On the semi-symmetric pseudo-Riemannian spaces

Four dimensional

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Goal

To study the homogeneous neutral semi-symmetric manifolds of dimension 4

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Introduction

A pseudo-Riemannian manifold (M, g) is said to be semi-symmetric if its Riemannian curvature tensor R satisfies $R.R = 0$. This is equivalent to

$$[R(X, Y), R(Z, T)] = R(R(X, Y)Z, T) + R(Z, R(X, Y)T), \qquad (1)$$

for any vector fields X, Y, Z, T.

Introduction

A pseudo-Riemannian manifold (M, g) is said to be semi-symmetric if its Riemannian curvature tensor R satisfies $R.R = 0$. This is equivalent to

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for any vector fields X, Y, Z, T.
Semi-symmetric pseudo-Riemannian manifolds generalize obviously locally symmetric manifolds ($\nabla R = 0$). They also generalize second-order locally symmetric manifolds ($\nabla^2 R = 0$ and $\nabla R \neq 0$) où ∇ est la connexion de Levi-Civita.

Semi-symmetric Riemannian manifolds have been first investigated by E. Cartan [Cartan] in his study of locally symmetrical Riemannian manifolds. These are defined as Riemannian manifolds for which the curvature tensor is invariant under all parallel translations. E. Cartan set himself the problem of giving a complete classification of these spaces. In an ingeneous manner he gave the problem two different group-theoretic formulations [Cartan2]. One of these is particularly effective and strikingly enough reduces the problem to the classification of simple Lie algebras over $\mathbb{R}$, a problem which Cartan himself had solved already in 1914.

Cartan's first method was based on the so-called holonomy group. If o is a point in a Riemannian manifold M, then the holonomy group of M is the group of all linear transformations of the tangent space T_oM obtained by parallel translation along closed curves starting at o. Of course each element of the holonomy group leaves the Riemannian structure g invariant; if M is locally symmetric the curvature tensor R_o is also left invariant, that each element of the holonomy group induces an isometry of a neighborhood of o in M onto itself leaving o fixed. This leads to algebraic relations between the Lie algebra $\mathcal{H}_0$ of the identity component of the holonomy group and the tensors g_o and R_o namely,

$$\begin{array}{ccl} g_o(AX,Y) & + & g_o(X,AY)=0, \\ [A,R_o(X,Y)] & = & R_o(AX,Y)+ \quad R_o(X,AY), \\ R_o(X,Y)\in\mathcal{H}_0 & & \end{array} \tag{2}$$

$A\in\mathcal{H}_0$, $X,Y\in T_oM$.

Cartan showed ([Cartan2], p. 225) that if for a given Lie algebra $\mathcal{H}_0$ a tensor and R of type $(1,3)$ satisfies these formulas 2, then there exists a locally symmetric space for which it is the curvature tensor at a point o.

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-From 1983 to 1985, Szabo [Zabo1, Zabo2] gave a complete description of these manifolds.

-In 2018, the author with M. Boucetta and A. Ikemakhen give a complet classification of four-dimensional homogeneous semi-symmetric Lorentzian manifolds [Benroummane].

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-Recently, A. Haji-Badali and A. Zaeim give a complet classification of four-dimensional semi-symmetric neutral Lie groups [Ali].

Main results

- Let $(V, \langle\ ,\ \rangle)$ be a vector space with metric of signature $(2, n)$, $\mathrm{K} : V \wedge V \longrightarrow V \wedge V$ a semi-symmetric algebraic curvature tensor and $\mathrm{Ric}_K : V \longrightarrow V$ its Ricci operator. The main result here (see Proposition 2.2) is that Ric_K has at most two non-real complex eigenvalues and if we note z et $\bar{z}$ such that eigenvalues, we get

$$\dim(\ker(\mathrm{Ric}_K^2 - (z + \bar{z})\mathrm{Ric}_K + z\bar{z}\mathrm{Id}_V)) = 4$$

- In second main proposition (see Proposition3.1), we give the list of the semi-symmetric curvature tensor on a four dimensional neutral space $(V, \langle\ ,\ \rangle)$.

Theorem

A four-dimensional Einstein neutral manifold with non null scalar curvature is semi-symmetric if and only if it is locally symmetric.

Theorem

Let M be a simply connected homogeneous semi-symmetric 4-dimensional neutral manifold. If the Ricci tensor of M has a non zero eigenvalue in $\mathbb{C}$, then M is symmetric and in this case it is a product of a space of constant curvature and a Cahen-Wallach space or it admits a complex structure.

- In the end, we give the list of 4-homogenous semisymmetric notsymmetric neutral manifolds.

Tools

Let $(V, \langle\ ,\ \rangle)$ be a n-dimensional pseudo-Riemannian vector space. We identify V and its dual V^* by the means of $\langle\ ,\ \rangle$. This implies that the Lie algebra $V \otimes V^*$ of endomorphisms of V is identified with $V \otimes V$, the Lie algebra $\mathrm{so}(V, \langle\ ,\ \rangle)$ of skew-symmetric endomorphisms is identified with $V \wedge V$ and the space of symmetric endomorphisms is identified with $V \vee V$ (the symbol $\wedge$ is the outer product and $\vee$ is the symmetric product). For any $u, v \in V$,

$$(u \wedge v)w = \langle v, w\rangle u - \langle u, w\rangle v \tag{3}$$

$$(u \vee v)w = \frac{1}{2}\left(\langle v, w\rangle u + \langle u, w\rangle v\right). \tag{4}$$

We denote $A_{u,v} := u \wedge v$.

The space $V \wedge V$ can be provided with a metric also denoted by $\langle\ ,\ \rangle$ and given by

$$\langle u \wedge v, w \wedge t\rangle := \langle u \wedge v(w), t\rangle = \langle v, w\rangle\langle u, t\rangle - \langle u, w\rangle\langle v, t\rangle. \tag{5}$$

We identify $V \wedge V$ with its dual by means of this metric.

Bianchi application

B the linear Bianchi application on the space
$P = (\wedge^2 V) \vee (\wedge^2 V) = \vee^2(\wedge^2 V)$ given by:

$$B((a\wedge b)\vee(c\wedge d)) = (a\wedge b)\vee(c\wedge d)+(b\wedge c)\vee(a\wedge d)+(c\wedge a)\vee(b\wedge d). \quad (6)$$

Let be $\mathfrak{g}$ a subalgebra of $so(V)$ and the action of $\mathfrak{g}$ on P given by

$$A.T: \quad (u\wedge v) \mapsto A.T(u\wedge v) = [A, T(u\wedge v)] - T(A(u)\wedge v) - T(u\wedge A(v)), \quad (7)$$

for all $(A, T) \in \mathfrak{g} \times P$.
We pute:

$$R(\mathfrak{g}) := \ker(B_{/\mathfrak{g}}) = \{T \in \mathfrak{g}\vee\mathfrak{g}/ \ B(T) = 0\} \text{ and } \mathfrak{g}_{sym} = \{T \in R(\mathfrak{g})/\mathfrak{g}.T = 0$$

- $R(\mathfrak{g})$ is called space of *curvature tensor* of type $\mathfrak{g}$ and we say *curvature tensor* of V each element of $R(so(V))$.
- $\mathfrak{g}_{sym}$ is called space of symmetric curvature tensor of type $\mathfrak{g}$.

Semisymmetric curvature tensor

A *curvature tensor* on $(V, \langle\ ,\ \rangle)$ is a $\mathrm{K} \in R(so(V))$ (i.e. K is a symmetric endomorphism of $\wedge^2 V$ and $B(\mathrm{K}) = 0$). Then, it is satisfying the algebraic Bianchi's identity:

$$\mathrm{K}(u,v)w + \mathrm{K}(v,w)u + \mathrm{K}(w,u)v = 0, \quad u, v, w \in V.$$

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The Ricci curvature tensor associated to K is the symmetric bilinear form on V given by:

$$\mathrm{ric}_K(u, v) = \mathrm{tr}(\tau(u, v)), \quad \text{where } \tau(u, v)(a) = \mathrm{K}(u, a)v. \tag{8}$$

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The Ricci operator is the symmetric endomorphism $\mathrm{Ric}_\mathrm{K} : V \longrightarrow V$ given by $\langle \mathrm{Ric}_K(u), v\rangle = \mathrm{ric}_\mathrm{K}(u, v)$.
K is called Einstein (resp. Ricci isotropic) if $\mathrm{Ric}_K = \lambda \mathrm{Id}_V$ (resp. $\mathrm{Ric}_\mathrm{K} \neq 0$ and $\mathrm{Ric}_\mathrm{K}^2 = 0$).

Example:

If $\mathrm{K} = (u \wedge v) \vee (w \wedge t)$
then,

$$\mathrm{ric}_K = \langle u, w \rangle t \vee v + \langle v, t \rangle u \vee w - \langle v, w \rangle t \vee u - \langle u, t \rangle v \vee w. \qquad (9)$$

Primitive holonomy algebra

We denote by $\mathfrak{h}(\mathrm{K})$ the vector subspace of $V \wedge V$ image of K, i.e., $\mathfrak{h}(\mathrm{K}) = \mathrm{span}\{\mathrm{K}(u, v)/\ u,\ v \in V\}$. The Lie algebra genrated by $\mathfrak{h}(\mathrm{K})$ is called *primitive holonomy algebra* of K.

