Affine submanifolds and affine maps

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Example from linear algebra

Let $V \subset R^n$ be a vector subspace, and $p_0 \in \mathbb{R}^n$.

Question

Does there exists an affine submanifold $N \subset \mathbb{R}^n$ (endowed with its canonical affine structure) around p_0 such that $T_{p_0}N = V$?

Solution

We can take $N = V + p_0 \subset \mathbb{R}^n$ clearly N is an affine submanifold of $\mathbb{R}^n$ which satisfy $T_{p_0}N = V$.

Example from linear algebra

Let

$$A : \mathbb{R}^n \to \mathbb{R}^p$$

be a linear map, $p_1 \in \mathbb{R}^n$ and $p_2 \in \mathbb{R}^p$.

Question

Does there exists an affine map

$$F : \mathbb{R}^n \to \mathbb{R}^p$$

such that $F(p_1) = p_2$ *and* $T_{p_1} F = A$

Solution

We can take $F(x) = Ax + p_2 - Ap_1$ *which satisfy* $F(p_1) = p_2$ *and* $T_{p_1} F = A$.

Affine manifold

Definition

An affine manifold is a smooth manifold (M, ∇) endowed with a mapping (called affine connection or Koszul connection)

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which satisfies the following properties:

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

- Every smooth manifold admit an affine connection.
- Every pseudo Riemannian manifold admit an unique torsion-free affine connection which preserves the metric (called Levi-Civita connection).

Covariant derivative

Proposition

Let (M, ∇) be an affine connection ∇. There exists an unique correspondence which associates to a vector field V along a smooth curve $\gamma : I \to M$ another vector field $D_t V$ along γ, called the covariant derivative of V along γ, such that:

- *i)* $D_t(V + W) = D_t V + D_t W$, *where W is a vector field along γ.*
- *ii)* $D_t(fV) = f'(t)V + fD_t V$, *where f is a smooth function on I.*
- *iii)* *If V is induced by a vector field $X \in \mathfrak{X}(M)$, i.e., $V(t) = X(\gamma(t))$, then $D_t V = \nabla_{\gamma'} X$.*

- $\gamma : I \to M$ is geodesic ($\nabla_{\gamma'}\gamma' = 0$) if and only if $D_t\gamma' = 0$.

Parallel translation

Definition

Let (M, ∇) be an affine manifold. A vector field along a curve $\gamma : I \to M$ is called parallel when $D_t V = 0$.

Proposition

Let (M, ∇) be an affine manifold. Let $\gamma : I \to M$ be a smooth curve and $V_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field V along γ, such that $V(t_0) = V_0$. $V(t)$ is called the parallel translation of V_0 along γ.

Let (M, ∇) be an affine manifold. Consider the mapping

$$\tau^\gamma : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$$

defined by: $\tau^\gamma(u)$ is the vector obtained by parallel translating the vector u along γ. The linear map τ^γ is called the parallel translate along the curve γ. And one can show that τ^γ is an isomorphism.

Affine map

Let (M^1, ∇^1) and (M^2, ∇^2) be two affine manifolds. Let $F : M^1 \to M^2$ a smooth map. We say that F is an affine map if $TF : TM^1 \to TM^2$ commute with covariant derivative. Consequently, the image $F \circ \gamma$ of each geodesic γ of M^1 is a geodesic of M^2, and the image $F_* X$ of a parallel vector field along a curve γ on M^1 is a parallel vector field along $F \circ \gamma$. Therefore, the parallel translation and the tangent map $TF : TM^1 \to TM^2$ commute. Conversely if TF commute with the parallel translation then F is an affine map.

Question 1

Let $p_1 \in M^1$, $p_2 \in M^2$ and $A : T_{p_1}M^1 \to T_{p_2}M^2$ a linear map.

