Introduction to affine geometry

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Outline

Let M denote an n-dimensional connected manifold, $n \geq 1$.
An affine atlas Φ on M is a covering of M by coordinate charts $\{U_i, \phi_i\}_{i \in S}$ such that, for any $i, j \in S$ with $U_i \cap U_j \neq \emptyset$,

$$\phi_i \circ \phi_j^{-1}(x) = A_{ij}x + b_{ij}, \quad A_{ij} \in \mathrm{GL}(n, \mathbb{R}), b_{ij} \in \mathbb{R}^n.$$

A maximal affine atlas is an affine structure on M, and M together with an affine structure is an affine manifold. Each chart in the affine structure defines affine coordinates.

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Given an affine structure on M, we define on M a covariant connection by putting for any vector fields X, Y and any $m \in M$

$$(\nabla_X Y)(m) = \sum_{i,j} X_i(m) \frac{\partial Y_j}{\partial x_i}(m) \partial_{x_j} \quad (1)$$

where $X = \sum_{i=1}^n X_i \partial_{x_i}$ and $Y = \sum_{i=1}^n X_i \partial_{x_i}$ are the expressions of X and Y in an affine chart $(x_1, \ldots, x_n)$ in an open neighborhood U of m.

If $(y_1, \dots, y_n)$ is another affine chart in an open neighborhood V of m, we have on $U \cap V$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + b,$$

where $A \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. So the components of X in the local frame $(\partial_{y_1}, \dots, \partial_{y_n})$ are given by

$$\begin{pmatrix} \widetilde{X}_1 \\ \vdots \\ \widetilde{X}_n \end{pmatrix} = \begin{pmatrix} X(y_1) \\ \vdots \\ X(y_n) \end{pmatrix} = A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

On the other hand the passage matrix from $(\partial_{x_1}, \dots, \partial_{x_n})$ to $(\partial_{y_1}, \dots, \partial_{y_n})$ is A^{-1}. Hence

$$\sum_{i,j} X_i(m) \frac{\partial Y_j}{\partial x_i}(m) \partial_{x_j} = \sum_{i,j} \widetilde{X}_i(m) \frac{\partial \widetilde{Y}_j}{\partial y_i}(m) \partial_{y_j}.$$

Thus ∇ is well defined. Since $\nabla_{\partial_{x_i}}\partial_{x_j} = 0$, we deduce that ∇ is torsion free and its curvature vanishes.

Let us study the converse. At first, we have the following result.

Proposition

Let (M, ∇) be a manifold endowed with a torsion free and flat connection. Then for any $m \in M$ and $v \in T_mM$ there exists an open neighborhood U of m and a unique vector field X^v defined on U such that

$$\nabla X^v = 0 \quad and \quad X^v(m) = v. \tag{2}$$

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$$\nabla X^v = 0 \quad \text{and} \quad X^v(m) = v. \tag{2}$$

Proof.

The problem is local so we can suppose that $M = \mathbb{R}^n$ and $m = 0$. Denote by $(x_1, \ldots, x_n)$ the canonical coordinates of $\mathbb{R}^n$ and put

$$X^v = \sum_{i=1}^{n} \alpha_i \partial_{x_i} \quad \text{and} \quad \nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_{x_k}.$$

The vanishing of the curvature is equivalent to

$$\partial_j \left(\Gamma_{il}^h\right) - \partial_i \left(\Gamma_{jl}^h\right) = \sum_{k=1}^{n} \left(\Gamma_{jl}^k \Gamma_{ik}^h - \Gamma_{il}^k \Gamma_{jk}^h\right), \quad i, j, l, h = 1, \ldots, n, \ i \neq j. \tag{3}$$

□

Proof Continued.

The equation (2) is equivalent to

$$\nabla_{\partial_{x_j}} X^v = 0, \quad j = 1, \ldots, n \quad \text{and} \quad X^v(0) = v$$

thus

$$\frac{\partial \alpha}{\partial x_j} = F_j(x, \alpha(x)), \quad i, j = 1, \ldots, n, \tag{4}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{and} \quad F_j^k(x, \alpha) = -\sum_{l=1}^{n} \alpha_l \Gamma_{jl}^k(x). \qquad (*)$$

□

Theorem (See Spivak volume I. pp.187)

