Special bi-invariant linear connections on Lie groups and finite dimensional Poisson algebras

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Outline

A *Poisson algebra* is a (finite dimensional) **Lie algebra** $(\mathfrak{g}, [\ ,\ ])$ endowed with a **commutative and associative** product $\circ$ such that, for any $u, v, w \in \mathfrak{g}$,

$$[u, v \circ w] = [u, v] \circ w + v \circ [u, w]. \tag{1}$$

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$$[u, v \circ w] = [u, v] \circ w + v \circ [u, w]. \tag{1}$$

In this case the product given by

$$u.v = \frac{1}{2}[u, v] + u \circ v$$

is Lie admissible, i.e.,

$$[u, v] = u.v - v.u,$$

and satisfies

$$[u, v.w] = [u, v].w + v.[u, w].$$

An algebra $(\mathrm{A}, .)$ is called *Poisson admissible* if $(\mathrm{A}, [\ ,\ ], \circ)$ is a Poisson algebra, where

$$[u, v] = u.v - v.u \quad \text{and} \quad u \circ v = \frac{1}{2}(u.v + v.u). \tag{2}$$

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Proposition.

$(A, .)$ is Poisson admissible iff

1. *$(A, [\ ,\ ])$ is a Lie algebra and $(A, \circ)$ is an associative commutative algebra.*
2. $[u, v.w] = [u, v].w + v.[u, w]$.

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Denote by $\mathcal{P}$ the set of Poisson admissible algebras and $\mathcal{AP}$ the set of associative Poisson admissible algebras.

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3. to show that the set $\mathcal{SL}$ of symmetric Leibniz algebras satisfies
$$\mathcal{SL} \subset \mathcal{P}^p,$$
4. to introduce a subclass $\mathcal{SYP} \subset \mathcal{AP}$ we call symplectic Poisson admissible algebras and give some geometric properties of this subclass.

Proposition.

Let $(\mathrm{A}, .)$ *be an algebra. Then*

1. *the following conditions are equivalent:*
   1. $(\mathrm{A}, .)$ *is a Poisson admissible algebra.*
   2. *For any* $u, v \in \mathrm{A}$,

$$[\mathrm{R}_u, \mathrm{R}_v] + \mathrm{L}_{[u,v]} + 3[\mathrm{L}_u, \mathrm{R}_v] = 0.$$

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2. *If* $(A, .)$ *is associative then* $(\mathrm{A}, .)$ *is a Poisson admissible algebra iff* $\mathfrak{g}^A$ *is a 2-nilpotent Lie algebra.*

In particular, any Poisson admissible algebra is flexible.

The proofs and details can be found in:

S. Benayadi M. Boucetta, *Special bi-invariant linear connections on Lie groups and finite dimensional Poisson structures*, Journal of Differential Geometry and its Applications, Volume 36, October 2014, Pages 66-89.

Geometric interpretation of finite dimensional Poisson structures

Connections, holonomy Lie algebra and parallel connections

Recall that a linear connection ∇ on a smooth manifold M is a $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

satisfying

$$\nabla_{fX}Y = f\nabla_X Y \quad \text{and} \quad \nabla_X fY = f\nabla_X Y + X(f)Y.$$

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Let T^∇ and K^∇ be, respectively, the torsion and the curvature of ∇ given by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$K^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

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The totality of these h^τ for all closed curves forms the **holonomy group** $H(p) \subset \mathrm{GL}(T_pM)$.
The **restricted holonomy group** $H(p)^0$ is the subgroup consisting of h^τ with τ homotopic to zero. Its Lie algebra is called **holonomy Lie algebra**.

On the other hand, consider linear endomorphisms of T_pM of the form $K^\nabla(X,Y)$, $(\nabla_Z K^\nabla)(X,Y)$, $(\nabla_W \nabla_Z K^\nabla)(X,Y)$, . . . (all covariant derivatives), where $X, Y, Z, W,$ . . . are arbitrary tangent vectors at p. They span a subalgebra $\mathfrak{h}_p^\nabla$ of $\mathrm{End}(T_pM, \mathbb{R})$ called **infinitesimal holonomy Lie algebra**.

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The Lie subgroup $\mathrm{Exp}(\mathfrak{h}_p^\nabla)$ of $\mathrm{GL}(T_pM, \mathbb{R})$ generated by $\mathfrak{h}_p^\nabla$ is the **infinitesimal holonomy group** at p.

The main result in this theory is that:

Theorem.

$$\mathrm{Exp}(\mathfrak{h}_p^\nabla) = H(p)^0.$$

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Another connection $\overline{\nabla}$ is **rigid** with respect to ∇ if $S = \overline{\nabla} - \nabla$ is parallel with respect to ∇, i.e., $\nabla S = 0$.