Question (1)

Does there exists an affine map

$$F : (M^1, \nabla^1) \to (M^2, \nabla^2)$$

such that $F(p_1) = p_2$ *and* $T_{p_1}F = A$*?*

Affine submanifold

Definition

We say that an (immersed) submanifold N of an affine manifold (M, ∇) is affine, if it can be equipped with an affine connection ∇^N such that the inclusion map $(N, \nabla^N) \hookrightarrow (M, \nabla)$ becomes affine, in other terms, for all vectors fields X, Y on M which are tangent to N, the vector field $\nabla_X Y$ is tangent to N.

Affine submanifold

• A submanifold N is affine if and only if its tangent bundle TN is invariant under parallel translation in M along curves on N.
• If N is an affine submanifold, then each geodesic γ of N is also a geodesic of M, in other words, affine submanifolds are totally geodesic.
• The image $F(N)$ of every injective affine immersion $F : N \to M$ is an affine submanifold of M.

Affine submanifold

• In a Riamannian manifold the affine submanifold are exactly the totally geodesic ones, but in affine manifolds this equivalence is not true as is demonstrated by the following example due to E. Cartan:
Let ∇° be the canonical affine connection of $\mathbb{R}^3$ and denote by $v \wedge w$ the cross product in this space. Then

$$\nabla_X Y = \nabla^\circ_X Y + X \wedge Y$$

defines another affine connection on $\mathbb{R}^3$, and in the affine manifold $(\mathbb{R}^3, \nabla)$ the (usual) planes are totally geodesic, but not affine not affine submanifolds; notice that the geodesics of $(\mathbb{R}^3, \nabla)$ are again the straight lines.

Question 2

Let $p_0 \in M$ and $V \subset T_{p_0}M$ a vector subspace.

Question (2)

Does there exists an affine submanifold $N \subset (M, \nabla)$ around p_0 such that $T_{p_0}N = V$?

Affine submanifold and affine map

- The torsion tensor of an affine connection is given by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

- The curvature tensor of an affine connection is given by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Definition

Let points $p_1 \in M^1$, $p_2 \in M^2$, a linear subspace $V \subset T_{p_1}M^1$ and a linear map $A : T_{p_1}M^1 \to T_{p_2}M^2$ be given, We say that V is torsion and curvature invariant if and only if V satisfies

$$T^1(V,V) \subset V \text{ and } R^1(V,V)V \subset V,$$

and that A preserves the torsion and the curvature tensor fields if and only if $A(T^1(u,v)) = T^2(Au,Av)$ and $A(R^1(u,v)w) = R^2(Au,Av)Aw$ for all $u,v,w \in T_{p_1}M^1$.

Affine submanifold and affine map

• We put $V(A) := \{(u, Au) | \, u \in T_{p_1}M^1\} \subset T_{(p_1,p_2)}(M^1 \times M^2)$.
• The tangent map of an affine map preserve the torsion and the curvature tensor fields.
• Denote by $M^\times := M^1 \times M^2$ the affine product endowed with the unique affine connection $\nabla^\times$ such that the canonical projections $p_i : M^\times \to M^i$ are affine maps.
• The geodesic of $(M^\times, \nabla^\times)$ are precisely the curves $(\gamma_1, \gamma_2) : I \to (M^\times$ where $\gamma_1 : I \to M^1$ and $\gamma_2 : I \to M^2$ are geodesics of M^1 resp. M^2.

Proposition

1. *A linear map $A : T_{p_1}M^1 \to T_{p_2}M^2$ preserves the torsion and curvature tensor fields if and only if $V(A)$ is torsion and curvature invariant with respect to $\nabla^\times$.*
2. *A smooth map $F : M^1 \to M^2$ is affine if and only if its graph is an affine submanifold of the affine product $M^\times$.*

Proof of proposition

Proof.

