An Introduction to the Geometry of Symmetric Spaces - I -

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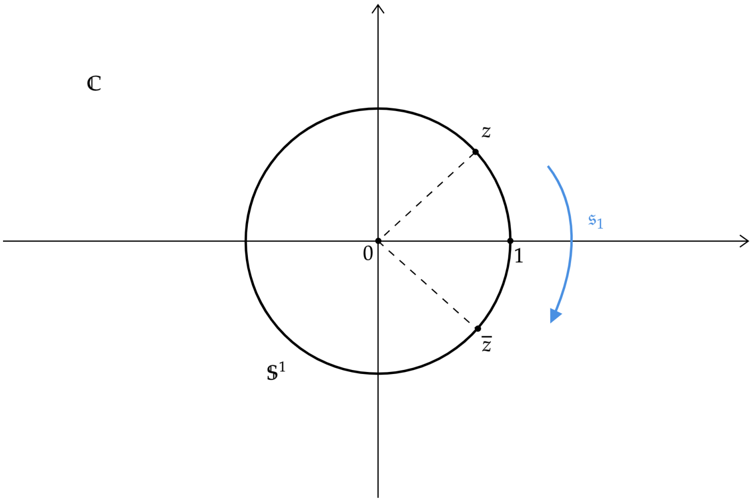
Lie groups

For any Lie group G we have a canonical involution:

$$\mathfrak{s}_e : G \to G, \quad \text{written} \quad g \mapsto g^{-1}.$$

1. $\mathfrak{s}_e(e) = e$.
2. $\mathfrak{s}_e \circ \mathfrak{s}_e = \mathrm{Id}_G$.

Example: The unit circle $\mathbb{S}^1 \subset \mathbb{C}$



Symmetries $(\mathfrak{s}_a)$ in a Lie group G

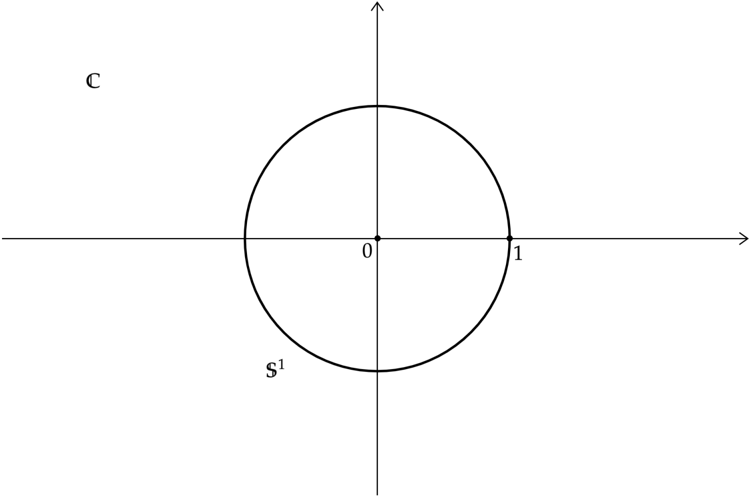
For any fixed point a in G, there exists a smooth involution of G such that a is an isolated fixed point for it.
It is defined by:

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{s}_e} & G \\ \uparrow{\scriptstyle l_{a^{-1}}} & & \downarrow{\scriptstyle l_a} \\ G & \xrightarrow{\mathfrak{s}_a} & G, \end{array}$$

where $l_g : G \to G$ is the left translation by the element $g \in G$.
More precisely, the map $\mathfrak{s}_a$ is given by:

$$\mathfrak{s}_a(b) := ab^{-1}a.$$

Example: The unit circle $\mathbb{S}^1 \subset \mathbb{C}$



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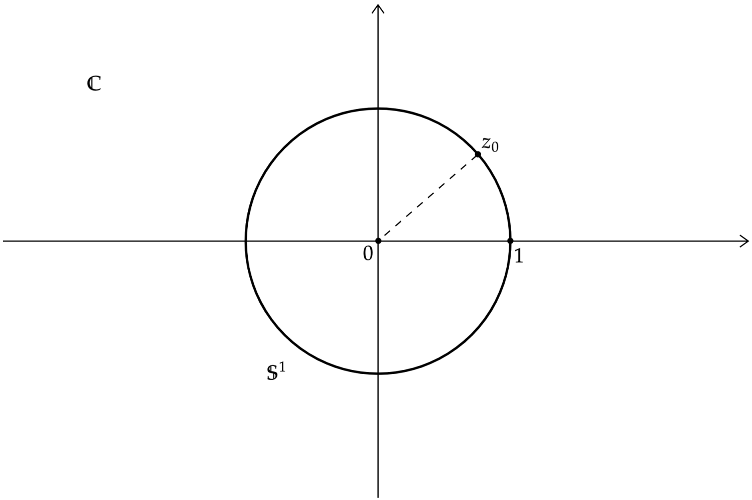

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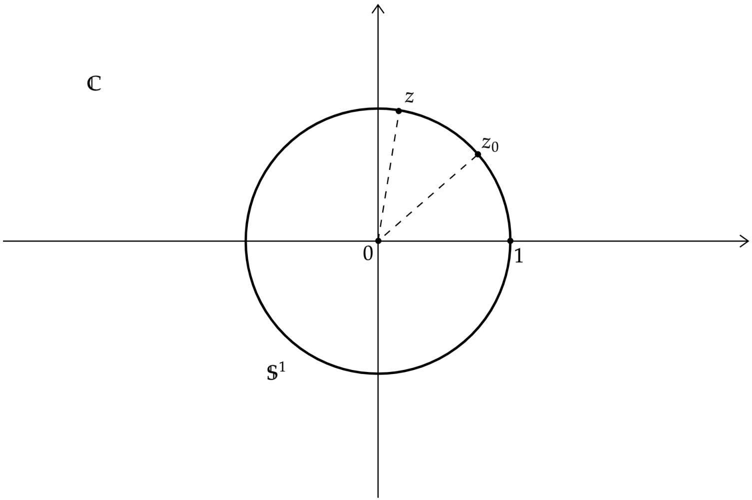

