UNIVERSITE HASSAN 2
FACULTE DES SCIENCES AIN-CHOCK
ROUTE D'ELJADIDA

On k-para-Kähler Lie algebras a subclass of k-symplectic Lie algebras

Aitbrik Ilham
Supervise by Pr. M. Boucetta

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Characterization of k-para-Kähler Lie algebras

Definition 1.1

Let $\mathfrak{g}$ be a $n(k+1)$-dimensional Lie algebra over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\theta^1, \ldots, \theta^k$ 2-forms of $\Lambda^2(\mathfrak{g})$ and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$ of codimension n. We recall that $(\theta^1, \ldots, \theta^k; \mathfrak{h})$ is a k-symplectic structure on $\mathfrak{g}$ if the following conditions are satisfied:

(i) *The family $(\theta^1, \ldots, \theta^k)$ is nondegenerate, i.e., $\bigcap_{i=1}^k \ker \theta^i = \{0\}$,*

(ii) *for $i = 1, \ldots, k$, θ^i is closed, i.e.,*
$d\theta^i(u, v, w) := \theta^i([u, v], w) + \theta^i([v, w], u) + \theta^i([w, u], v) = 0$,

(iii) *$\mathfrak{h}$ is totally isotropic with respect to $(\theta^1, \ldots, \theta^k)$, i.e., $\theta^i(u, v) = 0$ for any $u, v \in \mathfrak{h}$ and for $i = 1, \ldots, k$.*

Characterization of *k*-para-Kähler Lie algebras

$(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ is called a k-symplectic Lie algebra.

Characterization of *k*-para-Kähler Lie algebras

Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ be a k-symplectic Lie algebra where $\mathfrak{g}$ is a $n(k+1)$-dimensional Lie algebra and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ of dimension nk.
There exists always an isotropic supplementary $\mathfrak{p}$ of $\mathfrak{h}$ of dimension n (i.e., $\theta^\alpha |_{\mathfrak{p}} = 0$ for any $\alpha = 1, \ldots, k$.) such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.
In general there is not an isotropic Lie subalgebra supplementry $\mathfrak{p}$ of $\mathfrak{h}$.

Characterization of k-para-Kähler Lie algebras

Definition 1.2

Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ *be a k-symplectic Lie algebra. If* $\mathfrak{h}$ *admits an isotropic supplementary* $\mathfrak{p}$ *such that* $\mathfrak{p}$ *is a Lie subalgebra of* $\mathfrak{g}$, *then* $(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ *is a k-para-Kähler Lie algebra.*

Characterization of k-para-Kähler Lie algebras

Example 1

Let $(\mathfrak{g}, \mathfrak{h}, \theta^1)$ *be a 1-symplectic Lie algebra of dimension* $2n$ $(k = 1)$ *where* $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$ *of dimension n, then* $\mathfrak{h}$ *is lagrangian.*
Suppose that $\mathfrak{h}$ *admits an isotropic Lie subalgebra supplementary* $\mathfrak{p}$ *of dimension n, that is,* $\mathfrak{p}$ *is lagrangian. Hence* $(\mathfrak{g}, \mathfrak{h}, \theta^1)$ *is a para-Kähler Lie algebra.*

Characterization of k-para-Kähler Lie algebras

Definition 1.3

A left symmetric algebra is an algebra $(A, \bullet)$ *such that for any* $a, b, c \in A$,

$$\mathrm{ass}(a, b, c) = \mathrm{ass}(b, a, c) \quad \textit{where} \quad \mathrm{ass}(a, b, c) = (a \bullet b) \bullet c - a \bullet (b \bullet c).$$

Characterization of k-para-Kähler Lie algebras

Let $(\mathfrak{g}, \mathfrak{h}, \mathfrak{p}, \theta^\alpha)$ be a k-para-Kaheler Lie algebra for any $\alpha = 1, \ldots, k$.

1. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, then for any $p \in \mathfrak{p}$ and any $h \in \mathfrak{h}$ the Lie bracket $[p, h]$ can be written

$$[p, h] = -[h, p] = \phi_{\mathfrak{p}}(h) - \phi_{\mathfrak{h}}(p), \tag{1}$$

   where $\phi_p(h) \in \mathfrak{h}$ and $\phi_h(p) \in \mathfrak{p}$.

Characterization of k-para-Kähler Lie algebras

2. $\mathfrak{h} = \oplus_{\alpha=1}^{k} \mathfrak{h}^{\alpha}$ where, $\mathfrak{h}^{\alpha} = \bigcap_{\beta \neq \alpha} \ker \theta^{\beta}$.

Characterization of k-para-Kähler Lie algebras

2. $\mathfrak{h} = \oplus_{\alpha=1}^{k} \mathfrak{h}^{\alpha}$ where, $\mathfrak{h}^{\alpha} = \bigcap_{\beta \neq \alpha} \ker \theta^{\beta}$.
3. $\mathfrak{h}$ has a structure of left symmetric algebra such that $\mathfrak{h} \bullet \mathfrak{h}^{\alpha} \subset \mathfrak{h}^{\alpha}$ where, the left symmetric product $\bullet$ on $\mathfrak{h}$ is given by

$$\theta^{\alpha}(h_1 \bullet h_2, p) = -\theta^{\alpha}(h_2, [h_1, p]), \tag{2}$$

   for any $h_1, h_2 \in \mathfrak{h}$, for any $p \in \mathfrak{g}$

4. $i_\alpha : \mathfrak{h}^\alpha \longrightarrow \mathfrak{p}^*$ given by

$$i_\alpha(h)(p) = \theta^\alpha(h, p).$$

The linear map i_α is an isomorphism.

Characterization of k-para-Kähler Lie algebras

4. $i_\alpha : \mathfrak{h}^\alpha \longrightarrow \mathfrak{p}^*$ given by

$$i_\alpha(h)(p) = \theta^\alpha(h, p).$$

The linear map i_α is an isomorphism.

