Supra-Flat Riemannian manifolds

Mohamed Boucetta-Hasna Essoufi

Cadi-Ayyad University

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Outline

Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold,

$$\mathrm{so}(TM) = \bigcup_{x \in M} \mathrm{so}(T_xM)$$

where $\mathrm{so}(T_xM)$ is the vector space of skew-symmetric endomorphisms of T_xM and $k > 0$. The Levi-Civita connection ∇^M of $(M, \langle\ ,\ \rangle_{TM})$ defines a connection on the vector bundle $\mathrm{so}(TM)$ which we will denote in the same way and it is given, for any $X \in \Gamma(TM)$ and $F \in \Gamma(\mathrm{so}(TM))$, by

$$\nabla^M_X F(Y) = \nabla^M_X(F(Y)) - F(\nabla^M_X Y).$$

R^M is the curvature tensor of ∇^M given by

$$R^M(X,Y) = \nabla^M_{[X,Y]} - \left(\nabla^M_X \nabla^M_Y - \nabla^M_Y \nabla^M_X\right).$$

Let $k > 0$. The Atiyah Euclidean vector bundle associated to $(M, \langle\ ,\ \rangle_{TM}, k)$ is the triple $(E(M,k), \langle\ ,\ \rangle_k, \nabla^E)$ where

1. $E(M,k) = TM \oplus \mathrm{so}(TM) \longrightarrow M$,
2. $\nabla^E_X Y = \nabla^M_X Y + H_X Y$, $\nabla^E_X F = H_X F + \nabla^M_X F$,
3. $\langle X + F, Y + G\rangle_k = \langle X, Y\rangle_{TM} - k\,\mathrm{tr}(F \circ G)$,

where

$$H_X Y = -\frac{1}{2} R^M(X,Y) \quad \text{and} \quad \langle H_X F, Y\rangle_{TM} = -\frac{1}{2} k\,\mathrm{tr}(F \circ R^M(X,Y)). \tag{1}$$

We have

$$\boxed{\nabla^E(\langle\ ,\ \rangle_k) = 0.}$$

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1. $E(M,k)=TM\oplus\mathrm{so}(TM)\longrightarrow M$,
2. $\nabla^E_XY=\nabla^M_XY+H_XY,\ \nabla^E_XF=H_XF+\nabla^M_XF$,
3. $\langle X+F,Y+G\rangle_k=\langle X,Y\rangle_{TM}-k\,\mathrm{tr}(F\circ G)$,

where

$$H_XY=-\frac{1}{2}R^M(X,Y)\quad\text{and}\quad\langle H_XF,Y\rangle_{TM}=-\frac{1}{2}k\,\mathrm{tr}(F\circ R^M(X,Y)).\tag{1}$$

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Definition 1.1

The supra-curvature[1] *of* $(M, \langle\ ,\ \rangle_{TM}, k)$ *is the curvature of* $(E(M,k), \nabla^E)$ *given by*

$$\boxed{R^{\nabla^E}(X,Y) = \nabla^E_{[X,Y]} - \left(\nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X\right).}$$

Remark 1

The Atiyah vector bundle doesn't depend on k. *However, the metric and the connection do. We will see that for different values of* k *the situation can change drastically.*

[1] This notion has been introduced in the PhD thesis of H. Essoufi.

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$$\begin{aligned}
R^{\nabla^E}(X,Y)Z &= \Big\{R^M(X,Y)Z + H_Y H_X Z - H_X H_Y Z\Big\} \\
&\quad + \left\{-\frac{1}{2}\nabla^M_Z(K^M)(X\wedge Y)\right\}, \\
R^{\nabla^E}(X,Y)F &= \Big\{(R^{\nabla^E}(X,Y)F)_{TM}\Big\} \\
&\quad + \Big\{[R^M(X,Y),F] + H_Y H_X F - H_X H_Y F\Big\}, \\
\langle (R^{\nabla^E}(X,Y)F)_{TM}, Z\rangle_k &= -\langle R^{\nabla^E}(X,Y)Z, F\rangle_k,
\end{aligned}$$

$X, Y, Z \in \Gamma(TM)$, $F \in \Gamma(\mathrm{so}(TM))$, $K^M : \mathrm{so}(TM) \longrightarrow \mathrm{so}(TM)$ be the curvature operator given by $K^M(X\wedge Y) = R^M(X,Y)$ where $X \wedge Y(Z) = \langle Y, Z\rangle_{TM} X - \langle X, Z\rangle_{TM} Y$.