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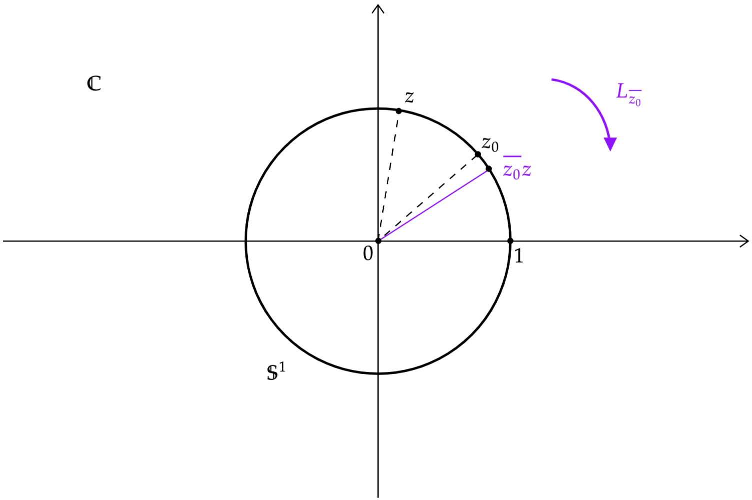

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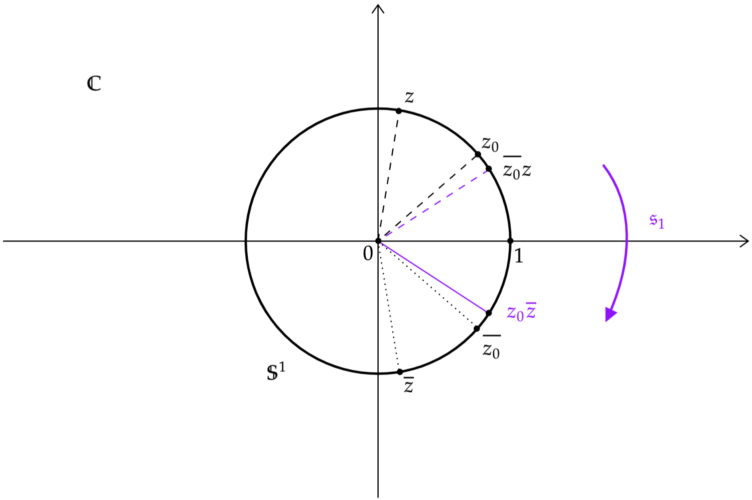

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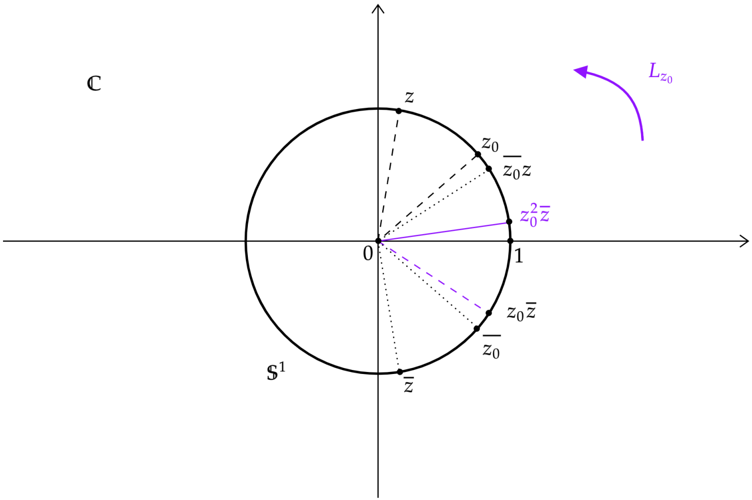
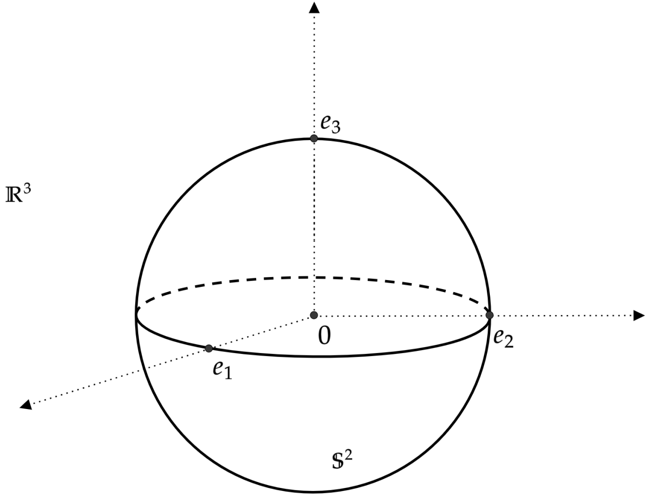
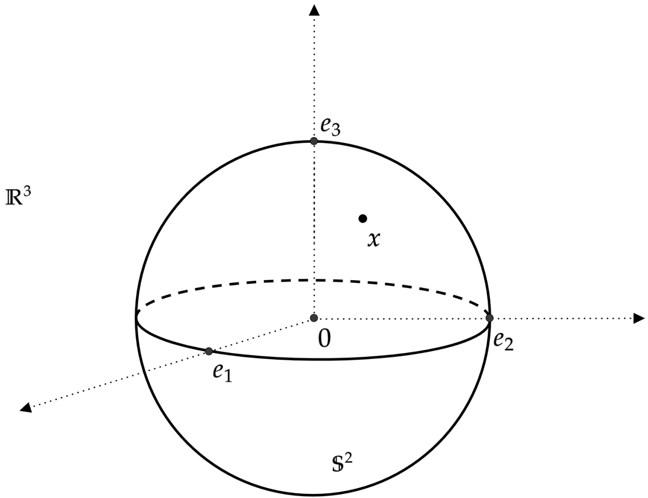


What about the unit sphere $\mathbb{S}^2$?