5. A family of products $\star_{\alpha,\beta}$ on $\mathfrak{p}$, given by

$$\theta^\alpha(p \star_{\alpha,\beta} q, h) = -\theta^\beta(q, [p, h]). \qquad (3)$$

Where, for any $\alpha, \beta \in \{1, \dots, k\}$ with $\alpha \neq \beta$ and for any $p_1, p_2 \in \mathfrak{p}$,

$$[p_1, p_2] = p_1 \star_{\alpha,\alpha} p_2 - p_2 \star_{\alpha,\alpha} p_1, \quad p_1 \star_{\alpha,\beta} p_2 = p_2 \star_{\alpha,\beta} p_1.$$

Characterization of *k*-para-Kähler Lie algebras

6. A family of products $\bullet_{\alpha,\beta}$ on $\mathfrak{p}^*$ given by

$$a \bullet_{\alpha\beta} b = i_\beta(i_\alpha^{-1}(a) \bullet i_\beta^{-1}(b)). \tag{4}$$

where , for any α, β, γ, $\bullet_{\alpha\beta} = \bullet_{\alpha\gamma}$ and if we denote $\bullet_{\alpha\beta} = \bullet_\alpha$, we have, for any $a, b, c \in \mathfrak{p}^*$,

$$a \bullet_\alpha (b \bullet_\beta c) - (a \bullet_\alpha b) \bullet_\beta c = b \bullet_\beta (a \bullet_\alpha c) - (b \bullet_\beta a) \bullet_\alpha c. \tag{5}$$

Characterization of k-para-Kähler Lie algebras

We consider $\Phi(\mathfrak{p}, k) = \mathfrak{p} \oplus (\mathfrak{p}^*)^k$

Characterization of *k*-para-Kähler Lie algebras

We consider $\Phi(\mathfrak{p}, k) = \mathfrak{p} \oplus (\mathfrak{p}^*)^k$

1. We endow $(\mathfrak{p}^*)^k$ with the product $\circ$ given by

$$(a_1, \ldots, a_k) \circ (b_1, \ldots, b_k) = \left(\sum_{\alpha=1}^{k} a_\alpha \bullet_\alpha b_1, \ldots, \sum_{\alpha=1}^{k} a_\alpha \bullet_\alpha b_k \right). \quad (6)$$

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2. We define $\phi : (\mathfrak{p}^*)^k \otimes \mathfrak{p}^* \longrightarrow \mathfrak{p}^*$ and $\psi : \mathfrak{p} \otimes \mathfrak{p}^k \longrightarrow \mathfrak{p}^k$ by

$$\begin{cases} \phi((a_1, \ldots, a_k), b) = \phi_{(a_1, \ldots, a_k)} b = \displaystyle\sum_{\alpha=1}^{k} \mathrm{L}^{\alpha}_{a_\alpha} b, \\ \psi(q, (p_1, \ldots, p_k)) = \psi_q(p_1, \ldots, p_k) = \displaystyle\sum_{\alpha=1}^{k} \left(\mathrm{L}^{\alpha,1}_q p_\alpha, \ldots, \mathrm{L}^{\alpha,k}_q p_\alpha \right). \end{cases} \quad (7)$$

where $\mathrm{L}^{\alpha}_a : \mathfrak{p}^* \longrightarrow \mathfrak{p}^*$, $b \mapsto a \bullet_\alpha b$ and $\mathrm{L}^{\alpha,\beta}_q : \mathfrak{p} \longrightarrow \mathfrak{p}$, $p \mapsto q \star_{\alpha,\beta} p$,

Characterization of k-para-Kähler Lie algebras

3. We endow $\Phi(\mathfrak{p}, k)$ with the bracket

$$\begin{cases} [a, b]_n = a \circ b - b \circ a, & \text{if } a, b \in (\mathfrak{p}^*)^k \\ [p, q]_n = [p, q], & \text{if } p, q \in \mathfrak{p} \\ [a, p]_n = \phi_a^*(p) - \psi_p^* a, & \text{if } a \in (\mathfrak{p}^*)^k, p \in \mathfrak{p} \end{cases} \tag{8}$$

where

$$\prec b, \phi_a^*(p) \succ = - \prec \phi_a b, p \succ \quad \text{and}$$

$$\prec \psi_p^* a, (p_1, \dots, p_k) \succ = - \prec a, \psi_p(p_1, \dots, p_k) \succ .$$

Characterization of k-para-Kähler Lie algebras

3. We endow $\Phi(\mathfrak{p}, k)$ with the bracket

$$\begin{cases} [a,b]_n = a \circ b - b \circ a, & \text{if } a,b \in (\mathfrak{p}^*)^k \\ [p,q]_n = [p,q], & \text{if } p,q \in \mathfrak{p} \\ [a,p]_n = \phi_a^*(p) - \psi_p^* a, & \text{if } a \in (\mathfrak{p}^*)^k, p \in \mathfrak{p} \end{cases} \tag{8}$$

where

$$\prec b, \phi_a^*(p) \succ = - \prec \phi_a b, p \succ \quad \text{and}$$

$$\prec \psi_p^* a, (p_1, \ldots, p_k) \succ = - \prec a, \psi_p(p_1, \ldots, p_k) \succ .$$

4. We define a family of 2-forms ρ^α, $\alpha = 1, \ldots, k$ by

$$\rho^\alpha(p + (a_1, \ldots, a_k), q + (b_1, \ldots, b_k)) = \prec a_\alpha, q \succ - \prec b_\alpha, p \succ . \tag{9}$$

Characterization of k-para-Kähler Lie algebras

Theorem 1.1

$(\Phi(\mathfrak{p}, k), [\ ,\ ]_n, (\mathfrak{p}^*)^k, \rho^1, \ldots, \rho^k)$ is a k-para-Kähler Lie algebra and $F : \mathfrak{g} \longrightarrow \Phi(\mathfrak{p}, k), (h_1 + \ldots + h_k + p) \mapsto (p, i_1(h_1), \ldots, i_k(h_k))$ is an isomorphism of k-para-Kähler Lie algebras.

