# An Introduction to the Geometry of Symmetric Spaces - II -

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# A Symmetric Space

$(M, \{\mathfrak{s}_x\}_{x\in M})$

1. $\mathfrak{s}_x(x) = x;$
2. $\mathfrak{s}_x \circ \mathfrak{s}_x = Id_M;$
3. $\mathfrak{s}_x \circ \mathfrak{s}_y \circ \mathfrak{s}_x = \mathfrak{s}_{\mathfrak{s}_x(y)};$
4. $\exists\, U_x \subseteq M,$ such that:

$$\begin{cases} \mathfrak{s}_x(y) = y; \\ y \in U_x \end{cases} \Rightarrow \quad y = x.$$

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2. $x \cdot (x \cdot y) = y;$
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z);$
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$$\begin{cases} x \cdot y = y; \\ y \in U_x \end{cases} \Rightarrow \quad y = x.$$

# É. Cartan Theorem

**Theorem**

*A pseudo-Riemannian manifold $(M, g)$ is locally pseudo-Riemannian symmetric if and only if $\nabla R = 0$, where $\nabla$ is the Levi-Civita connection of $g$ and $R$ its curvature tensor field. If $M$ is simply-connected and complete, then $(M, g)$ is pseudo-Riemannian symmetric if and only if $\nabla R = 0$.*

# Ambrose-Singer Theorem

**Theorem**

*Let $(M, g)$ be a simply-connected and complete pseudo-Riemannian manifold. The following properties are equivalent:*

1. *$(M, g)$ is reductive homogeneous pseudo-Riemannian manifold.*
2. *$(M, g)$ admits a linear connection $\nabla'$ satisfying*

$$\nabla' g = 0, \quad \nabla' R = 0, \quad \nabla' S = 0,$$

*where $S := \nabla - \nabla'$, $\nabla$ the Levi-Civita connection of $g$, and $R$ its curvature tensor field.*

For a proof and other similar theorems, one can see Calvaruso, G., & López, M. C. (2019). Pseudo-Riemannian Homogeneous Structures (Vol. 59). New York, NY, USA: Springer.

# B. Kostant Theorem [1]

**Theorem**

*Let $\nabla$ be a connection on a simply-connected manifold $M$. The following properties are equivalent:*

1. *$M$ is reductive with respect to a connected Lie subgroup $G \subset \mathrm{Aff}(M, \nabla)$.*
2. *There exists a complete connection $\nabla'$ satisfying*

$$\nabla' T = 0, \quad \nabla' R = 0, \quad \nabla' S = 0,$$

*where $S := \nabla - \nabla'$, $T$ the torsion of $\nabla$, and $R$ its curvature tensor field.*

[1] Kostant, Bertram. "A characterization of invariant affine connections." Nagoya Mathematical Journal 16 (1960): 35-50.

# Jordan algebras

A Jordan algebras $\mathcal{A}$ is a finite dimensional vector space with a bilinear multiplication $xy$ satisfying

$$xy = yx, \qquad x(x^2y) = x^2(xy),$$

and has a unit element $e$.

**Proposition**

*The set $M$ of inversible elements of $\mathcal{A}$ is open in $\mathcal{A}$ and becomes a symmetric space with the product*

$$\mathfrak{s}_x(y) := 2x(y^{-1}x) - x^2y^{-1}.$$

# Main Examples: Symmetric Pairs and Affine Symmetric Spaces

$(G, H, \sigma)$ a symmetric pair

- $\sigma \in \mathrm{Aut}(G)$ such that:

$$\sigma \circ \sigma = Id_G,$$

and $\mathrm{Fix}^0(\sigma) \subseteq H \subseteq \mathrm{Fix}(\sigma).$

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$$\Downarrow$$

$$(G/H, \mu_\sigma)$$

is a symmetric space, where

$$\overline{a} \cdot \overline{b} := \overline{a\sigma\left(a^{-1}b\right)}, \quad \forall\, a, b \in G.$$

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$(M, \nabla)$ an affine symmetric space

- $\nabla$ is a connection and

  $\forall\, x \in M,\ \exists!\ \mathfrak{s}_x \in \mathrm{Aff}(M, \nabla)$

  such that: $\mathfrak{s}_x(\gamma(t)) = \gamma(-t),$

  where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic of $\nabla$ and $\gamma(0) = x.$

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where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic of $\nabla$ and $\gamma(0) = x.$

$$\Downarrow$$

$$(M, \{\mathfrak{s}_x\}_{x \in M})$$

is a symmetric space.