The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$



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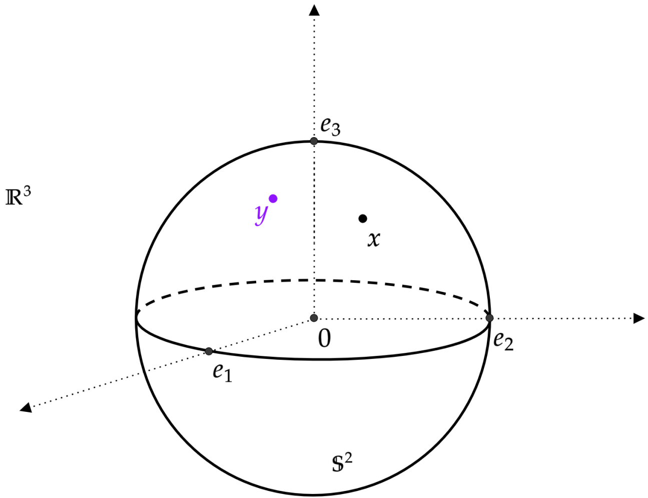

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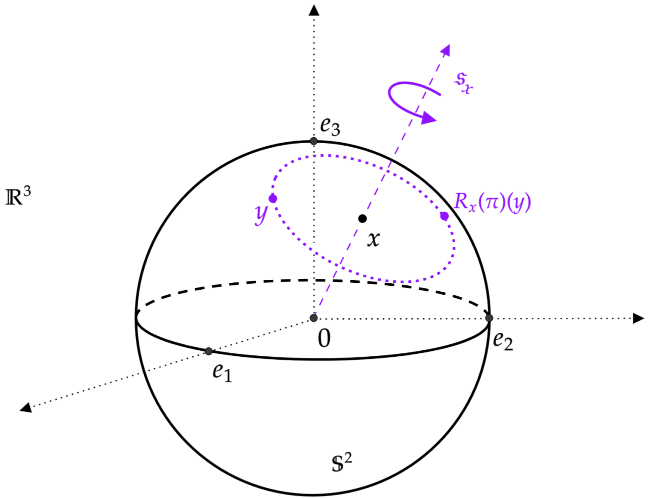


The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$

For each $x \in \mathbb{S}^2$, we have an involution:

$$\begin{array}{rcccc} \mathfrak{s}_x & : & \mathbb{S}^2 & \longrightarrow & \mathbb{S}^2 \\ & & y & \longmapsto & R_x(\pi)(y), \end{array}$$

where $R_x(\pi)$ is the rotation around the x-axis by angle π.
Explicitly, it is given by:

$$R_x(\pi)(y) := 2\langle x, y\rangle x - y.$$

Symmetries on the Poincaré Half-Plane

Consider $\mathbb{H} := \{z = x + iy \in \mathbb{C} / \ y > 0\}$ with the metric $ds^2 := \frac{1}{y^2}(dx^2 + dy^2)$. The group $\mathrm{SL}(2, \mathbb{R})$ acts transitively and isometrically on $\mathbb{H}$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

Moreover, a symmetry (an isometry of $\mathbb{H}$) at i is given by:

$$\mathfrak{s}_i(z) := -\frac{1}{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z.$$

Hence, under conjugaison by elements in $\mathrm{SL}(2, \mathbb{R})$ we get symmetries: $\mathfrak{s}_a : \mathbb{H} \to \mathbb{H}$ for any $a \in \mathbb{H}$.

What is a Symmetric Space?

Definition

A *symmetric space* is a connected smooth manifold M with a smooth family of involutions $\{\mathfrak{s}_x\}_{x\in M}$, in the sense that

$$\begin{array}{ccc} M\times M & \longrightarrow & M \\ (x,y) & \longmapsto & \mathfrak{s}_x(y), \end{array}$$

is smooth, and which satisfies the following properties:

1. $\mathfrak{s}_x(x) = x, \quad \forall\, x \in M;$
2. $\mathfrak{s}_x \circ \mathfrak{s}_y \circ \mathfrak{s}_x = \mathfrak{s}_{\mathfrak{s}_x(y)}, \quad \forall\, x, y \in M;$
3. For each $x \in M$, there exists a neighborhood $U_x \subseteq M$ of x such that x is the only fixed point of $\mathfrak{s}_x$ in U_x.

An equivalent definition

Definition

A *symmetric space* is a connected smooth manifold M with a smooth map $\mu : M \times M \to M$ written as $\mu(x, y) := x \cdot y$, which satisfies the following properties:

1. $x \cdot x = x, \quad \forall\, x \in M;$
2. $x \cdot (x \cdot y) = y, \quad \forall\, x, y \in M;$
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z), \quad \forall\, x, y, z \in M;$
4. For each $x \in M$, there exists a neighborhood $U_x \subseteq M$ of x such that

$$\begin{cases} x \cdot y = y \\ y \in U_x \end{cases} \quad \Rightarrow \quad y = x.$$