Definition 1.4

A k-left symmetric algebra is a real vector space $\mathcal{A}$ endowed with k left symmetric products $\bullet_1, \ldots, \bullet_k$ such that one of the following equivalent assertions hold:

1. *For any $\alpha, \beta \in \{1, \ldots, k\}$ and for any $a, b, c \in \mathcal{A}$,*

$$a \bullet_\alpha (b \bullet_\beta c) - (a \bullet_\alpha b) \bullet_\beta c = b \bullet_\beta (a \bullet_\alpha c) - (b \bullet_\beta a) \bullet_\alpha c. \qquad (10)$$

2. *$(\mathcal{A}^k, \circ)$ is a left symmetric algebra where $\circ$ is given by*

$$(a_1, \ldots, a_k) \circ (b_1, \ldots, b_k) = \left(\sum_{\alpha=1}^{k} a_\alpha \bullet_\alpha b_1, \ldots, \sum_{\alpha=1}^{k} a_\alpha \bullet_\alpha b_k \right). \qquad (11)$$

Definition 1.5

A $(k \times k)$-left symmetric algebra is a vector space $\mathcal{B}$ endowed with a $k \times k$-matrix $(\star_{\alpha,\beta})_{1 \leq \alpha,\beta \leq k}$ of products such that:

1. *For any α, β and for any $p, q \in \mathcal{B}$,*

$$p \star_{\alpha,\alpha} q - q \star_{\alpha,\alpha} p = p \star_{\beta,\beta} q - q \star_{\beta,\beta} p = [p, q].$$

2. *$\star_{\alpha,\beta}$ are commutative when $\alpha \neq \beta$,*

Example 2

1. *The notions of* 1*-left symmetric algebra and* (1×1)*-left symmetric algebra are the same and correspond to the classical notion of left symmetric algebra.*

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2. *Let* $(\mathcal{A}, \bullet)$ *be a left symmetric algebra. For any* $k \geq 1$*, endow* $\mathcal{A}$ *with the k-left symmetric structure given by* $\bullet_\alpha = \mu_\alpha \bullet$*, where* $\mu_\alpha \in \mathbb{R}$*. Then* $(\mathcal{A}, \bullet_1, \ldots, \bullet_k)$ *is a k-left symmetric algebra.*

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3. *If* $\bullet_1, \ldots, \bullet_k$ *are left symmetric products on* $\mathcal{B}$ *such that* $a \bullet_\alpha b - b \bullet_\alpha a = a \bullet_\beta b - b \bullet_\beta a$ *for any* α, β *then* $(\mathcal{B}, (\star_{\alpha,\beta})_{1 \leq \alpha \leq \beta \leq k})$ *is* $(k \times k)$*-left symmetric algebra where* $\star_{\alpha,\beta} = 0$ *if* $\alpha \neq \beta$ *and* $\star_{\alpha,\alpha} = \bullet_\alpha$.

Characterization of k-para-Kähler Lie algebras

Let study the converse.

Characterization of k-para-Kähler Lie algebras

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We consider:

1. A vector space $\mathfrak{p}$ of dimension n.
2. A k-left symmetric structure $(\bullet_1, \dots, \bullet_k)$ on $\mathfrak{p}^*$. This defines a left symmetric product $\circ$ on $(\mathfrak{p}^*)^k$ and hence a Lie algebra structure on $(\mathfrak{p}^*)^k$
$$[a, b] = a \circ b - b \circ a$$
3. A $(k \times k)$-left symmetric structure $\star_{\alpha,\beta}$ on $\mathfrak{p}$. This defines a Lie algebra structure on $\mathfrak{p}$ by
$$[p, q]_{\mathfrak{p}} = p \star_{\alpha,\alpha} q - q \star_{\alpha,\alpha} q.$$

Characterization of k-para-Kähler Lie algebras

We consider $\phi(\mathfrak{p}, k) = \mathfrak{p} \oplus (\mathfrak{p}^*)^k$.

Characterization of k-para-Kähler Lie algebras

We consider $\phi(\mathfrak{p},k)=\mathfrak{p}\oplus(\mathfrak{p}^*)^k$.
We define on $\phi(\mathfrak{p},k)$:

1. The bracket

$$[a,b]=a\circ b-b\circ a,\ [p,q]=[p,q]_{\mathfrak{p}}\quad \text{and}\quad [a,p]=\phi_a^*(p)-\psi_p^*a,\quad a,b\in(\mathfrak{p}^*)^k \tag{12}$$

2. The family $(\rho^1,\ldots,\rho^k)$ of 2-forms given by

$$\rho^\alpha(p+(a_1,\ldots,a_k),q+(b_1,\ldots,b_k))=\prec a_\alpha,q\succ-\prec b_\alpha,p\succ.$$

Characterization of k-para-Kähler Lie algebras

We denote by $\phi^T : \mathfrak{p} \longrightarrow \mathfrak{p}^k \otimes \mathfrak{p}$ and $\psi^T : (\mathfrak{p}^*)^k \longrightarrow (\mathfrak{p}^*) \otimes (\mathfrak{p}^*)^k$ the dual of $\phi : (\mathfrak{p}^*)^k \otimes \mathfrak{p}^* \longrightarrow \mathfrak{p}^*$ and $\psi : \mathfrak{p} \otimes \mathfrak{p}^k \longrightarrow \mathfrak{p}$.

Characterization of *k*-para-Kähler Lie algebras

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Question: Under which condition the bracket given in (12) is a Lie bracket?

