Completeness of left invariant pseudo-Riemannian metrics

Mohamed Boucetta

Cadi-Ayyad University

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Outline

Warning

Let $(V, \langle\ ,\ \rangle)$ be a real vector space endowed with a bilinear symmetric nondegenerate form. Then there exists (p, q) two integers and an orthonormal basis $(e_1, \ldots, e_p, f_1, \ldots, f_q)$ such that

$$\langle e_i, e_i \rangle = -1 \quad \text{and} \quad \langle f_j, f_j \rangle = 1,\ i = 1, \ldots, p, j = 1, \ldots, q.$$

(p, q) is the signature of $\langle\ ,\ \rangle$.

$$\begin{cases} \langle\ ,\ \rangle = \text{Euclidean if } (p, q) = (0, n), \\ \langle\ ,\ \rangle = \text{Lorentzian if } (p, q) = (1, n-1), \\ \langle\ ,\ \rangle = \text{pseudo-Euclidean else.} \end{cases}$$

Let (M, g) be a manifold endowed with a bilinear symmetric nondegenerate tensor field.

$$\begin{cases} (M,g) = \text{Riemannian if } g_x = \text{Euclidean } \forall x \\ (M,g) = \text{Lorentzian if } \ g_x = \text{Lorentzian } \forall x, \\ (M,g) = \text{pseudo-Riemannian else.} \end{cases}$$

Let X be a vector field on a manifold M. For any $m \in M$ there exists and a unique maximal interval $]a(m), b(m)[$ and a unique curve $c_m :]a(m), b(m)[\longrightarrow M$ such that

$$c_m(0) = m, \; X(c_m(t)) = \frac{d}{dt} c_m(t). \qquad (1)$$

This curve is called the **maximal integral curve** passing through m.

The vector field is **complete** if for any $m \in M$, $]a(m), b(m)[= \mathbb{R}$.

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Example

The vector field $X = x^2\partial_x$ on $\mathbb{R}^2$ is not complete. Indeed, for any $(a, b) \in \mathbb{R}^2$, the integral curve passing through (a, b) is given by

$$c(t) = \begin{cases} (0, b), t \in \mathbb{R} & \text{if } a = 0, \\ \left(\frac{a}{1-at}, b\right), t \in]-\infty, \frac{1}{a}[& \text{if } a > 0, \\ \left(\frac{a}{1-at}, b\right), t \in]\frac{1}{a}, +\infty[& \text{if } a < 0. \end{cases}$$

Theorem

Let X be a vector field and $c : I \longrightarrow M$ be a maximal integral curve of X. If for any $a, b \in I$, $c(]a, b[)$ is contained in a compact then $I = \mathbb{R}$.

Corollary

Every vector field with a compact support is complete. In particular, every vector field in a compact manifold is complete.

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Theorem (Abraham-Marsden, Foundations of Mechanics, pp.71)

Let X be a vector field. If there exists a **proper** *function $f \in C^\infty(M)$ and $a, b \in \mathbb{R}^*$ such that for any $m \in M$*

$$|X(f)(m)| \leq a|f(m)| + b$$

then X is complete.

Corollary

If $X(f)$ is bounded and f is proper then X is complete. In particular, if X has a proper first integral then X is complete.

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Let M be a manifold and ∇ is connection on M. A geodesic of ∇ is a curve $\gamma : I \longrightarrow M$ defined on an interval I such that

$$\nabla_{\gamma'(t)}\gamma'(t) = 0 \quad \text{and} \quad (\gamma(0) = m, \gamma'(0) = u \in T_m M).$$

The connection ∇ is called complete if any geodesic can be extended to $\mathbb{R}$.

We associate to ∇ a vector field $G \in \Gamma(TTM)$ by

$$\boxed{G(u) = \frac{d}{dt}_{|t=0} \tau_c^t(u)}$$

where $c :]-\varepsilon, \varepsilon[\longrightarrow M$ is a curve such that $c'(0) = u$ and $\tau_c^t : T_mM \longrightarrow T_{c(t)}M$ is the parallel transport with respect to ∇ and associated to c.

The vector field G is a called the **geodesic flow** associated to ∇.

Proposition

If $\gamma : I \longrightarrow M$ a geodesic of ∇ then $\gamma' : I \longrightarrow M$ is an integral curve of G passing through $\gamma'(0)$. In particular, ∇ is complete if and only if G is complete.

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Proposition

Let $(M, \langle\ ,\ \rangle)$ be a pseudo-Riemannian manifold. Then the geodesic flow associated to the Levi-Civita connection is tangent to

$$\boxed{U_r M = \{u \in TM, \langle u, u\rangle = r^2\}.}$$

In particular, if the metric is Riemannian and M is compact then G is complete.