Primitive holonomy algebra

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A curvature tensor K is called *semi-symmetric* if it is invariant by $\mathfrak{h}(\mathrm{K})$, i.e.,

$$\mathrm{K}(u,v).\mathrm{K} = 0, \quad \forall (u,v) \in V^2. \tag{10}$$

This is equivalent to

$$[\mathrm{K}(u,v), \mathrm{K}(a,b)] = \mathrm{K}(\mathrm{K}(u,v)a, b) + \mathrm{K}(a, \mathrm{K}(u,v)b), \quad \forall (u,v,a,b) \in V^4. \tag{11}$$

In this case, $\mathfrak{h}(\mathrm{K})$ is a Lie subalgebra of $\mathrm{so}(V, \langle\ ,\ \rangle)$ and it is *primitive holonomy algebra* of K.

Remark

If K is semi-symmetric, then its Ricci operator is also invariant by $\mathfrak{h}(\mathrm{K})$, i.e.,

$$\mathrm{K}(u,v) \circ \mathrm{Ric}_{\mathrm{K}} = \mathrm{Ric}_{\mathrm{K}} \circ \mathrm{K}(u,v), \quad \forall (u,v) \in V^2. \tag{12}$$

Theorem

([Boubel]) Let (M, g) a pseudo-Riemannian manifold with parallel Ricci (i.e, $\nabla.\mathrm{Ric} = 0$) and let χ be minimal polynomial of Ric. Then, the following properties are checked:

1. $\chi = \Pi_i P_i$ *with :*
   - $\forall i \neq j$, $P_i \wedge P_j = 1$ *(i.e, P_i and P_j are mutually prime),*
   - $\forall i$, P_i *is irreducible or* $P_i = X^2$.
2. *There is a canonical family $(M_i)_i$ of pseudo-Riemannian manifolds such that the minimal polynomial of $\mathrm{Ric}_i = \mathrm{Ric}_{M_i}$ on each M_i is P_i, and a local isometry f mapping the Riemarmian product ΠM_i onto M. f is unique up to composition with a product of isometries of each factor M_i. If M is complete and simply connected, f is an isometry.*

In the proof of the first result of this theorem, **C. Boubel** used only the following algebraic hypothesis: On each tangent space T_xM at the point $x \in M$, the Ricci operator Ric_x commutes with each endomorphisms $R_x(u, v)$ for all u, $v \in T_xM$. that is said Ricci operator is semi-symmetric which was verified for spaces with semi-symmetric curvature.

Proposition

Let K be a semi-symmetric curvature tensor on the pseudo-Riemannian space $(V, \langle\ ,\ \rangle)$ and let χ be a minimal polynomial of $\mathrm{Ric}_{\mathrm{K}}$. Then, the following properties are checked:

1. $\chi = \Pi_i P_i$, *with;*
   - $\forall i \neq j$, $P_i \wedge P_j = 1$ *(i.e, P_i and P_j are mutually prime),*
   - $\forall i$, P_i *is irreducible or* $P_i = X^2$.
2. V *splits orthogonally:*

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_r, \tag{13}$$

where $V_0 = \ker((\mathrm{Ric}^2))$ *and* $V_i = \ker(P_i(\mathrm{Ric}))$.
Moreover, the following situations is verified:

a) *for all* $u, v \in V$ *and* $i \in \{0, \ldots, r\}$, V_i *is* $\mathfrak{h}(\mathrm{K})$*-invariant,,*
b) *for all* $i, j \in \{0, \ldots, r\}$ *with* $i \neq j$, $\mathrm{K}_{|V_i \wedge V_j} = 0$,
c) *for all* $i = 1, \ldots, r$, $\dim V_i \geq 2$.

Proposition

Let K *be a semi-symmetric curvature tensor on the pseudo-Riemannian space* $(V, \langle\ ,\ \rangle)$ *with metric of signature* $(2, n)$ *such that* $n \geq 2$. *Then the Ricci curvature* $\mathrm{Ric}_{\mathrm{K}}$ *admits at most two non-real eigenvalues. Denote by* $\alpha_1, \ldots, \alpha_r$ *the non-zero real eigenvalues and* $V_1, \ldots, V_r$ *the corresponding eigenspaces. Then one of the following situations is verified:*

1. $\mathrm{Ric}_{\mathrm{K}}$ *has two non-real eigenvalues* z *and* $\bar{z}$ *and* V *splits orthogonally*

$$V = V_c \oplus V_0 \oplus \ldots \oplus V_r,$$

where $V_0 = \ker(\mathrm{Ric})$ *and* $V_c = \ker(\mathrm{Ric}_{\mathrm{K}}^2 - (z + \bar{z})\mathrm{Ric}_{\mathrm{K}} + |z|^2 I)$. *Moreover,* $\dim(V_c) = 4$ *and* V_i *is a Riemannian semi-symmetric space for all* $i \geq 0$.
In this case, $\mathrm{Ric}_{\mathrm{K}}$ *is said complex Ricci.*

2. $\mathrm{Ric}_{\mathrm{K}}$ *has only real eigenvalues and* V *splits orthogonally:*

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_r, \quad \textit{where} \quad V_0 = \ker(\mathrm{Ric}_{\mathrm{K}}^2).$$

proof

We will only show the following result: If $\mathrm{Ric} = \mathrm{Ric}_{\mathrm{K}}$ admits a non real eigenvalue z, then, z and $\bar{z}$ are the only non-real eigenvalues of Ric and $V_c = \ker(\mathrm{Ric}^2 - (z+\bar{z})\mathrm{Ric} + |z|^2 I)$ is the neutral space of dimension 4. The other results are easy to proof.
Now, we suppose that Ric admits a non-real eigenvalue z with the associated caractestic subspace $V_c = \ker(\mathrm{Ric}^2 - (z+\bar{z})\mathrm{Ric} + |z|^2 I)$, that subspace is a pseudo-Riemannian semi-symmetric of even dimension and necessarily, V_c admits the metric of signature $(2, 2p)$, otherwise, we have V_c is a non degenerete space then, one and only one of two following situations is checked:

1. V_c is a Riemannian space . So Ric is a symmetric endomorphism and it's diagonalizable admitting only the real eigenvalues,
2. V_c is a Lorentzian semi-symmetric space. According to [Benroummane], Ric has only the real eigenvalues.

Which is impossible in both situations.

proof

So, $(V_c)^\perp$ will be a Riemannian space which Ric has only real eigenvalues. Now, we choose a non-zero isotropic vector e in V_c. Then, $(e, \mathrm{Ric}(e))$ is a free family in V_c, otherwise, e will be an eigenvector associated of a real eigenvalue of Ric on V_c, which is impossible.
On the other hand, the subspace $V_c^e = span\{e, \mathrm{Ric}(e)\}$ generated by e and its image $\mathrm{Ric}(e)$, is totally isotropic if and only if e and $\mathrm{Ric}(e)$ are orthogonal. Then, one of the two following situations is verified:

a) If e and $\mathrm{Ric}(e)$ are not orthogonl (i.e $\langle e, \mathrm{Ric}(e)\rangle \neq 0$), then, V_c^e will be a Lorentzian space stable by Ric and let $\overline{V_c^e}$ be a subspace such that V_c splits orthogonally $V_c = V_c^e \oplus \overline{V_c^e}$. Then $\overline{V_c^e}$ is a Lorentzian subspace invariant by Ric then it's semi-symmetric and necessarily, $\overline{V_c^e}$ is of dimension 2 and $\dim(V_c) = 4$.

b) Now, if $V_c^e = span\{e, \mathrm{Ric}(e)\}$ is totally isotropic.
We choose $\bar{e}$ a dual vector of $\mathrm{Ric}(e)$ in V_c. So, $\mathrm{Ric}(\bar{e})$ is a dual vector of e and one of the two following situations is verified:

b1) $\bar{e}$ and $\mathrm{Ric}(\bar{e})$ are duals vectors. Then, $\overline{V_c^e} = vect\{\bar{e},\ \mathrm{Ric}(\bar{e})\}$ is a Lorentzian subspace of V_c, stable by Ric and we come back to the case (a).

b2) $\bar{e}$ and $\mathrm{Ric}(\bar{e})$ are orthogonals. Then, $\overline{V_c^e} = vect\{\bar{e},\ \mathrm{Ric}(\bar{e})\}$ is a totally isotropic subspace of V_c and a dual subspace of V_c^e and $V_c = V_c^e \oplus \overline{V_c^e}$. This completes the proof of the proposition.

Remark

This proposition reduces the determination of semi-symmetric curvature tensors on vector space equiped with metric of signature $(2, n)$ to the determination of three classes of semi-symmetric curvature tensors: Einstein semi-symmetric curvature tensors, semi-symmetric curvature with Ricci isotropic, semi-symmetric curvature tensor with complex Ricci on four dimensional neutral space.

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Neutral 4-spaces

Let K be curvature tensor on pseudo-Riemannian vector space $(V, \langle\ ,\ \rangle)$. K is semi-symmetric curvature tensors if only if $\mathfrak{h}(K)$ is a Lie subalgebra of $\mathfrak{so}(V, \langle\ ,\ \rangle)$ and $\mathfrak{h}(\mathrm{K}).\mathrm{K} = 0$.

