Flat Riemannian manifolds: The famous Bieberbach's Theorem

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Seminar Algebra, Geometry, Topology and Applications

Outline

Let (M, g) be a complete flat Riemannian manifold. For any $m \in M$, we consider

$$\exp_m : (T_m M, \widetilde{g}) \longrightarrow (M, g) \quad \widetilde{g} = \exp_m^*(g).$$

1. $\exp_m : (T_m M, \widetilde{g}) \longrightarrow (M, g)$ is a local isometry and hence a covering.
2. $\widetilde{g} = g_m$ and hence the universal Riemannian covering of (M, g) is isomorphic to $(\mathbb{R}^n, \langle\ ,\ \rangle)$ with its canonical metric.
3. Let $p : (\mathbb{R}^n, \langle\ ,\ \rangle) \longrightarrow (M, g)$ the universal covering of (M, g) and Γ its group covering. Then M is isometric to $\mathbb{R}^n/\Gamma$ where

$$\Gamma \subset E(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$$

is **discrete and acts freely and properly discontinuously** on $\mathbb{R}^n$. If M is compact then Γ is **cocompact**.

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The classification of compact flat Riemannian manifolds reduces to the classification of subgroups $\Gamma \subset E(n)$ which are discrete, cocompact and acts freely on $\mathbb{R}^n$.

Problem (Hilbert's eighteenth problem (1900))

Show that there are only finitely many types of subgroups of the group $E(n)$ with compact fundamental domain.

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Definition

1. *A crystallographic group of dimension n is a discrete and cocompact subgroup of $E(n)$.*
2. *A Bieberbach group of dimension n is a crystallographic group of dimension n which acts freely (torsion free) on $\mathbb{R}^n$.*

Example

1. $\mathbb{Z}^n = \{(I, \sum_{i=1}^n n_i e_i) | n_i \in \mathbb{Z}, (e_1, \ldots, e_n)$ *a basis of* $\mathbb{R}^n\}$.
2. $\Gamma = < \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (\frac{1}{2}, 0) \right), (I, (0,2)) > \subset E(2)$ *is a Bieberbach group and* $\mathbb{R}^2/\Gamma$ *is the Klein bottle. (Exercise)*

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Let $\Gamma \subset E(n)$. *Then*

1. Γ *is discrete if for any* $\gamma \in \Gamma$ *there exists an open set* $U_\gamma \subset E(n)$ *such that* $U_\gamma \cap \Gamma = \{\gamma\}$. *This is equivalent to every convergent sequence of* Γ *is eventually constant.*
2. Γ *acts properly discontinuously on* $\mathbb{R}^n$ *if for any* $x \in \mathbb{R}^n$ *there exists an open set* U_x *such that*
$$\{\gamma \in \Gamma, U_x \cap (\gamma U_x) \neq \emptyset\}$$
is finite.
3. Γ *acts freely on* $\mathbb{R}^n$ *if for any* $x \in \mathbb{R}^n$, $\Gamma_x = \{\gamma, |\gamma.x = x\} = \{(I_n, 0)\}$.

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Proposition (Exercise)

Let $\Gamma \subset E(n)$ be a subgroup.

1. *If Γ is discrete then Γ is closed in $E(n)$.*
2. *If Γ is discrete then Γ acts freely on $\mathbb{R}^n$ if and only if Γ is torsion free (has no element of finite order):*

$$\gamma \in \Gamma,\ \gamma^k = I \Longrightarrow \gamma = I.$$

3. *The following assertions are equivalent:*
   1. *Γ is discrete.*
   2. *Γ acts properly discontinuously on $\mathbb{R}^n$.*
   3. *For any $x \in \mathbb{R}^n$, $\Gamma.x$ is discrete in $\mathbb{R}^n$.*
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1. *$E(n)/\Gamma$ is compact.*
2. *$\mathbb{R}^n/\Gamma$ is compact.*
3. *There exists a compact subset $D \subset E(n)$ such that $E(n) = D\Gamma$.*
4. *There exists a an open connected bounded set $F \subset \mathbb{R}^n$ such that*
$$\mathbb{R}^n = \bigcup_{\gamma \in \Gamma} \gamma.\overline{F}$$
and $\gamma_1 F \cap \gamma_2 F = \emptyset$ if $\gamma_1 \neq \gamma_2$. $\overline{F}$ is called a fundamental domain.