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Another connection $\overline{\nabla}$ is **rigid** with respect to ∇ if $S = \overline{\nabla} - \nabla$ is parallel with respect to ∇, i.e., $\nabla S = 0$. In this case, we have the following formula

$$K^{\overline{\nabla}}(X,Y) = K^{\nabla}(X,Y) + [S_X, S_Y] + S_{T^{\nabla}(X,Y)}. \tag{3}$$

Theorem.

(**Kostant**) *Let ∇ be an affine connection on a simply connected manifold M. Then M is a reductive homogeneous space with respect to a connected Lie group G whose action leaves ∇ invariant if and only if there exists an affine connection ∇^0 on M such that:*

1. *∇^0 is invariant under parallelism,*
2. *∇ is rigid with respect to ∇^0,*
3. *M is complete with respect to ∇^0.*

The case of a Lie group considered as a reductive homogeneous space

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If $\mathfrak{g} = T_eG$ is the Lie algebra, for any $u \in \mathfrak{g}$ we denote by u^l (resp. u^r) the left invariant (resp. the right invariant) vector field associated to u.

Proposition.

The linear connection ∇^0 on G given by

$$\nabla^0_{u^l} v^l = \frac{1}{2}[u^l, v^l],$$

for any $u, v \in \mathfrak{g}$, is bi-invariant, torsion free, parallel, complete and its curvature and holonomy Lie algebra are given by

$$K^{\nabla^0}(u^l, v^l)w^l = -\frac{1}{4}[[u^l, v^l], w^l], \quad u, v, w \in \mathfrak{g}. \tag{4}$$

$$\mathfrak{h}_e^{\nabla^0} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}. \tag{5}$$

Lemma.

Let ∇ be a linear connection on G. Then the following assertions are equivalent:

1. ∇ *is a bi-invariant linear connection.*
2. ∇ *is left invariant and rigid with respect to* ∇^0.
3. *For any* $X, Y \in \mathcal{X}^\ell(G)$, $\nabla_X Y \in \mathcal{X}^\ell(G)$ *and the product* $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *given by*

$$u.v = (\nabla_{u^\ell} v^\ell)(e)$$

satisfies

$$[u, v.w] = [u, v].w + v.[u, w]. \tag{6}$$

Let $(A, .)$ be a real Lie admissible algebra, $(\mathfrak{g}^A, [\ ,\ ])$ its associated Lie algebra and G^A the associated Lie group. The product on A defines a left invariant connection ∇ on G^A by $\nabla_{u^l} v^l = (u.v)^l$.

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The relation $[u,v.w]=[u,v].w+v.[u,w]$ is equivalent to ∇ is bi-invariant and hence ∇ is rigid with respect to ∇^0. So, if $S=\nabla-\nabla^0$ then

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The relation $[u, v.w] = [u, v].w + v.[u, w]$ is equivalent to ∇ is bi-invariant and hence ∇ is rigid with respect to ∇^0. So, if $S = \nabla - \nabla^0$ then

$$K^{\nabla}(u, v) = K^{\nabla^0}(u, v) + [S_u, S_v].$$

$(A, \circ)$ is associative and commutative iff $[S_u, S_v] = 0$.

Proposition.

$(A, .)$ is Poisson admissible iff $(A, [\ ,\ ])$ is a Lie algebra, ∇ is bi-invariant and $K^{\nabla} = K^{\nabla^0}$.

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Definition.

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Proposition.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra and ∇ be a left invariant torsion free linear connection on G. Define on $\mathfrak{g}$ the product given by

$$u.v = (\nabla_{u^l} v^l)(e).$$

Then $(\mathfrak{g}, .)$ is Poisson admissible iff ∇ is special.

Proposition.

Any special connection ∇ on G is semi-symmetric, i.e.,

$$K.K(X,Y) := \nabla_X \nabla_Y K^{\nabla} - \nabla_Y \nabla_X K^{\nabla} - \nabla_{[X,Y]} K^{\nabla} = 0. \tag{7}$$

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Lemma.

Let ∇ be a special connection on G. Then the holonomy Lie algebra of ∇ is given by

$$\mathfrak{h}_e^\nabla = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{L}_{[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}]} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{R}_{[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}]},$$

where $\mathrm{L}, \mathrm{R} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ *are given by* $\mathrm{L}_u v = u.v$ *and* $\mathrm{R}_u v = v.u$ *and* $u.v = (\nabla_{u^l} v^l)(e)$.

Definition.

Let ∇ be a special connection on a Lie group.