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1. First step: According of the classification of Lie subalgebra of $\mathfrak{so}(2,2)$ given by Komrakov in[Komrakov] and for each subalgebra $\mathfrak{g}$ of $\mathfrak{so}(2,2)$, we give all curvature tensor and all symmetric curvature of type $\mathfrak{g}$.
2. In second step: We give a list of non flat semi-symmetric curvature tensors on the four dimensional neutral vector space $(V, \langle\ ,\ \rangle)$ by giving the list of Lie subalgebras $\mathfrak{g} \neq \{0\}$ of $\mathfrak{so}(V)$ such that $\mathfrak{g} = \mathfrak{g}_{sym}$.

theorem

For each Lie subalgebra $\mathfrak{g}$ of $\mathfrak{so}(2,2)$, the space $R(\mathfrak{g})$ of all curvature tensor of type $\mathfrak{g}$ and the space $\mathfrak{g}_{sym}$ of all symmetric curvature tensor of type $\mathfrak{g}$ are the following:

- $\dim \mathfrak{g} = 1$:
  - $\mathfrak{g} : 1.1^1 = \mathbb{R}\{A_{x,z} + a.A_{y,t}\}$, with $\langle x, z\rangle = \langle y, t\rangle = 1$, $a \in [0,1]$
    If $a = 0$, we get; $R(\mathfrak{g}) = \mathfrak{g}_{sym} = \mathbb{R}\{A_{x,z} \vee A_{x,z}\}$.
    Otherwise, $R(\mathfrak{g}) = 0$,
  - $\mathfrak{g} : 1.1^2 = \mathbb{R}\{A_{x,z} + a.A_{y,t}\}$, with
    $\langle x, x\rangle = -\langle y, y\rangle = \langle z, z\rangle = -\langle t, t\rangle = 1$, $a \in [0,1]$
    If $a = 0$, we get; $R(\mathfrak{g}) = \mathfrak{g}_{sym} = \mathbb{R}\{A_{x,z} \vee A_{x,z}\}$.
    Otherwise, $R(\mathfrak{g}) = 0$,
  - $\mathfrak{g} : 1.2^1 = \mathbb{R}\{A_{x,z} + A_{x,t} + A_{y,t}\}$, with $\langle x, z\rangle = \langle y, t\rangle = 1$,
    $R(\mathfrak{g}) = 0$.
  - $\mathfrak{g} : 1.2^2 = \mathbb{R}\{A_{x,y} + A_{x,t} + A_{y,z}\}$, with $\langle x, z\rangle = \langle y, t\rangle = 1$,
    $R(\mathfrak{g}) = 0$.
  - $\mathfrak{g} : 1.3^1 = \mathbb{R}A_{x,y}$, with $\langle x, t\rangle = \langle y, z\rangle = 1$,
    $R(\mathfrak{g}) = \mathfrak{g}_{sym} = \mathbb{R}.A_{x,y} \vee A_{x,y}$.
  - $\mathfrak{g} : 1.4^1 = \mathbb{R}A_{x,y}$, with $\langle x, z\rangle = -\langle y, y\rangle = \langle t, t\rangle = 1$,
    $R(\mathfrak{g}) = \mathfrak{g}_{sym} = \mathbb{R}A_{x,y} \vee A_{x,y}$.

- $\dim \mathfrak{g} = 1$:
  - $\mathfrak{g} : 1.1^5 = \mathbb{R}\{\cos(\phi)(A_{x,t} + A_{y,z}) + \sin(\phi)(A_{y,z} + A_{t,y})\}$ with $\langle x, t\rangle = \langle y, z\rangle = 1$ and $\phi \in]0, \frac{\pi}{4}]$,
    $R(\mathfrak{g}) = 0$,
  - $\mathfrak{g} : 1.1^6 = \mathbb{R}\{\cos(\phi)(A_{x,t} + A_{z,y}) + \sin(\phi)(A_{x,z} + A_{y,t})\}$ with $\langle x, z\rangle = \langle y, t\rangle = 1$ and $\phi \in]0, \frac{\pi}{4}[$,
    $R(\mathfrak{g}) = 0$,
- $\dim \mathfrak{g} = 2$:
  - $\mathfrak{g} : 2.1^1 = vect\{A_{x,z}, \quad A_{y,t}\}$, with $\langle x, z\rangle = \langle y, t\rangle = 1$,
    $R(\mathfrak{g}) = \mathfrak{g}_{sym} = span\{A_{x,z} \vee A_{x,z}, \ A_{y,t} \vee A_{y,t}\}$.
  - $\mathfrak{g} : 2.1^3 = span\{A_{x,z}, \quad A_{y,t}\}$, with
    $\langle x, x\rangle = -\langle y, y\rangle = \langle z, z\rangle = -\langle t, t\rangle = 1$,
    $R(\mathfrak{g}) = \mathfrak{g}_{sym} = span\{A_{x,z} \vee A_{x,z}, \ A_{y,t} \vee A_{y,t}\}$.
  - $\mathfrak{g} : 2.1^4 = span\{\pi_1 = A_{x,z} + A_{t,y}, \quad \pi_2 = A_{x,t} + A_{y,z}\}$, with
    $\langle x, t\rangle = \langle y, z\rangle = 1$,
    $R(\mathfrak{g}) = \mathfrak{g}_{sym} = span\{(\pi_1 \vee \pi_1 - \pi_2 \vee \pi_2), \ \pi_1 \vee \pi_2\}$.
  - $\mathfrak{g} : 2.2^1 = span\{\pi_1 = A_{x,z} + aA_{y,t}, \quad \pi_2 = A_{x,t}\}$ with
    $\langle x, z\rangle = \langle y, t\rangle = 1$ and $a \in [-1, 1]$,
    If $a = 0$, we get $R(\mathfrak{g}) = span\{\pi_1 \vee \pi_1, \ \pi_2 \vee \pi_2, \ \pi_1 \vee \pi_2\}$, and $\mathfrak{g}_{sym} = 0$.
    If $a = 1$, we get $R(\mathfrak{g}) = span\{\pi_2 \vee \pi_2, \quad \pi_1 \vee \pi_2\} = \mathfrak{g}_{sym}$.
    If $a \neq 1$ and $a \neq 0$, we get $R(\mathfrak{g}) = span\{\pi_2 \vee \pi_2, \quad \pi_1 \vee \pi_2\}$ and
    $\mathfrak{g}_{sym} = 0$.

- $\dim \mathfrak{g} = 3$:
  - $\mathfrak{g} : 3.1^1 = span\{A_{x,z}, A_{x,t}, A_{y,t}\}$, with $\langle x, z\rangle = \langle y, t\rangle = 1$,
    $R(\mathfrak{g}) =$
    $span\{A_{x,z} \vee A_{x,z}, \quad A_{x,t} \vee A_{x,t}, \quad A_{y,t} \vee A_{y,t}, \quad A_{x,t} \vee A_{y,t}, \quad A_{x,z}, A_{x,t}\}$,
    and $\mathfrak{g}_{sym} = 0$
  - $\mathfrak{g} : 3.2^1 = span\{\pi_1 = A_{x,z} + aA_{y,t}, \quad \pi_2 = A_{x,t}, \quad \pi_3 = A_{x,y}\}$, with
    $\langle x, z\rangle = \langle y, t\rangle = 1$ and $a \geq 0$,
    If $a = 0$, we get
    $R(\mathfrak{g}) = span\{\pi_1 \vee \pi_1, \quad \pi_2 \vee \pi_2, \quad \pi_3 \vee \pi_3, \quad \pi_2 \vee \pi_3, \quad \pi_1 \vee \pi_2, \quad \pi_1 \vee \pi_3\}$
    and $\mathfrak{g}_{sym} = 0$.
    Otherwise, $R(\mathfrak{g}) = span\{\pi_2 \vee \pi_2, \quad \pi_3 \vee \pi_3, \quad \pi_2 \vee \pi_3, \pi_1 \vee \pi_2, \pi_1 \vee \pi_3\}$,
    and if more $a \neq 1$, we get $\mathfrak{g}_{sym} = \mathbb{R}\{\pi_2 \vee \pi_2\}$, otherwise, we get;
    $\mathfrak{g}_{sym} = 0$,
  - $\mathfrak{g} : 3.3^1 = span\{\pi_1 = A_{y,t}, \quad \pi_2 = A_{x,t}, \quad \pi_3 = A_{x,y}\}$ with
    $\langle x, z\rangle = \langle y, t\rangle = 1$,
    $R(\mathfrak{g}) = span\{\pi_1 \vee \pi_1, \quad \pi_2 \vee \pi_2, \quad \pi_3 \vee \pi_3, \quad \pi_2 \vee \pi_3, \quad \pi_1 \vee \pi_2, \quad \pi_1 \vee \pi_3\}$,
    $\mathfrak{g}_{sym} = \mathbb{R}.\pi_2 \vee \pi_3$.
  - $\mathfrak{g} : 3.4^1 = span\{\pi_1 = A_{x,z} + A_{t,y}, \quad \pi_2 = A_{x,t}, \quad \pi_3 = A_{y,z}\}$ with
    $\langle x, z\rangle = \langle y, t\rangle = 1$,
    $R(\mathfrak{g}) = span\{(\pi_1 \vee \pi_1 - 2\pi_2 \vee \pi_3), \quad \pi_2 \vee \pi_2, \quad \pi_3 \vee \pi_3, \quad \pi_1 \vee \pi_2, \quad \pi_1 \vee \pi_3\}$,
    $\mathfrak{g}_{sym} = 0$.
  - $\mathfrak{g} : 3.5^1 = span\{\pi_1 = A_{x,z}, \quad \pi_2 = A_{x,y}, \quad \pi_3 = A_{y,z}\}$, with