It is obvious that if the curvature of M vanishes then its supra-curvature vanishes. Hence it is natural to ask of there is non flat Riemannian manifolds with are supra-flat.

Proposition 2.1

Suppose that $(M, \langle\ ,\ \rangle_{TM})$ has constant sectional curvature c, i.e,

$$R^M(X,Y) = -cX \wedge Y$$

and put $\varpi = \frac{1}{4}c(2-ck)$. Then, for any $X, Y \in \Gamma(TM)$ and $F \in \Gamma(\mathrm{so}(TM))$,

$$R^{\nabla^E}(X,Y)Z = -2\varpi X \wedge Y(Z) \quad \text{and} \quad R^{\nabla^E}(X,Y)F = -2\varpi[X \wedge Y, F].$$

So if $c > 0$ then the supra-curvature of $(M, \langle\ ,\ \rangle_{TM}, \frac{2}{c})$ vanishes.

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So if $c > 0$ *then the supra-curvature of* $(M, \langle\ ,\ \rangle_{TM}, \frac{2}{c})$ *vanishes.*

Corollary 2.1

Any sphere has an Euclidean vector bundle with a flat metric connection.

This defines a cohomology on the differential forms on the sphere with values in the sections of this vector bundle. It will be interesting to study this cohomology.

One can ask if there are other supra-flat Riemannian manifolds.

Theorem 2.1 (M. Boucetta-H. Essoufi, Mediterr. J. Math. (2020).)

Let $(M, \langle\ ,\ \rangle_{TM})$ be a connected Riemannian manifold. Then the supra-curvature of $(M, \langle\ ,\ \rangle_{TM}, k)$ vanishes if and only if the Riemannian universal covering of $(M, \langle\ ,\ \rangle_{TM})$ is isometric to the Riemannian product $(\mathbb{R}^n, \langle\ ,\ \rangle_0) \times \mathbb{S}^{n_1}\left(\sqrt{\frac{k}{2}}\right) \times \ldots \times \mathbb{S}^{n_p}\left(\sqrt{\frac{k}{2}}\right)$ where $\mathbb{S}^{n_i}\left(\sqrt{\frac{k}{2}}\right)$ is the Riemannian sphere of dimension n_i, of radius $\sqrt{\frac{k}{2}}$ and constant curvature $\frac{2}{k}$.

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Corollary 2.2

Let $(M, \langle\ ,\ \rangle_{TM})$ be a connected supra-flat Riemannian manifold. Then $M = N/\Gamma$ where

$$N = (\mathbb{R}^n, \langle\ ,\ \rangle_0) \times \mathbb{S}^{n_1}\left(\sqrt{\frac{k}{2}}\right) \times \ldots \times \mathbb{S}^{n_p}\left(\sqrt{\frac{k}{2}}\right)$$

and

$$\Gamma = \Gamma_0 \times \Gamma_1 \times \ldots \times \Gamma_p,$$

where $\Gamma_0 \subset \mathrm{O}(n) \ltimes \mathbb{R}^n$ is a discrete group and $\Gamma_i \subset \mathrm{O}(n_i + 1)$ is a finite group, for $i = 1, \ldots, p$.