1. *We call ∇ flat if $K^{\nabla} = 0$.*
2. *We call ∇ parallel if $\nabla K^{\nabla} = 0$.*
3. *We call ∇ strongly special if its holonomy Lie algebra $\mathfrak{h}_e^{\nabla} = \mathfrak{h}_e^{\nabla^0} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$.*

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Let $(A, .)$ be a (real) Poisson admissible algebra and ∇ the corresponding special connection on G^A.

1. ∇ is flat iff $(A, .)$ an associative algebra.
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We have

$$\mathcal{AP} \subset \mathcal{P}^p \subset \mathcal{P}^s \subset \mathcal{P}.$$

Leibniz algebras

A **left Leibniz algebra** is an algebra $(\mathrm{A}, .)$ such that for any $u \in \mathrm{A}$, the left multiplication L_u is a derivation, i.e., for any $v, w \in \mathrm{A}$,

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A **right Leibniz algebra** is an algebra $(\mathrm{A}, .)$ such that, for any $u \in \mathrm{A}$, the right multiplication R_u is a derivation, i.e., for any $v, w \in \mathrm{A}$,

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$$(v.w).u = (v.u).w + v.(w.u).$$

An algebra which is left and right Leibniz is called **symmetric Leibniz algebra**.

Remark.

Any Lie algebra is a symmetric Leibniz algebra and any symmetric Leibniz algebra is Lie-admissible. However, the class of symmetric Leibniz algebras contains strictly the class of Lie algebras. Leibniz algebras were introduced by Loday in [20].

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We can state now one of our main results.

Theorem.

Let $(\mathrm{A}, .)$ *be a symmetric Leibniz algebra. Then*

$$(\mathrm{A}, .) \in \mathcal{P}^p.$$

By using the geometric interpretation of Poisson structures, we get the following interesting corollary.

Corollary.

Let $(\mathrm{A}, .)$ *be a real symmetric Leibniz algebra which is not a Lie algebra and* G *any connected Lie group associated to* $(\mathfrak{g}^{\mathrm{A}}, [\ ,\ ])$. *Then the left invariant connection on* G *given by*

$$\nabla_{u^l} v^l = (u.v)^l$$

is different from ∇^0, *strongly special and its curvature is parallel.*

Example.

$A = R^4$ with the symmetric Leibniz product

$$e_1.e_1 = e_4,\ e_2.e_1 = e_3,\ e_3.e_1 = e_4,\ e_1.e_2 = -e_3,\ e_1.e_3 = -e_4.$$

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The underlying Lie algebra say $\mathfrak{g} = \mathbb{R}^4$ has its non-vanishing Lie brackets given by

$$[e_1, e_2] = -2e_3 \quad \textit{and} \quad [e_1, e_3] = -2e_4.$$

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$$[e_1, e_2] = -2e_3 \quad \textit{and} \quad [e_1, e_3] = -2e_4.$$

The associated connected and simply connected Lie group is $G = \mathbb{R}^4$ with the multiplication given by Campbell-Baker-Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]].$$

Example.

We consider the two torsion free linear connections on G defined by the formulas

$$\nabla^0_{x^l} y^l = \frac{1}{2}[x^l, y^l] \quad \textit{and} \quad \nabla_{x^l} y^l = (x.y)^l. \tag{8}$$

The dot here is the symmetric Leibniz product.

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The dot here is the symmetric Leibniz product. We get that the only non-vanishing Christoffel symbols are given by

$$\nabla^0{}_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} = -\frac{2}{3} x_2 \frac{\partial}{\partial x_4} \quad \textit{and} \quad \nabla^0{}_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} = \frac{1}{3} x_1 \frac{\partial}{\partial x_4},$$

and

$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} = (1 - \frac{2}{3} x_2) \frac{\partial}{\partial x_4} \quad \textit{and} \quad \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} = \frac{1}{3} x_1 \frac{\partial}{\partial x_4}.$$

Theorem.

Let $(\mathrm{A}, .)$ *be a Poisson admissible algebra and* U *an associative* LR-*algebra. Then the product on* $\mathrm{A} \otimes \mathrm{U}$ *given by*

$$(u \otimes a) \star (v \otimes b) = \frac{1}{2}[u, v] \otimes (ab + ba) + \frac{1}{2} u.v \otimes (3ab + ba)$$

induces on $\mathrm{A} \otimes \mathrm{U}$ *a Poisson admissible algebra structure. Moreover, if* $(A, .) \in \mathcal{P}^s$ *then* $(\mathrm{A} \otimes \mathrm{U}, \star) \in \mathcal{P}^s$.

Theorem.