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Every homogeneous Riemannian manifold is geodesically complete.

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Example (Clifton-Pohl torus.)

Example of a compact Lorentzian manifold that is not geodesically complete.
Consider the manifold $\mathrm{M} = \mathbb{R}^2 \setminus \{0\}$ *with the metric*

$$g = \frac{dx\,dy}{\frac{1}{2}(x^2 + y^2)}$$

Any homothety is an isometry of M*, in particular including the map:*

$$\lambda(x, y) = 2 \cdot (x, y)$$

Example

Let Γ be the subgroup of the isometry group generated by λ . Then Γ has a proper, discontinuous action on M. Hence the quotient $T = M/\Gamma$, which is topologically the torus, is a Lorentz surface that is called the Clifton-Pohl torus.
It can be verified that the curve

$$\sigma(t) := \left(\frac{1}{1-t}, 0\right)$$

is a geodesic of M that is not complete.

Proposition

Let (G,∇) be a Lie group endowed with a left invariant connection and $\gamma : I \longrightarrow G$ a curve. Let $V : I \longrightarrow TG$ be a vector field along γ. We define $\mu : I \longrightarrow \mathfrak{g}$ and $W : I \longrightarrow \mathfrak{g}$ by

$$\mu(t) = T_{\gamma(t)}L_{\gamma(t)^{-1}}(\gamma'(t)) \quad \textit{and} \quad W(t) = T_{\gamma(t)}L_{\gamma(t)^{-1}}(V(t)).$$

Then V is parallel along γ with respect ∇ if and only if

$$W'(t) + \mu(t) \bullet W(t) = 0,$$

where $u \bullet v = (\nabla_{u^+}v^+)(e)$. In particular, γ is a geodesic if and only if

$$\mu'(t) = -\mu(t) \bullet \mu(t).$$

Proof.

We consider $(u_1, \ldots, u_n)$ a basis of $\mathfrak{g}$ and $(X_1, \ldots, X_n)$ the corresponding left invariant vector fields. Then

$$\begin{cases} \mu(t) = \sum_{i=1}^{n} \mu_i(t) u_i, \ W(t) = \sum_{i=1}^{n} W_i(t) u_i, \\ \gamma'(t) = \sum_{i=1}^{n} \mu_i(t) X_i, \ V(t) = \sum_{i=1}^{n} W_i(t) X_i. \end{cases}$$

□

Continued.

Then

$$\begin{aligned}\nabla_t V(t) &= \sum_{i=1}^n W_i'(t)X_i + \sum_{i=1}^n W_i(t)\nabla_{\gamma'(t)}X_i \\ &= \sum_{i=1}^n W_i'(t)X_i + \sum_{i,j=1}^n W_i(t)\mu_j(t)\nabla_{X_j}X_i \\ &= \sum_{i=1}^n W_i'(t)X_i + \sum_{i,j=1}^n W_i(t)\mu_j(t)(u_j \bullet u_i)^+ \\ &= \left(W'(t) + \mu(t) \bullet W(t)\right)^+\end{aligned}$$

and the result follows having in mind that u^+ is the left invariant vector field associated to $u \in \mathfrak{g}$.

Let (G, ∇) be a Lie group endowed with a left invariant connection and $\bullet : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ the associated product given by

$$\boxed{u \bullet v = (\nabla_{u^+} v^+)(e).}$$

The **Euler vector field** associated to ∇ is the (quadratic) vector field $\Gamma : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$\boxed{\Gamma(u) = -u \bullet u.}$$

Theorem

A curve $\gamma : I \longrightarrow G$ is a geodesic of ∇ if and only if its hodograph curve $\mu : I \longrightarrow \mathfrak{g}$ given by $\mu(t) = T_{\gamma(t)} L_{\gamma(t)^{-1}}(\gamma'(t))$ is an integral curve of the Euler vector field Γ. In particular, ∇ is complete if and only if Γ is complete.

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Proposition

Let (G, ∇) be a Lie group endowed with a left invariant connection. If the product $\bullet$ has an idempotent, i.e., a vector $u \in \mathfrak{g}$ such that $u \neq 0$ and $u \bullet u = u$ then ∇ is not complete.

Proof.