Proposition 3.1

Let $(M, \langle\ ,\ \rangle_{TM})$ be the Riemannian product of p Riemannian manifolds $(M_1, \langle\ ,\ \rangle_1), \ldots, (M_p, \langle\ ,\ \rangle_p)$. Then the supra-curvature of $(M, \langle\ ,\ \rangle_{TM}, k)$ at a point $x = (x_1, \ldots, x_p)$ is given by

$$\begin{cases} R^{\nabla^E}[(X_1, \ldots, X_p), (Y_1, \ldots, Y_p)](Z_1, \ldots, Z_p) \\ = \left(R^{\nabla^{E_1}}(X_1, Y_1)Z_1, \ldots, R^{\nabla^{E_p}}(X_p, Y_p)Z_p\right), \\ R^{\nabla^E}[(X_1, \ldots, X_p), (Y_1, \ldots, Y_p)](F) \\ = \left(R^{\nabla^{E_1}}(X_1, Y_1)F_1, \ldots, R^{\nabla^{E_p}}(X_p, Y_p)F_p\right), \end{cases}$$

where $X_i, Y_i, Z_i \in T_{x_i}M_i$, $F \in \mathrm{so}(T_xM)$, $F_i = \mathrm{pr}_i \circ F_{|TM_i}$, $R^{\nabla^{E_i}}$ is the supra-curvature of $(M_i, \langle\ ,\ \rangle_i, k)$ and $i = 1, \ldots, p$.

Proposition 3.2

If $(M, \langle\ ,\ \rangle_{TM}, k)$ *is supra-flat then* $(M, \langle\ ,\ \rangle_{TM})$ *is locally symmetric, i.e.,* $\nabla^M(R^M) = 0$ *and for any* $X, Y \in \Gamma(TM)$,

$$\langle R^M(X,Y)X, Y\rangle_{TM} = \langle H_X Y, H_X Y\rangle_k \geq 0.$$

Thus $(M, \langle\ ,\ \rangle_{TM})$ *has non-negative sectional curvature.*

Proof.

It is an immediate consequence of the expression of the supra-curvature given in Proposition 1.1.

□

A Riemannian manifold $(M, \langle\ ,\ \rangle_{TM})$ is called symmetric if for any $x \in M$ there exists (a unique) an isometry f_x of $(M, \langle\ ,\ \rangle)$ such that $f_x(x) = x$ and $T_x f_x = -\mathrm{Id}_{T_xM}$.

Then

$$T_x f_x[\nabla^M_{u_1}(R^M)(u_2, u_3, u_4)] = \nabla_{T_x f_x(u_1)}(R^M)(T_x f_x(u_2), T_x f_x(u_3), T_x f_x(u_4))$$

and hence

$$\nabla^M(R^M) = 0.$$

Moreover, if we consider the local symmetry $s_x : \exp_x(v) \longrightarrow \exp_x(-v)$ then $s_x(x) = x$ and $T_x s_x = -\mathrm{Id}_{T_xM}$. Then s_x coincides with f_x in a neighborhood of x.

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Proposition 3.3

Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold. Then the following assertions are equivalent:

1. $\nabla^M(R^M) = 0$.
2. *for any $x \in M$, the local symmetry s_x, $\exp_x(v) \mapsto \exp_x(-v)$ is an isometry.*

A Riemannian manifold $(M, \langle\ ,\ \rangle_{TM})$ satisfying one of the conditions above is called locally symmetric.

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A Riemannian symmetric space is homogeneous.

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A Riemannian symmetric space is homogeneous.

Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian symmetric manifold. We denote by G the connected component of the identity in the group of isometries of $(M, \langle\ ,\ \rangle)$ and by K the isotropy subgroup at some point x (fixed one of all).

The symmetry f_x around x belongs to K and generates an involutive automorphism σ of G

$$\sigma(f) = f_x \circ f \circ f_x^{-1}.$$

We denote by $G^\sigma = \{f \in G, \sigma(f) = f\}$ and G_0^σ the connected component of G^σ in G.

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Theorem 3.1 (E. Cartan)

1. *Given any connected symmetric space* $(M, \langle\ ,\ \rangle_{TM})$ *and any* $x \in M$, *then the corresponding involutive automorphism* σ *of* G *satisfies* $G_0^\sigma \subset K \subset G^\sigma$.
2. *Conversely, if* G *is a Lie group,* σ *an involutive automorphism of* G *and* K *a compact subgroup of* G *such that* $G_0^\sigma \subset K \subset G^\sigma$. *Then any invariant Riemannian metric in* G/K *is symmetric.*
3. *A simply-connected complete locally symmetric space is a symmetric space.*

Let (G, σ, K) be a symmetric space. Let $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K and $\rho = T_e\sigma$ the automorphism of $\mathfrak{g}$ associated to σ. We have $\mathfrak{k} = \{X \in \mathfrak{g}, \rho(X) = X\}$ and let $\mathfrak{p} = \{X \in \mathfrak{g}, \rho(X) = -X\}$.