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Γ *is called cocompact if it satisfies one the equivalent conditions above*

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Theorem (Bieberbach (1910-1912))

1. *If* $\Gamma \subset E(n)$ *is a crystallographic group then the set of translations* $\Gamma \cap (I \times \mathbb{R}^n)$ *is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index, i.e.,*

$$L(\Gamma) = \left\{ \left(I, \sum_{i=1}^{n} n_i e_i \right) | n_i \in \mathbb{Z}, (e_1, \ldots, e_n) \text{ a basis of } \mathbb{R}^n \right\} \simeq \mathbb{Z}^n,$$

$\Gamma / L(\Gamma) \simeq p_1(\Gamma) \subset O(n)$ *is finite,* $p_1 : \Gamma \longrightarrow O(n)$.

2. *For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.*

3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group* $A(n) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

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$\mathbb{R}^n$ is endowed with its canonical Euclidean product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

and $\mathbb{C}^n$ with its canonical Hermitian product

$$\langle z, u\rangle = \sum_{i=1}^{n} z_i \bar{u}_i.$$

For any endomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we denote by

$$||f|| = \sup_{|x|=1} |f(x)|.$$

We have $||f \circ g|| \leq ||f||||g||$ and if $g \in O(n)$,

$$||f \circ g|| = ||g \circ f|| = ||f||.$$

Lemma

There is a neighborhood of the identity $U \subset O(n)$ such that for any $h \in U$ if $g \in O(n)$ commutes with $[g,h] = ghg^{-1}h^{-1}$, then g commutes with h.

Proof.

We consider $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. It is a normal isomorphism and hence it is diagonalizable. So

$$\mathbb{C}^n = V_1 \oplus \ldots \oplus V_r$$

and for any $x \in V_i$, $gx = \lambda_i x$, $\lambda_i \in S^1$. The relation $g[g,h] = [g,h]g$ is equivalent to $ghg^{-1}h^{-1} = hg^{-1}h^{-1}g$. So $hg^{-1}h^{-1}$ commutes with g and hence $hg^{-1}h^{-1}V_i \subset V_i$ thus $h^{-1}V_i = gh^{-1}V_i$. So

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Continued.

Suppose that $i \neq j$ and $x = h^{-1}y \in (h^{-1}V_i \cap V_j)$ with $y \in V_i$ and $|x| = 1$. Since $\langle x, y\rangle = 0$ then $|x - y| = \sqrt{2}$. But

$$|x - y| = |hy - y| \leq ||h - I||.$$

If we take $h \in U = B(I, \sqrt{2} - 1) \cap O(n)$, we get that, for $i \neq j$, $(h^{-1}V_i \cap V_j) = \{0\}$ and hence for any i, $hV_i = V_i$ and hence h commutes with g. □

Lemma

For some neighborhood of the identity $U \subset O(n)$ and for any $g, h \in U$ the sequence

$$[g,h], [g,[g,h]], [g,[g,[g,h]]], \dots$$

converges to the identity.

Proof.

Let $U = B(I,\varepsilon) \cap O(n)$ with $\varepsilon < \frac{1}{4}$. We have

$$\begin{aligned} ||[g,h] - I|| &= ||ghg^{-1}h^{-1} - I|| \\ &= ||gh - hg|| = ||gh - g - h + I - (hg - g - h + I)|| \\ &= ||(g-I)(h-I) - (h-I)(g-I)|| \\ &\leq 2||g-I||||h-I|| \leq \frac{||h-I||}{2}. \end{aligned}$$

Continued.

By induction, we deduce that

$$||\underbrace{[g,[g,[g,\ldots,[g}_{n},h]\ldots]]]-I|| \leq \frac{||h-I||}{2^n}$$

and the result follows. □

Lemma

Let $G \subset O(n)$ be a connected subgroup and U be a neighborhood of the identity, then the group $< G \cap U >$ generated by the set $G \cap U$ is equal to G.

Proof.