Let $\tilde{\gamma} : [0,1] \to \mathrm{Graph}(F)$, $\tilde{\gamma} = (\gamma, F(\gamma))$ where γ is a curve on M^1 such that $\gamma(0) = p$ and $\gamma(1) = q$ hence for any $u \in T_pM^1$ we have

$$\tau^{\tilde{\gamma}}(u, T_pF(u)) = (\tau^{\gamma}(u), \tau^{F(\gamma)}(T_pF(u)))$$

Hence $T_qF(\tau^{\gamma}(u)) = \tau^{F(\gamma)}(T_pF(u))$ if and only if $\tau^{\tilde{\gamma}}(u, T_pF(u)) \in T_{(q,F(q))}\mathrm{Graph}(F)$. This show that F is an affine map if and only if commute with parrallel transport if and only if $\mathrm{Graph}(F)$ is an affine submanifold of $(M^{\times}, \nabla^{\times})$. □

Jacobi vector field

Definition

Let (M, ∇) be an affine manifold and $\gamma : I \to M$ be a geodesic. A vector field J on M along γ is said to be a Jacobi vector field if it satisfies the Jacobi equation:

$$D_t^2 J = R(\gamma', J)\gamma' + D_t(T(\gamma', J))$$

where R and T are the curvature and the torsion of ∇.

• A Jacobi vector field is determined by its initial conditions: $J(0)$ and $D_t J(0)$.

Jacobi vector field

Now let (M, ∇) be an affine manifold. Fix $p_0 \in M$ and we choose a normal neighbourhood $\exp_{p_0} : U^T \to U$ of p_0 in M; where U^T is a star shaped neighbourhood of 0 in $T_{p_0}M$ on which the exponential map is a diffeomorphism into M.

Lemma (1)

Let $u \in U^T \setminus \{0\}$ and $v \in T_{p_0}M$. Denote by $Q = \{(s,t) \in \mathbb{R}^2 \setminus t(u+sv) \in U^T\}$ which an open of $\mathbb{R}^2$. Then the infinitesimal variation

$$J^v : t \mapsto f_*(\partial_s)|_{(0,t)}$$

of the geodesic variation $f : (s,t) \in Q \mapsto \gamma_{u+sv}(t) = \exp_{p_0}(t.(u+sv))$ is the Jacobi vector field along the geodesic γ_u satisfying the initial data: $J^v(0) = 0$ and $D_t J^v(0) = v$.

Proof of Lemma 1

Proof.

Let $S = f_*(\partial_s)|_{(s,t)}$ and $W = f_*(\partial_t)|_{(s,t)}$. Then we have

$$D_t W = 0,\ [S, W] = 0,\ \text{and } T(W, S) = D_t S - D_s W$$

Thus

$$R(W, S)W = D_t D_s W = D_t^2 W - D_t(T(W, S)),$$

and evaluating at $(0, t)$ shows that J^v is a Jacobi along the geodesic γ_u. And we have $J^v(t) = d_{tu} \exp_{p_0}(tv)$ hence we get that $J^v(0) = 0$. In the same we get that $D_t J^v(t) = d_{tu} \exp_{p_0}(v) + tD_t(d_{tu} \exp_{p_0}(v))$ and evaluating at $t = 0$ shows that $D_t J^v(0) = v$. □

Jacobi vector field

Lemma (2)

Let $u \in U^T \setminus \{0\}$, J a Jacobi vector field along γ_u satisfying $J(0) = 0$ and X a vector field on U, which is parallel along every geodesic γ_v ($v \in T_{p_0}M$). Then the vector field $Y : t \mapsto \nabla_{J(t)}X$ satisfies the defferential equation

$$D_t Y = R(\gamma_u', J)(X \circ \gamma_u).$$

Proof of Lemma 2

Proof.

If we put $v := D_t J(0)$ and define f and J^v as in Lemma 1, then we have $J = J^v$, hence

$$Y(t) = D_s(X \circ f)(0, t).$$

Because of $X \circ f(s, t) = X \circ \gamma_{u+sv}(t)$ the parallelity of X along the geodesic rays implies

$$D_t(X \circ f)(s, t) = 0.$$

Therefore, the structure equation for the curvature tensor gives

$$\begin{aligned} D_t Y(t) &= D_t D_s(X \circ f)(0, t) \\ &= R(W, S)(X \circ f)(0, t) + D_s D_t(X \circ f)(0, t) \\ &= R(\gamma_u'(t), J(t))(X \circ \gamma_u). \end{aligned}$$