- $\dim \mathfrak{g} = 4$:
  - $\mathfrak{g} : 4.1^1 = span\{A_{x,y}, A_{x,z}, A_{x,t}, A_{y,t}\}$ with $\langle x, z\rangle = \langle y, t\rangle = 1$, $R(\mathfrak{g}) = span\{A_{x,y} \vee A_{x,y},\ A_{x,z} \vee A_{x,z},\ A_{x,y} \vee A_{x,z},\ A_{x,t} \vee A_{x,z},\ A_{x,t} \vee A_{x,t},\ A_{y,t} \vee A_{x,t},\ A_{x,y} \vee A_{x,t},\ A_{y,t} \vee A_{y,t},\ A_{x,y} \vee A_{y,t}\}$, $\mathfrak{g}_{sym} = 0$
  - $\mathfrak{g} : 4.2^1 = span\{A_{x,t}, A_{x,z}, A_{y,t}, A_{y,z}\}$ with $\langle x, z\rangle = \langle y, t\rangle = 1$, $R(\mathfrak{g}) = span\{A_{x,z} \vee A_{x,z},\ A_{x,t} \vee A_{x,t},\ A_{y,z} \vee A_{y,z},\ A_{y,t} \vee A_{y,t},\ A_{x,z} \vee A_{x,t},\ A_{x,z} \vee A_{y,z},\ (A_{x,z} \vee A_{y,t} + A_{x,t} \vee A_{y,z}),\ A_{x,t} \vee A_{y,t},\ A_{y,z} \vee A_{y,t}\}$, and $\mathfrak{g}_{sym} = \mathbb{R}\{A_{x,z} \vee A_{x,z} + A_{y,t} \vee A_{y,t} + A_{x,z} \vee A_{y,t} + A_{x,t} \vee A_{y,z}\}$
  - $\mathfrak{g} : 4.3^1 = span\{\pi_1 = A_{x,y}, \pi_2 = A_{x,t}, \pi_3 = A_{t,y} + A_{x,z}, \pi_4 = A_{y,z}\}$ with $\langle x, z\rangle = \langle y, t\rangle = 1$, $R(\mathfrak{g}) = span\{\pi_1 \vee \pi_1,\ \pi_2 \vee \pi_2,\ (\pi_3 \vee \pi_3 - 2\pi_2 \vee \pi_4),\ \pi_4 \vee \pi_4,\ \pi_1 \vee \pi_2,\ \pi_1 \vee \pi_3,\ \pi_1 \vee \pi_4,\ \pi_2 \vee \pi_3,\ \pi_3 \vee \pi_4\}$, and $\mathfrak{g}_{sym} = \mathbb{R}\{\pi_1 \vee \pi_1\}$.
- $\dim \mathfrak{g} \in \{5,\ 6\}$:
  - $\mathfrak{g} : 5.1^1 = span\{A_{x,y}, A_{x,z}, A_{x,t}, A_{y,z}, A_{y,t}\}$ with $\langle x, z\rangle = \langle y, t\rangle = 1$, $\dim(R(\mathfrak{g})) = 14$ and $\mathfrak{g}_{sym} = 0$.
  - $\mathfrak{g} : 6.1^1 = so(2,2)$, we get $\dim(R(\mathfrak{g})) = 19$ $\mathfrak{g}_{sym} = \mathbb{R}(A_{x,z}\vee A_{x,z}+A_{y,t}\vee A_{y,t}+A_{x,z}\vee A_{y,t}+2.A_{x,t}\vee A_{y,z}+2.A_{x,y}\vee A_{t,z})$ with $\langle x, z\rangle = \langle y, t\rangle = 1$

Proposition

Let K *be a semi-symmetric curvature tensor on the four dimensional neutral vector space* $(V, \langle\ ,\ \rangle)$. *Then, there is the basis* (x, y, z, t) *of* V *such that the one of following situations is checked:*

- $\dim \mathfrak{h}(\mathrm{K}) = 1$ *and* K *has one of the following forms:*
  - $\mathrm{K} = b.A_{x,z} \vee A_{x,z}$ *and* $\mathrm{Ric} = -2b.z \vee x$, *where* $b \in \mathbb{R}^*$ *with* $\langle x, z\rangle = \langle y, t\rangle = 1$.
  - $\mathrm{K} = b.A_{x,z} \vee A_{x,z}$ *and* $\mathrm{Ric} = b.(x \vee x + z \vee z)$ *where* $b \in \mathbb{R}^*$, *with* $\langle x, x\rangle = -\langle y, y\rangle = \langle z, z\rangle = -\langle t, t\rangle = 1$.
  - $\mathrm{K} = a.A_{x,y} \vee A_{x,y}$, $\mathrm{Ric} = 0$ *and* $a \in \mathbb{R}^*$, *with* $\langle x, t\rangle = \langle y, z\rangle = 1$.
  - $\mathrm{K} = aA_{x,y} \vee A_{x,y}$ *and* $\mathrm{Ric} = -a.x \vee x$, *with* $a \in \mathbb{R}^*$ *and* $\langle x, z\rangle = -\langle y, y\rangle = \langle t, t\rangle = 1$.

Proposition

- $\dim \mathfrak{h}(\mathrm{K}) = 2$ and K has one of the following forms:
  - $\mathrm{K} = a.A_{x,z} \vee A_{x,z} + bA_{y,t} \vee A_{y,t}$, $\mathrm{Ric} = -2(a.x \vee z + b.y \vee t)$, with $(a,b) \in \mathbb{R}^* \times \mathbb{R}^*$ and $\langle x, z\rangle = \langle y, t\rangle = 1$.
  - $\mathrm{K} = a.A_{x,z} \vee A_{x,z} + bA_{y,t} \vee A_{y,t}$, $\mathrm{Ric} = a(x \vee x + z \vee z) - b(y \vee y + t \vee t)$, with $(a,b) \in \mathbb{R}^* \times \mathbb{R}^*$ and $\langle x, x\rangle = -\langle y, y\rangle = \langle z, z\rangle = -\langle t, t\rangle = 1$,
  - $\mathrm{K} = a(\pi_1 \vee \pi_1 - \pi_2 \vee \pi_2) + b.\pi_1 \vee \pi_2$, $\mathrm{Ric} = -4a(x \vee t + y \vee z) + 2b(y \vee t - x \vee z)$, with $(a,b) \in \mathbb{R}^* \times \mathbb{R}^*$, $\pi_1 = A_{x,z} + A_{t,y}$, $\pi_2 = A_{x,t} + A_{y,z}$ and $\langle x, t\rangle = \langle y, z\rangle = 1$,
  - $\mathrm{K} = c\pi_2 \vee \pi_2 + d\pi_1 \vee \pi_2$, $\mathrm{Ric} = -2d.x \vee t$, with $(c,d) \in \mathbb{R}^* \times \mathbb{R}^*$ and $\pi_1 = A_{x,z} + A_{y,t}$, $\pi_2 = A_{x,t}$ and $\langle x, z\rangle = \langle y, t\rangle = 1$.
  - $\mathrm{K} = a.A_{x,y} \vee A_{x,y} + b.A_{x,t} \vee A_{x,t} + c.A_{x,y} \vee A_{x,t}$, $\mathrm{Ric} = c.x \vee x$, with $(a,b,c) \in \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}^*$ and $\langle x, z\rangle = \langle y, t\rangle = 1$.

Proposition

- $\dim \mathfrak{h}(\mathrm{K}) = 3$ and there is $a \in \mathbb{R}^*$ such that :
  $\mathrm{K} = a(A_{x,z} \vee A_{x,z} + 2A_{x,y} \vee A_{y,z})$ and $\mathrm{Ric} = -2a(2x \vee z + y \vee y)$,
  with $\langle x, z\rangle = \langle y, y\rangle = -\langle t, t\rangle = 1$,
- $\dim \mathfrak{h}(\mathrm{K}) = 4$ and there is $a \in \mathbb{R}^*$ such that:
  $\mathrm{K} = a(A_{x,z} \vee A_{x,z} + A_{y,t}\vee A_{y,t} + A_{x,z}\vee A_{y,t} + A_{x,t}\vee A_{y,z})$,
  $\mathrm{Ric} = -3a(.x \vee z + y \vee t)$,
  with $\langle x, z\rangle = \langle y, t\rangle = 1$.
- $\dim \mathfrak{h}(\mathrm{K}) = 6$ then, $\mathfrak{h}(\mathrm{K}) = so(2,2)$ and there is $a \in \mathbb{R}^*$ such that:

  $$\mathrm{K} = a(A_{x,z}\vee A_{x,z}+A_{y,t}\vee A_{y,t}+A_{x,z}\vee A_{y,t}+2.A_{x,t}\vee A_{y,z}+2.A_{x,y}\vee A_{t,z}), \quad a$$

  with $\langle x, z\rangle = \langle y, t\rangle = 1$

In the following corollary, we give all curvature tensors on the four dimensional neutral vector space in the some particuler cases: Einstein, isotropic Ricci or complex Ricci.