Example: Lie groups

For a connected Lie group G, we have

$$\mu : G \times G \to G, \quad \text{written} \quad a \cdot b := ab^{-1}a.$$

Only the last property requires verification. Using the fact that $\exp_G : \mathfrak{g} \to G$ is a local diffeomorphism on a neighborhood of $0 \in \mathfrak{g}$, there exists a neighborhood $U_e \subseteq G$ of $e \in G$ such that

$$\begin{cases} g^2 = e \\ g \in U_e \end{cases} \quad \Rightarrow \quad g = e.$$

So for each $a \in G$, we define $U_a := U_e a$, then we get

$$\begin{cases} a \cdot b = b \\ b \in U_a \end{cases} \Leftrightarrow \begin{cases} (ba^{-1})^2 = e \\ b \in U_a \end{cases} \quad \Rightarrow \quad b = a.$$

Example: Symmetric Pairs

Definition

A *symmetric pair* is a triple (G, H, σ) such that:

1. G is a connected Lie group and H a closed subgroup;
2. $\sigma : G \to G$ is an involutive automorphism of G satisfying the following condition

$$\mathrm{Fix}^0(\sigma) \subseteq H \subseteq \mathrm{Fix}(\sigma),$$

where $\mathrm{Fix}(\sigma) := \{g \in G \mid \sigma(g) = g\}$.

Main example: $(\mathrm{GL}^+(n, \mathbb{R}), \mathrm{SO}(n), \sigma)$, where the involution automorphism σ is

$$\sigma : \mathrm{GL}^+(n, \mathbb{R}) \to \mathrm{GL}^+(n, \mathbb{R}), \quad A \mapsto \left(A^{-1}\right)^T.$$

From Symmetric Pairs to Symmetric Spaces

Theorem

Let (G, H, σ) be a symmetric pair, then $M := G/H$ is a symmetric space.

Proof. Let $\pi : G \to M, g \mapsto \overline{g}$ be the projection map. Define:

$$\mathfrak{s}_{\overline{e}}(\overline{b}) := \overline{\sigma(b)}, \qquad \mathfrak{s}_{\overline{a}} := \lambda_a \circ \mathfrak{s}_{\overline{e}} \circ \lambda_{a^{-1}},$$

where $\lambda_a : M \to M, \quad \overline{b} \mapsto \overline{ab}$;

$$\mu : M \times M \to M, \qquad \overline{a} \cdot \overline{b} := \mathfrak{s}_{\overline{a}}(\overline{b}) = \overline{a\sigma(a^{-1}b)}.$$

This is well defined since $H \subseteq \mathrm{Fix}(\sigma)$. For $a, b \in G$, we have

$$\begin{aligned} \overline{a} \cdot (\overline{a} \cdot \overline{b}) &= \overline{a} \cdot \overline{a\sigma(a^{-1}b)} \\ &= \overline{a\sigma\left(a^{-1}a\sigma(a^{-1}b)\right)} \\ &= \overline{b}. \end{aligned}$$

Moreover, for $g \in G$, we have

$$\begin{aligned}\overline{g} \cdot (\overline{a} \cdot \overline{b}) &= \overline{g\sigma\left(g^{-1}a\sigma(a^{-1}b)\right)} \\ &= \overline{g\sigma(g^{-1}a)\sigma\left(\sigma(a^{-1}g)\sigma(g^{-1}b)\right)} \\ &= \overline{g\sigma(g^{-1}a)\sigma\left(\sigma(a^{-1}g)g^{-1}g\sigma(g^{-1}b)\right)} \\ &= (\overline{g} \cdot \overline{a}) \cdot (\overline{g} \cdot \overline{b}).\end{aligned}$$

Finally, let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be the canonical decomposition of $\mathfrak{g}$. Consider a symmetric neighborhood $V_0 \subseteq \mathfrak{m}$ of 0 in $\mathfrak{m}$ such that $\pi \circ \exp_{G|_{V_0}} : V_0 \to U_{\overline{e}} \subset M$ is a diffeomorphism. For $u \in V_0$, we get

$$\begin{aligned}\overline{e} \cdot \overline{\exp_G(u)} &= \overline{\sigma\left(\exp_G(u)\right)} \\ &= \overline{\exp_G\left(\sigma'(u)\right)} \\ &= \overline{\exp_G(-u)}.\end{aligned}$$

Thus

$$\begin{cases} \overline{e} \cdot \overline{a} = \overline{a} \\ \overline{a} \in U_{\overline{e}} \end{cases} \quad \Rightarrow \quad \overline{a} = \overline{e}.$$

Similarly, for each $a \in G$, we put $U_{\overline{a}} := \lambda_a \left(U_{\overline{e}} \right)$, then we obtain

$$\begin{cases} \overline{a} \cdot \overline{b} = \overline{b} \\ \overline{b} \in U_{\overline{a}} \end{cases} \quad \Rightarrow \quad \overline{b} = \overline{a}. \quad \blacksquare$$

Examples of Symmetric Pairs

- $(G \times G, \Delta G, \sigma)$, where G is a connected Lie group and

$$\sigma : G \times G \to G \times G, \quad (a,b) \mapsto (b,a).$$

- $(\mathrm{SO}(p+q), \mathrm{SO}(p) \times \mathrm{SO}(q), \sigma)$, with σ given by:

$$\sigma : \mathrm{SO}(p+q) \to \mathrm{SO}(p+q), \quad A \mapsto I_{p,q} A I_{p,q},$$

where $I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

Examples of Symmetric Pairs

- $(\mathrm{SO}^0(p,q), \mathrm{SO}(p) \times \mathrm{SO}(q), \sigma)$, where $\mathrm{SO}^0(p,q)$ is the identity component of the generalized orthogonal group defined by:

$$\mathrm{O}(p,q) := \left\{ A \in \mathrm{GL}(p+q, \mathbb{R}) \mid A^T I_{p,q} A = I_{p,q} \right\},$$

  and σ is the following automorphism:

$$\sigma : \mathrm{SO}^0(p,q) \to \mathrm{SO}^0(p,q), \quad A \mapsto \left(A^{-1}\right)^T = I_{p,q} A I_{p,q}.$$