The curve $\mu(t) = \lambda(t)u$ is an integral curve of Γ if and only if

$$\lambda'(t) = -\lambda(t)^2.$$

Thus

$$\lambda(t) = \frac{1}{t+c}.$$

□

Let $(G, \langle\ ,\ \rangle)$ be a Lie group endowed with a left invariant pseudo-Riemannian metric and ∇ its Levi-Civita connection. The product associated to ∇ is called the Levi-Civita product and its given by

$$2\langle u.v, w\rangle = \langle [u,v], w\rangle + \langle [w,u], v\rangle + \langle [w,v], u\rangle.$$

The Euler vector field associated to ∇ is given by

$$\Gamma(u) = -u.u = \mathrm{ad}_u^* u$$

where ad^* is adjoint of ad_u with respect to metric.
Thus $(G, \langle\ ,\ \rangle)$ is complete if and only if Γ is complete.

We have $\langle \Gamma(u), u \rangle = 0$ and Γ is tangent to the "spheres" $\{\langle u, u \rangle = r^2\}$.

Theorem

Any left invariant Riemannian metric on a Lie group is complete.

Example

We consider the non-abelian 2-dimensional Lie algebra $\mathbb{R}^2$ with the Lie bracket $[e_1, e_2] = e_1$ and the Lorentzian metric

$$\langle \, , \, \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\Gamma(x, y) = -xy\partial_x + y^2\partial_y$$

and the integral curve of Γ passing through (λ, μ) is given by

$$\mu(t) = \left(\lambda(1 - t\mu), \frac{\mu}{1 - tu}\right).$$

So the metric is not complete.

Example

We consider the 3-dimensional Heisenberg Lie algebra $\mathbb{R}^3$ with the Lie bracket $[e_1, e_2] = e_3$ and the Lorentzian metric

$$\langle\ ,\ \rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then

$$\Gamma(x, y, z) = zy\partial_x - zx\partial_y$$

and the integral curve of Γ passing through (λ, μ, ν) is given by

$$\mu(t) = (\mu\sin(\nu t) + \lambda\cos(\nu t), \mu\cos(\nu t) - \lambda\sin(\nu t), \nu).$$

So the metric is complete.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra. It is Euler vector field is given by

$$\Gamma(u) = \mathrm{ad}_u^* u.$$

Let π be the Poisson tensor on $\mathfrak{g}$ given by

$$\pi(\alpha, \beta)(u) = \langle u, [\alpha(u)^{\#}, \beta(u)^{\#}] \rangle$$

where $\alpha, \beta \in \Omega^1(\mathfrak{g})$ and $\# : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ is the isomorphism associated to $\langle\ ,\ \rangle$. For any $f \in C^\infty(\mathfrak{g})$ we denote by X_f the Hamiltonian vector field associated to f and given by

$$X_f(g) = \pi(df, dg).$$

Proposition

We have $\Gamma = X_\mu$ *where* $\mu : \mathfrak{g} \longrightarrow \mathbb{R}$ *is given by*

$$\mu(u) = \frac{1}{2}\langle u, u\rangle.$$

Proof.

Let $f \in \mathfrak{g}^*$, $u \in \mathfrak{g}$ and c_u the integral curve of Γ passing through u. Note first that

$$d_u\mu(v) = \frac{1}{2}\frac{d}{dt}_{t=0}\langle u+tv, u+tv\rangle = \langle u, v\rangle = u^\flat(v).$$

We have

$$\begin{aligned}\Gamma(f)(u) &= \frac{d}{dt}_{t=0} f(c_u(t)) = \frac{d}{dt}_{t=0}\langle (d_{c_u(t)}f)^\#, c_u(t)\rangle \\ &= \frac{d}{dt}_{t=0}\langle f^\#, c_u(t)\rangle = \langle (f)^\#, \Gamma(u)\rangle = \langle [u, (f)^\#], u\rangle \\ &= \pi(u^\flat, df)(u) = \pi(d\mu, df)(u) = X_\mu(f).\end{aligned}$$

So for any $f \in \mathfrak{g}^*$, $X_\mu(f) = \Gamma(f)$. Since this relation depend only on df we get the desired result.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. The map $\Phi : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(g, u) \mapsto \mathrm{Ad}^t_{g^{-1}} u$ is an action and the orbits of this action are the symplectic leaves of π. Thus Γ is tangent to the orbits of this action.

Proposition (Hermann-1972)

If G is compact then any left invariant pseudo-Riemannian metric on G is complete.

Theorem (Marsden 1973)

Let M be a compact pseudo-Riemannian manifold. If there exists a Lie group G which acts transitively on M by isometries then M is geodesically complete.

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Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra. For any $u \in \mathfrak{g}$, $J_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by $J_u(v) = \mathrm{ad}_v^* u$.
J_u is skew-symmetric and $J_u = 0$ if and only if $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$.

Lemma

If $\mathfrak{g}$ is 2-step nilpotent then $J_u(v) \in [\mathfrak{g}, \mathfrak{g}]^\perp$ for any $u, v \in \mathfrak{g}$ and hence

$$J_{J_u(v)} = 0$$

for any $u, v \in \mathfrak{g}$.