# From Affine Symmetric Spaces to Symmetric Pairs

Let $(M, \nabla)$ be an affine symmetric space. Then we have:

- $\mathrm{Aff}^0(M, \nabla)$ acts transitively on $M$.
- Let $x_0 \in M$ fixed, and denote by $H_{x_0}$ the isotropy group of $x_0$ in $\mathrm{Aff}^0(M, \nabla)$.
- Define an involutive automorphism of $\mathrm{Aff}^0(M, \nabla)$ by:

  $$\sigma^\nabla : \mathrm{Aff}^0(M, \nabla) \to \mathrm{Aff}^0(M, \nabla), \quad F \mapsto \mathfrak{s}_{x_0} \circ F \circ \mathfrak{s}_{x_0},$$

  where $\mathfrak{s}_{x_0} : M \to M$ is the geodesic symmetry about $x_0$.
- The following inclusions hold

  $$\mathrm{Fix}^0(\sigma^\nabla) \subseteq H_{x_0} \subseteq \mathrm{Fix}(\sigma^\nabla).$$

In summary:

$$(M, \nabla) \quad \Longrightarrow \quad \left(\mathrm{Aff}^0(M,\nabla), H_{x_0}, \sigma^\nabla\right)$$

An affine symmetric space $\qquad$ A symmetric pair

• The next step: Expression of the canonical connection $\nabla$ associated to a symmetric pair $(G, H, \sigma)$? i.e. $G$-invariant connection on $G/H$ for which $\overline{\sigma} : G/H \to G/H$ is an affine map.

# Reductive homogeneous $G$-spaces

A homogeneous $G$-space $G/H$ is called *reductive* if there exists a vector subspace $\mathfrak{m} \subset \mathfrak{g}$ such that:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \quad \text{and} \quad \mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m},$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$ respectively.

**Remark.** Not all homogeneous spaces are reductive.
For example:

$$G := \mathrm{GL}^+(2, \mathbb{R}), \quad \text{and} \quad H := \left\{ \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \mid y > 0, \, x \in \mathbb{R} \right\}.$$

One can easily check that $G/H \cong \mathbb{R}^2 \backslash \{0\}$ is not reductive.

Similarly, not all reductive homogeneous spaces are symmetric. For example the Stiefel manifolds $\mathrm{SO}(n)/\mathrm{SO}(n-k)$ are not symmetric spaces for $2 \leq k \leq n-2$. To see why, consider the matrices $I_{p,q}$ and $J_{n'}$ defined by:

$$I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad \text{and} \quad J_{n'} := \begin{pmatrix} 0 & -I_{n'} \\ I_{n'} & 0 \end{pmatrix},$$

where $p+q=n$ and $J_{n'}$ is defined only if $n$ is even, in which case $n' := \frac{n}{2}$.

It is known (cf. S. Helgason pp. $453$) that up to conjugation, the only involutive automorphisms of $\mathfrak{so}(n)$ are given by:

- $\tau_{p,q}(X) := I_{p,q} X I_{p,q}$, in which case we have

$$\ker(\tau_{p,q} - \mathrm{Id}) \cong \mathfrak{so}(p) \times \mathfrak{so}(q) \neq \mathfrak{so}(n-k).$$

- $\theta(X) := J_{n'} X J_{n'}^T$, in which case we have

$$\ker(\theta - \mathrm{Id}) \cong \mathfrak{u}(n') \neq \mathfrak{so}(n-k).$$

# Nomizu Theorem

**Theorem**

*Let $M := G/H$ be a reductive homogeneous $G$-space with a fixed reductive decomposition, i.e*

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \quad \text{and} \quad \mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m}.$$

*Then there exists a one-to-one correspondence between the set of $G$-invariant connections on $M$ and the set of bilinear maps $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ which are $\mathrm{Ad}(H)$-invariant, i.e*

$$\mathrm{Ad}_h \alpha(u, v) = \alpha\left(\mathrm{Ad}_h u, \mathrm{Ad}_h v\right),$$

*for $u, v \in \mathfrak{m}$ and $h \in H$.*

Let $M := G/H$ be a reductive homogeneous $G$-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. For each $u \in \mathfrak{g}$, we define a vector field $u^* \in \mathfrak{X}(M)$, called the *fundamental vector field* associated to $u$ by:

$$u^*_{\overline{a}} := \frac{d}{dt}_{|t=0} \overline{\exp_G(tu)a}, \qquad \forall\, \overline{a} \in M.$$

Moreover, we have a linear isomorphism between $\mathfrak{m}$ and $T_{\overline{e}}M$, given by:

$$\begin{array}{rccc} \mathrm{I}_{\overline{e}} & : & \mathfrak{m} & \xrightarrow{\cong} & T_{\overline{e}}M \\ & & u & \longmapsto & u^*_{\overline{e}}. \end{array}$$

If $\nabla$ is a $G$-invariant connection on $M$, then its associated bilinear map $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined as follows[2]:

$$\alpha(u,v) := \mathrm{I}_{\overline{e}}^{-1}\Big( (\nabla_{u^*} v^*)_{\overline{e}} \Big) + [u,v]_{\mathfrak{m}}.$$

[2]For $w \in \mathfrak{g}$, we denote by $w_{\mathfrak{m}}$ the projection of $w$ on $\mathfrak{m}$.