We show first that $< G \cap U >$ is open in G. Let $\underline{x \in < G \cap U >}$ and $\varepsilon > 0$ such that $B(I, \varepsilon) \cap O(n) \subset U$. Then for any $\underline{y \in B(x, \varepsilon) \cap G}$,

$$||yx^{-1} - I|| = ||yx^{-1} - xx^{-1}|| = ||y - x|| < \varepsilon.$$

So $yx^{-1} \in B(I, \varepsilon) \cap G \subset U \cap G$ and hence

$$y = yx^{-1}x \in < G \cap U > .$$

Thus $\underline{B(x, \varepsilon) \cap G \subset < G \cap U >}$ and hence $< G \cap U >$ is open in

Continued.

Let us show also that $S = G \setminus < G \cap U >$ is also open. For any $y \in S$, $B(y, \varepsilon) \cap G \subset S$. Indeed, if $x \in B(y, \varepsilon) \cap G$ then by what above if $xy^{-1} \in U \cap G$ and $x \in S$ otherwise $y \in < U \cap G >$. □

Lemma

Let $V = B(I, \varepsilon)$. Then for any $g \in O(n)$, $gVg^{-1} = V$.

Proof.

For any $g \in O(n)$ and $h \in V$,

$$||ghg^{-1} - I|| = ||g(h - I)g^{-1}|| = ||h - I||,$$

so for any $g \in O(n)$, $gVg^{-1} \subset V$ and $g^{-1}Vg \subset V$. Thus $V = g(g^{-1}Vg)g^{-1} \subset gVg^{-1}$ and hence $gVg^{-1} = V$. □

Lemma

Let Γ be a crystallographic group and $x \in \mathbb{R}^n$. Then the linear space generated by the set $\{\gamma(x), \gamma \in \Gamma\}$ is equal to $\mathbb{R}^n$.

Proof.

In the contrary, suppose that there exists $x_0 \in \mathbb{R}^n$ such that,

$$W = \operatorname{span}\{\gamma(x_0) | \gamma \in \Gamma\}$$

is a proper subspace of $\mathbb{R}^n$.

By using a conjugation by the translation $T_{x_0} = (I, x_0)$ we can suppose that $x_0 = 0$. So

$$W = \operatorname{span}\{a | \gamma = (A, a) \in \Gamma\}.$$

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Continued.

Moreover, since Γ is a group then W is invariant by $p_1(\Gamma)$[1] and hence $W^\perp$ is also invariant by $W^\perp$. Let $x \in W^\perp$, then for any $\gamma = (A, a) \in \Gamma$,

$$\langle \gamma(x), \gamma(x) \rangle = \langle Ax + a, Ax + a \rangle = \langle x, x \rangle + \langle a, a \rangle \geq |x|^2.$$

But Γ has a compact fundamental domain D, so for any $x \in W^\perp$ there exists γ_0 such that $\gamma_0(x) \in D \subset B(0, r)$. This is impossible if we take $x \in W^\perp$ with $|x| > r$. This completes the proof. □

[1] $p_1 : \Gamma \longrightarrow O(n), (A, a) \mapsto A$.

Lemma

Let Γ be an abelian crystallographic group; then Γ contains only pure translations.

Proof.

Suppose that Γ contains $\gamma_0 = (A, a)$ with $A \neq I$. Then there exists $P \in O(n)$ such that

$$P^{-1}AP = \begin{pmatrix} I_r & 0 \\ 0 & B' \end{pmatrix} = B$$

where $B' - I$ is invertible. □

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On the other hand, for any $x \in \mathbb{R}^n$, we have

$$(I,-x)(B,b)(I,x) = (I,-x)(B,Bx+b) = (B,Bx+b-x).$$

But

$$Bx+b-x = (x_1+b_1-x_1, B'x_2+b_2-x_2).$$

Let x_2 such that $(B'-I)x_2 = -b_2$. So we can suppose that $\gamma_0 = (B,b)$ with $b = (b',0,\ldots,0)$ and $b' \in \mathbb{R}^r$.

So we can suppose that $\gamma_0 = (B,(b',0,\ldots,0)) \in \Gamma$ with

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Continued.