- $(\mathrm{SO}(2n), \mathrm{U}(n), \sigma)$, with σ given by:

$$\sigma : \mathrm{SO}(2n) \to \mathrm{SO}(2n), \quad A \mapsto J_n A J_n^T,$$

  where $J_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Proposition

Let (G, H, σ) be a symmetric pair, then $M := G/H$ is a reductive homogeneous G-space. More precisely we have:

1. *The Lie algebra of H is* $\mathfrak{h} := \{u \in \mathfrak{g} \ / \ \sigma'(u) = u\}$.
2. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, *where* $\mathfrak{m} := \{u \in \mathfrak{g} \ / \ \sigma'(u) = -u\}$.
3. $\mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$.

Definition

Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of a Lie algebra $\mathfrak{g}$.

- The pair $(\mathfrak{g}, \tau)$ is called an *involutive* Lie algebra.
- The canonical decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} := \ker(\tau - \mathrm{Id}_{\mathfrak{g}})$ and $\mathfrak{m} := \ker(\tau + \mathrm{Id}_{\mathfrak{g}})$.

The following relations hold:

$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}.$

Conversely, we have

Proposition

Any involutive automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ *of a Lie algebra* $\mathfrak{g}$ *gives rise to a symmetric pair* $(\widetilde{G}, \widetilde{H}, \widetilde{\sigma})$, *where*

- $\widetilde{G}$ *is a simply connected Lie group having* $\mathfrak{g}$ *as Lie algebra;*
- $\widetilde{H} := \langle \exp_{\widetilde{G}}(\mathfrak{h}) \rangle$, *with* $\mathfrak{h} := \ker(\tau - \mathrm{Id}_{\mathfrak{g}})$*;*
- $\widetilde{\sigma} \in \mathrm{Aut}(\widetilde{G})$ *such that* $\widetilde{\sigma}' = \tau$.

Moreover, if (G, H, σ) *is a symmetric pair with* $(\mathfrak{g}, \tau)$ *as associated involutive Lie algebra and such that* G/H *is simply connected, then* $\widetilde{G}/\widetilde{H} \cong G/H$.

Proof. Since $\mathrm{Lie}\,(\mathrm{Fix}(\widetilde{\sigma})) = \mathfrak{h}$, it suffices to check that $\widetilde{H} \subseteq \mathrm{Fix}(\widetilde{\sigma})$. But this is obvious by using the following commutative diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\ \widetilde{\sigma}\ } & \widetilde{G} \\ {\scriptstyle \exp_{\widetilde{G}}}\big\uparrow & & \big\uparrow{\scriptstyle \exp_{\widetilde{G}}} \\ \mathfrak{g} & \xrightarrow{\ \tau\ } & \mathfrak{g}. \end{array}$$

The last assertion follows from the uniqueness of the universal covering manifold. ∎

The Canonical decomposition of $\mathfrak{g}$ for the previous Examples

- For $(\mathrm{GL}^+(n,\mathbb{R}), \mathrm{SO}(n), \sigma)$, we have

$$\mathfrak{h} = \mathfrak{so}(n), \quad \text{and} \quad \mathfrak{m} = \mathrm{Sym}(n,\mathbb{R}).$$

- For $(G \times G, \Delta G, \sigma)$, we have

$$\mathfrak{h} = \{(u,u) \mid u \in \mathfrak{g}\}, \quad \text{and} \quad \mathfrak{m} = \{(u,-u) \mid u \in \mathfrak{g}\}.$$

- For $(\mathrm{SO}(p+q), \mathrm{SO}(p) \times \mathrm{SO}(q), \sigma)$, we have

$$\mathfrak{h} = \mathfrak{so}(p) \times \mathfrak{so}(q), \quad \text{and} \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix} \mid X \in \mathcal{M}_{p,q}(\mathbb{R}) \right\}.$$

The Canonical decomposition of $\mathfrak{g}$ for the previous Examples

- For $(\mathrm{SO}^0(p,q), \mathrm{SO}(p) \times \mathrm{SO}(q), \sigma)$, we have

$$\mathfrak{h} = \mathfrak{so}(p) \times \mathfrak{so}(q), \quad \text{and} \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \mid X \in \mathcal{M}_{p,q}(\mathbb{R}) \right\}.$$

- For $(\mathrm{SO}(2n), \mathrm{U}(n), \sigma)$, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \mid X \in \mathfrak{so}(n), \, Y \in \mathrm{Sym}(n, \mathbb{R}) \right\},$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X, Y \in \mathfrak{so}(n) \right\}.$$

Homomorphism of Symmetric Spaces

A *homomorphism* between two symmetric spaces (M_1, μ_1) and (M_2, μ_2) is a smooth map $\Phi : M_1 \to M_2$ such that:

$$\Phi(x \cdot y) = \Phi(x) \cdot \Phi(y), \qquad \forall\, x, y \in M_1.$$

An *isomorphism* between two symmetric spaces (M_1, μ_1) and (M_2, μ_2) is both a homomorphism and diffeomorphism.

An *automorphism* of a symmetric space (M, μ) is just an isomorphism from M to itself. We denote by $\mathrm{Aut}(M)$ the group of all automorphisms of M.