Characterization of k-para-Kähler Lie algebras

Theorem 1.2

$(\Phi(\mathfrak{p}, k), [\ ,\ ])$ is a Lie algebra if and only if

1. $\phi^T : \mathfrak{p} \longrightarrow \mathfrak{p}^k \otimes \mathfrak{p}$ *is a 1-cocycle of* $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}})$ *and the representation* $\psi \otimes \mathrm{ad}$, *i.e.*,

$$\begin{aligned}\phi^T([p,q]_{\mathfrak{p}})((a_1,\ldots,a_k),b) &= \phi^T(p)((a_1,\ldots,a_k),\mathrm{ad}_q^* b)\\ &+\phi^T(p)(\psi_q^*(a_1,\ldots,a_k),b)\\ &-\phi^T(q)((a_1,\ldots,a_k),\mathrm{ad}_p^* b)\\ &-\phi^T(q)(\psi_p^*(a_1,\ldots,a_k),b).\end{aligned}$$

2. $\psi^T : (\mathfrak{p}^*)^k \longrightarrow (\mathfrak{p}^*) \otimes (\mathfrak{p}^*)^k$ *is a 1-cocycle of* $((\mathfrak{p}^*)^k, [\ ,\ ])$ *and the representation* $\phi \otimes \mathrm{ad}$ *and* $[\ ,\ ]$ *is given by* $[a,b] = a \circ b - b \circ a$, *i.e.*,

Characterization of k-para-Kähler Lie algebras

$$\begin{aligned}\psi^T([a,b])(p,(q_1,\dots,q_k)) &= \psi^T(a)(p,\mathrm{ad}_b^*(q_1,\dots,q_k))\\ &+\psi^T(a)(\phi_b^*p,(q_1,\dots,q_k))\\ &-\psi^T(b)(p,\mathrm{ad}_a^*(q_1,\dots,q_k))\\ &-\psi^T(b)(\phi_a^*p,(q_1,\dots,q_k)).\end{aligned}$$

In this case $(\Phi(\mathfrak{p},k),[\ ,\ ],(\mathfrak{p}^*)^k,\rho^1,\dots,\rho^k)$ is a k-para-Kähler Lie algebra. Moreover, all k-para-Kähler Lie algebras are obtained in this way.

Characterization of k-para-Kähler Lie algebras

Definition 1.6

A $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ and a k-left symmetric algebra structure on $\mathfrak{p}^*$ are called compatible if they satisfy the conditions of the previous Theorem.

Example 3

1. *Any k-left symmetric algebra structure on $\mathfrak{p}^*$ is compatible with the trivial $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$.*
2. *Any $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ is compatible with the trivial k-left symmetric algebra structure on $\mathfrak{p}^*$.*

Characterization of *k*-para-Kähler Lie algebras

Conclusion

k-para-Kahler Lie algebras are obtained by:

1. A vector space $\mathfrak{p}$ of dimension n.
2. A k-left symmetric structure $(\bullet_1, \ldots, \bullet_k)$ on $\mathfrak{p}^*$.
3. A $(k \times k)$-left symmetric structure $\star_{\alpha,\beta}$ on $\mathfrak{p}$.
4. and the 2 structures are compatible.

Characterization of k-para-Kähler Lie algebras

Proposition 1.1

Let $(A, .)$ be a commutative associative algebra and $(D_1, \dots, D_k)$ the derivations of $(A, .)$ which commute. Then for any $\alpha = 1, \dots k$, the products

$$a \bullet_\alpha b = a.D_\alpha b$$

are left symmetric and define a k-left symmetric structure on A.

Example 4

We consider $\mathbb{R}^4$ endowed with the associative commutative product

$$e_1.e_1 = e_1,\ e_1.e_2 = e_2.e_1 = e_2,\ e_1.e_3 = e_3.e_1 = e_3,\ e_1.e_4 = e_4.e_1 = e_4.$$

We consider the two derivations

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad and \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

These two derivations commute and, according to the previous Proposition , they define a 2-left symplectic structure on $\mathbb{R}^4$ by

$$e_1 \bullet_1 e_i = e_i, i = 2, 3, 4, \quad e_1 \bullet_2 e_3 = ae_2 \quad and \quad e_1 \bullet_2 e_4 = be_2 + ce_3.$$

Exact k-para-Kähler Lie algebras

We consider :

1. A k-left symmetric structure $(\bullet_1, \dots, \bullet_k)$ on $\mathfrak{p}^*$.
2. $\psi = \delta(\mathbf{r})$ is a coboundary, i.e., for $\mathbf{r} \in \mathfrak{p}^* \otimes (\mathfrak{p}^*)^k$ $\psi : \mathfrak{p} \otimes (\mathfrak{p})^k \longrightarrow \mathfrak{p}^k$ is given by

$$\prec a, \psi(p, u) \succ = -\mathbf{r}(\phi_a^* p, u) - \mathbf{r}(p, \mathrm{ad}_a^* u), \qquad p \in \mathfrak{p}, u \in \mathfrak{p}^k, a \in (\mathfrak{p}^*)^k. \tag{13}$$

Exact k-para-Kähler Lie algebras

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We define $\mathbf{r}_\# : \mathfrak{p} \longrightarrow (\mathfrak{p}^*)^k$ by $\prec \mathbf{r}_\#(p), u \succ = \mathbf{r}(p, u)$

Exact k-para-Kähler Lie algebras

Problem: Under which condition ψ defines a $k \times k$-left symmetric structure on $\mathfrak{p}$ compatible with the k-left symmetric structure on $\mathfrak{p}^*$.