1. *Let* $\mathfrak{g}$ *be a perfect Lie algebra, i.e.,* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$*. Then the product* $u.v = \frac{1}{2}[u, v]$ *is the only strongly Poisson admissible product on* $\mathfrak{g}$ *whose underline Lie algebra is* $\mathfrak{g}$*.*
2. *Let* $\mathfrak{g}$ *be a semi-simple Lie algebra. Then the product* $u.v = \frac{1}{2}[u, v]$ *is the only Poisson admissible product on* $\mathfrak{g}$ *whose underline Lie algebra is* $\mathfrak{g}$*.*

Theorem.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and ∇ a torsion free bi-invariant connection on G. Then:

1. *If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ then ∇ has the same curvature and holonomy as ∇^0 if and only if $\nabla = \nabla^0$.*
2. *If $\mathfrak{g}$ is semi-simple then ∇ has the same curvature as ∇^0 if and only if $\nabla = \nabla^0$.*

Proposition.

Let $(A, .) \in \mathcal{P}^s$ *and* $\mathfrak{g}$ *its underlining Lie algebra. Then* $\mathfrak{g}^3 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ *is two sided ideal of* $(A, .)$, $(\mathfrak{g}^3, .)$ *is a symmetric Leibniz algebra,* $(\mathfrak{g}/\mathfrak{g}^3, .)$ *is associative and the sequence*

$$0 \longrightarrow (\mathfrak{g}^3, .) \longrightarrow (A, .) \longrightarrow (A/\mathfrak{g}^3, .) \longrightarrow 0$$

is an exact sequence of Poisson admissible algebras.

In dimension 2, the only non trivial Poisson admissible algebras are the associative commutative algebras.

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In dimension 3, the only non trivial Poisson admissible algebras are:

1. The associative commutative algebras,
2. Two associative non commutative algebras whose underlying Lie algebra is the 3-dimensional Heisenberg Lie algebra,
3. One Poisson admissible algebra which is not strong whose underlying Lie algebra is $E(2)$.
4. One Poisson admissible algebra which is parallel and not Leibniz symmetric whose underlying Lie algebra is $E(2)$.
5. One Leibniz symmetric algebra whose underlying Lie algebra is $E(2)$.

Example.

Let $n \in \mathbb{N}^*$ *and* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *with* $0 < \lambda_1 \leq \ldots \leq \lambda_n$. *The oscillator Lie algebra denoted by* $\mathfrak{g}_\lambda$*, admits a basis* $\mathbb{B} = \{e_{-1}, e_0, e_i, \check{e}_i,\}_{i=1,\ldots,n}$ *where the non vanishing brackets are given by*

$$[e_{-1}, e_j] = \lambda_j \check{e}_j, \qquad [e_{-1}, \check{e}_j] = -\lambda_j e_j, \qquad [e_j, \check{e}_j] = e_0.$$

The Poisson structures on $\mathfrak{g}_\lambda$ *are of three types:*

1. $e_{-1}.e_{-1} = be_{-1} + ae_0,\ e_{-1}.v = \frac{1}{2}[e_{-1}, v] + bv,$
   $v.e_{-1} = \frac{1}{2}[v, e_{-1}] + bv,\ u.v = \frac{1}{2}[u, v],\ b \neq 0,$
   $u, v \in \mathrm{span}\{e_0, e_i, \check{e}_i\},$
2. $e_{-1}.e_{-1} = be_{-1} + ae_0, e_{-1}.v = \frac{1}{2}[e_{-1}, v],\ v.e_{-1} = \frac{1}{2}[v, e_{-1}],$
   $u.v = \frac{1}{2}[u, v],\ b \neq 0,\ u, v \in \mathrm{span}\{e_0, e_i, \check{e}_i\},$
3. $e_{-1}.e_{-1} = ae_0,\ e_{-1}.v = \frac{1}{2}[e_{-1}, v],\ v.e_{-1} = \frac{1}{2}[v, e_{-1}],$
   $e_0.w = w.e_0 = 0,\ u.v = \frac{1}{2}[u, v] + \langle u, v\rangle e_0,$
   $u, v \in \mathrm{span}\{e_i, \check{e}_i\},\ w \in \mathfrak{g}_\lambda,$

Symplectic Poisson algebras

Let (G, Ω) be a symplectic Lie group. It is well-known that the linear connection given by the formula

$$\Omega(\nabla^{\mathrm{a}}_{u^l} v^l, w^l) = -\Omega(v^l, [u^l, w^l]), \tag{9}$$

where $u, v, w \in \mathfrak{g}$, defines a left invariant flat and torsion free connection ∇^{a}. Moreover, $\nabla^{\mathrm{a}}\Omega$ never vanishes unless G is abelian.