Further, the torsion $T^{\nabla}$ of the $G$-invariant connection $\nabla$ gives rise to a bilinear map $T^{\alpha} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ written as

$$T^{\alpha}(u, v) := \alpha(u, v) - \alpha(v, u) - [u, v]_{\mathfrak{m}}.$$

Hence

**Corollary**

*Let $\nabla$ be a $G$-invariant connection on $M$ and $\alpha$ its associated bilinear map. Then $\nabla$ is torsion-free if and only if for any $u, v \in \mathfrak{m}$*

$$\alpha(u, v) = \frac{\alpha(u, v) + \alpha(v, u)}{2} + \frac{1}{2}[u, v]_{\mathfrak{m}},$$

*i.e. the bilinear map $\alpha_{\mathrm{sym}}(u, v) := \alpha(u, v) - \frac{1}{2}[u, v]_{\mathfrak{m}}$ is symmetric.*

# Particular $G$-invariant connections on $M$

- The natural connection $\nabla^0$ given by:

$$\alpha^0(u,v) = \frac{1}{2}[u,v]_{\mathfrak{m}}, \qquad \forall\, u,v \in \mathfrak{m}.$$

It is **torsion-free**.

- The canonical connection $\nabla^c$ given by:

$$\alpha^c(u,v) = 0, \qquad \forall\, u,v \in \mathfrak{m}.$$

It is invariant under parallelism i.e the torsion and the curvature tensors of $\nabla^c$ are both parallel.

**Remark.** $\nabla^c = \nabla^0$ if and only if $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$.

Nomizu's Theorem allows us to transfer geometric conditions to algebra, or algebraic conditions to geometry.

**Proposition**

*Let $M := G/H$ be a reductive homogeneous $G$-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\nabla$ a $G$-invariant connection on $M$ with $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ its associated bilinear map. For each $u \in \mathfrak{m}$, we have*

$$\alpha(u,u) = 0 \qquad \Leftrightarrow \qquad t \mapsto \overline{\exp_G(tu)} \text{ is a geodesic of } \nabla.$$

**Proof.** Let $u \in \mathfrak{m}$ and $\gamma : \mathbb{R} \to M$, $t \mapsto \overline{\exp_G(tu)}$. Since $\dot{\gamma}(t) = u^*_{\gamma(t)}$, then a direct computation yields

$$\nabla_{\dot{\gamma}}\dot{\gamma}(t) = \left(\lambda_{\exp_G(tu)}\right)_* \alpha(u,u)^*_{\bar{e}}. \quad \blacksquare$$

Notice that if $\nabla$ is a $G$-invariant connection on $M$ whose geodesics through $\overline{e}$ are exactly the curves $t \mapsto \overline{\exp_G(tu)}$ for any $u \in \mathfrak{m}$, then the geodesics through another point $\overline{a}$ of $M$ are exactly the curves $t \mapsto \overline{\exp_G(t\mathrm{Ad}_a u)a}$, with $u \in \mathfrak{m}$.

**Corollary**

*On a reductive homogeneous $G$-space $M := G/H$ with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, the natural connection $\nabla^0$ is the only $G$-invariant torsion-free connection whose geodesics are exactly the curves $t \mapsto \overline{\exp_G(t\mathrm{Ad}_a u)a}$, with $u \in \mathfrak{m}$ and $\overline{a} \in M$.*

**Example.** A connected Lie group $G$, viewed as a reductive homogeneous $(G \times G)$-space, endowed with its natural bi-invariant connection!

# From Symmetric Pairs to Affine Symmetric Spaces

**Theorem**

*Let $(G, H, \sigma)$ be a symmetric pair, then $M := G/H$ is an affine symmetric space.*

**Proof.** Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be the canonical decomposition of $\mathfrak{g}$ and $\nabla^0$ the natural torsion-free $G$-invariant connection on $M$ associated to the bilinear map $\alpha^0 \equiv 0$. Consider the following smooth map on $M$

$$\mathfrak{s}^0 : M \to M, \qquad \overline{a} \mapsto \overline{\sigma(a)}.$$

This is well defined because $H \subseteq \mathrm{Fix}(\sigma)$, and satisfies

$$\mathfrak{s}^0 \circ \mathfrak{s}^0 = \mathrm{Id}_M .$$

## **Proof.** $\mathfrak{s}^0 \in \mathrm{Aff}(M, \nabla^0)$

Define a connection $\nabla$ on $M$ by:

$$\nabla_X Y := \mathfrak{s}^0_* \left( \nabla^0_{\mathfrak{s}^0_* X} \mathfrak{s}^0_* Y \right), \qquad \forall\, X, Y \in \mathfrak{X}(M).$$

Let us show that $\nabla = \nabla^0$. First, for each $a \in G$, we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{s}^0} & M \\ {\scriptstyle \lambda_a}\downarrow & & \downarrow{\scriptstyle \lambda_{\sigma(a)}} \\ M & \xrightarrow{\mathfrak{s}^0} & M \end{array} .$$

Thus $\nabla$ is $G$-invariant. Let $\alpha$ be its associated bilinear map.