Corollary

Let K *be a semi-symmetric curvature tensor on the four dimensional neutral vector space* $(V, \langle\ ,\ \rangle)$*. Then, there is the basis* (x, y, z, t) *of* V *such that:*

- *If* K *is the Einstein curvature tensor with non zero scalar curvature. Then, one of the following situations is checked::*
  - $\dim \mathfrak{h}(\mathrm{K}) = 6$, *then;*
    $\mathrm{K} = a(A_{x,z} \vee A_{x,z} + A_{y,t} \vee A_{y,t} + A_{x,z} \vee A_{y,t} + 2.A_{x,t} \vee A_{y,z} + 2.A_{x,y} \vee A_{t,z})$,
    $\mathrm{Ric} = -6a(.x \vee z + y \vee t)$,
    with $a \in \mathbb{R}^*$ *and* $\langle x, z\rangle = \langle y, t\rangle = 1$
  - $\dim \mathfrak{h}(\mathrm{K}) = 4$, *then,*
    $\mathrm{K} = a(A_{x,z} \vee A_{x,z} + A_{y,t} \vee A_{y,t} + A_{x,z} \vee A_{y,t} + A_{x,t} \vee A_{y,z})$,
    with $a \in \mathbb{R}^*$ *and* $\langle x, z\rangle = \langle y, t\rangle = 1.$
  - $\dim \mathfrak{h}(\mathrm{K}) = 2$, *then;*
    $\mathrm{K} = a.(A_{x,z} \vee A_{x,z} + A_{y,t} \vee A_{y,t})$ *or* $\mathrm{K} = a.(A_{x,y} \vee A_{x,y} + A_{z,t} \vee A_{z,t})$,
    with $a \in \mathbb{R}^*$ *and* $\langle x, x\rangle = -\langle y, y\rangle = \langle z, z\rangle = -\langle t, t\rangle = 1,$

Corollary

- If K is a Ricci flat, then, there is $(a,b)\in\mathbb{R}^*\times\mathbb{R}^*$ such that ;
  $\mathrm{K}=a.A_{x,y}\vee A_{x,y}+b.A_{x,t}\vee A_{x,t}$,
  with $\langle x,z\rangle=\langle y,t\rangle=1$.
- If K is an isotropic Ricci, Then the one of following situations is checked:
  - $\mathrm{K}=aA_{x,y}\vee A_{x,y}$ and $\mathrm{Ric}=-a.x\vee x$,
    with $a\in\mathbb{R}^*$ and $\langle x,z\rangle=-\langle y,y\rangle=\langle t,t\rangle=1$.
  - $\mathrm{K}=c\pi_2\vee\pi_2+d(\pi_1\vee\pi_2)$, $\mathrm{Ric}=-2d.x\vee t$,
    with $(c,d)\in\mathbb{R}\times\mathbb{R}^*$, $\pi_1=A_{x,z}+A_{y,t}$, $\pi_2=A_{x,t}$ and
    $\langle x,z\rangle=\langle y,t\rangle=1$.
  - $\mathrm{K}=a.A_{x,y}\vee A_{x,y}+b.A_{x,t}\vee A_{x,t}+c.A_{x,y}\vee A_{x,t}$, $\mathrm{Ric}=c.x\vee x$,
    with $(a,b,c)\in\mathbb{R}^2\times\mathbb{R}^*$ and $\langle x,z\rangle=\langle y,t\rangle=1$
- If Ricci has a non-real eigenvalue, then, there is $(a,b)\in\mathbb{R}\times\mathbb{R}^*$ such that

  $\mathrm{K}=a(\pi_1\vee\pi_1-\pi_2\vee\pi_2)+b.\pi_1\vee\pi_2$ and $\mathrm{Ric}=-4a(x\vee t+y\vee z)+2b(y\vee t-$

  with $\pi_1=A_{x,z}+A_{t,y}$, $\pi_2=A_{x,t}+A_{y,z}$ and $\langle x,t\rangle=\langle y,z\rangle=1$.

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Semi-symmetric manifolds

Let (M, g) be pseudo-Riemannian manifolds of dimension n with Levi-Civita connexion ∇ and Riemannian curvature R, $\mathfrak{ric}$ and Ric are the Ricci tensor and the Ricci operator respectively and let $\mathcal{X}(M)$ be the space of all vetor fields on M.

Definition

(M, g) has said semi-symmetric if, $R.R = 0$, i.e, R verifies

$$[R(X,Y), R(Z,T)] = R(R(X,Y)Z,T) + R(Z, R(X,Y)T), \quad X,Y,Z,T \in \mathcal{X} \tag{14}$$

Let (M, g) be pseudo-Riemannian semi-symmetric manifolds. Then, for each point $m \in M$, the restriction R_m of R on tangent space $T_m M$ is semi-symmetric curvature tensor. So, the minimal polynomial of Ric_m is of the form $\chi = \prod_i P_i$ such that the polynomials $(P_i)_i$ are mutually prim between them and for all i, P_i is irreducible or $P_i = X^2$. We can suppose that P_i is irreducible for all $i \geq 1$ and define the distributions:

$$V_0(m) := \ker(\mathrm{Ric}_m^2) \quad \text{et} \quad V_i(m) := \ker(P_i(\mathrm{Ric}_m)) \quad \text{for all} \quad i \geq 1,$$

and we get the following proposition:

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and we get the following proposition:

Proposition

The distributions $(V_i)_i$ have the following proprietes:
For all $i \neq j$, we get:

$$\nabla_{V_j} V_i \subset V_i, \;\; \nabla_{V_i} V_i \subset V_0 + V_i, \;\; \nabla_{V_0} V_i \subset V_i, \;\; \nabla_{V_0} V_0 \subset V_0, \;\; \nabla_{V_i} V_0 \subset V_0 + V \tag{15}$$

proof

Let $m \in M$. According the propositon.2.1, we get:

$$V_m := T_m M = V_0(m) \oplus V_1(m) \oplus ... \oplus V_r(m). \tag{16}$$

proof

Let $m \in M$. According the propositon.2.1, we get:

$$V_m := T_m M = V_0(m) \oplus V_1(m) \oplus ... \oplus V_r(m). \tag{16}$$

In first, we show that for $i \geq 1$ and $X \in V_i^{\perp}$, $\nabla_X V_i \subset V_i$:
We choose $i \geq 1$ and $X \in V_i^{\perp}$. Then, we get $\nabla_X(R(V_i, V_i)V_i) \subset V_i$.
Indeed:
Let $Y,\ Z,\ T \in V_i$. According the second identity of Bianchi, we get:

$$
\begin{aligned}
\nabla_X R(Y,Z,T) &:= (\nabla_X R)(Y,Z)T \\
&= -\nabla_Y R(Z,X,T) - \nabla_Z R(X,Y,T) \\
&= -\nabla_Y(R(Z,X)T) + R(\nabla_Y Z, X)T + R(Z,\nabla_Y X)T \\
&\quad + R(Z,X)\nabla_Y T - \nabla_Z(R(Y,X)T) + R(\nabla_Z Y, X)T \\
&\quad + R(Y,\nabla_Z X)T + R(Y,X)\nabla_Z T \\
&= R(\nabla_Y Z, X)T + R(Z,\nabla_Y X)T \\
&\quad + R(\nabla_Z Y, X)T + R(Y,\nabla_Z X)T.
\end{aligned}
$$

According the proposition.2.1, we take: $R(V,V)(V_i) \subset V_i$ and $\nabla_X R(Y,Z,T) \in V_i$.

The otherwise,

$$
\begin{aligned}
\nabla_X R(Y,Z,T) &= \nabla_X(R(Y,Z)T) - R(\nabla_X Y, Z)T \\
&\quad - R(Y,\nabla_X Z)T - R(Y,Z)\nabla_X T \\
&= \nabla_X(R(Y,Z)T) - R(\nabla_X Y, Z)T - R(Y,\nabla_X Z)T \\
&\quad + R(Z,\nabla_X T)Y + R(\nabla_X T, Y)Z.
\end{aligned}
$$

Then, $\nabla_X(R(Y,Z)T) \in V_i$.

Now, we will show that $\nabla_X \mathrm{Ric}(Y) \in V_i$.
We choose a pseudo-orthonormally basis $(e_1, \dots, e_n)$ adapted to the decomposition (16) and we put $\epsilon_k = \langle e_k, e_k \rangle$. Let $Z \in V_i^\perp$.
If $e_k \in V_i$, we have seen that $\nabla_X(R(Y, e_k)e_k) \in V_i$ and if $e_k \in V_i^\perp$, we get $R(Y, e_k) = 0$. Then,

$$\begin{aligned} \langle \nabla_X(\mathrm{Ric}(Y)), Z \rangle &= -\langle \mathrm{Ric}(Y), \nabla_X Z \rangle \\ &= \sum_{k=1}^{n} \epsilon_k \langle R(Y, e_k)e_k, \nabla_X z \rangle \\ &= -\sum_{k=1}^{n} \epsilon_k \langle \nabla_X(R(Y, e_k)e_k), Z \rangle \\ &= 0. \end{aligned}$$

So, $\nabla_X \mathrm{Ric}(Y) \in V_i$ and;

So, $\nabla_X \mathrm{Ric}(Y) \in V_i$ and;
If $P_i(t) = t^2 + at + b$ with $b \neq 0$. Then for all $Y \in V_i$, we get

$$Y = -\frac{1}{b}(\mathrm{Ric}^2(Y) + a\mathrm{Ric}(Y)) \quad \text{and,} \quad \nabla_X Y \in V_i.$$

So, $\nabla_X \mathrm{Ric}(Y) \in V_i$ and;
If $P_i(t) = t^2 + at + b$ with $b \neq 0$. Then for all $Y \in V_i$, we get

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If $P_i(t) = t - \lambda_i$ with $\lambda_i \neq 0$. Then, for all $Y \in V_i$, we get

$$Y = \frac{1}{\lambda_i}\mathrm{Ric}(Y) \quad \text{and ,} \quad \nabla_X Y \in V_i.$$

So, $\nabla_X \mathrm{Ric}(Y) \in V_i$ and;
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$$Y = \frac{1}{\lambda_i}\mathrm{Ric}(Y) \quad \text{and ,} \quad \nabla_X Y \in V_i.$$

So, $\nabla_X V_i \subset V_i$, this shows that $\nabla_{V_j} V_i \subset V_i$ and $\nabla_{V_0} V_i \subset V_i$, for all $i,\ j \geq 1$ with $i \neq j$.
The other results are obtained immediatly because the metric g is parallel $(\nabla g = 0)$.