Let (M, μ) be a symmetric space. Define the *group of displacements* of M by:

$$G(M) := \langle \mathfrak{s}_x \circ \mathfrak{s}_y \,;\; x, y \in M \rangle.$$

If $\Phi \in \mathrm{Aut}(M)$, then using the following commutative diagram

$$\begin{CD} M @>{\mathfrak{s}_x}>> M \\ @V{\Phi}VV @VV{\Phi}V \\ M @>{\mathfrak{s}_{\Phi(x)}}>> M, \end{CD}$$

we get that

$$\Phi \circ \left(\mathfrak{s}_x \circ \mathfrak{s}_y\right) \circ \Phi^{-1} = \mathfrak{s}_{\Phi(x)} \circ \mathfrak{s}_{\Phi(y)}.$$

Hence $G(M)$ is a normal subgroup of $\mathrm{Aut}(M)$.

Let $o \in M$ be a fixed point, then for each $x, y \in M$ we have

$$\mathfrak{s}_x \circ \mathfrak{s}_y = \mathfrak{s}_x \circ \mathfrak{s}_o \circ \mathfrak{s}_o \circ \mathfrak{s}_y = \left(\mathfrak{s}_x \circ \mathfrak{s}_o\right) \circ \left(\mathfrak{s}_y \circ \mathfrak{s}_o\right)^{-1}.$$

Thus

$$G(M) = \langle \mathfrak{s}_x \circ \mathfrak{s}_o\,;\ x \in M \rangle.$$

For each $u \in T_oM$, we define a vector field $\widetilde{u} \in \mathfrak{X}(M)$ by:

$$\widetilde{u}_x := \frac{1}{2}\, u \cdot (o \cdot x), \qquad \forall\, x \in M.$$

From Symmetric Spaces to Symmetric Pairs

Theorem

Let (M, μ) be a symmetric space and $o \in M$ a fixed point, then the following properties hold.

1. $\mathrm{Aut}(M)$ *is a Lie transformation group with Lie algebra*

$$\mathrm{Der}(M) := \Big\{ X \in \mathfrak{X}(M) \,|\, X_{x \cdot y} = x \cdot X_y + X_x \cdot y, \ \forall\, x, y \in M \Big\}.$$

2. $G(M)$ *is a connected Lie subgroup of* $\mathrm{Aut}(M)$ *with Lie algebra*

$$\mathfrak{g}(M) := \big\{ \widetilde{u} \,|\, u \in T_o M \big\}.$$

3. $G(M)$ *acts transitively on* M*.*

From Symmetric Spaces to Symmetric Pairs

Theorem (Continued)

4. $(G(M), H_o, \sigma_o)$ *is a symmetric pair, where* H_o *denotes the isotropy group of* o, *and* σ_o *given by:*

$$\sigma_o : G(M) \to G(M), \quad F \mapsto \mathfrak{s}_o \circ F \circ \mathfrak{s}_o.$$

   Moreover, M *is isomorphic to* $G(M)/H_o$.

5. $G(M)$ *is the smallest subgroup of* $\mathrm{Aut}(M)$ *which is transitive on* M *and stable under* σ_o.
6. *The canonical decomposition of* $\mathfrak{g}(M)$ *corresponding to* σ_o *is*

$$\mathfrak{g}(M) = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}],$$

   where $\mathfrak{m} := \ker(\sigma_o' + \mathrm{Id}_{\mathfrak{g}})$.

For a full proof one can see Loos, Ottmar. *Symmetric spaces: General theory.* Vol. 1. WA Benjamin, 1969.

Affine Symmetric Spaces

Definition

An *affine symmetric space* is a connected smooth manifold M endowed with a connection ∇ which satisfies the following:

- For each $x \in M$, there exists an affine map $\mathfrak{s}_x : M \to M$ such that:

$$\mathfrak{s}_x (\gamma(t)) = \gamma(-t),$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic of ∇ with $\gamma(0) = x$.

The affine map $\mathfrak{s}_x : M \to M$ is called the *geodesic symmetry* about x.

$\mathfrak{s}_x : M \to M$, $\mathfrak{s}_x(\gamma(t)) = \gamma(-t)$.

Clearly a geodesic symmetry $\mathfrak{s}_x$ is different from Id_M and admits x as an isolated fixed point.
Moreover, if $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic of ∇ with $\gamma(0) = x$ and $u_x = \dot{\gamma}(0)$, then

$$\begin{aligned} T_x\mathfrak{s}_x(u_x) &= \frac{d}{dt}_{|t=0} \mathfrak{s}_x(\gamma(t)) \\ &= \frac{d}{dt}_{|t=0} \gamma(-t) \\ &= -u_x. \end{aligned}$$

Thus

$$T_x\mathfrak{s}_x = -\mathrm{Id}_{T_xM}.$$

Furthermore, using the following Lemma

Lemma

Let M be a connected smooth manifold, and ∇ a connection on it. If $F_1, F_2 : M \to M$ are two affine maps such that:

$$F_1(x_0) = F_2(x_0), \quad \text{and} \quad T_{x_0}F_1 = T_{x_0}F_2.$$

Then $F_1 = F_2$.

We deduce that

- $\mathfrak{s}_x \circ \mathfrak{s}_x = \mathrm{Id}_M, \quad \forall\, x \in M;$
- $\mathfrak{s}_x \circ \mathfrak{s}_y \circ \mathfrak{s}_x = \mathfrak{s}_{\mathfrak{s}_x(y)}, \quad \forall\, x, y \in M.$

$$(M, \nabla)$$

An affine symmetric space

(M, ∇)
An affine symmetric space

$\Longrightarrow$

$(M, \{\mathfrak{s}_x\}_{x \in M})$
A symmetric space

Properties

Proposition

Let (M, ∇) be an affine symmetric space.