# **Proof.** $\mathfrak{s}^0 \in \mathrm{Aff}(M, \nabla^0)$

For each $u \in \mathfrak{m}$ and $a \in G$ we have

$$\begin{aligned}
\left(\mathfrak{s}^0_* u^*\right)_{\overline{a}} &= \frac{d}{dt}_{|t=0} \mathfrak{s}^0 \left(\overline{\exp_G(tu)\sigma(a)}\right) \\
&= \frac{d}{dt}_{|t=0} \overline{\exp_G(-tu)a} \\
&= -u^*_{\overline{a}}.
\end{aligned}$$

Thus

$$\mathfrak{s}^0_* u^* = -u^*, \qquad \forall\, u \in \mathfrak{m}.$$

# **Proof.** $\mathfrak{s}^0 \in \operatorname{Aff}(M, \nabla^0)$

Hence for $u, v \in \mathfrak{m}$ we have

$$\begin{aligned} \alpha(u,v) &= \mathrm{I}_{\overline{e}}^{-1}\Big( (\nabla_{u^*} v^*)_{\overline{e}} \Big) \\ &= \mathrm{I}_{\overline{e}}^{-1}\Big( s^0_* \left(\nabla^0_{u^*} v^*\right)_{\overline{e}} \Big) \\ &= -\mathrm{I}_{\overline{e}}^{-1}\Big( \alpha^0(u,v)^*_{\overline{e}} \Big) \\ &= 0, \end{aligned}$$

which implies that $\nabla = \nabla^0$ and therefore $\mathfrak{s}^0 \in \operatorname{Aff}(M, \nabla^0)$.

# Proof. $\mathfrak{s}^0$ is a geodesic symmetry about $\overline{e}$

Now it only remains to check that $\mathfrak{s}^0$ is a geodesic symmetry about $\overline{e}$. Let $t \mapsto \overline{\exp_G(tu)}$ be a geodesic through $\overline{e}$ with $u \in \mathfrak{m}$, then

$$\begin{aligned} \mathfrak{s}^0\left(\overline{\exp_G(tu)}\right) &= \overline{\sigma\left(\exp_G(tu)\right)} \\ &= \overline{\exp_G(-tu)}. \end{aligned}$$

Thus $\mathfrak{s}^0$ is a geodesic symmetry about $\overline{e}$.

Finally, for any $\overline{a} \in M$ we define the geodesic symmetry about $\overline{a}$ as follow

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{s}^0} & M \\ {\scriptstyle \lambda_{a^{-1}}}\uparrow & & \downarrow{\scriptstyle \lambda_a} \\ M & \xrightarrow{\mathfrak{s}_{\overline{a}}} & M. \end{array}$$

One can check easily that $\mathfrak{s}_{\overline{a}}$ satisfies all the conditions required for a geodesic symmetry. ■

$(G, H, \sigma)$
A symmetric pair

$\Longrightarrow$

$\left(G/H, \nabla^0\right)$
An affine symmetric space

# Invariant Pseudo-Riemannian Metrics on a Reducitve Homogeneous $G$-space

**Theorem**

*Let $M := G/H$ be a reductive homogeneous $G$-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. There is a natural one-to-one correspondence between the set of $G$-invariant pseudo-Riemannian metrics on $M$ and the set of $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear forms on $\mathfrak{m}$.*

For the sake of simplicity, we shall use the same notation $\langle \cdot \, , \cdot \rangle$ to denote both the $G$-invariant pseudo-Riemannian metric on $M$, and its associated $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$.

**Proposition**

*Let $M := G/H$ be a reductive homogeneous $G$-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and let $\langle \cdot \, , \cdot \rangle$ be a $G$-invariant pseudo-Riemannian metric on $M$. The Levi-Civita connection $\nabla^{\mathrm{LC}}$ of $\langle \cdot \, , \cdot \rangle$ is $G$-invariant and its associated bilinear map $\alpha^{\mathrm{LC}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is given by:*

$$\alpha^{\mathrm{LC}}(u, v) := \frac{1}{2}[u, v]_{\mathfrak{m}} + \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u, v),$$

*where $\alpha^{\mathrm{LC}}_{\mathrm{sym}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the symmetric bilinear map defined by:*

$$\langle \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u, v), w \rangle = \frac{1}{2}\Big\{ \langle [w, u]_{\mathfrak{m}}, v \rangle + \langle u, [w, v]_{\mathfrak{m}} \rangle \Big\},$$