Corollary

Let (M, g) *be pseudo-Riemannian semi-symmetric manifolds. Let* $\chi = \prod_i P_i$ *be the minimal polynomial of* Ric*. If we put* $V_0 = \ker(\mathrm{Ric}^2)$ *and* $\forall i \geq 1$, $V_i = \ker(P_i(\mathrm{Ric}))$*. Then, for all* $i \geq 1$*, the distribution* V_0 *and* $V_0 + V_i$ *are involutives.*

Corollary

Let (M, g) be pseudo-Riemannian semi-symmetric manifolds. Let $\chi = \prod_i P_i$ be the minimal polynomial of Ric. If we put $V_0 = \ker(\mathrm{Ric}^2)$ and $\forall i \geq 1$, $V_i = \ker(P_i(\mathrm{Ric}))$. Then, for all $i \geq 1$, the distribution V_0 and $V_0 + V_i$ are involutives.

Remark

The distribution V_0 and $V_0 + V_i$ are involutives spaces and not necessarily parallels.

Proof of Theorem 2.1

Let (M, g) be a four dimensional neutral manifold and the tensor curvature R is considered a symmetric endomorphism in the space $\Lambda^2 TM$;

$$\begin{array}{rccc} R: & \Lambda^2 TM & \to & \Lambda^2 TM \\ & x \wedge y & \mapsto & R(x \wedge y) := R(x, y). \end{array} \tag{17}$$

Let $J: \quad \Lambda^2 TM \to \Lambda^2 TM$ be a `Hodge` morphism given by:

$$\alpha \wedge \beta = \langle J\alpha, \beta \rangle_1 \omega,$$

for all $m \in M$, α, $\beta \in \Lambda^2 T_m M$, $\omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$, such that (e_1, e_2, e_3, e_4) is a positive-oriented orthonormal basis of $T_m M$ and $\langle \ , \ \rangle_1$ is the metric of $\Lambda^2 T_m M$ induced by g. It's easy to proof that $J^2 = id_{\Lambda^2 TM}$ and we put $\Lambda^+ T_m M$ and $\Lambda^- T_m M$ the eigenspaces of J_m associated respectively to eigenvalues 1 and -1, they are the same dimension 3.

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$$J \circ R = R \circ J.$$

Therefore, $\Lambda^+ TM$ and $\Lambda^- TM$ are invariant by R.
Moreover,

$$e_1 \wedge e_2 \pm e_3 \wedge e_4, \;\; e_1 \wedge e_3 \mp e_2 \wedge e_4, \quad \text{and} \quad e_1 \wedge e_4 \pm e_2 \wedge e_3$$

is a basis of $\Lambda^\pm T_m M$. The proof is based on Corollary 3.1 and the following theorem proved in [Derdzinski]

Theorem

[Derdzinski] Let the self-dual curvature operator $R^+ : \Lambda^+ TM \to \Lambda^+ TM$ *of an oriented four-dimensional Einstein manifold* (M, g) *of the metric signature* $(2, 2)$ *be complex-diagonalizable at every point, with complex eigenvalues forming constant functions* $M \longrightarrow \mathbb{C}$*. If* $\nabla R^+ \neq 0$ *somewhere in* M*, then* (M, g) *is locally homogeneous, namely, locally isometric to a Lie group with a left-invariant metric. More precisely,* (M, g) *then is locally isometric to one of Petrov's Ricci-flat manifolds.[Derdzinski]*

Proof of Theorem2.1

Let (M, g) be a four dimensional Einstein semi-symmetric neutral manifold. Then the Ricci tensor satisfies the following relationship:

$$\mathfrak{ric} = \frac{\mathfrak{s}}{4} g, \tag{18}$$

such that $\mathfrak{s}$ is the scalar curvature.

Proof of Theorem2.1

Let (M, g) be a four dimensional Einstein semi-symmetric neutral manifold. Then the Ricci tensor satisfies the following relationship:

$$\mathfrak{ric} = \frac{\mathfrak{s}}{4} g, \tag{18}$$

such that $\mathfrak{s}$ is the scalar curvature. On the other way, the Ricci tensor satisfies the following relationship:

$$\delta(\mathfrak{ric}) = -\frac{1}{2} d(\mathfrak{s}), \tag{19}$$

where δ and d are the contravariant and the covariant differential on M respectively (See [Besse],Proposition1.94, page 43).
This induces that the scalar curvature $\mathfrak{s}$ is a constant function.

So, (M, g) is semi-symmetric and according of the corollary3.1 we get that morphism $R^+ : \Lambda^+ TM \longrightarrow \Lambda^+ TM$ is diagonalizable and it satisfies of one of the following situations:

1. R^+ is an homothety with a report $\frac{\mathfrak{s}}{4}$,
2. R^+ is diagonalizable as $\mathbb{C}$-linear endomorphism of $\wedge^+ T_p M$ with eigenvalues 0 and $\frac{\mathfrak{s}}{4}$ of multiplicity 2 and 1 respectively, where $\mathfrak{s}$ is the scalar curvature.

Then M is a no Ricci flat. According to theorem4.1, we get that M is localy symmetric. This completes the proof of the theorem.2.1.

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Four dimensional semi-symmetric neutral Lie groups

In this section, we give some general properties of semi-symmetric neutral Lie groups and we prove Theorem 2.2 when M is a neutral Lie group.

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A Lie group G together with a left-invariant pseudo-Riemannian metric g is called a *pseudo-Riemannian Lie group*.

Four dimensional semi-symmetric neutral Lie groups

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A Lie group G together with a left-invariant pseudo-Riemannian metric g is called a *pseudo-Riemannian Lie group*.
The metric g defines a pseudo-Euclidean product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g} = T_eG$ of G, and conversely, any pseudo-Euclidean product on $\mathfrak{g}$ gives rise to an unique left-invariant pseudo-Riemannian metric on G.

We will refer to a Lie algebra endowed with a pseudo-Euclidean product as a *pseudo-Euclidean Lie algebra*. The Levi-Civita connection of (G, g) defines a product $\mathrm{L} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the Levi-Civita product and given by Koszul's formula:

$$2\langle \mathrm{L}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle. \tag{20}$$

For any $u, v \in \mathfrak{g}$, $\mathrm{L}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is skew-symmetric and $[u, v] = \mathrm{L}_u v - \mathrm{L}_v u$. We will also write $u.v = \mathrm{L}_u v$.

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The Riemannian curvature on $\mathfrak{g}$ is given by:

$$\mathrm{K}(u, v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v]. \tag{21}$$

It is well-known that K is a curvature tensor on $(\mathfrak{g}, \langle\ ,\ \rangle)$ and, moreover, it satisfies the differential Bianchi identity

$$\mathrm{L}_u(\mathrm{K})(v,w) + \mathrm{L}_v(\mathrm{K})(w,u) + \mathrm{L}_w(\mathrm{K})(u,v) = 0, \quad u,v,w \in \mathfrak{g} \tag{22}$$

where

$$\mathrm{L}_u(\mathrm{K})(v,w) = [\mathrm{L}_u, \mathrm{K}(v,w)] - \mathrm{K}(\mathrm{L}_u v, w) - \mathrm{K}(v, \mathrm{L}_u w). \tag{23}$$

Denote by $\mathfrak{h}(\mathfrak{g})$ the holonomy Lie algebra of (G, g). It is the smallest Lie algebra containing $\mathfrak{h}(\mathrm{K}) = \mathrm{span}\{\mathrm{K}(u,v) : u, v \in \mathfrak{g}\}$ and satisfying $[\mathrm{L}_u, \mathfrak{h}(\mathfrak{g})] \subset \mathfrak{h}(\mathfrak{g})$, for any $u \in \mathfrak{g}$.

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(G, g) is semi-symmetric iff K is a semi-symmetric curvature tensor of $(\mathfrak{g}, \langle\ ,\ \rangle)$.
Without reference to any Lie group, we call a pseudo-Euclidean Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ semi-symmetric if its curvature is semi-symmetric.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a semi-symmetric Lie algebra with metric $\langle\ ,\ \rangle$ of signature $(2, n)$ such that $n \geqslant 2$. According to Proposition 2.2, $\mathfrak{g}$ splits orthogonally as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r, \tag{24}$$

where $\mathfrak{g}_0 = \ker(\mathrm{Ric}^2)$ and $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ are the eigenspaces associated to the non zero eigenvalues of Ric,
or

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r, \tag{25}$$

where $\mathfrak{g}_0 = \ker(\mathrm{Ric})$, $\mathfrak{g}_c = \ker(\mathrm{Ric}^2 - (z + \bar{z})\mathrm{Ric} + \mid z \mid^2 I)$ such that z is non real eigenvalue of Ric and $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ are the eigenspaces associated to the non zero real eigenvalues of Ric.
Moreover, $\mathrm{K}(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for any $i \neq j$, $\dim(\mathfrak{g}_c) = 4$ and $\dim \mathfrak{g}_i \geq 2$ if $i \neq 0$. According the Proposition4.1, the following proposition gives more properties of the $\mathfrak{g}_i$'s involving the Levi-Civita product.