- *If ∇' a connection on M such that the geodesic symmetries of ∇ also preserve the connection ∇', then $\nabla' = \nabla$.*
- *The torsion T of ∇ is 0, and the curvature tensor R is parallel (i.e. $\nabla R = 0$).*

Proof. Let $\mathrm{D} \in \Gamma\left(T^{(1,2)}M\right)$ be the difference tensor between ∇ and ∇', i.e

$$\mathrm{D}(X,Y) := \nabla'_X Y - \nabla_X Y, \qquad \forall\, X, Y \in \mathfrak{X}(M).$$

Let $x \in M$ and $\mathfrak{s}_x$ its associated geodesic symmetry, then for any $u, v \in T_xM$ we have

$$\begin{aligned} \mathrm{D}_x(u,v) &= \mathrm{D}_x\left(T_x\mathfrak{s}_x(u), T_x\mathfrak{s}_x(v)\right) \\ &= T_x\mathfrak{s}_x\Big(D_x(u,v)\Big) \\ &= -\,\mathrm{D}_x(u,v). \end{aligned}$$

Hence $\mathrm{D}_x(u,v) = 0$ and therefore $\mathrm{D} \equiv 0$. ■

Properties

Proposition

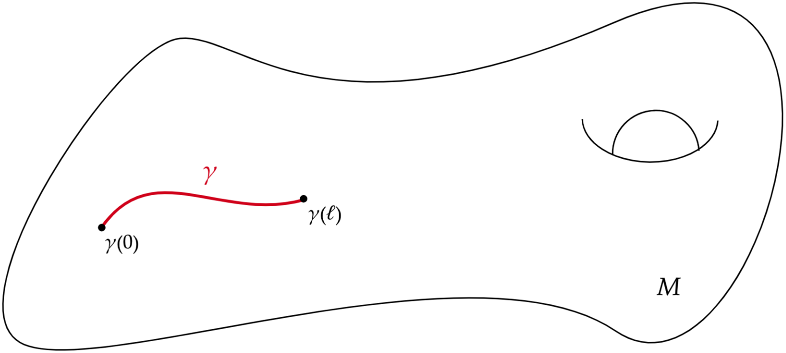
Let (M, ∇) be an affine symmetric space, then we have

1. ∇ *is complete.*
2. $\mathrm{Aff}(M, \nabla)$ *acts transitively on M, and the same is true for its identity component $\mathrm{Aff}^0(M, \nabla)$.*

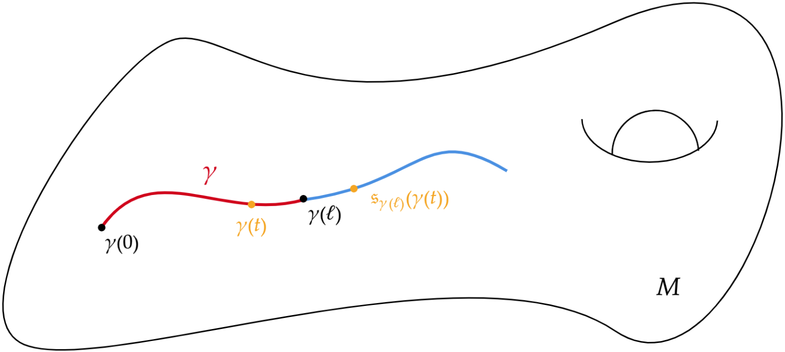
Proof. For the first assertion, if $\gamma : [0, \ell] \to M$ is a geodesic segment, then it can be extended to $[0, 2\ell]$ by reflecting γ about the point $\gamma(\ell)$:

$$\widetilde{\gamma}(t) := \begin{cases} \gamma(t) & \text{for} \quad t \in [0, \ell] \\ \mathfrak{s}_{\gamma(\ell)} \left(\gamma(2\ell - t)\right) & \text{for} \quad t \in [\ell, 2\ell]. \end{cases}$$

Proof. ∇ is complete



Proof. ∇ is complete



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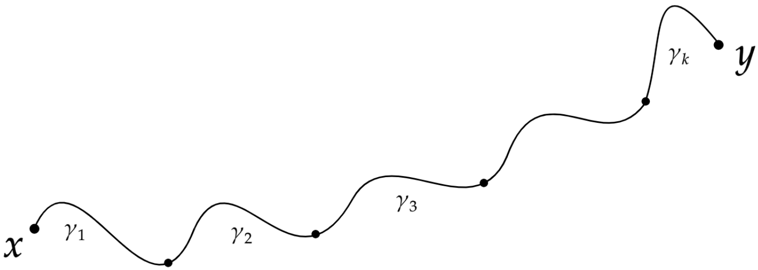

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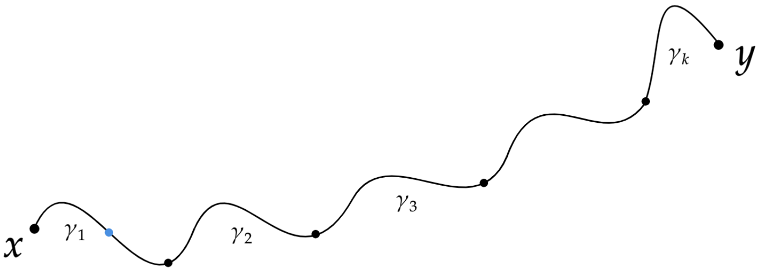

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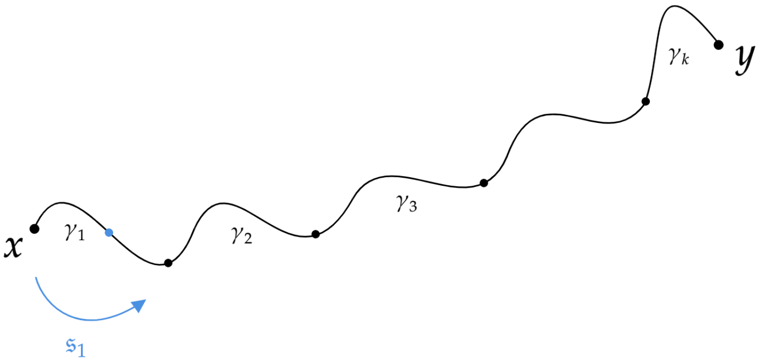