Theorem 2.1

Let $\mathfrak{p}$ be a vector space of dimension n such that $\mathfrak{p}^*$ is endowed with a k-left symmetric algebra structure $(\bullet_1, \ldots, \bullet_k)$ and
$\mathbf{r} = (\mathbf{s}_1 + \mathbf{a}_1, \ldots, \mathbf{s}_k + \mathbf{a}_k) \in \mathfrak{p}^* \otimes (p^*)^k$ *such that, for any $\alpha \neq \beta$ and for any $\rho \in \mathfrak{p}^*$,*

$$\mathrm{L}^{\alpha}_{\rho}(\mathbf{a}_{\beta}) = 0 \quad \textit{and} \quad \mathrm{L}^{\alpha}_{\rho}(\mathbf{a}_{\alpha}) = \mathrm{L}^{\beta}_{\rho}(\mathbf{a}_{\beta}) =: \mathrm{L}(\mathbf{a})(\rho, ., .).$$

Then ψ given by (13) defines a $(k \times k)$-left symmetric structure on $\mathfrak{p}$ compatible with the k-left symmetric structure of $(\mathfrak{p}^*)^k$ if and only if, for any $a \in (p^*)^k$ and $p, q \in \mathfrak{p}$,

$$[a, \Delta(\mathbf{r})(p,q)] + \mathrm{L}_a(\Delta(\mathbf{r}))(p,q) = 0, \quad \Delta(\mathbf{r})(p,q) = \mathbf{r}_{\#}([p,q]_{\mathfrak{p}}) - [\mathbf{r}_{\#}(p), \mathbf{r}_{\#}(q)]$$

and, for any $a \in (\mathfrak{p}^*)^k$, $\rho \in \mathfrak{p}^*$, $p, q \in \mathfrak{p}$,

$$\mathrm{L}(\mathbf{a})(\mathrm{L}_a\rho, p, q) + \mathrm{L}(\mathbf{a})(\rho, \mathrm{L}^*_a p, q) + \mathrm{L}(\mathbf{a})(\rho, p, \mathrm{L}^*_a q) = 0.$$

Exact k-para-Kähler Lie algebras

Corollary 2.1

Let $\mathbf{r} = (\mathbf{s}_1, \ldots, \mathbf{s}_k)$ *be a family of symmetric elements of* $\mathfrak{p}^* \otimes \mathfrak{p}^*$. *Then* ψ *defines a* $(k \times k)$*-left symmetric structure on* $\mathfrak{p}$ *compatible with the k-left symmetric structure of* $(\mathfrak{p}^*)^k$ *if and only if, for any* $a \in (p^*)^k$ *and* $p, q \in \mathfrak{p}$,

$$[a, \Delta(\mathbf{r})(p, q)] + \mathrm{L}_a(\Delta(\mathbf{r}))(p, q) = 0, \; \Delta(\mathbf{r})(p, q) = \mathbf{r}_{\#}([p, q]_{\mathfrak{p}}) - [\mathbf{r}_{\#}(p), \mathbf{r}_{\#}(q)]$$

Definition 2.1

Let $\mathbf{r} = (\mathbf{r}^1, \ldots, \mathbf{r}^k)$ *be a family of symmetric elements of* $\mathcal{A} \otimes \mathcal{A}$ *where* $\mathcal{A}$ *has a structure of k-left symmetric algebra* $(\bullet_1, \ldots, \bullet_k)$*. We call* $\mathbf{r}$ *a* S_k*-matrix if* $\Delta(\mathbf{r}) = 0$ *where* $\Delta(\mathbf{r})(p,q) = \mathbf{r}_\#([p,q]_\mathfrak{p}) - [\mathbf{r}_\#(p), \mathbf{r}_\#(q)]$*, i.e., for any* $\alpha = 1, \ldots, k$, $p, q \in \mathcal{A}^*$,

$$\mathbf{r}^\alpha_\#([p,q]_*) = \sum_{\beta=1}^{k} \left[\mathbf{r}^\beta_\#(p) \bullet_\beta \mathbf{r}^\alpha_\#(q) - \mathbf{r}^\beta_\#(q) \bullet_\beta \mathbf{r}^\alpha_\#(p) \right],$$

where

$$[p,q]_* = \sum_{\beta=1}^{k} \left[(\mathrm{L}^\beta_{\mathbf{r}^\beta_\#(p)})^* q - (\mathrm{L}^\beta_{\mathbf{r}^\beta_\#(q)})^* p \right].$$

Example 5

1. *Let $(\mathcal{A}, \bullet)$ be a left symmetric algebra and $\mathbf{r} \in \mathcal{A} \otimes \mathcal{A}$ be a classical S-matrix, i.e.,* $\mathbf{r}$ *satisfies*

$$\mathbf{r}\left(\mathrm{L}^*_{\mathbf{r}_\#(p)}q - \mathrm{L}^*_{\mathbf{r}_\#(p)}q\right) = \mathbf{r}_\#(p) \bullet \mathbf{r}_\#(q) - \mathbf{r}_\#(q) \bullet \mathbf{r}_\#(p),$$

   for any $p, q \in \mathcal{A}^*$ (see [6, 9]). For any $k \geq 1$, endow $\mathcal{A}$ with the k-left symmetric structure given by $\bullet_\alpha = \mu_\alpha \bullet$, where $\mu_\alpha \in \mathbb{R}$. Then $\mathbf{r}^k = (\mathbf{r}, \ldots, \mathbf{r})$ is a S_k-matrix.

2. *Consider the 2-left symmetric on $\mathbb{R}^4$ given in the previous Example , then one can check by a direct computation that*

$$\mathbf{r}^1 = r_{2,4}e_2 \odot e_4 + r_{2,2}e_2 \odot e_2 + r_{4,4}e_4 \odot e_4 \quad \text{and} \quad \mathbf{r}^2 = s_{1,1}e_1 \odot e_1 + s_{1,2}e_1 \odot e_2$$

   constitute a S_2-matrix on $\mathbb{R}^4$ ($\odot$ is the symmetric product).

k-symplectic Lie algebras of dimension $(k+1)$

Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ be a k-symplectic Lie algebra of dimension $(k+1)$.

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Then the $\dim \mathfrak{h} = k$, $\dim \mathfrak{h}^\alpha = 1$ and $\dim \mathfrak{p}^* = 1$.