*for all $u, v, w \in \mathfrak{m}$.*

**Proof.** A direct computation using Koszul's formula shows that $\nabla^{\mathrm{LC}}$ is $G$-invariant. Moreover, for $u, v, w \in \mathfrak{m}$ we have

$$\begin{aligned}
\langle \alpha^{\mathrm{LC}}(u,v), w\rangle &= \langle \nabla^{\mathrm{LC}}_{u^*} v^*, w^*\rangle_{\overline{e}} + \langle [u,v]^*, w^*\rangle_{\overline{e}} \\
&= \frac{1}{2}\Big\{\langle [u,v]^*, w^*\rangle_{\overline{e}} + \langle [w,u]^*, v^*\rangle_{\overline{e}} + \langle u^*, [w,v]^*\rangle_{\overline{e}}\Big\} \\
&= \frac{1}{2}\Big\{\langle [u,v]_{\mathfrak{m}}, w\rangle + \langle [w,u]_{\mathfrak{m}}, v\rangle + \langle u, [w,v]_{\mathfrak{m}}\rangle\Big\} \\
&= \langle \frac{1}{2}[u,v]_{\mathfrak{m}} + \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u,v), w\rangle,
\end{aligned}$$

where

$$\langle \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u,v), w\rangle := \frac{1}{2}\Big\{\langle [w,u]_{\mathfrak{m}}, v\rangle + \langle u, [w,v]_{\mathfrak{m}}\rangle\Big\}. \quad \blacksquare$$

**Corollary**

*With the notations of the previous proposition, The Levi-Civita connection* $\nabla^{\mathrm{LC}}$ *of* $\langle\cdot\,,\cdot\rangle$ *coincides with the natural connection* $\nabla^0$ *associated to the decomposition* $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ *if and only if*

$$\langle[u,v]_{\mathfrak{m}},w\rangle+\langle v,[u,w]_{\mathfrak{m}}\rangle=0,\qquad\forall\,u,v,w\in\mathfrak{m}.$$

**Corollary**

*Let* $(G,H,\sigma)$ *be a symmetric pair. A* $G$*-invariant pseudo-Riemannian metric on* $G/H$*, if there exists any, induces the canonical connection.*

# Semi-simple Lie Algebras

**Definition**

Let $(\mathfrak{g}, [\,,])$ be a Lie algebra.

- $\mathfrak{g}$ is *simple* if it is nonabelian and does not contain any ideal distinct from $\{0\}$ and $\mathfrak{g}$.
- $\mathfrak{g}$ is *semi-simple* if does not contain any nonzero solvable ideal. ( $\mathfrak{a}$ is solvable i.e. there exists $n$ s.t. $\mathcal{D}^n(\mathfrak{a}) = \{0\}$).

Let $(\mathfrak{g}, [\,,])$ be a Lie algebra. Then the following statements are equivalent:

1. $\mathfrak{g}$ is semi-simple.
2. $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, where the $\mathfrak{g}_i$'s are ideals of $\mathfrak{g}$ which are simple (as Lie algebras).
3. $\mathfrak{g}$ has no nonzero abelian ideal.
4. The Killing form $B_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ of $\mathfrak{g}$ is non-degenerate.

# Cartan involution

Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an automorphism with $\tau^2 = \mathrm{Id}_{\mathfrak{g}}$. Then, the bilinear form

$$B^\tau(u, v) := -B_{\mathfrak{g}}(u, \tau(v)),$$

is symmetric, where $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. $\tau$ is called a *Cartan involution* if $B^\tau$ is an inner product on $\mathfrak{g}$.

**Proposition**

*$\theta(A) := -A^t$ is an involution of $M_n(\mathbb{R})$. If $\mathfrak{g} \subset M_n(\mathbb{R})$ is a subalgebra such that*

$$\theta(\mathfrak{g}) \subset \mathfrak{g}, \quad \text{and} \quad Z(\mathfrak{g}) = \{0\},$$

*then, $\tau := \theta_{|_{\mathfrak{g}}}$ is a Cartan involution of $\mathfrak{g}$.*

It is the case, for example, of the subalgebras $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(p, q)$.

**Proof.** We have to show that for any $X \in \mathfrak{g}$, s.t. $X \neq 0$

$$B^\tau(X, X) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_{X^t}) > 0 \quad ?$$

Consider the canonical inner product on $\mathfrak{g}$:

$$\langle X, Y \rangle := \operatorname{tr}(X^t Y),$$

this induces an inner product on $\operatorname{End}(\mathfrak{g})$:

$$\langle\langle f_1, f_2 \rangle\rangle := \operatorname{tr}(f_1^T \circ f_2),$$

where $f_1^T : \mathfrak{g} \to \mathfrak{g}$ is the transpose defined through $\langle \cdot\,, \cdot \rangle$.
A small computation shows that $\operatorname{ad}_{X^t} = (\operatorname{ad}_X)^T$. ■

**Theorem**

*Let $(G, H, \sigma)$ be a symmetric pair such that $G$ is semi-simple. Then the canonical connection on $G/H$ is induced by a $G$-invariant pseudo-Riemannian metric. If moreover $\sigma'$ is a Cartan involution, then the canonical connection on $G/H$ is induced by a $G$-invariant Riemannian metric.*