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where $u, v, w \in \mathfrak{g}$, defines a left invariant flat and torsion free connection ∇^{a}. Moreover, $\nabla^{\mathrm{a}}\Omega$ never vanishes unless G is abelian.
So we can define a tensor field N by the relation

$$\nabla^{\mathrm{a}}_{u^l}\Omega(v^l, w^l) = \Omega(\mathrm{N}(u^l, v^l), w^l).$$

The linear connection given by

$$\nabla^{\mathrm{s}}_{u^l} v^l = \nabla^{\mathrm{a}}_{u^l} v^l + \frac{1}{3}\mathrm{N}(u^l, v^l) + \frac{1}{3}\mathrm{N}(v^l, u^l)$$

is left invariant torsion free and symplectic, i.e., $\nabla^{\mathrm{s}}\Omega = 0$.

A straightforward computation gives that ∇^{s} can be defined by the following formula

$$\Omega(\nabla^{\mathrm{s}}_{u^l} v^l, w^l) = \frac{1}{3}\Omega([u^l, v^l], w^l) + \frac{1}{3}\Omega([u^l, w^l], v^l). \tag{10}$$

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Let $(\mathfrak{g}, \omega)$ be the Lie algebra of G endowed with the value of Ω at e. We denote by α^{a} and α^{s} the product on $\mathfrak{g}$ induced, respectively, by ∇^{a} and ∇^{s}. We have, for any $u, v \in \mathfrak{g}$,

$$\alpha^{\mathrm{a}}(u, v) = -\mathrm{ad}^*_u v \quad \text{and} \quad \alpha^{\mathrm{s}}(u, v) = \frac{1}{3}\left(\mathrm{ad}_u v - \mathrm{ad}^*_u v\right), \qquad (11)$$

where ad^*_u is the adjoint of ad_u with respect to ω.

Proposition.

Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and α^{a}, α^{s} the products given by (11). Then the following assertions are equivalent:

1. α^{a} *is special.*
2. α^{s} *is special.*
3. $\mathfrak{g}$ *is 2-nilpotent and, for any* $u, v \in \mathfrak{g}$, $[\mathrm{ad}_u, \mathrm{ad}_v^*] = 0$.

Moreover, if one of the conditions above holds then $(\mathfrak{g}, \alpha^{\mathrm{a}})$ *and* $(\mathfrak{g}, \alpha^{\mathrm{s}})$ *are both associative* LR*-algebras.*

Definition.

A **symplectic Poisson algebra** *is a 2-nilpotent symplectic Lie algebra* $(\mathfrak{g}, \omega)$ *satisfying, for any* $u, v \in \mathfrak{g}$,

$$[\mathrm{ad}_u, \mathrm{ad}_v^*] = 0. \tag{12}$$

Proposition.

Let $(\mathfrak{g}, \omega)$ be 2-nilpotent symplectic Lie algebra which carries a bi-invariant pseudo-Euclidean product B. Then $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra.

Proposition.

Let $(\mathfrak{g}, \omega)$ be 2-nilpotent symplectic Lie algebra which carries a bi-invariant pseudo-Euclidean product B. Then $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra.

Let $(\mathfrak{g}, \omega)$ be a non abelian real symplectic Poisson algebra and G a connected Lie group having $\mathfrak{g}$ as its Lie algebra. The symplectic form ω defines on G a symplectic left invariant form Ω. Consider the two linear connections ∇^{a} and ∇^{s} defined on G by (9)-(10). These two connections are bi-invariant, flat, complete and $\nabla^{\mathrm{s}}\Omega = 0$. It was shown in [3] that Ω is polynomial of degree at most $\dim G - 1$ in any affine coordinates chart associated to ∇^{a}. The following result gives a more accurate statement on the polynomial nature of Ω.

Theorem.

With the hypothesis and the notations above we have

$$(\nabla^{\mathrm{a}})^3\Omega = 0.$$

In particular, Ω is polynomial of degree at least one and at most 2 in any affine coordinates chart associated to ∇^{a}. Moreover, if the restriction of ω to $[\mathfrak{g}, \mathfrak{g}]$ does not vanish then the degree is 2.

Theorem.

Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra. Then $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra if and only if it is a symplectic double extension of a symplectic Poisson algebra $(\mathcal{H}, \overline{\omega})$ of dimension $\dim \mathfrak{g} - 2$ by the one dimensional Lie algebra by means of an admissible element $(D, z) \in \mathrm{Der}(\mathcal{H}) \times \mathcal{H}$.

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