Theorem 3.1

Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \theta^2)$ be a 2-symplectic Lie algebra of dimension 3. Then one of the following situations holds:

1. *$\mathfrak{h}$ is an abelian ideal and there exists a basis (e, f, g) of $\mathfrak{g}$ and D an endomorphism of $\mathfrak{h}$ such that $[h, e] = D(h)$ for any $h \in \mathfrak{h}$, $\theta^1 = e^* \wedge f^*$ and $\theta^2 = e^* \wedge g^*$.*
2. *$(\mathfrak{g}, \mathfrak{h}, \theta^1, \theta^2)$ is isomorphic to $(\mathrm{sl}(2, \mathbb{R}), \mathfrak{h}_0, \rho^1, \rho^2)$ with $\mathfrak{h}_0 = \mathrm{span}\{h, g\}$, $\rho^1 = h^* \wedge f^* + bg^* \wedge f^*$ and $\rho^2 = g^* \wedge f^*$.*
3. *$(\mathfrak{g}, \mathfrak{h}, \theta^1, \theta^2)$ is isomorphic to $(\mathfrak{sol}, \mathfrak{h}_0, \rho^1, \rho^2)$ with $\mathfrak{h}_0 = \mathrm{span}\{u_1, u_2\}$, $\rho^1 = u_1^* \wedge u_3^* + bu_2^* \wedge u_3^*$ and $\rho^2 = cu_1^* \wedge u_3^* + u_2^* \wedge u_3^*$.*

Theorem 3.2

Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ be a k-symplectic Lie algebra such that $\dim \mathfrak{h} = k \geq 3$*. Then one of the following situation holds:*

1. *$\mathfrak{h}$ is an abelian ideal and there exists a basis $(e, f_1, \ldots, f_k)$ of $\mathfrak{g}$ and an endomorphism D of $\mathfrak{h}$ such that $\mathfrak{h} = \mathrm{span}\{f_1, \ldots, f_k\}$, $[e, h] = D(h)$ for any $h \in \mathfrak{h}$ and, for $\alpha = 1, \ldots, k$, $\theta^\alpha = f_\alpha^* \wedge e^*$*
2. *There exists $(f_1, \ldots, f_k, e)$ a basis of $\mathfrak{g}$, a family of constants $(a_1, \ldots, a_k) \in \mathbb{R}^k$, $a_1 \neq 0$, $(b_2, \ldots, b_k) \in \mathbb{R}^{k-1}$ and $\lambda \in \mathbb{R}$ such that $\mathfrak{h} = \mathrm{span}\{f_1, \ldots, f_k\}$,*

$$\theta^1 = f_1^* \wedge e^* - \sum_{i=2}^{k} a_i f_i^* \wedge e^* \quad \text{and} \quad \theta^i = a_1 f_i^* \wedge e^*, i = 2, \ldots, k,$$

and the non vanishing Lie brackets are given by

$$[e, f_1] = a_1 e + \lambda f_1 + \sum_{l=2}^{k} b_l f_l, \;\; [e, f_i] = -\lambda f_i, \;\; [f_1, f_i] = a_1 f_i, i = 2, \ldots, k.$$

Six dimensional 2-para-Kähler Lie algebras

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2. In Table 2, we give for each 2-left symmetric algebra in Table 1 its compatible 2×2-left symmetric algebras.
3. In Table 3, we give for each couple of compatible structures in Table 2 the corresponding 2-para-Kähler Lie algebra.
4. All our computations were checked by using the software Maple.

Six dimensional 2-para-Kähler Lie algebras

Name of the 2-LSS	First left symmetric product	Second left symmetric product
$\mathbf{b}_{1,\alpha}, (\alpha \neq 1, \alpha \neq \frac{1}{2})$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = \alpha e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_{1,\frac{1}{2}}$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = \frac{1}{2} e_2$	$e_2 \bullet_2 e_1 = a e_1, e_2 \bullet_2 e_2 = \frac{1}{2} a e_2 + b e_1$
$\mathbf{b}_{1,1}$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = e_2$	$e_1 \bullet e_1 = a e_1, e_1 \bullet_2 e_2 = a e_2, e_2 \bullet_2 e_1 = b e_1,$ $e_2 \bullet_2 e_2 = b e_2$
$\mathbf{b}_2$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = e_1 + e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_{3,\alpha}, \alpha \neq 1, \alpha \neq 0,$	$e_1 \bullet_1 e_2 = e_1, e_2 \bullet_1 e_1 = (1 - \frac{1}{\alpha}) e_1, e_2 \bullet_1 e_2 = e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_{3,1}$	$e_1 \bullet_1 e_2 = e_1, e_2 \bullet_1 e_2 = e_2$	$e_1 \bullet_2 e_1 = a e_1, e_1 \bullet_2 e_2 = b e_1, e_2 \bullet_2 e_1 = a e_2,$ $e_2 \bullet_2 e_2 = b e_2$
$\mathbf{b}_4$	$e_1 \bullet_1 e_2 = e_1, e_2 \bullet_1 e_2 = e_1 + e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_5^+$	$e_1 \bullet_1 e_1 = e_2, e_2 \bullet_1 e_1 = -e_1, e_2 \bullet_1 e_2 = -2e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_5^-$	$e_1 \bullet_1 e_1 = -e_2, e_2 \bullet_1 e_1 = -e_1, e_2 \bullet_1 e_2 = -2e_2$	$\bullet_2 = a \bullet_1$