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a semi-symmetric Lie algebra with metric* $\langle\ ,\ \rangle$ *of signature* $(2, n)$ *such that* $n \geqslant 2$. *Then, for any* $i, j \in \{c, 1, \ldots, r\}$ *and* $i \neq j$,

$$\mathfrak{g}_j.\mathfrak{g}_i \subset \mathfrak{g}_i,\ \mathfrak{g}_i.\mathfrak{g}_i \subset \mathfrak{g}_0 + \mathfrak{g}_i,\ \mathfrak{g}_0.\mathfrak{g}_i \subset \mathfrak{g}_i,\ \mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0,\ \mathfrak{g}_i.\mathfrak{g}_0 \subset \mathfrak{g}_0 + \mathfrak{g}_i.$$

Let (G, g) be a four dimensional semi-symmetric neutral Lie group with Ricci curvature having a non zero eigenvalue and, according to (24) , (25) and Proposition 5.1, the Lie algebra $\mathfrak{g}$ of G has one of the following types:

- $(S4\lambda)$: $\dim \mathfrak{g} = 4$ and $\mathfrak{g} = \mathfrak{g}_\lambda$ with $\lambda \neq 0$.
- $(S4\mu\lambda)$: $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_\mu = \dim \mathfrak{g}_\lambda = 2$, $\lambda \neq \mu$, $\lambda \neq 0$, $\mu \neq 0$, $\mathfrak{g}_\mu.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_\lambda.\mathfrak{g}_\mu \subset \mathfrak{g}_\mu$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu.\mathfrak{g}_\mu \subset \mathfrak{g}_\mu$.
- $(S4\lambda 0^1)$: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_0 = 1$, $\mathfrak{g}_0.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and $\lambda \neq 0$.
- $(S4\lambda 0^2)$: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_\lambda = 2$, $\mathfrak{g}_0.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and $\lambda \neq 0$,
- $(Sz\bar{z})$: $\mathfrak{g} = \ker(\mathrm{Ric}^2 - (z + \bar{z})\mathrm{Ric} + \mid z \mid^2 I)$ with $z \in \mathbb{C} - \mathbb{R}$.

where $\mathfrak{g}_\lambda = \ker(\mathrm{Ric} - \lambda \mathrm{Id}_{\mathfrak{g}})$ and $\mathfrak{g}_0 = \ker(\mathrm{Ric})$.

Proposition

Ricci is neither flat nor isotropic

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a four dimensional semi-symmetric neutral Lie algebra with Ricci curvature admitting a non-zero eigenvalue. Then* $\mathfrak{g}$ *is a symmetric space. Precisely, one of the following cases occurs:*

- $\mathfrak{g}$ *is of type* $(S4\mu\lambda)$. *Then* $\mathfrak{g}_\lambda.\mathfrak{g}_\mu = \mathfrak{g}_\mu.\mathfrak{g}_\lambda = 0$ *and* $\mathfrak{g}$ *is a product of two Lie algebras with the same metric et and the same dimension* 2.
- $\mathfrak{g}$ *is of type* $(S4\lambda 0^1)$. *Then* $\mathfrak{g}.\mathfrak{g}_0 = 0$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ *and hence* $\mathfrak{g}$ *is the semi-direct product of* $\mathfrak{g}_0$ *with the three dimensional Lorentzian Lie algebra* $\mathfrak{g}_\lambda$ *of constant curvature and the action of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_\lambda$ *is by a skew-symmetric derivation.*

Proposition

- $\mathfrak{g}$ is of type $(S4\lambda 0^2)$. Then $\mathfrak{g}_0.\mathfrak{g} = 0$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_\lambda.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and hence $\mathfrak{g}$ is the semi-direct product of the pseudo-Euclidean Lie algebra $\mathfrak{g}_\lambda$ with the abelian Lie algebra $\mathfrak{g}_0$ and the action of $\mathfrak{g}_\lambda$ on $\mathfrak{g}_0$ is given by skew-symmetric endomorphisms.
- $\mathfrak{g}$ is of type $(S4\lambda)$ with $\lambda \neq 0$. In this case, we get $\dim(\mathfrak{h}(\mathrm{K})) \in \{2, 4, 6\}$.
- $\mathfrak{g}$ is of type $(S4z\bar{z})$. In this case, $\dim(\mathfrak{h}(\mathrm{K})) = 2$.

proof

-For types $(S4\mu\lambda)$, $(S4\lambda0^1)$ and $(S4\lambda0^2)$, Ric admits two real eigenvalues. Then each eigenspace is either Lorentzian or Riemannian and the demonstration of similar cases in [Benroummane] remains valid in the current situation.
-For type $(S4\lambda)$, it is a result of the theorem(2.1).
-For type $(S4z\bar{z})$, it is the same proof in the case Ricci complex for a homogeneous semi-symmetric manifolds.

According to this proposition, we get the following theorem:

Theorem

Let G be a four-dimensional connexe simply connected neutral Lie group. If (G, g) is semi-symmetric space admitting a left invariant metric g and it's Ricci cuvature admits no zero eigenvalue, then G is localy symmetric.

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Corollary

Let $\mathfrak{g}$ be a four-dimensional semi-symmetric nonsymmetric neutral Lie algebra. Then, its Ricci operator satisfies the condition

$$\mathrm{Ric}^2 = 0$$

i.e; Ricci is plat or istrope. Precisely, Ricci has only 0 eigenvalue.

Remark

There are some four dimensional neutral semi-symmetric non-symmetric Lie algebras with Ricci plat:

Example: *Let* $\mathfrak{g} = vect(x, y, z, t)$ *be a Lie algebra equipped with a metric* $\langle\ ,\ \rangle$ *given by:* $\langle x, z\rangle = \langle y, t\rangle = 1$ *and the non zero brackets are:*
$[x, y] = Ax + Bt$, $[x, z] = 2Dx$, $[y, z] = Cx - Dy + Az$,
$[y, t] = -2At$ *and* $[z, t] = -\frac{AD}{B}x - Dt$.
Then $\mathfrak{g}$ *is semi-symmetric non symmetric with Ricci plat and the courvature:*

$$R = 4AC.A_{x,t} \vee A_{x,t}.$$

This example makes the difference between the Lorentzian case and the case of the signature $(2, 2)$*: In the first case, the semi-symmetric Lie algebras of Ricci flat are flat and locally symmetrical.*

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Remark

let $\mathfrak{g}$ *be a four-dimensional semi-symmetric neutral Lie algebra with isotropic Ricci. Then, two cases are possibles:* $rank(\mathrm{Ric}) = 1$ *or* $rank(\mathrm{Ric}) = 2$ *and* $\dim(\mathfrak{h}(\mathrm{K})) \in \{1, 2\}$

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a four dimensional semi-symmetric neutral Lie algebra with* K *the curvature tensor and* Ric *the Ricci operator.*

1. *If* $\mathrm{K} \neq 0$ *and* $\mathrm{Ric} = 0$*, then, there is a basis* (x, y, z, t) *such that* $\langle x, t\rangle = \langle y, z\rangle = 1$ *and* $\mathrm{K} = aA_{x,z} \vee A_{x,y}$*, where* $a \in \mathbb{R}^*$.
2. *If* $\mathrm{Ric} \neq \mathrm{Ric}^2 = 0$*. Then, one of the following situations is checked:*
   - $\mathfrak{h}(\mathrm{K})$ *is of type* 1.4^1*. Then, there is a basis* (x, y, z, t) *such that* $\langle x, z\rangle = -\langle y, y\rangle = \langle t, t\rangle = 1$ *and* $\mathrm{K} = qA_{x,y} \vee A_{x,y}$, $\mathrm{Ric} = -q(x \vee x)$, $q \neq 0$.
   - $\mathfrak{h}(\mathrm{K})$ *is of type* 2.5^1*. Then, there is a basis* (x, y, z, t) *such that* $\langle x, z\rangle = \langle y, t\rangle = 1$ *and* $\mathrm{K} = rA_{x,y} \vee A_{x,y} + p.A_{x,t} \vee A_{x,t} + q.A_{x,y} \vee A_{x,t}, \quad \mathrm{Ric} = q(x \vee x)$, $p \neq 0 \neq q$ *and* $r \neq 0$.
   - $\mathfrak{h}(\mathrm{K})$ *is of type* 2.2^1*. Then,* $\dim \mathfrak{h}(\mathrm{K}) = 2$ *and there is a basis* (x, y, z, t) *such that* $\langle x, z\rangle = \langle y, t\rangle = 1$ *and* $\mathrm{K} = sA_{x,t} \vee A_{x,t} + p.((A_{x,z} + A_{y,t}) \vee .A_{x,t})$, $\mathrm{Ric} = -p(x \vee t)$, $p \neq 0 \neq s$.
   - $\mathfrak{h}(\mathrm{K})$ *is of type* 2.2^2*. Then, there is a basis* (x, y, z, t)*such that* $\langle x, t\rangle = \langle y, z\rangle = 1$ *and*

In [Ali], A. Haji-Badali and A. Zaeim give a complet classification of four-dimensional semi-symmetric nonsymmetric neutral Lie algebras.