By Lemma 3.5, $\mathbb{R}^n$ is spanned by the $\gamma(0)$ so there exists

$$\gamma_1 = (C, (t_1, t_2)) \in \Gamma,\ t_2 \in \mathbb{R}^{n-r} \quad \text{and} \quad t_2 \neq 0.$$

Now the relation $\gamma_0\gamma_1 = \gamma_1\gamma_0$ is equivalent to

$$(BC, (t_1, B' t_2) + (b', 0)) = (CB, Cb + (t_1, t_2)).$$

Then $BCb = CBb = Cb$ and hence $Cb \in \mathbb{R}^r$ and we get $B' t_2 = t_2$ which is impossible. This completes the proof. □

Lemma

Let Γ be a crystallographic group. Let $p_1 : E(n) \longrightarrow O(n)$ be the projection onto the first factor. Then $\overline{p_1(\Gamma)}_0$ is an abelian group.

Proof.

Let $\gamma_1 = (A_1, a_1) \in \Gamma$ and $\gamma_2 = (A_2, a_2) \in \Gamma$ such that $A_1, A_2 \in B(I, \frac{1}{4})$ and define the sequence γ_i by

$$\gamma_{i+1} = [\gamma_1, \gamma_i] = \gamma_1 \gamma_i \gamma_1^{-1} \gamma_i^{-1}.$$

We have

$$[\gamma_1, \gamma_i] = ([A_1, A_i], (I - A_1 A_i A_1^{-1}) a_1 + A_1 (I - A_i A_1^{-1} A_i^{-1}) a_i).$$

By Lemma 3.2, $\lim_{i \longrightarrow \infty} [A_1, A_i] = I$. □

Continued.

Moreover,

$$|a_{i+1}| \leq ||A_i - I|||a_1| + \frac{1}{4}|a_i|.$$

Thus $\lim_{i \longrightarrow \infty} a_i = 0$.
Since Γ is discrete then $\gamma_i = (I, 0)$ and hence $[A_1, A_i] = I$ for i large. So A_1 commutes with $[A_1, A_{i-1}]$ and hence commutes with A_{i-1} by virtue of Lemma 3.1. By induction we deduce that $A_1 A_2 = A_2 A_1$.
Let $A, B \in \overline{p_1(\Gamma)}_0 \cap U$. Then there exists two sequences $A_n, B_n \in p_1(\Gamma) \cap U$ such $A = \lim A_n$ and $B = \lim B_n$. But $A_n B_n = B_n A_n$ and hence $AB = BA$. Now according to Lemma 3.3, $\overline{p_1(\Gamma)}_0 \cap U$ generates $\overline{p_1(\Gamma)}_0$ and hence $\overline{p_1(\Gamma)}_0$ is abelian. □

Continued.

Moreover,

$$|a_{i+1}| \leq ||A_i - I|||a_1| + \frac{1}{4}|a_i|.$$

Thus $\lim_{i\longrightarrow\infty} a_i = 0$.
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Lemma

Let Γ be a crystallographic group. Then Γ contains a pure translation.

Proof.

Assume that $\Gamma \cap (I \times \mathbb{R}^n)$ is trivial. Then p_1 is an isomorphism of Γ into $O(n)$. Since $O(n)$ is compact, the closure of $p_1(\Gamma)$ can have only a finite number of components:

$$\overline{p_1(\Gamma)} = \overline{p_1(\Gamma)}_0 \cup A_1.\overline{p_1(\Gamma)}_0 \cup \ldots \cup A_r.\overline{p_1(\Gamma)}_0.$$

Hence, since by Lemma 3.7 $\overline{p_1(\Gamma)}_0$ is abelian, Γ contains a subgroup $\Gamma_1 = \Gamma \cap \overline{p_1(\Gamma)}_0$ of finite index which is abelian. But then Γ_1, being of finite index in Γ, is also a crystallographic group. Hence, by Lemma 3.6, Γ_1 consists of pure translations. Thus we see that $\Gamma \cap (I \times \mathbb{R}^n)$ is nonempty. □

Consider

$$W = \mathrm{span}\left\{a \middle| (I,a) \in \Gamma\right\}.$$

We will show that $W = \mathbb{R}^n$. Note first that W is invariant by $p_1(\Gamma)$. Indeed, for any $\gamma = (I,b) \in \Gamma$ and $\gamma_1 = (A,a) \in \Gamma$,

$$\gamma_1 \gamma \gamma_1^{-1} = (I, Ab) \in \Gamma$$

and hence $Ab \in W$. This implies that $W^\perp$ is also invariant by $p_1(\Gamma)$.