**Proof.** Define an $\mathrm{Ad}(H)$-invariant symmetric bilinear form on $\mathfrak{m}$ by:

$$\langle \cdot\,, \cdot \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, \quad \text{written} \quad \langle u, v \rangle := -B_{\mathfrak{g}}(u, v),$$

where $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the Killing form of $\mathfrak{g}$. Furthermore, since $\mathfrak{g}$ is semi-simple and $B_{\mathfrak{g}}(\mathfrak{h}, \mathfrak{m}) = 0$, we deduce that $\langle \cdot\,, \cdot \rangle$ is non-degenerate. ■

# Irreducible Symmetric Spaces

In what follows, $(G, H, \sigma)$ will be a symmetric pair, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ the canonical decomposition of $\mathfrak{g}$ corresponding to $\sigma$, and

$$\mathrm{ad}^{\mathfrak{m}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m}), \qquad u \mapsto [u, \cdot],$$

the adjoint representation of $\mathfrak{h}$ in $\mathfrak{m}$. Moreover, we put $M := G/H$ and we assume that the action of $G$ on $M$ is almost effective, i.e. the representation $\mathrm{ad}^{\mathfrak{m}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is injective.

**Definition**

$M$ is called *irreducible* if $\mathrm{ad}^{\mathfrak{m}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is irreducible.

**Proposition (1)**

*If $M$ is irreducible, then either*

$\mathfrak{g}$ *is semi-simple*, or $[\mathfrak{m}, \mathfrak{m}] = \{0\}$.

**Proof.** Let $\mathfrak{m}' := \mathrm{rad}(B_{\mathfrak{g}}) \cap \mathfrak{m}$. It is clear that $\mathfrak{m}'$ is an $\mathfrak{h}$-submodule of $\mathfrak{m}$ and hence either $\mathfrak{m}' = \{0\}$ or $\mathfrak{m}' = \mathfrak{m}$.

1. **If $\mathfrak{m}' = \{0\}$:** We shall prove that $\mathfrak{g}$ is semi-simple. Let $u \in \mathrm{rad}(B_{\mathfrak{g}})$, then write $u = u_{\mathfrak{m}} + u_{\mathfrak{h}}$ for $u_{\mathfrak{m}} \in \mathfrak{m}$ and $u_{\mathfrak{h}} \in \mathfrak{h}$. Since $B_{\mathfrak{g}}(\mathfrak{h}, \mathfrak{m}) = 0$, we have for $v \in \mathfrak{m}$

$$B_{\mathfrak{g}}(u_{\mathfrak{m}}, v) = B_{\mathfrak{g}}(u, v) = 0.$$

Thus $u_{\mathfrak{m}} \in \mathfrak{m}' = \{0\}$ and therefore $u \in \mathfrak{h} \cap \mathrm{rad}(B_{\mathfrak{g}})$. Hence $[u, v] = 0$ for all $v \in \mathfrak{m}$. Now, using the fact that $\mathrm{ad}^{\mathfrak{m}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is injective we deduce that $u = 0$, and it follows that $\mathfrak{g}$ is semi-simple.

2. **If $\mathfrak{m}' = \mathfrak{m}$:** In this case we have $\mathfrak{m} \subseteq \mathrm{rad}(B_{\mathfrak{g}})$. Recall that a nil ideal of $\mathfrak{g}$ is an ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathrm{ad}_u$ is nilpotent for all $u \in \mathfrak{n}$. We denote by $\mathrm{nilrad}(\mathfrak{g})$ the unique maximal nil ideal of $\mathfrak{g}$, then the following inclusion holds[3]

$$[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{nilrad}(\mathfrak{g}).$$

Hence, we have

$$\mathfrak{m} = [\mathfrak{h}, \mathfrak{m}] \subseteq [\mathfrak{h}, \mathrm{rad}(B_{\mathfrak{g}})] \subseteq [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{nilrad}(\mathfrak{g}).$$

Since $\mathrm{nilrad}(\mathfrak{g})$ is nilpotent, there exists a positive integer $k$ such that $\mathrm{nilrad}(\mathfrak{g})^k = \{0\}$ and therefore $\mathfrak{m}^k = \{0\}$. If $k = 1$ then we are done. Suppose that $k \geq 2$ and $k$ is odd, then it is clear that $\mathfrak{m}^{k-1}$ is an $\mathfrak{h}$-submodule of $\mathfrak{m}$.

[3] For more details about $\mathrm{rad}(\mathfrak{g})$ and $\mathrm{nilrad}(\mathfrak{g})$, we refer the interested reader to the book of V.S. Varadarajan (Ref.).