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Proof of Theorem 2.2

Let (M, g) be a four dimensional simply connected homogeneous neutral semi-symmetric manifold with Ricci curvature having a non zero eigenvalue. According to proposition3.1. If the Ricci curvature has a non zero eigenvalue in $\mathbb{C}$, Then, M is one of the following types: $(S4\lambda)$, $(S4\lambda\mu)$, $(S4\lambda 0^1)$, $(S4\lambda 0^2)$ or $(Sz\bar{z})$.

Lemma

Let (M, g) be a four dimensional simply connected homogeneous neutral semi-symmetric manifold of type $(S4\lambda\mu)$ or $(S4\lambda 0^1)$ or $(S4\lambda 0^2)$ then (M, g) is either Ricci-parallel or locally isometric to a Lie group equipped with a left invariant neutral metric.

Proof of Lemma 6.1

If M is one of the following types $(S4\lambda\mu)$ or $(S4\lambda 0^2)$, the proof of theorem 4.1 in [Calvaruso-Zaeim] remains valid in the current situation. For the case $(S4\lambda 0^1)$ and according to [Komrakov], we find the same homogeneous manifolds as the theorem 4.6 in [Calvaruso-Zaeim] and consequently its proof remains valid in the current situation. This completes the proof of the Lemma6.1.■
So if (M, g) is of type $(S4\lambda)$ such that $\lambda \neq 0$, that is, M is the Einstein space with non null scalar curvature and we can apply Theorem 2.1 to get that M is locally symmetric.
If (M, g) is Ricci-parallel and the Ricci operator has two distinct real eigenvalues then, according to Theorem 7.3 [Boubel] and the Proposition4.1 , (M, g) is a product of two Einstein homogeneous semi-symmetric pseudo-Riemannian manifolds of dimension less or equal 3 and according the some results of same situation in [Benroummane] we get that (M, g) is localy symmetric.

If M is locally isometric to a Lie group equipped with a left invariant neutral metric, we have shown in section 5 that M is locally symmetric. Suppose now that (M, g) is of type $(Sz\bar{z})$. Let $z = a + ib$ and $\bar{z} = a - ib$ be the eigenvalues of Ric such that $b \neq 0$. Then there is a pseudo-orthonormal frame $\mathbb{B} = (e, f, u, v)$ such that $g(e, u) = g(f, v) = 1$ in which the matrix of Ric has the following form:

$$[\mathrm{Ric}]_{\mathbb{B}} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

Let J be the operator given by $J := \frac{1}{b}(\mathrm{Ric} - a.I)$. We have $J^2 = -I$ such that I is the identity operator of the tangent fibre of M. Then, by complexification, we get that the complex semi-symmetric manifolds $(M^{\mathbb{C}}, g^{\mathbb{C}})$ which $\mathrm{Ric}^{\mathbb{C}}$ the complex Ricci is semi-symmetric operator admitting two eigenvalues $z = a + ib$ and $\bar{z} = a - ib$ which are the constant functions because M is homogeneous. Moreover, $\mathrm{Ric}^{\mathbb{C}}$ must be diagonalizable in $\mathbb{C}$. More precisely, $\ker(\mathrm{Ric}^{\mathbb{C}} - z.I) = \ker(J - i.I)$ and $\ker(\mathrm{Ric}^{\mathbb{C}} - \bar{z}.I) = \ker(J + i.I)$. Applying the procedure of the proof of the proposition4.1, we find that the two two-dimensional orthogonaly eigenspaces of $\mathrm{Ric}^{\mathbb{C}}$ are parallel and consequently, they are locally symmetric. So $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is locally symmetric. As a result, (M, g) is locally symmetric. This completes the proof of the Teorem2.2.

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Ricci flat or isotropic

In this section, we deal with non flat semi-symmetric four-dimensional neutral manifolds with ithe Ricci curvature is either isotropic or flat . We use Komrakov's classification [Komrakov] of four-dimensional homogeneous pseudo-Riemannian manifolds and we apply the following algorithm to find among Komrakov's list the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to four-dimensional Ricci flat or Ricci isotropic homogeneous semi-symmetric neutral manifolds which are not locally symmetric.

Let $M = \overline{G}/G$ be an homogeneous manifold with G connected and $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{m}$, where $\overline{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{m}$ an arbitrary complementary of $\mathfrak{g}$ (not necessary $\mathfrak{g}$-invariant). The pair $(\overline{\mathfrak{g}}, \mathfrak{g})$ uniquely defines the isotropy representation $\rho : \mathfrak{g} \longrightarrow \mathrm{gl}(\mathfrak{m})$ by $\rho(x)(y) = [x, y]_{\mathfrak{m}}$, for all $x \in \mathfrak{g}$, $y \in \mathfrak{m}$. Let $\{e_1, \dots, e_r, u_1, \dots, u_n\}$ be a basis of $\overline{\mathfrak{g}}$ where $\{e_i\}$ and $\{u_j\}$ are bases of $\mathfrak{g}$ and $\mathfrak{m}$, respectively. The algorithm goes as follows.

- Determination of invariant pseudo-Riemannian metrics on M:
It is well-known that invariant pseudo-Riemannian metrics on M are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms on $\mathfrak{m}$. A symmetric bilinear form on $\mathfrak{m}$ is determined by its matrix B in $\{u_i\}$ and its invariant if $\rho(e_i)^t \circ B + B \circ \rho(e_i) = 0$ for $i = 1, \dots, r$.
- Determination of the Levi-Civita connection:
Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{m}$. It defines uniquely an invariant linear Levi-Civita connection $\nabla : \bar{\mathfrak{g}} \longrightarrow \mathrm{gl}(\mathfrak{m})$ given by

$$\nabla(x) = \rho(x),\ \nabla(y)(z) = \frac{1}{2}[y, z]_{\mathfrak{m}} + \nu(y, z),\ x \in \mathfrak{g}, y, z \in \mathfrak{m},$$

where $\nu : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ is given by the formula

$$2B(\nu(a, b), c) = B([c, a]_{\mathfrak{m}}, b) + B([c, b]_{\mathfrak{m}}, a),\ a, b, c \in \mathfrak{m}.$$

- Determination of the curvature:
The curvature of B is the bilinear map $\mathrm{K} : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathrm{gl}(\mathfrak{m})$ given by

$$\mathrm{K}(a, b) = [\nabla(a), \nabla(b)] - \nabla([a, b]_{\mathfrak{m}}) - \rho([a, b]_{\mathfrak{g}}),\ a, b \in \mathfrak{m}.$$

- Determination of the Ricci curvature:
  It is given by its matrix in $\{u_i\}$, i.e., $\mathrm{ric} = (\mathrm{ric}_{ij})_{1\leq i,j\leq n}$ where

$$\mathrm{ric}_{ij} = \sum_{r=1}^{n} \mathrm{K}_{ri}(u_r, u_j).$$

- Determination of the Ricci operator:
  We have $\mathrm{Ric} = B^{-1}\mathrm{ric}$.
- Checking the semi-symmetry condition.

The following theorem gives the list of four dimensional homogeneous neutral semi-symmetric manifolds non flat which Ricci is either isotropic or flat.

Theorem

Let $M = \bar{G}/G$ be a 4-homogeneous neutral semi-symmetrique no symmetric manifolds.
Let $\bar{\mathfrak{g}} = span\{e_1, .., e_n, u_1, u_2, u_3, u_4\}$ and $\mathfrak{g} = span\{e_1, .., e_n\}$ the Lie algebras associted respectively to $\bar{G}$ and G. Then, M is isometric to one of the following types:

I) $\bar{\mathfrak{g}} = span\{e_1, u_1, u_2, u_3, u_4\}$;

- $\langle u_1, u_4\rangle = -\langle u_2, u_3\rangle = a$, $\langle u_3, u_3\rangle = -\langle u_4, u_4\rangle = b$, $\langle u_3, u_4\rangle = c$
  $1.3^1 : 2, 3, 4, 6, 7, 10, 15, 16, 24, 26 - 30,$
  $1.3^1 : 5$ with $(\lambda, \mu) \neq (0, 2)$,
  $1.3^1 : 8, 19, 20, 22$ with $b \neq 0$,
  $1.3^1 : 9$ with $b\lambda(\lambda + 1) \neq 0$,
  $1.3^1 : 12$ with $(\lambda - \mu - 1)(\lambda - \mu + 1) \neq 0$,
  $1.3^1 : 13$ with $\lambda \neq \frac{1}{2}$,
  $1.3^1 : 21$ with $b\lambda(\lambda - 1) \neq 0$,
  $1.3^1 : 25$ with $(b, \lambda) \neq (0, 2)$,

Theorem

- $\langle u_1, u_3\rangle = -\langle u_2, u_2\rangle = a, \ \langle u_3, u_3\rangle = b, \ \langle u_3, u_4\rangle = d, \ \langle u_4, u_4\rangle = a - b,$
  1.4^1 : 2 with $b \neq 0$ and $p = 1$,
  1.4^1 : $9 - 11, 13, 15 - 20$.

II) $\bar{\mathfrak{g}} = span\{e_1, e_2, u_1, u_2, u_3, u_4\}$ with $\langle u_1, u_3\rangle = \langle u_2, u_4\rangle = a$;

- 2.2^1 : 2 with $\lambda(\lambda^2 - 4) \neq 0$
- 2.2^1 : 3.
- 2.5^1 : $3 - 6$.

III) $\bar{\mathfrak{g}} = span\{e_1, e_2, e_3, u_1, u_2, u_3, u_4\}$;
3.3^1 : 1 with $\langle u_1, u_3\rangle = \langle u_2, u_4\rangle = a$:

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Thank you for watching me