Lemma 3.1 (Fundamental Lemma)

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p}$ is $\mathrm{Ad}_G(K)$-invariant and

$$\boxed{[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad \textit{and} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.}$$

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The curvature of a symmetric space is easy to compute.

Proposition 3.5

Let G/K be a symmetric space and $X, Y \in T_{\pi(e)}G/K = \mathfrak{p}$. Then

$$\langle R(X,Y)X, Y\rangle = -\langle \mathrm{ad}_X \circ \mathrm{ad}_X Y, Y\rangle,$$
$$\mathrm{ric} = -\frac{1}{2}B_{|\mathfrak{p}},$$

where B is the Killing form of $\mathfrak{g}$.

Corollary 3.1

Let $(G/K, \langle\ ,\ \rangle)$ be a symmetric space. Then $\langle\ ,\ \rangle$ is Einstein if and only if the restriction of B to $\mathfrak{p}$ is

1. *either identically zero,*
2. *or definite and proportional to $\langle\ ,\ \rangle$.*

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Proposition 3.6

If the adjoint representation $\mathrm{Ad}_G(K)$ *of* K *in* $\mathfrak{p}$ *is irreducible then* $\langle\ ,\ \rangle$ *is Einstein.*

Definition 3.1

A symmetric space $(G/K, \langle\ ,\ \rangle)$ *is called irreducible if the adjoint representation* $\mathrm{Ad}_G(K)$ *of* K *in* $\mathfrak{p}$ *is irreducible.*

Theorem 3.2

A simply-connected symmetric space is the Riemannian product of a Euclidean space and a finite number of irreducible Riemannian symmetric spaces.

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Let G be a compact connected Lie group with $\mathfrak{g}$ its Lie algebra and K be a closed subgroup of G with $\mathfrak{k}$ its Lie algebra. Denote by $\pi : G \longrightarrow G/K$ the canonical projection. Suppose that:

1. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is Ad_K-invariant, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$,
2. the restriction of the Killing form B of $\mathfrak{g}$ to $\mathfrak{p}$ is negative definite.

The scalar product $\langle \ , \ \rangle_{\mathfrak{p}} = \lambda B_{|\mathfrak{p}\times\mathfrak{p}}$ with $\lambda < 0$ defines a G-invariant Riemannian metric $\langle \ , \ \rangle_{G/K}$ on G/K which is locally symmetric.

For any $X \in \mathfrak{k}$, we denote by Φ_X the restriction of ad_X to $\mathfrak{p}$.
$\Phi : \mathfrak{k} \longrightarrow \mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}})$ is a representation and

$$\boxed{\mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}}) = \Phi_{\mathfrak{k}} \oplus (\Phi_{\mathfrak{k}})^{\perp},}$$

where $(\Phi_{\mathfrak{k}})^{\perp}$ is the orthogonal with respect to the invariant scalar product on $\mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}})$, $(A, B) \mapsto -\mathrm{tr}(AB)$.

Remark 2

$[\mathfrak{p}, \mathfrak{p}]$ *is an ideal of* $\mathfrak{k}$ *and hence*

$$\boxed{[\Phi_{[\mathfrak{p},\mathfrak{p}]}, \Phi_{\mathfrak{k}}] \subset \Phi_{[\mathfrak{p},\mathfrak{p}]}.}$$