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Theorem (Guediri 1994)

Any left invariant pseudo-Riemannian metric on a 2-step nilpotent Lie group is complete.

Proof.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be the Lie algebra of G and the Euler vector field is given by

$$\Gamma(u) = J_u.u.$$

Then

$$u(t) = \exp(tJ_{u_0})(u_0)$$

is the integral curve of Γ passing through u_0. We have

$$u'(t) = J_{u_0}u(t)$$

and by virtue of the Lemma above

$$J_{u(t)}(u(t)) = J_{u_0}(u(t)).$$

Example

We consider the counter-example of Guediri. Let $\mathfrak{g}$ be the four dimensional 3-nilpotent Lie algebra given by

$$[e_1, e_2] = e_3 \quad \textit{and} \quad [e_1, e_3] = e_4$$

endowed with the Lorentzian metric for which the basis $(e_1, \ldots, e_4)$ is orthonormal and $\langle e_1, e_1 \rangle = -1$. The integral curves of Γ satisfies

$$\begin{cases} x_1' = x_3(x_2 + x_4), \\ x_2' = x_1 x_3, \\ x_3' = x_1 x_4, \\ x_4' = 0, \\ -x_1^2 + x_2^2 + x_3^2 + x_4^2 = e. \end{cases}$$

Example

We have $x_4 = C$ *and* $Cx_2' - x_3x_3' = 0$ *so* $2Cx_2 - x_3^2 = m$. *Suppose* $C \neq 0$ *and* $e = m = 0$, *we get*

$$x_1^2 = \left(\frac{x_3^2}{2C}\right)^2 + x_3^2 + C^2 = \left(\frac{x_3^2}{2C} + C\right)^2.$$

We deduce that

$$x_3' = \pm\left(\frac{x_3^2}{2} + C^2\right)$$

and hence

$$x_3(t) = \pm C\tan(C(t+a))$$

and the metric is incomplete

An affine Lie group is a Lie group G endowed with a left invariant torsionless and flat connection ∇. This is equivalent to the product on $\mathfrak{g} = T_e G$

$$u \bullet v = (\nabla_{u^+} v^+)(e)$$

is Lie-admissible, i.e.,

$$u \bullet v - v \bullet u = [u, v]$$

and left symmetric, i.e.,

$$\mathrm{ass}(u, v, w) = \mathrm{ass}(v, u, w)$$

where

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There is a correspondence between affine structure on $\mathfrak{g}$ and Lie-admissible left symmetric product on $\mathfrak{g}$.

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Theorem

Let (G, ∇) is an affine Lie group and $\bullet$ the associated product on $\mathfrak{g}$. Then the following are equivalent:

1. (G, ∇) *is geodesically complete.*
2. *For any* $u \in \mathfrak{g}$, $\det(\mathrm{I}_{\mathfrak{g}} + R_u) \neq 0$.
3. *For any* $u \in \mathfrak{g}$, R_u *is nilpotent.*
4. *For any* $u \in \mathfrak{g}$, $\mathrm{tr}(R_u) = 0$,

where $R_u : \mathfrak{g} \longrightarrow \mathfrak{g}$, $v \mapsto v \bullet u$.

Recall that a product $\bullet$ on a vector space is called A is called left Leibniz if, for any $u, v, w \in A$,

$$u \bullet (v \bullet w) = (u \bullet v) \bullet w + v \bullet (u \bullet w).$$

In this case the vector space $N = \mathrm{span}\{u \bullet v + v \bullet u\}$ is two-side ideal and for any $u \in N$,

$$\mathrm{L}_u = 0$$

where $L_u(v) = u \bullet v$.
$(A, \bullet)$ is Lie algebra if and only if $N = \{0\}$.

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Theorem (Benayadi-Boucetta.)

Let (G, ∇) be a Lie group and $\bullet$ the product on $\mathfrak{g}$ associated to ∇. If $\bullet$ is left Leibniz then (G, ∇) is complete.

Proof.

Recall that (G, ∇) is complete if and only if the Euler vector field $\Gamma(u) = -u \bullet u$ is complete. We claim that the curve

$$u(t) = \exp(-t\mathrm{L}_{u_0})(u_0)$$

is the integral curve of Γ passing through u_0. It is a consequence of the fact that

$$\mathrm{L}_{u_0}^n(u_0) \in N$$

for any $n \geq 1$ and hence

$$\mathrm{L}_{u(t)} = \mathrm{L}_{u_0}.$$

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An open problem

Problem

Study the completeness of left invariant pseudo-Riemannian (Lorentzian) metrics on nilpotent Lie groups.