Thus either

$$\mathfrak{m}^{k-1} = \mathfrak{m}, \qquad \text{or} \qquad \mathfrak{m}^{k-1} = \{0\}.$$

In the first case, we get

$$[\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}, \mathfrak{m}^{k-1}] = \mathfrak{m}^k = \{0\}.$$

In the second case, we have

$$[\mathfrak{m}^{k-2}, \mathfrak{m}] = \mathfrak{m}^{k-1} = \{0\}.$$

Since $\mathfrak{m}^{k-2} \subset \mathfrak{h}$ and $\mathrm{ad}^{\mathfrak{m}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is injective we get that $\mathfrak{m}^{k-2} = \{0\}$. This argument shows that

$$[\mathfrak{m}, \mathfrak{m}] = \{0\}. \quad \blacksquare$$

**Proposition (2)**

*If $\mathfrak{g}$ is semi-simple, then $[\mathfrak{m},\mathfrak{m}] = \mathfrak{h}$.*

**Proof.** It is straightforward to see that $[\mathfrak{m},\mathfrak{m}] \neq \{0\}$, because otherwise $\mathfrak{m}$ will be an abelian ideal of $\mathfrak{g}$. Moreover, we can easily check that $\mathfrak{m} \oplus [\mathfrak{m},\mathfrak{m}]$ is an ideal of $\mathfrak{g}$ and therefore since $\mathfrak{g}$ is semi-simple, there exists a supplementary ideal $\mathfrak{a} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{m} \oplus [\mathfrak{m},\mathfrak{m}] \oplus \mathfrak{a}.$$

We will prove that $\mathfrak{a} = \{0\}$. Using that $\mathrm{ad}^{\mathfrak{m}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is injective, it is sufficient to show that $\mathfrak{a}$ is contained in $\mathfrak{h}$. Let $u \in \mathfrak{a}$, then write $u = u_{\mathfrak{m}} + u_{\mathfrak{h}}$ for $u_{\mathfrak{m}} \in \mathfrak{m}$ and $u_{\mathfrak{h}} \in \mathfrak{h}$.

For $v \in \mathfrak{m}$ one has

$$\underbrace{[u, v]}_{\in \mathfrak{a}} = \underbrace{[u_{\mathfrak{m}}, v]}_{\in [\mathfrak{m}, \mathfrak{m}]} + \underbrace{[u_{\mathfrak{h}}, v]}_{\in \mathfrak{m}}.$$

Thus $[u_{\mathfrak{m}}, v] = [u, v] = [u_{\mathfrak{h}}, v] = 0$. Similarly, for $v \in [\mathfrak{m}, \mathfrak{m}]$ we have

$$\underbrace{[u, v]}_{\in \mathfrak{a}} = \underbrace{[u_{\mathfrak{m}}, v]}_{\in \mathfrak{m}} + \underbrace{[u_{\mathfrak{h}}, v]}_{\in [\mathfrak{m}, \mathfrak{m}]}.$$

Hence $[u_{\mathfrak{m}}, v] = [u, v] = [u_{\mathfrak{h}}, v] = 0$. Let $v \in \mathfrak{g}$ and write $v = v_{\mathfrak{m}} + v_{[\mathfrak{m},\mathfrak{m}]} + v_{\mathfrak{a}}$ for $v_{\mathfrak{m}} \in \mathfrak{m}$, $v_{[\mathfrak{m},\mathfrak{m}]} \in [\mathfrak{m}, \mathfrak{m}]$, $v_{\mathfrak{a}} \in \mathfrak{a}$, then

$$[u_{\mathfrak{m}}, v] = [u_{\mathfrak{m}}, v_{\mathfrak{m}}] + [u_{\mathfrak{m}}, v_{[\mathfrak{m},\mathfrak{m}]}] + [u_{\mathfrak{m}}, v_{\mathfrak{a}}] = 0.$$

Thus $u_{\mathfrak{m}} \in Z(\mathfrak{g}) = \{0\}$, and it follows that $u = u_h \in \mathfrak{h}$. ■

# Irreducible Symmetric Spaces

**Theorem**

*Let $M := G/H$ be an irreducible symmetric space where the action of $G$ on $M$ is effective and $\mathfrak{g}$ is semi-simple. Then $\mathrm{Aff}^0(M, \nabla^0) = G$.*

**Proof.** First, since the action is effective, we can identify $G$ and $H$ with their images under the homogeneous action $\lambda$:

$$G \cong \lambda(G) \subseteq G^1 := \mathrm{Aff}^0(M, \nabla^0),$$

then $H \subseteq H^1 := G^1_{\overline{e}}$, the isotropy group of $\overline{e}$ in $G^1$. Let $\mathfrak{g}^1 = \mathfrak{m}^1 \oplus \mathfrak{h}^1$ be the canonical decomposition of the symmetric pair $(G^1, \sigma^1, H^1)$, where $\sigma^1(f) = \overline{\sigma} \circ f \circ \overline{\sigma}$.