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Proposition 3.7

The supra-curvature of $(G/K, \langle\ ,\ \rangle_{G/K}, k)$ *at* $\pi(e)$ *is given by*

$$\begin{aligned} R^{\nabla^E}(X,Y)Z &= [[X,Y],Z] + \frac{k}{4}\left([Y, U(\Phi_{[X,Z]})] - [X, U(\Phi_{[Y,Z]})]\right), \\ R^{\nabla^E}(X,Y)F &= [\Phi_{[X,Y]}, \Phi_{X^F - \frac{k}{4}U(F)}] + [\Phi_{[X,Y]}, F^{\perp}], \end{aligned}$$

where $X, Y, Z \in T_{\pi(e)}G/K = \mathfrak{p}$,
$F = \Phi_{X^F} + F^{\perp} \in \mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}}) = \Phi_{\mathfrak{k}} \oplus (\Phi_{\mathfrak{k}})^{\perp}$ *and* $U(F)$ *is the element of* $\mathfrak{k}$ *given by*

$$U(F) = \sum_{i=1}^{n} [X_i, F(X_i)],$$

where $(X_1, \ldots, X_n)$ *is an orthonormal basis of* $\mathfrak{p}$.

Corollary 3.2

If the supra-curvature of $(G/K, \langle\ ,\ \rangle_{G/K}, k)$ vanishes then $\Phi_{[\mathfrak{p},\mathfrak{p}]}$ is an ideal of $\mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}})$.

Remark 3

For every $n \neq 4$, $\mathrm{so}(n)$ is a simple Lie algebra.

Theorem 3.3 (Jensen)

A simply-connected four dimensional homogeneous Einstein manifold with positive scalar curvature is isometric to $\mathbb{S}^4(r)$, $\mathbb{S}^2(r) \times \mathbb{S}^2(r)$ or $P^2(\mathbb{C})$.

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Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow P^n(\mathbb{C})$ be the natural projection and $\pi_s : S^{2n+1} \longrightarrow P^n(\mathbb{C})$ its restriction to $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$. For any $m \in S^{2n+1}$, put $F_m = \ker((\pi_s)_*)_m$ and let $F_m^\perp$ be the orthogonal complementary subspace to F_m in $T_m(S^{2n+1})$;

$$T_m(S^{2n+1}) = F_m \oplus F_m^\perp.$$

We introduce the Riemannian metric $\langle\ ,\ \rangle_{P^n(\mathbb{C})}$ on $P^n(\mathbb{C})$ so that the restriction of $(\pi_s)_*$ to $F_m^\perp$ is an isometry onto $T_{\pi(m)}(P^n(\mathbb{C}))$. Let J_0 be the canonical complex structures on $\mathbb{C}^{n+1}$ and the standard complex structures J on $P^n(\mathbb{C})$ is given by

$$J(\pi_s)_* v = (\pi_s)_* J_0 v,\ \ v \in F_m^\perp.$$

Proposition 3.8

The curvature and the supra-curvature of $(P^n(\mathbb{C}), g, k)$ *are given by*

$$\begin{aligned} R^{P^n(\mathbb{C})}(X,Y)Z &= \langle X,Z\rangle_{P^n(\mathbb{C})}Y - \langle Y,Z\rangle_{P^n(\mathbb{C})}X - 2\langle JY,X\rangle_{P^n(\mathbb{C})}JZ \\ &\quad + \langle JZ,Y\rangle_{P^n(\mathbb{C})}JX - \langle JZ,X\rangle_{P^n(\mathbb{C})}JY, \\ R^{\nabla^E}(X,Y)Z &= (k-1)\left(\langle Y,Z\rangle_{P^n(\mathbb{C})}X - \langle X,Z\rangle_{P^n(\mathbb{C})}Y + 2\langle JY,X\rangle_{P^n(\mathbb{C})}JZ\right. \\ &\quad + ((2n+3)k-1)\left(\langle JZ,X\rangle_{P^n(\mathbb{C})}JY - \langle JZ,Y\rangle_{P^n(\mathbb{C})}JX\right), \\ R^{\nabla^E}(X,Y)F &= \left(\frac{k}{2}-1\right)[F, X\wedge Y + JX\wedge JY] + 2\langle JY,X\rangle_{P^n(\mathbb{C})}[F,J] \\ &\quad + \frac{k}{2}\left([J\circ F\circ J, X\wedge Y] - J\circ F(X)\wedge JY - JX\wedge J\circ F(Y)\right. \end{aligned}$$

where $X, Y, Z \in \Gamma(TP^n(\mathbb{C}))$ *and* $F \in \Gamma(\mathrm{so}(TP^n(\mathbb{C})))$.