Let $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n$ be open, where U is a neighborhood of 0 in $\mathbb{R}^m$, and let $F_i : U \times V \longrightarrow \mathbb{R}^n$ be C^∞ functions, for $i = 1, \ldots, m$. Then for every $x_0 \in V$ there is at most one function

$$\alpha : W \longrightarrow V$$

defined in a neighborhood of 0 in $\mathbb{R}^m$, satisfying

$$\frac{\partial \alpha}{\partial x^j} = F_j(x, \alpha(x)) \quad \text{for all } x \in W \quad \text{and} \quad \alpha(0) = x_0.$$

Moreover, such a function exists and automatically C^∞ in some neighborhood W if and only if there is a neighborhood of $(0, x) \in U \times V$ on which

$$\frac{\partial F_j}{\partial x^i} - \frac{\partial F_i}{\partial x^j} + \sum_{k=1}^{n} \frac{\partial F_j}{\partial \alpha^k} F_i^k - \sum_{k=1}^{n} \frac{\partial F_i}{\partial \alpha^k} F_j^k = 0.$$

According to this theorem, to complete the proof one needs to show that $F_1, \dots, F_n$ given by $(*)$ satisfy the conditions of the theorem. Indeed, we have

$$\begin{aligned} \frac{\partial F_j^h}{\partial x_i} - \frac{\partial F_i^h}{\partial x_j} &= -\sum_{l=1} \alpha_l \left(\frac{\partial \Gamma_{jl}^h}{\partial x_i} - \frac{\partial \Gamma_{il}^h}{\partial x_j} \right), \\ \sum_{k=1}^{n} \left(\frac{\partial F_j^h}{\partial \alpha^k} F_i^k - \frac{\partial F_i^h}{\partial \alpha^k} F_j^k \right) &= \sum_{k,l=1}^{n} \alpha_l \left(\Gamma_{jk}^h \Gamma_{il}^k - \Gamma_{ik}^h \Gamma_{jl}^k \right). \end{aligned}$$

By using (3) we can conclude. □

Let (M, ∇) be a manifold endowed with a torsion free and flat connection. For any $m \in M$, choose a basis $(v_1, \ldots, v_n)$ of T_mM. According to Proposition 1.1, there exists an open set U containing m and $(X^1, \ldots, X^n)$ a family of vector fields such that $\nabla X^i = 0$ and $X^i(m) = v_i$ for $i = 1, \ldots, n$. Since ∇ is torsion free, we have, for any $i, j = 1, \ldots, n$,

$$[X^i, X^j] = \nabla_{X^i} X^j - \nabla_{X^j} X^i = 0.$$

Moreover, since $(X^1(m), \ldots, X^n(m))$ are linearly independent, we can choose U such that $(X^1(x), \ldots, X^n(x))$ are linearly independent for any $x \in U$. By applying Frobenius's Theorem, we get a coordinate system $(x_1, \ldots, x_n)$ around m such that

$$\partial_{x_i} = X^i, \quad i = 1, \ldots, n.$$

Theorem (Frobenius's Theorem)

Let M be a n-dimensional smooth manifold and $X^1, \ldots, X^p$ $(1 \leq p \leq n)$ a family of linearly independent vector fields on a neighborhood of a point $m \in M$. Then there exists a coordinates system $(x_1, \ldots, x_n)$ around m such that

$$\partial_{x_i} = X^i, \quad i = 1, \ldots, p.$$

By varying the point m on the construction above we get a differential atlas Φ of M such that, for any chart $(x_1, \dots, x_n)$ in Φ,

$$\nabla \partial_{x_i} = 0, \quad i = 1, \dots, n.$$

Let us show that Φ is an affine atlas. Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ two charts in Φ over, respectively, two open sets U and V such that $U \cap V \neq 0$. On $U \cap V$, we have

$$\partial_{y_i} = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \partial_{x_k} \quad i = 1, \dots, n.$$

Since both $(\partial_{x_1}, \ldots, \partial_{x_n})$ $(\partial_{y_1}, \ldots, \partial_{y_n})$ are parallel with respect to ∇ we get that

$$\frac{\partial y_i}{\partial x_h \partial x_k} = 0, \quad i, k, h = 1, \ldots, n.$$

This show that, for any $i = 1, \ldots, n$,

$$y_i = \sum_{k=1}^{n} a_{ik} x_k + b_i,$$

where $A = (a_i j)$ is an invertible matrix and $(b_1, \ldots, b_n) \in \mathbb{R}^n$. This shows that the change of coordinates from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$ is given by an affine map of $\mathbb{R}^n$.

To summarize all what above, we have proved the following theorem.