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} G^1 & \xrightarrow{\sigma^1} & G^1 \\ \uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} \\ G & \xrightarrow{\sigma} & G, \end{array}$$

where $\iota : G \hookrightarrow G^1$ is the canonical injection. This implies $\mathfrak{m} \subseteq \mathfrak{m}^1$ and $\mathfrak{h} \subseteq \mathfrak{h}^1$. But since $M = G/H = G^1/H^1$ we have $\mathfrak{m} = \mathfrak{m}^1$. Then, $\mathfrak{h}^1 \to \mathrm{End}(\mathfrak{m}^1)$ is irreducible because $\mathfrak{h} \subseteq \mathfrak{h}^1$. Now, $\mathfrak{g}$ is semi-simple, then $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}^1, \mathfrak{m}^1]$, which implies (Proposition (1)) $\mathfrak{g}^1$ is semi-simple, and therefore $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}^1$ (Proposition (2)). Thus $\mathfrak{h} = \mathfrak{h}^1$, which proves that $\mathfrak{g}^1 = \mathfrak{g}$ and finally $G^1 = G$. ■

**Corollary**

*Let $(M := G/H, \langle \cdot\,, \cdot \rangle)$ be an irreducible (pseudo)Riemannian symmetric space where $G$ is effective on $M$ and $\mathfrak{g}$ is semi-simple. Then $G = \mathrm{Iso}^0(M, \langle \cdot\,, \cdot \rangle) = \mathrm{Aff}^0(M, \nabla^0)$.*

Now we return to our question:

**Given a symmetric space $(M, \mu)$, how can we define directly from $\mu$ a torsion-free connection on $M$ such that it becomes an affine symmetric space?**

The answer is complicated, so we will just sketch out the idea.

Starting from a symmetric space $(M, \mu)$, we will construct a torsion-free connection on $M$. But first we need to introduce some constructions.

Let $F \in C^\infty(M \times M)$ be a smooth function. For each $x \in M$ we define two smooth functions $F^\ell_x, F^r_x \in C^\infty(M)$ by:

$$F^\ell_x(y) := F(x, y), \quad \text{and} \quad F^r_x(y) := F(y, x).$$

We can use this to associated to each vector field $X \in \mathfrak{X}(M)$, two vector fields $X_\ell, X_r \in \mathfrak{X}(M \times M)$, defining they action on an arbitrary smooth fuction $F \in C^\infty(M \times M)$ by:

$$(X_\ell F)(x, y) := (XF^\ell_x)(y), \quad \text{and} \quad (X_r F)(x, y) := (XF^r_y)(x).$$

Let $X, Y \in \mathfrak{X}(M)$, the construction above allows as to define an operator

$$X \cdot Y : C^\infty(M) \to C^\infty(M),$$

by setting

$$(X \cdot Y)f := X_r Y_\ell (f \circ \mu) \circ \Delta,$$

where $\mu : M \times M \to M$ is the multiplication map and

$$\Delta : M \to M \times M, \quad x \mapsto (x, x),$$

is the diagonal mapping.

**Lemma**

*Let* $(U, x^i)$ *be a local chart of* $M$ *centered at* $x_0 \in M$*, and* $X, Y \in \mathfrak{X}(M)$ *two vector fields on* $M$*. Then if we write* $X = X^i \partial_{x^i}$ *and* $Y = Y^j \partial_{x^j}$ *on* $U$*, we have*

$$XY = X^i Y^j \frac{\partial^2}{\partial x^i \partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j},$$

*and*

$$\frac{1}{2} X \cdot Y = -X^i Y^j \frac{\partial^2}{\partial x^i \partial x^j} + \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

*where* $\Gamma^k_{ij}$ *are smooth functions defined on* $U$*.*

# The Canonical Connection on Symmetric Spaces

**Theorem**

*Let $(M, \{\mathfrak{s}_x\}_{x \in M})$ be a symmetric space, then there exists a unique torsion-free connection on $M$ such that each involution $\mathfrak{s}_x$ is a geodesic symmetry about $x$.*

**Sketch of the Proof.** For $X, Y \in \mathfrak{X}(M)$, we define

$$\nabla^0_X Y := XY + \frac{1}{2} X \cdot Y. \quad \blacksquare$$

For a full proof one can see Loos, Ottmar. *Symmetric spaces: General theory.* Vol. 1. WA Benjamin, 1969.

$$(M, \mu)$$

A symmetric space

$$\Updownarrow \qquad \qquad \Updownarrow$$

$(G, H, \sigma)$ A symmetric pair $\quad \Longleftrightarrow \quad$ $(M, \nabla^0)$ An affine symmetric space

# Example: Lie groups

Let $G$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then

$$(G, \mu)$$

$$a \cdot b := ab^{-1}a \quad \forall\, a, b \in G$$

$$(G \times G, \Delta G, \sigma) \Longleftrightarrow (G, \nabla^0)$$

$$\sigma(a,b) := (b,a) \quad \forall\, a, b \in G$$

$$\nabla^0_{u^+} v^+ := \frac{1}{2}[u^+, v^+] \quad \forall\, u, v \in \mathfrak{g}$$

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