Six dimensional 2-para-Kähler Lie algebras

$\mathbf{c}_2$	$e_2 \bullet_1 e_2 = e_2$	$e_1 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = be_2$
$\mathbf{c}_3^1$	$e_2 \bullet_1 e_2 = e_1$	$e_2 \bullet_2 e_1 = 2ae_1, e_2 \bullet_2 e_2 = be_1 + ae_2$
$\mathbf{c}_3^2$	$e_2 \bullet e_2 = e_1$	$e_1 \bullet_2 e_2 = ae_1, e_2 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = be_1 + ae_2$
$\mathbf{c}_4$	$e_2 \bullet e_2 = e_2, e_1 \bullet_1 e_2 = e_2 \bullet_1 e_1 = e_1$	$e_1 \bullet_2 e_2 = ae_1, e_2 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = be_1 + ae_2$
$\mathbf{c}_5^+$	$e_1 \bullet_1 e_1 = e_2 \bullet_1 e_2 = e_2, e_1 \bullet_1 e_2 = e_2 \bullet_1 e_1 = e_1$	$e_1 \bullet_2 e_2 = e_2 \bullet_2 e_1 = be_1 + ae_2$ $e_1 \bullet_2 e_1 = e_2 \bullet_2 e_2 = ae_1 + be_2$
$\mathbf{c}_5^-$	$e_1 \bullet_1 e_1 = -e_2 \bullet_1 e_2 = -e_2, e_1 \bullet_1 e_2 = e_2 \bullet_1 e_1 = e_1$	$e_1 \bullet_2 e_2 = e_2 \bullet_2 e_1 = be_1 + ae_2$ $e_1 \bullet_2 e_1 = -e_2 \bullet_2 e_2 = ae_1 - be_2$

Table 1: Two dimensional 2-left symmetric structures, $(a, b) \in \mathbb{R}^2$.

Six dimensional 2-para-Kähler Lie algebras

Name	2-left symmetric structure	Compatible (2×2)-left symmetric structure	conditions
$\mathbf{bb}_{1,\alpha}$	$\mathbf{b}_{1,\alpha}, (\alpha \neq 1, \alpha \neq \frac{1}{2})$	$L^{1,1}_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & -ac \end{pmatrix}, L^{1,2}_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & -ad \end{pmatrix}, L^{2,1}_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, L^{2,2}_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	$a \in \mathbb{R},\ \alpha = 0$
$\mathbf{bb}_{1,1}$	$\mathbf{b}_{1,1}$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1, 2\}$	$a = 0,\ b \in \mathbb{R}$
$\mathbf{bb}_2$	$\mathbf{b}_2$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1, 2\}$	$a \neq 1$
		$L^{1,1}_{e_2} = L^{1,2}_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, L^{2,1}_{e_2} = L^{2,2}_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$	$a = 1$
$\mathbf{bb}_{3,1}$	$\mathbf{b}_{3,1}$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1, 2\}$	$a \neq 0,\ b \in \mathbb{R}$
$\mathbf{bb}_4$	$\mathbf{b}_4$	$L^{1,1}_{e_1} = \begin{pmatrix} 0 & 0 \\ -ac & 0 \end{pmatrix},\ L^{1,2}_{e_1} = \begin{pmatrix} 0 & 0 \\ -a^2c & 0 \end{pmatrix},\ L^{2,1}_{e_1} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},\ L^{2,2}_{e_1} = \begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix}$	$a \in \mathbb{R}$
$\mathbf{cc}^1_3$	$\mathbf{c}^1_3$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1, 2\}$	$a \neq 0,\ b \in \mathbb{R}$
		$L^{1,1}_{e_1} = \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \end{pmatrix},\ L^{1,2}_{e_1} = \begin{pmatrix} bc_1 & 0 \\ d_1 & 0 \end{pmatrix},\ L^{2,1}_{e_1} = \begin{pmatrix} g_1 & 0 \\ g_2 & 0 \end{pmatrix},\ L^{2,2}_{e_1} = \begin{pmatrix} bg_1 & 0 \\ d_2 & 0 \end{pmatrix}$	$a = 0,\ b \in \mathbb{R}$
$\mathbf{cc}^2_3$	$\mathbf{c}^2_3$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1, 2\}$	$a \neq 0,\ b \in \mathbb{R}$
		$L^{1,1}_{e_1} = \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \end{pmatrix},\ L^{1,2}_{e_1} = \begin{pmatrix} bc_1 & 0 \\ d & 0 \end{pmatrix},\ L^{2,1}_{e_1} = \begin{pmatrix} g_1 & 0 \\ g_2 & 0 \end{pmatrix},\ L^{2,2}_{e_1} = \begin{pmatrix} bg_1 & 0 \\ h & 0 \end{pmatrix}$	$a = 0,\ b \in \mathbb{R}$
$\mathbf{cc}^+_5$	$\mathbf{c}^+_5$	$L^{1,1}_{e_1} = L^{1,1}_{e_2} = \begin{pmatrix} -c & -c \\ -c & -c \end{pmatrix},\ L^{1,2}_{e_1} = L^{1,2}_{e_2} = \begin{pmatrix} -c & -c \\ -c & -c \end{pmatrix}$ $L^{2,1}_{e_1} = L^{2,1}_{e_2} = \begin{pmatrix} c & c \\ c & c \end{pmatrix},\ L^{2,2}_{e_1} = L^{2,2}_{e_2} = \begin{pmatrix} c & c \\ c & c \end{pmatrix}$	$a \in \mathbb{R},\ b \in \mathbb{R}$

Table 2: Compatible two dimensional 2-left symmetric and (2×2)-left symmetric structures.