Theorem 4.1 (M. Boucetta-H. Essoufi, Mediterr. J. Math. (2020).)

Let $(M, \langle\ ,\ \rangle_{TM})$ be a connected Riemannian manifold. Then the supra-curvature of $(M, \langle\ ,\ \rangle_{TM}, k)$ vanishes if and only if the Riemannian universal cover of $(M, \langle\ ,\ \rangle_{TM})$ is isometric to $(\mathbb{R}^n, \langle\ ,\ \rangle_0) \times \mathbb{S}^{n_1}\left(\sqrt{\frac{k}{2}}\right) \times \ldots \times \mathbb{S}^{n_p}\left(\sqrt{\frac{k}{2}}\right)$ where $\mathbb{S}^{n_i}\left(\sqrt{\frac{k}{2}}\right)$ is the Riemannian sphere of dimension n_i, of radius $\sqrt{\frac{k}{2}}$ and constant curvature $\frac{2}{k}$.

Proof.

(1) If the supra-curvature of $(M, \langle\ ,\ \rangle_{TM}, k)$ vanishes then the supra-curvature of the Riemannian covering $(N, \langle\ ,\ \rangle_{TN})$ of $(M, \langle\ ,\ \rangle_{TM})$ vanishes since $\pi : N \longrightarrow M$ is a local isometry.

(2) Then $(N, \langle\ ,\ \rangle_{TN})$ is locally symmetric and hence symmetric. Moreover, its sectional curvature is non-negative. (See Proposition 3.2)

(3) $(N, \langle\ ,\ \rangle_{TN}) = (E, \langle\ ,\ \rangle_0) \times (N_1, \langle\ ,\ \rangle_1) \times \ldots \times (N_p, \langle\ ,\ \rangle_p)$ where $(E, \langle\ ,\ \rangle_0)$ is flat and the $(N_i, \langle\ ,\ \rangle_i)$ are irreducible symmetric spaces with non-negative sectional curvature. (See Theorem 3.2)

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□

Continued.

(4) For each $i = 1, \dots, p$, N_i is an irreducible symmetric space with non-negative sectional curvature and hence it is Einstein with positive scalar curvature and hence compact. (See Proposition 3.6 and Meyer's Theorem.)

(5) The vanishing of the supra-curvature of $(N, \langle\ ,\ \rangle_{TN}, k)$ implies the vanishing of the supra-curvature of $(N_i, \langle\ ,\ \rangle_i, k)$ for $i = 1, \dots, p$ (See Proposition 3.1).

(6) If $\dim N_i = 4$ then, according to Jensen's Theorem, N_i is isometric to $\mathbb{S}^4(r)$, $\mathbb{S}^2(r) \times \mathbb{S}^2(r)$. ($P^2(\mathbb{C})$ has a non vanishing supra-curvature by virtue of Proposition 3.8).

(7) If $\dim N_i \neq 4$ then $N_i = G/K$ as in Proposition 3.7 and the vanishing of the supra-curvature of N_i implies that $\Phi_{[\mathfrak{p},\mathfrak{p}]}$ is an ideal of $\mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}})$.

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(8) Since the curvature of N_i is different from zero then $\Phi_{[\mathfrak{p},\mathfrak{p}]} \neq 0$ and hence $\Phi_{[\mathfrak{p},\mathfrak{p}]} = \mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}})$.

(9) Let $n_i = \dim N_i = \dim \mathfrak{p}$. Then

$$\dim \mathfrak{k} \geq \dim \Phi_{\mathfrak{k}} \geq \dim \mathrm{so}(\mathfrak{p}) = \frac{n_i(n_i - 1)}{2}.$$

So

$$\dim G = \dim \mathfrak{k} + n_i \geq \frac{n_i(n_i + 1)}{2}.$$

But the dimension of the group of isometries is always less or equal to $\frac{n_i(n_i+1)}{2}$ with equality when the manifold has constant curvature. Thus $\dim G = \frac{n(n+1)}{2}$ and hence N_i has constant curvature. This completes the proof.

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