\mathfrak{h}$ associated to the vector field u^* ?
Solution.

$$\begin{aligned} F^u(g) &= g^{-1} \cdot u^*_g \\ &= \Phi_e^{-1}(g^{-1} \frac{d}{dt}_{|_{t=0}} (\exp(-tu))gH) \\ &= \Phi_e^{-1}(\frac{d}{dt}_{|_{t=0}} (\exp(-tAd_{g^{-1}} u))H) \\ &= -Ad_{g^{-1}}(u) + \mathfrak{h}, \end{aligned}$$

hence

$$\boxed{F^u : g \mapsto -Ad_{g^{-1}}(u) + \mathfrak{h}}.$$

Invariant sections

Let $\pi : E \to G/H$ be a G-vector bundle, and $\Gamma(E)$ the vector set of all sections of E. We have a linear action of G on $\Gamma(E)$:

$$g \cdot s := \tilde{\lambda}_g \circ s \circ \lambda_{g^{-1}},$$

where $\tilde{\lambda}_g$ and λ_g are the left actions on E and G/H respectively, that is $g \cdot s(x) = g \cdot (s(g^{-1} \cdot x))$.

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where $l_{g^{-1}}$ is the left translation in G. Hence we obtain

Theorem

There is a natural isomorphism between the space $(\Gamma(E))^G$ of G-invariant sections of E and the vector space $(E_o)^H$ of H-invariant vectors in E_o. In particular $\dim(\Gamma(E))^G < +\infty$.

Corollary

The space $(\Omega^k(G/H))^G$ *of* G*-invariant differential forms is identified with* $(\wedge^k(\mathfrak{g}/\mathfrak{h})^*)^H$*. The action of* H *on* $\wedge^k(\mathfrak{g}/\mathfrak{h})^*$ *is given by :*

$$(a\cdot\varphi)(u_1+\mathfrak{h},\cdots,u_k+\mathfrak{h}) =$$

$$\varphi(\mathrm{Ad}_{(a^{-1})}(u_1)+\mathfrak{h},\cdots,\mathrm{Ad}_{(a^{-1})}(u_k)+\mathfrak{h}),$$

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for any $a \in H$.

Corollary

A homogeneous space G/H admits a G-invariant Riemannian metric if and only if the image $H_1 \subset GL(\mathfrak{g}/\mathfrak{h})$ of H under the isotropy representation $\overline{Ad} : H \to GL(\mathfrak{g}/\mathfrak{h})$ has compact closure in $GL(\mathfrak{g}/\mathfrak{h})$.

Invariant connections

Let $M = G/H$, we recall that a connection on $TM \to M$ is said to be invariant if for any $X, Y \in \Gamma(TM)$ and $g \in G$,

$$g \cdot (\nabla_X Y) = \nabla_{g \cdot X} g \cdot Y.$$

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This means that for any $g \in G$, the transformation $\lambda_g : M \to M$ is an affine map, and then in particular the canonicanl action of the group $\mathrm{Aff}(M, \nabla)$ on M is transitive. The existence of such structures was studied by K. Nomizu (1954).

Nomizu's result constitutes a nice bridge between the two areas : *"Differential Geometry and nonassociative algebras"*, which was the main prupose of CIMPA research school in Marrakech (April 13-24, 2015). In this school, Alberto Elduque had given a course on the Nomizu theorem and he recently published this course in *Communications in Mathematics (2020)*.

Another formulation

An invariant connection ∇ on $TM \to M$ could be seen as a G-operator

$$\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$$

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$$\Gamma(TM) \stackrel{\cong}{\to} C^\infty(G;\mathfrak{g}/\mathfrak{h})^H,\ \Gamma(T^*M\otimes TM) \stackrel{\cong}{\to} C^\infty(G;\mathfrak{g}/\mathfrak{h}^*\otimes\mathfrak{g}/\mathfrak{h})^H.$$

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This leads us to define an invariant connection as a G-operator

$$D : C^\infty(G; \mathfrak{g}/\mathfrak{h})^H \to C^\infty(G; (\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}/\mathfrak{h})^H$$

which satisfies a Leibniz formula : $D(fF) = fDF + df \otimes F$, where $f \in (C^\infty(G))^H$ and $df \otimes F(g) := \Phi_g^*(df_g) \otimes F(g)$. Here the action of G on a smooth map F is given by $g \cdot F = F \circ l_{g^{-1}}$.

Therefore, the question becomes to determine the G-operators $D : C^\infty(G;\mathfrak{g}/\mathfrak{h})^H \to C^\infty(G; End(\mathfrak{g}/\mathfrak{h}))^H$ satisfying the Leibniz formula : $D(fF) = fDF + df \otimes F$, where $f \in (C^\infty(G))^H$ and $(df \otimes F)(g)(u + \mathfrak{h}) := df_g(u_g^+)F(g)$ (here u^+ is the left invariant vector field on G associated to $u \in \mathfrak{g}$).