Six dimensional 2-para-Kähler Lie algebras

Structure	Associated 2-para-Kähler Lie algebra	Conditions
$\mathbf{bb}_{1,\alpha}$	$[f_1, f_2] = -f_1,\ [f_1, f_4] = -af_1,\ [f_2, f_3] = f_3,\ [f_3, f_4] = -af_3,$ $[f_2, e_1] = -e_1,\ [f_2, e_2] = -c(af_2 - f_4),\ [f_4, e_1] = -ae_1,\ [f_4, e_2] = -d(af_2 - f_4).$	$a \in \mathbb{R},$
$\mathbf{bb}_{1,1}$	$[f_1, f_2] = -f_1,\ [f_1, f_4] = -bf_1,\ [f_2, f_3] = f_3,\ [f_2, f_4] = -bf_2 + f_4,\ [f_3, f_4] = -bf_3,$ $[f_2, e_1] = -e_1,\ [f_2, e_2] = -e_2,\ [f_4, e_1] = -be_1,\ [f_4, e_2] = -be_2.$	$b \in \mathbb{R}$
$\mathbf{bb}_2$	$[f_1, f_2] = -f_1,\ [f_1, f_4] = -af_1,\ [f_2, f_3] = f_3,\ [f_2, f_4] = -a(f_1 + f_2) + f_3 + f_4,$ $[f_3, f_4] = -af_3,\ [f_2, e_1] = -e_1 - e_2,\ [f_2, e_2] = -e_2,\ [f_4, e_1] = -a(e_1 + e_2), [f_4, e_2] = -ae_2.$	$a \neq 1$
	$[f_1, f_2] = -f_1,\ [f_1, f_4] = -f_1,\ [f_2, f_3] = f_3,\ [f_2, f_4] = -f_1 - f_2 + f_3 + f_4,$ $[f_3, f_4] = -f_3,\ [f_2, e_1] = -e_1 - e_2,\ [f_2, e_2] = -c(f_2 - f_4) - e_2,\ [f_4, e_1] = -e_1 - e_2,$ $[f_4, e_2] = -c(f_2 - f_4) - e_2.$	$c \in \mathbb{R}$
$\mathbf{bb}_{3,1}$	$[f_1, f_2] = f_1,\ [f_1, f_3] = -af_1,\ [f_1, f_4] = -af_2 + f_3,\ [f_2, f_3] = -bf_1,$ $[f_2, f_4] = -bf_2 + f_4,\ [f_3, f_4] = bf_3 - af_4,\ [f_1, e_1] = -e_2,\ [f_2, e_2] = -e_2,$ $[f_3, e_1] = -ae_1 - be_2,\ [f_4, e_2] = -ae_1 - be_2.$	$a \neq 0,\ b \in \mathbb{R}$
$\mathbf{bb}_4$	$[f_1, f_2] = f_1,\ [f_1, f_4] = f_3,\ [f_2, f_3] = -af_1,\ [f_2, f_4] = -a(f_1 + f_2) + f_3 + f_4,$ $[f_3, f_4] = af_3,\ [f_1, e_1] = -e_2,\ [f_2, e_1] = -c(af_1 - f_3) - e_2,\ [f_2, e_2] = -e_2,$ $[f_3, e_1] = -ae_2, [f_4, e_1] = -ac(af_1 - f_3) - ae_2,\ [f_4, e_2] = -ae_2.$	$a \in \mathbb{R}$

Six dimensional 2-para-Kähler Lie algebras

$\mathfrak{cc}_3^1$	$[f_1,f_4]=-2af_1,\ [f_2,f_4]=-bf_1-af_2+f_3,\ [f_3,f_4]=-2af_3,\ [f_2,e_1]=-e_2,$ $[f_4,e_1]=-2ae_1-be_2,\ [f_4,e_2]=-ae_2.$	$a\neq 0,\ b\in\mathbb{R}$
	$[f_2,f_4]=-bf_1+f_3,\ [f_1,e_1]=c_1f_1+g_1f_3,\ [f_2,e_1]=c_2f_1+g_2f_3-e_2,$ $[f_3,e_1]=bc_1f_1+bg_1f_3,\ [f_4,e_1]=d_2f_1+hf_3-be_2.$	$b\in\mathbb{R}$
$\mathfrak{cc}_3^2$	$[f_1,f_4]=-af_1,\ [f_2,f_3]=-af_1,\ [f_2,f_4]=-bf_1-af_2+f_3,\ [f_2,e_1]=-e_2,\ [f_3,e_1]=-ae_2$ $[f_4,e_1]=-ae_1-be_2,\ [f_4,e_2]=-ae_2.$	$a\neq 0,\ b\in\mathbb{R}$
	$[f_2,f_4]=-bf_1+f_3,\ [f_1,e_1]=c_1f_1+g_1f_3,\ [f_2,e_1]=c_2f_1+g_2f_3-e_2,$ $[f_3,e_1]=bc_1f_1+bg_1f_3,\ [f_4,e_1]=df_1+hf_3-be_2.$	$b\in\mathbb{R}$
$\mathfrak{cc}_5^+$	$[f_1,f_3]=-af_1-bf_2+f_4,\ [f_1,f_4]=-bf_1-af_2+f_3,\ [f_2,f_3]=-bf_1-af_2+f_3,$ $[f_2,f_4]=-af_1-bf_2+f_4,\ [f_1,e_1]=-c(f_1+f_2-f_3-f_4)-e_2,\ [f_1,e_2]=-c(f_1+f_2-f_3-f_4)-e_1,$ $[f_2,e_1]=-c(f_1+f_2-f_3-f_4)-e_1,\ [f_2,e_2]=-c(f_1+f_2-f_3-f_4)-e_2$ $[f_3,e_1]=-c(f_1+f_2-f_3-f_4)-ae_1-be_2,\ [f_3,e_2]=-c(f_1+f_2-f_3-f_4)-be_1-ae_2,$ $[f_4,e_1]=-c(f_1+f_2-f_3-f_4)-be_1-ae_2,\ [f_4,e_2]=-c(f_1+f_2-f_3-f_4)-ae_1-be_2.$	$a\in\mathbb{R},\ b\in\mathbb{R}$
	$\mathfrak{h}=\mathrm{span}\{f_1,f_2,f_3,f_4\},\ \theta^1=f_1^*\wedge e_1^*+f_2^*\wedge e_2^*$ and $\theta^2=f_3^*\wedge e_1^*+f_4^*\wedge e_2^*$	

Table 3: Six dimensional 2-para-Kähler Lie algebras.

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