Theorem

Let M be a smooth manifold. Then the following assertions are equivalent:

1. *There exists on M and affine atlas Φ.*
2. *M carries a torsion free flat linear connection.*

Let M be a manifold and ∇ a connection on TM.

A variation of curves is a smooth map $\Gamma = (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$. Any variation of curves defines two collections of curves: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting $s = constant$ and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $(-\varepsilon, \varepsilon)$ by setting $t = constant$.

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A vector field along Γ is a map $V : (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow TM$ such that

$$V(s, t) \in T_{\Gamma(s,t)}M.$$

The tangent vectors to these two families of curves are examples of vector fields along Γ, we denote them by

$$T(s, t) = \partial_t\Gamma(s, t) = \frac{d}{dt}\Gamma_s(t), \quad S(s, t) = \partial_s\Gamma(s, t) = \frac{d}{ds}\Gamma^{(t)}(s).$$

If V is a vector field along Γ, we can compute the covariant derivative of V either along the main curves or along the transverse curves, the resulting vector fields along Γ are denoted by D_tV and D_sV. The following lemma is classical.

Lemma

With the notation above, we have

$$D_sT - D_tS = T^{\nabla}(S,T). \tag{5}$$

and

$$D_sD_tY - D_tD_sY = -R^{\nabla}(S,T)Y, \tag{6}$$

$$T^{\nabla}(X,Y) = \nabla_XY - \nabla_YX - [X,Y]$$

$$R(X,Y) = \nabla_{[X,Y]} - \nabla_X\nabla_Y + \nabla_Y\nabla_X.$$

Let (M, ∇) be an affine manifold.
Let $H : [0,1] \times [0,1]$ a smooth homotopy with fixed end, i.e., $H(s,0) = x$ and $H(s,1) = y$ for any $s \in [0,1]$. Fix $v \in T_xM$.
For any $s \in [0,1]$ there exists an unique $V(s,.) : [0,1] \longrightarrow TM$ such that

$$V(s,t) \in T_{H(s,t)}M \quad \text{and} \quad V(s,0) = v$$

and $V(s,.)$ is parallel along $H(s,.)$. From (6) we get

$$\nabla_S \nabla_T V - \nabla_T \nabla_S V = 0,$$

where $S = \frac{\partial H}{\partial s}$ and $T = \frac{\partial H}{\partial t}$.

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where $S = \frac{\partial H}{\partial s}$ and $T = \frac{\partial H}{\partial t}$.

Since $\nabla_T V = 0$ we deduce that $\nabla_S V$ is parallel. But $H(s,0) = x$ and $V(s,0) = v$ so $(\nabla_S V)(0,0) = 0$ and hence $\nabla_S V = 0$. Moreover, $H(s,1) = y$ and hence $V(s,1)$ is constant.

Proposition

Let M be an affine manifold. Then the parallel displacement along a smooth curve γ depends only on the homotopy class of γ.

Let $\mathbf{p} : \widetilde{M} \longrightarrow M$ be an universal covering of M and $\mathrm{Aut}(\mathbf{p})$ the group of deck transformations of $\mathbf{p}$.
A diffeomorphism $F : \widetilde{M} \longrightarrow \widetilde{M}$ is an element of $\mathrm{Aut}(\mathbf{p})$ if $\mathbf{p} \circ F = \mathbf{p}$.
It is well-known that $\mathrm{Aut}(\mathbf{p})$ is isomorphic to $\pi_1(M)$. Indeed,

$$\widetilde{M} = \{[c], c : [0,1] \longrightarrow M, c(0) = x_0\} \quad \text{and} \quad \mathbf{p} : \widetilde{M} \longrightarrow M, [c] \mapsto c(1)$$

and if $[\gamma] \in \pi_1(M, x_0)$ then $F_\gamma : \widetilde{M} \longrightarrow \widetilde{M}$, $[c] \mapsto [\gamma].[c]$ is a deck transformation and the map $[\gamma] \longrightarrow F_\gamma$ is an isomorphism.