Therefore, the question becomes to determine the G-operators $D : C^\infty(G; \mathfrak{g}/\mathfrak{h})^H \to C^\infty(G; End(\mathfrak{g}/\mathfrak{h}))^H$ satisfying the Leibniz formula : $D(fF) = fDF + df \otimes F$, where $f \in (C^\infty(G))^H$ and $(df \otimes F)(g)(u + \mathfrak{h}) := df_g(u_g^+)F(g)$ (here u^+ is the left invariant vector field on G associated to $u \in \mathfrak{g}$).

Theorem

The G-operators as above are in bijective correspondence with linear maps $L : \mathfrak{g} \to End(\mathfrak{g}/\mathfrak{h})$ satisfying

1. $L(u)(v + \mathfrak{h}) = [u, v] + \mathfrak{h}$, *for any* $u \in \mathfrak{h}$.
2. $L(a \cdot u) = a.L(u)$, *for any* $a \in H$ *(where* $a \cdot u = Ad_a(u)$, $a.L(u) = \overline{Ad}_a \circ L(u) \circ \overline{Ad}_{a^{-1}}$*).*

The G-operator D corresponding to L is given by

$$(DF)(g)(u + \mathfrak{h}) = (dF)_g(u_g^+) + L(u)(F(g)) \tag{1}$$

Sketch of the proof

- **Step1** : For any $u \in \mathfrak{g}$ we consider the following linear operator $B_u : C^\infty(G; \mathfrak{g}/\mathfrak{h})^H \to C^\infty(G; \mathfrak{g}/\mathfrak{h})^H$ given by

$$B_u F : g \mapsto (DF)(g)(u + \mathfrak{h}) - (dF)_g(u_g^+)$$

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- **Step1** : For any $u \in \mathfrak{g}$ we consider the following linear operator $B_u : C^\infty(G;\mathfrak{g}/\mathfrak{h})^H \to C^\infty(G;\mathfrak{g}/\mathfrak{h})^H$ given by

$$B_u F : g \mapsto (DF)(g)(u+\mathfrak{h}) - (dF)_g(u_g^+)$$

- **Step2** : B_u is G-equivariant and $(C^\infty(G))^H$-linear.
This means that for any $g \in G$, $f \in (C^\infty(G))^H$ and $F \in C^\infty(G;\mathfrak{g}/\mathfrak{h})^H$ we can prove :

$$\boxed{B_u(g\cdot F) = g\cdot B_u F}, \quad \text{and} \quad \boxed{B_u(fF) = fB_u F}$$

- **Step3** : B_u is a local operator, i.e. if $F \in C^\infty(G;\mathfrak{g}/\mathfrak{h})^H$ vanishes on an open subset $U \subset G$ then so is $B_u F$.
Indeed, let $g_0 \in U$ and V be an open subset of U with compact closure and $g_0 \in V$. Consider then $\rho \in C_c^\infty(G)$ with $\rho = 1$ on V and $\mathrm{supp}(\rho) \subset U$. Denote by da a left Haar measure on H and define an H-invariant function $\overline{\rho} : G \to \mathbb{R}$ by

$$\overline{\rho}(g) := \frac{1}{\int_H \rho(g_0 a)da}. \int_H \rho(ga)da$$

which satisfies moreover $\overline{\rho}(g_0) = 1$ and $\mathrm{supp}(\overline{\rho}) \subset UH$.
Hence $\overline{\rho}F = 0$ and then $0 = B_u(\overline{\rho}F) = \overline{\rho}(g_0)(B_u F)(g_0)$.

- **Step4** : There is an open neighborhood $U \ni e$ and a family of functions $F_1, \ldots, F_r \in C^\infty(UH; \mathfrak{g}/\mathfrak{h})^H$ such that $\{F_1, \ldots, F_r\}$ is a basis of $C^\infty(UH; \mathfrak{g}/\mathfrak{h})^H$ as a $C^\infty(UH)^H$-module.
  Let $\{e_1 + \mathfrak{h}, \ldots, e_r + \mathfrak{h}\}$ be a basis of $\mathfrak{g}/\mathfrak{h}$ and consider the family $F_1, \ldots, F_r \in C^\infty(G; \mathfrak{g}/\mathfrak{h})^H$ given by $F_i(g) := Ad_{g^{-1}}(e_i) + \mathfrak{h}$. We have $\{F_1(e), \ldots, F_r(e)\}$ is a basis of $\mathfrak{g}/\mathfrak{h}$, then there is an open neighborhood $U \ni e$ such that for any $g \in U$ the family $\{F_1(g), \ldots, F_r(g)\}$ is a basis of $\mathfrak{g}/\mathfrak{h}$. Now, from the H-equivariance of the F_i we get that for any $g \in UH$ the family $\{F_1(g), \ldots, F_r(g)\}$ is a basis of $\mathfrak{g}/\mathfrak{h}$, which leads us to conclude.
- **Step5** : For any $F \in C^\infty(G; \mathfrak{g}/\mathfrak{h})^H$, if $F(e) = 0$ then $(B_u F)(e) = 0$.
  We will use the step before. Indeed, we can write locally $F = f_1 F_1 + \ldots + f_r F_r$ where $f_i \in C^\infty(UH)^H$ and $f_1(e) = \ldots = f_r(e) = 0$, then $(B_u F)(e) = f_1(e)(B_u F_1)(e) + \ldots + f_r(e)(B_u F_r)(e) = 0$.