Claim.

$p_1(\Gamma)_{|W}$ is a discrete group and hence finite.

In fact, let $(A_i, a_i) \in \Gamma, i \in \mathbb{N}$ be a sequence of elements such that $\lim_{i \longrightarrow \infty} A_i = I$. Put

$$(B_i, b_i) = (I, e_k)(A_i, a_i)(I, -e_k) = (A_i, (I - A_i)e_k + a_i),$$

where $e_k \in \Gamma \cap (I \times \mathbb{R}^n)$. Then the sequence

$$(B_i, b_i)(A_i^{-1}, -A_i^{-1}(a_i)) = (I, (I - A_i)e_k), i \in \mathbb{N}$$

defines a convergent sequence of Γ and hence must be constant. So $A_i e_k = e_k$ for i large and hence the sequence $(A_i)_{|W} = I$.

We consider now

$$\Gamma_1 = \{(A_{|W^\perp}, pr_2(a)) | (A, a) \in \Gamma, A_{|W} = I\}.$$

Let us show that Γ_1 is a crystallographic group of $W^\perp$. It is obvious that Γ_1 is a group of isometric transformations of $W^\perp$. It is also cocompact since there is a continuous map from $\mathbb{R}^n/\Gamma$ to $W^\perp/\Gamma_1$ which maps $[x]$ to $[p_2(x)]$.

We claim that Γ_1 is discrete. Suppose the contrary, i.e., Γ_1 is not discrete. This means that $\{\gamma(0)|, \Gamma \in \Gamma_1\}$ is not discrete. So there exists a sequence $\gamma_i = (A_i, y_i) \in \Gamma_1$ such that the sequence $y_i \in W^\perp$ has an accumulation point $y \in W$. We consider the sequence $\bar{\gamma}_i = (A_i, z_i + y_i) \in \Gamma$ defining γ_i.

Let $(e_1, \dots, e_r)$ be a basis of W such that $(I, e_k) \in \Gamma$ for $k = 1, \dots, r$. We have, for any i, $z_i = \sum t_k^i e_k$ and let $b_i = \sum [t_k^i] e_k$ where $[t]$ is the integer part of t. $\mu_i = (I, b_i)$ is a sequence of Γ and

$$\mu_i^{-1} \bar{\gamma}_i = (A_i, z_i - b_i + y_i)$$

and

$$|z_i - b_i| \leq r \max |e_k|.$$

So we can suppose that the sequence $z_i - b_i$ is convergent and the sequence $\mu_i^{-1} \bar{\gamma}_i$ is not eventually constant which contradicts the fact that γ is discrete. Now Γ_1 is crystallographic group which acts without translation and hence $W^\perp = \{0\}$ which completes the proof.

1. A compact flat Riemannian manifold is isometric to
$$\mathbb{R}^n/\Gamma$$
where Γ is a Bieberbach group. We have the exact sequence
$$I \longrightarrow L(\Gamma) \longrightarrow \Gamma \longrightarrow \Gamma/L(\Gamma) \longrightarrow I$$
where $L(\Gamma)$ is a lattice and $\Gamma/L(\Gamma)$ is finite.
2. $\mathbb{T}^n = \mathbb{R}^n/L(\Gamma) \longrightarrow \mathbb{R}^n/\Gamma$ is a finite covering.
3. Two compact flat Riemannian manifolds $\mathbb{R}^n/\Gamma_1$ and $\mathbb{R}^n/\Gamma_2$ are isometric if and only if Γ_1 and Γ_2 are isomorphic.

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1. There are only two compact flat Riemannian manifolds of dimension two: the torus and the Klein bottle.
2. It is known since 1933 that there are ten compact flat Riemannian manifolds in dimension 3.