For any vector field X on M there exists a unique vector field X^ℓ on $\widetilde{M}$ such that, for any $m \in \widetilde{M}$,::

$$T_m\mathbf{p}(X^\ell(m)) = X(\mathbf{p}(m)).$$

We define on $\widetilde{M}$ a linear connection $\widetilde{\nabla}$ which is the lift of ∇ by the following formula:

$$\widetilde{\nabla}_{X^\ell} Y^\ell = (\nabla_X Y)^\ell,$$

where X, Y are vector fields on M. This connection defines on $\widetilde{M}$ an affine structure for which the elements of $\pi_1(M)$ are affine transformations. Because, for any $X \in \Gamma(TM)$, and for any $F \in \pi_1(M)$

$$F_* X^\ell = X^\ell.$$

Moreover, since $\widetilde{M}$ is simply connected, for any $m, m' \in \widetilde{M}$, the parallel displacement defines an isomorphism

$$\tau_{mm'} : T_m M \longrightarrow T_{m'} M.$$

Fix a point $m \in \widetilde{M}$ and a basis $\mathbf{b} = (v_1, \ldots, v_n)$ of $T_m \widetilde{M}$. We can build from $\mathbf{b}$ a family

$$\mathbb{B} = (X_1^{\mathbf{b}}, \ldots, X_n^{\mathbf{b}})$$

of parallel vector fields on $\widetilde{M}$ linearly independent and satisfying $X_i^{\mathbf{b}}(m) = v_i$ for any i.

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of parallel vector fields on $\widetilde{M}$ linearly independent and satisfying $X_i^{\mathbf{b}}(m) = v_i$ for any i.

The dual basis

$$\Gamma^{\mathbf{b}} = (\alpha_1^{\mathbf{b}}, \ldots, \alpha_n^{\mathbf{b}})$$

of $\mathbb{B}$ consists of parallel 1-forms and hence closed. Since $\widetilde{M}$ is simply-connected $H^1(\widetilde{M}, \mathbb{R}) = 0$, so there exists a unique map

$$D^{\mathbf{b}} : \widetilde{M} \longrightarrow \mathbb{R}^n$$

such that

$$dD^{\mathbf{b}} = (\alpha_1^{\mathbf{b}}, \ldots, \alpha_n^{\mathbf{b}}) \quad \text{and} \quad D^{\mathbf{b}}(m) = 0. \tag{7}$$

Since $(\alpha_1^{\mathbf{b}}, \ldots, \alpha_n^{\mathbf{b}})$ is parallel, $D^{\mathbf{b}}$ is an affine map. We call it **the developing map** associated to **b**.

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Proposition

1. *If* $\mathbf{b}$ *and* $\mathbf{b}'$ *are two basis, respectively, of* $T_m\widetilde{M}$ *and* $T_{m'}\widetilde{M}$ *then*

$$D^{\mathbf{b}'} = A \circ D^{\mathbf{b}},$$

where $A = (P, -P \circ D^{\mathbf{b}}(m')) \in \mathrm{Aff}(\mathbb{R}^n)$ *with* $P = \mathrm{Mat}(\tau_{mm'}, \mathbf{b}, \mathbf{b}')$.

2. *For any affine transformations* f *of* $\widetilde{M}$

$$D^{\mathbf{b}} \circ f = D^{f_*\mathbf{b}} = \alpha_{\mathbf{b}}(f) \circ D^{\mathbf{b}},$$

where $\alpha_{\mathbf{b}}(f) = (P, -P \circ D^{\mathbf{b}}(f(m))) \in \mathrm{Aff}(\mathbb{R}^n)$ *with* $P = \mathrm{Mat}(\tau_{mf(m)}, \mathbf{b}, f_*\mathbf{b})$.

Proof.

From (7) we get that

$$d(D^{\mathbf{b}} \circ f) = f^*\Gamma_{\mathbf{b}}.$$

But from the fact that f is affine, one can show that

$$f^*\Gamma_{\mathbf{b}} = A \circ \Gamma_{\mathbf{b}},$$

where $A \in \mathrm{GL}(\mathbb{R}^n)$ is the passage matrix from **b** to $\tau_{f(m)m} \circ T_m f(\mathbf{b})$. So we get that

$$d(D^{\mathbf{b}} \circ f) = d(A \circ D^{\mathbf{b}}),$$

and the proposition follows. □

According to this proposition, we get a representation

$$\alpha^{\mathbf{b}} : \pi_1(M) \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$$

$$\alpha_{\mathbf{b}}(f) = (P, -P \circ D^{\mathbf{b}}(f(m))) \in \mathrm{Aff}(\mathbb{R}^n) \quad \text{and} \quad P = \mathrm{Mat}(\tau_{mf(m)}, \mathbf{b}, f_* \mathbf{b})$$

called *holonomy affine representation* associated to $D^{\mathbf{b}}$.