- **Step6** : For any $u \in \mathfrak{g}$, we define $L(u) \in End(\mathfrak{g}/\mathfrak{h})$ by $L(u)(v+\mathfrak{h}) = (B_u F)(e)$, where $F \in C^\infty(G;\mathfrak{g}/\mathfrak{h})^H$ satisfies $F(e) = v + \mathfrak{h}$. Then we show that $L(u)$ satisfies the properties :
  1. $L(u)(v+\mathfrak{h}) = [u,v] + \mathfrak{h}$, for any $u \in \mathfrak{h}$.
  2. $L(a \cdot u) = a.L(u)$, for any $a \in H$.
  3. $L(u)(F(g)) = B_u F(g) = (DF)(g)(u+\mathfrak{h}) - (dF)_g(u_g^+)$, for any $g \in G$.

Invariant connections

Theorem (Invariant connections)

The invariant connections ∇ on $TM \to M$ are in bijective correspondence with linear maps $L : \mathfrak{g} \to End(\mathfrak{g}/\mathfrak{h})$ satisfying

1. $L(u)(v + \mathfrak{h}) = [u, v] + \mathfrak{h}$, *for any* $u \in \mathfrak{h}$.
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   (where $a \cdot u = Ad_a(u)$, $a.L(u) = \overline{Ad}_a \circ L(u) \circ \overline{Ad}_{a^{-1}}$*).*

The connection ∇ corresponding to L is given by

$$(\nabla_X Y)_{\overline{g}} = \tilde{X}_g F^Y + L(g^{-1} \cdot \tilde{X}_g)(F^Y(g))$$

where $F^Y : G \to \mathfrak{g}/\mathfrak{h}$ is the H-equivariant function associated to Y and $\tilde{X}_g \in T_g G$ satisfying $p_*(\tilde{X}_g) = X_{\overline{g}}$.

If we use the fundamental vector fields u^*, the expression of the above invariant connection ∇ is given by

$$(\nabla_{u^*} v^*)_{\overline{e}} = \Phi_e(L(u)(v + \mathfrak{h}) - [u, v] + \mathfrak{h})$$

and

$$(\nabla_{u^*} v^*)_{\overline{g}} = (\lambda_g)_*((\nabla_{(Ad_{g^{-1}} u)^*}(Ad_{g^{-1}} v)^*)_{\overline{e}})$$

Moreover, the torsion T^∇ vanishes if and only if for any $u, v \in \mathfrak{g}$

$$L(u)(v + \mathfrak{h}) - L(v)(u + \mathfrak{h}) = [u, v] + \mathfrak{h}$$

The curvature R^∇ vanishes if and only if for any $u, v \in \mathfrak{g}$

$$L[u, v] = [L(u), L(v)] \in End(\mathfrak{g}/\mathfrak{h})$$

Theorem (Invariant flat connections)

There is a one-to-one correspondence between G-invariant flat connections on $M := G/H$ and Lie algebra representations $L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ which satisfy the following two conditions :

$$L(u)(v + \mathfrak{h}) - L(v)(u + \mathfrak{h}) = [u, v] + \mathfrak{h}, \quad \forall u, v \in \mathfrak{g}$$

and

$$L(Ad_a(u)) = \overline{Ad}_a \circ L(u) \circ \overline{Ad}_{a^{-1}}, \quad \forall u \in \mathfrak{g} \;\; \forall a \in H.$$

We will say that a Lie algebra $\mathfrak{g}$ have a compatible left symmetric algebra structure if there exists a product $\bullet$ on $\mathfrak{g}$ such that for any $u, v, w \in \mathfrak{g}$ we have $ass(u, v, w) = ass(v, u, w)$ and $[u, v] = u \bullet v - v \bullet u$, where $ass(u, v, w) := (u \bullet v) \bullet w - u \bullet (v \bullet w)$. This is equivalent to say that there exists a Lie algebra representation $L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ which satisfy $L(u)(v) - L(v)(u) = [u, v]$ for any $u, v \in \mathfrak{g}$.