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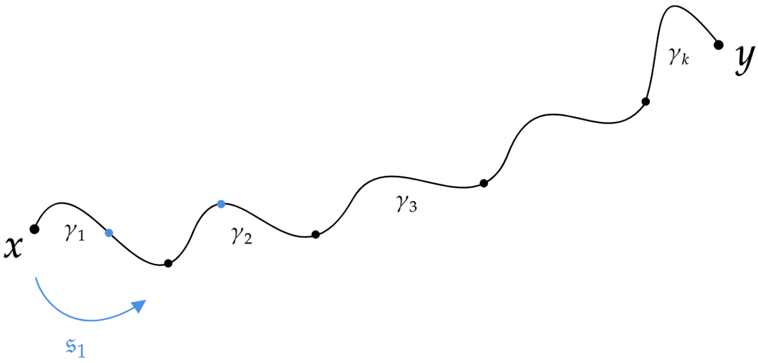

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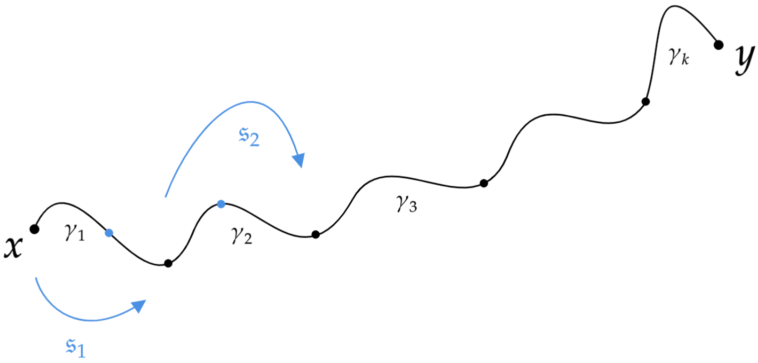

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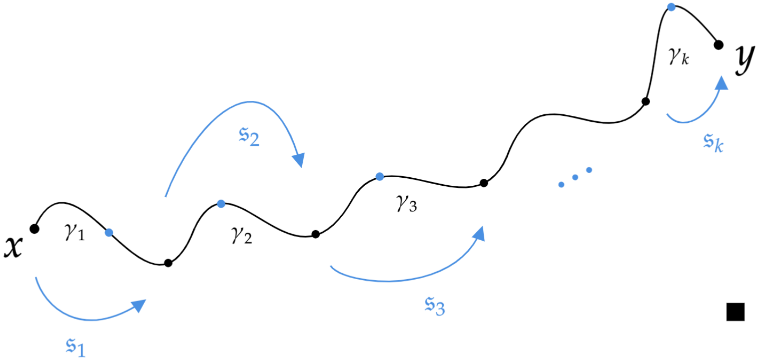

■

Symmetric Pair associated to Affine Symmetric Space

If (M, ∇) is an affine symmetric space, then we can write

$$M \cong \mathrm{Aff}^0(M, \nabla)/H_{x_0},$$

where H_{x_0} denotes the isotropy group of a point $x_0 \in M$ in $\mathrm{Aff}^0(M, \nabla)$. Let $\mathfrak{s}^0$ be the geodesic symmetry about x_0, and define a homomorphism

$$\sigma^\nabla : \mathrm{Aff}^0(M, \nabla) \to \mathrm{Aff}^0(M, \nabla), \quad \text{written} \quad F \mapsto \mathfrak{s}^0 \circ F \circ \mathfrak{s}^0.$$

In fact σ^∇ is an involutive automorphism of $\mathrm{Aff}^0(M, \nabla)$.

Symmetric Pair associated to Affine Symmetric Space

Proposition

With the notations above, the triple $\left(\mathrm{Aff}^0(M,\nabla), H_{x_0}, \sigma^\nabla\right)$ *is a symmetric pair.*

Proof. It only remains to check that

$$\mathrm{Fix}^0(\sigma^\nabla) \subseteq H_{x_0} \subseteq \mathrm{Fix}(\sigma^\nabla).$$

For the first inclusion, let $w \in \mathrm{Lie}\left(\mathrm{Fix}(\sigma^\nabla)\right)$ and put

$$\beta_t := \exp_{\mathrm{Aff}(M,\nabla)}(tw), \qquad \forall\, t \in \mathbb{R}.$$

Since β lies in $\mathrm{Fix}^0(\sigma^\nabla)$, we have $\mathfrak{s}^0 \circ \beta_t \circ \mathfrak{s}^0 = \beta_t$.

Thus

$$\mathfrak{s}^0\left(\beta_t(x_0)\right) = \beta_t(x_0).$$

Using the fact that x_0 is an isolated fixed point of $\mathfrak{s}^0$ we get that $\beta_t(x_0) = x_0$ for t lies in a small enough neighborhood of 0. Hence β_t lies in $H^0_{x_0}$ for t sufficiently small, which implies by taking the derivative of β_t with respect to $t = 0$ that $w \in \mathfrak{h}_{x_0}$.

For the second inclusion, let $F \in H_{x_0}$ be an element of H_{x_0}, then a direct computation yields

$$T_{x_0}\left(\mathfrak{s}^0 \circ F \circ \mathfrak{s}^0\right) = T_{x_0}F.$$

Since M is connected it follows that $\mathfrak{s}^0 \circ F \circ \mathfrak{s}^0 = F$ and hence $H_{x_0} \subseteq \mathrm{Fix}(\sigma^\nabla)$. ∎