Theorem

There exists an affine local diffeomorphism $D : \widetilde{M} \longrightarrow \mathbb{R}^n$ *called the* **developing map** *of* (M, ∇) *and a group homomorphism* $\alpha : \pi_1(M) \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$ *called the* **holonomy representation** *of* (M, ∇) *such that, for any* $f \in \pi_1(M)$*,*

$$D \circ f = \alpha(f) \circ D.$$

Proposition

For any $m' \in \widetilde{M}$ *and any curve* $\gamma : [a,b] \longrightarrow \widetilde{M}$ *such that* $\gamma(a) = m$ *and* $\gamma(b) = m'$*, we have*

$$D^{\mathbf{b}}(m') = \left(\int_a^b \alpha_1^{\mathbf{b}}(\dot\gamma(t))dt, \ldots, \int_a^b \alpha_1^{\mathbf{b}}(\dot\gamma(t))dt\right)$$
$$= \left(\int_a^b x_1(t)dt, \ldots, \int_a^b x_n(t)dt\right),$$

where, for any $t \in [a,b]$*,*

$$\dot\gamma(t) = \sum_{i=1}^{n} x_i(t) X_i^{\mathbf{b}}(\gamma(t)).$$

Let $\gamma : [a, b] \longrightarrow \widetilde{M}$ a geodesic. Then

$$\frac{d}{dt}\alpha_i^{\mathbf{b}}(\dot{\gamma}(t)) = \alpha_i^{\mathbf{b}}(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)) = 0.$$

Thus

$$\begin{aligned} D^{\mathbf{b}}(\gamma(t)) &= D^{\mathbf{b}}(\gamma(a)) + \left(\int_a^b \alpha_1^{\mathbf{b}}(\dot{\gamma}(t))dt, \ldots, \int_a^b \alpha_n^{\mathbf{b}}(\dot{\gamma}(t))dt\right) \\ &= D^{\mathbf{b}}(\gamma(a)) + tv. \end{aligned}$$

So the image of a geodesic is a line in $\mathbb{R}^n$.

In particular, if $\exp_m : \mathcal{E} \subset T_m\widetilde{M} \longrightarrow \widetilde{M}$ then

$$D^{\mathbf{b}} \circ \exp_m : T_m\widetilde{M} \longrightarrow \mathbb{R}^n, v \mapsto (\alpha_1^{\mathbf{b}}(v), \ldots, \alpha_n^{\mathbf{b}}(v)).$$

Thus $\exp_m$ is injective.

Theorem

If ∇ is complete then $D^{\mathbf{b}} : \widetilde{M} \longrightarrow \mathbb{R}^n$ is a diffeomorphism.

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Thus $\exp_m$ is injective.

Theorem

If ∇ is complete then $D^{\mathbf{b}} : \widetilde{M} \longrightarrow \mathbb{R}^n$ is a diffeomorphism.

Now consider a Lie group G with a left-invariant affine structure, i.e., G has an affine structure such that all the left translations ℓ_g, $g \in G$ are affine. By passing to the universal covering group, we will assume G is simply connected, if necessary, so that the existence of developing map is guaranteed. Then G has a developing map $D : G \longrightarrow \mathbb{R}^n$. Since the affine structure is left-invariant, for each ℓ_g, $g \in G$, there exists a unique $\phi(g) \in \mathrm{Aff}(\mathbb{R}^n)$ such that

$$D \circ \ell_g = \phi(g) \circ D.$$

Since

$$\phi(gh)\circ D = D\circ \ell_{gh} = D\circ \ell_g\circ \ell_h = \phi(g)\circ D\circ \ell_h = \phi(g)\phi(h)\circ D$$

$\phi(gh)$ and $\phi(g)\circ\phi(h)$ agree on an open set $D(G)$ and hence are equal. Hence we obtain a Lie group homomorphism

$$\phi_D = \phi : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n).$$

For a given point $x \in \mathbb{R}^n$, let

$$Ev_x : \mathrm{Aff}(\mathbb{R}^n) \longrightarrow \mathbb{R}^n$$

be the evaluation map given by $Ev_x(a) = a(x) = a.x$, and $ev_x : G \longrightarrow \mathbb{R}^n$ be the composition given by $ev_x = Ev_x \circ \phi$. Now observe that $D = ev_x$, where $x = D(e)$. Indeed,

$$D(g) = D(g.e) = \phi(g).D(e) = \phi(g).x = Ev_x(\phi(g)) = ev_x(g).$$

Furthermore, $d(ev_x)_{|e}$ should be non-singular since D is a local diffeomorphism.