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Corollary (I)

If $\Gamma \subset G$ is a discrete subgroup of G, then there is a one-to-one correspondence between the G-invariant flat connections on G/Γ and the compatible left symmetric algebras products $(\mathfrak{g}, \bullet)$ which are $Ad(\Gamma)$-invariant, that is

$$Ad_a(u \bullet v) = Ad_a(u) \bullet Ad_a(v), \quad \forall a \in \Gamma$$

Corollary (II)

If (G, H) is a reductive pair with the decomposition : $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $Ad(H)(\mathfrak{m}) = \mathfrak{m}$, *then there is a one-to-one correspondence between the G-invariant flat connections on G/H and the products $\mathfrak{m} \times \mathfrak{m} \xrightarrow{\bullet} \mathfrak{m}$ satisfying the following conditions :*

1. $u \bullet v - v \bullet u = [u, v]_{\mathfrak{m}}$,
2. $ass(u, v, w) - ass(v, u, w) = [[u, v]_{\mathfrak{h}}, w]$, [a]
3. $Ad_a(u \bullet v) = Ad_a(u) \bullet Ad_a(v)$, *for any* $a \in H$.

[a] We denote by $w_{\mathfrak{h}}$ (resp. $w_{\mathfrak{m}}$) the projection of w on $\mathfrak{h}$ (resp. on $\mathfrak{m}$).

Now it is clear that any Lie group could be seen as a $G \times G$-homogeneous space

$$(G \times G) \times G \to G, \quad (g_1, g_2) \cdot x = g_1 x g_2^{-1}$$

Hence we can apply the corollary (II)

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Hence we can apply the corollary (II) to prove :

Corollary (III)

Then there is a one-to-one correspondence between biinvariant flat connections on a connected Lie group G and compatible associative algebra structures on $\mathfrak{g}$.

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Hence we can apply the corollary (II) to prove :

Corollary (III)

Then there is a one-to-one correspondence between biinvariant flat connections on a connected Lie group G and compatible associative algebra structures on $\mathfrak{g}$.

In the proof of this corollary we use the following lemma (to do as exercise)

Lemma

A Lie algebra $\mathfrak{g}$ have a compatible associative algebra structure if and only if there exists a Lie algebra representation $L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ which satisfy $L(u)(v) - L(v)(u) = [u, v]$ and $L(Ad_a(u)) = Ad_a \circ L(u) \circ Ad_{a^{-1}}$, for any $a \in G$.

Examples of corollary (III)

1. An associative algebra structure on a two step nilpotent Lie algebra $\mathfrak{g}$ is defined by : $u \bullet v = \frac{1}{2}[u, v]$.
2. If (M, ∇) is an affine manifold, the an associative algebra structure on the Lie algebra of affine vector fields $\mathfrak{aff}(\mathrm{M}, \nabla)$ is defined by : $X \bullet Y = \nabla_X Y$.

Example of corollary (II)

Consider $M := SPD(n)$ the set of real symmetric positive definite $n \times n$ matrices, which is an open subset of $S(n)$: the vector space of real symmetric $n \times n$ matrices. The connected Lie group $G := \mathrm{GL}^+(n, \mathbb{R})$ of positive determinant $n \times n$ matrices acts transitively on M : $g \cdot x := gxg^T$, and the istropy subgroup in I_n is $H := SO(n)$.
The Lie algebra of H is $\mathfrak{h} = \mathfrak{so}(n, \mathbb{R}) = \{u \in \mathfrak{g} | u + u^t = 0\}$ and with $\mathfrak{m} := S(n)$ we have a canonical decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \text{ with } Ad(H)(\mathfrak{m}) \subset \mathfrak{m}.$$

Define a the following product :

$$\mathfrak{m} \times \mathfrak{m} \stackrel{\bullet}{\to} \mathfrak{m}, A \bullet B := AB + BA.$$

It is easy to see that $\bullet$ satisfies the conditions of the Corollary (II), so we get a G-invariant affine connection on M.

Corollary (K. Yagi 1970)

Let G be a connected Lie group and $H \subset G$ a closed subgroup such that :

(i) *There is a compatible associative algebra structure $\bullet$ on $\mathfrak{g}$,*

(ii) *$\mathfrak{h}$ is a left ideal of $(\mathfrak{g}, \bullet)$. (i.e. $u \bullet \mathfrak{h} \subset \mathfrak{h}$ for any $u \in \mathfrak{h}$)*

Then there exists a unique G-invariant flat connection on G/H such that

$$\nabla_{u^*} v^* = (v \bullet u)^*$$

for any $u, v \in \mathfrak{g}$.

Sketch of the proof . Consider $L(u)(v + \mathfrak{h}) := u \bullet v + \mathfrak{h}$, which is well defined because $\mathfrak{h}$ is a left ideal of $(\mathfrak{g}, \bullet)$...

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