From this observation, we immediately obtain:

Proposition

$D : G \longrightarrow \Omega = D(G) \subset \mathbb{R}^n$ is a overing map.

Proof. G acts on $\mathbb{R}^n$ as affine map through the representation $\phi : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$. Since $D = ev_x$ where $x = D(e)$,

$$\Omega = D(G) = ev_x(G) = G.x = \text{the orbit of } x,$$

and Ω can be canonically identified with G/G_x, where

$$G_x = \{g \in G | g.x = x\}$$

is discrete since $D = ev_x$ is a local diffeomorphism. This shows that the projection $p : G \longrightarrow G/G_x$ is a covering map and so is $D : G \longrightarrow G/G_x \simeq \Omega$. $\square$

Let's examine what happens if we choose a different developing map $D' = a \circ D, a \in \mathrm{Aff}(\mathbb{R}^n)$. If we denote the corresponding representation by $\phi' = \phi \circ D'$, then $\phi'(g).D' = D' \circ \ell_g$. From

$$\phi'(g) \circ a \circ D = a \circ D \circ \ell_g = a \circ \phi(g) \circ D,$$

we have $\phi'(g) = a \circ \phi(g) \circ a^{-1}$ and hence $\phi' = ca \circ \phi$, where ca is the conjugation by a. If we denote $x' = D'(e) = aD(e) = a(x)$, then $D' = ev_{x'}$.

We can now conclude that a left-invariant affine structure on G gives rise to a class of representations $[\phi] \in \mathrm{Aff}(\mathbb{R}^n)/\mathrm{Hom}(G, \mathrm{Aff}(\mathbb{R}^n))$, where $\mathrm{Aff}(\mathbb{R}^n)$ acts on $\mathrm{Hom}(G, \mathrm{Aff}(\mathbb{R}^n))$ by composition with conjugation. And the representation ϕ has the property that $d(ev_x)_{|e}$ is a linear isomorphism for some $x \in \mathbb{R}^n$.

Conversely, suppose we have a Lie group homomorphism $\phi : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$ with the property that $d(ev_x)_{|e}$ is a linear isomorphism for some $x \in \mathbb{R}^n$. Observe that $ev_x \circ \ell_g = \phi(g) \circ ev_x$ for all $g \in G$. Taking differential at $e \in G$ on both sides, we see that $d(ev_x)_{|g}$ is a linear isomorphism for all $g \in G$ and hence that ev_x is an immersion. In fact, non-singularity of $d(ev_x)_{|e}$ implies that the isotropy group G_x is discrete and the orbit map ev_x is a covering map onto its image.

Now the pull back affine structure under $D = ev_x$ is left-invariant since

$$D \circ \ell_g = ev_x \circ \ell_g = \phi(g) \circ ev_x = \phi(g) \circ D$$

and so $l_g, g \in G$, becomes an (affine map. Alternatively we can see as follows: Since ev_x is a covering projection and $ev_x \circ l_g = \phi(g) \circ ev_x$ holds, l_g is a lifting of $\phi(g)$ and hence l_g should be an affine map. Notice that if ϕ has the property that $d(ev_x)_{|e}$ has rank k for some $x \in \mathbb{R}^n$, then the above argument shows that $d(ev_x)_{|g}$ also has rank k for all $g \in G$, and hence the orbit map ev_x becomes a submersion onto its image which is just a quotient map: $G \longrightarrow G/G_x \simeq G.x$.

Theorem

Let G be a simply connected (hence connected) Lie group. Then the followings are equivalent.

1. *G admits a left-invariant affine structure.*
2. *There is a Lie group homomorphism $\phi : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$ such that $d(ev_x)_{|e}$ is an isomorphism for some $x \in \mathbb{R}^n$.*

Proposition

Let G be a Lie group with a left-invariant affine structure determined by a developing map D. Let $\phi : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$ be the associated representation with $ev_x = Ev_x \circ \phi, x \in \mathbb{R}^n$. Then the followings are equivalent.

1. $D : G \longrightarrow \mathbb{R}^n$ *is a diffeomorphism, i.e, the affine structure is complete.*
2. G *action on* $\mathbb{R}^n$ *determined by* ϕ *is transitive.*
3. $d(ev_x)_{|e}$ *is a linear isomorphism for all* $x \in \mathbb{R}^n$.