$\square$

Notation

Let us fix the fix the following notations:
• For every $u \in T_{p_0}M$ let $I_u = \{t \in \mathbb{R} | \, tu \in U^T\}$ and
$\gamma_u : t \in I_u \to \exp_{p_0}(tu)$.
• For every $u \in U^T$ let us abbreviate $p(u) := \exp_{p_0}(u)$, $\tau_u := \tau^{\gamma_u}$ the parallel translation along γ_u from $T_{p_0}M$ to $T_{p(u)}M$ and
$V_u := \tau_u(V) \subset T_{p(u)}M$.
• We call the regular submanifold $N := \exp_{p_0}(V \cap U^T)$ the geodesic umbrella associated to the data (p_0, V).

Theorem 1

Theorem (1)

N is an affine submanifold of (M, ∇) if and only if for every $u \in V \cap U^T$

$$T(\gamma_u'(1), V_u) \subset V_u \text{ and } R(\gamma_u'(1), V_u)V_u \subset V_u. \quad (1)$$

Of course, condition (1) is satisfied, if V_u is torsion and curvature invariant.

- The riemannian version of the theorem is attributed to E. Cartan.

Proof.

$\Longrightarrow$ If the geodesic umbrella N is affine, its construction implies $V_u = T_{p(u)}N$ for every $u \in V \cap U^T$ and therefore, the linear subspaces V_u satisfy (1).

$\Longleftarrow$ At the moment we fix three vectors $u, v, w \in V$ with $u \in U^T \setminus \{0\}$, use the Jacobi field J^v and the map f from Lemma 1. Because of $u + sv \in V$ the image of the map f lies in the geodesic umbrella N. Therefore, Lemma 1 shows

$$\forall t \in I_u : J^v(t) \in T_{\gamma_u(t)}N. \tag{2}$$

In the following we use the development $z : I_u \to T_{p_0}M$ of vectors fields Z along γ_u in the sense of Cartan given by

$$z(t) := \tau_{tu}^{-1}(Z(t)),$$

which satisfies

$$\forall t \in I_u : D_t Z(t) = \tau_{tu}(z'(t)). \tag{3}$$

Proof of Theorem 1

Besides, for $t \in I_u$ we also define the tensors

$$\widetilde{T}_u : I_u \to \mathrm{End}(T_{p_0}M) \text{ and } \widetilde{R}_u : I_u \to \mathrm{Hom}(\otimes^2 T_{p_0}M, T_{p_0}M)$$

by

$$\widetilde{T}_u(t)(x) := \tau_{tu}^{-1}\left(T(\gamma'_{tu}(1), \tau_{tu}(x))\right)$$

and

$$\widetilde{R}_u(t)(x, y) := \tau_{tu}^{-1}\left(R(\gamma'_{tu}(1), \tau_{tu}(x))\tau_{tu}(y)\right).$$

According to condition (1) the linear subspace V is invariant and with respect to these tensors, i.e.:

$$\widetilde{T}_u(t)(V) \subset V \text{ and } \widetilde{R}_u(t)(V, V) \subset V. \tag{4}$$

Proof of Theorem 1

If now j denotes the development of the Jacobi field J^v, then the Jacobi equation implies that j is a solution of the linear differential equation

$$j''(t) = \widetilde{R}_u(t)(j(t), u) + (\widetilde{T}_u)'(t)(j(t))$$

with the initial values $j(0) = 0$ and $j'(0) = v \in V$. Combining this with (4) we obtain

$$j(I_u) \subset V, \text{ hence } \forall t \in I_u : \; J^v(t) \in V_{tu}. \tag{5}$$

Now let us vary the vector v: Since U is a normal neighbourhood U, there are no conjugate points along $\gamma_u : I_u \to U$, combining (2) and (5) we therefore obtain

$$V_u = T_{p(u)}N \text{ for all } u \in V \cap U^T. \tag{6}$$

Proof of Theorem 1

We continue the argumentation with the Jacobi field J^v and apply formula (3) to the development z of the vector field

$$Z : t \mapsto \nabla_{J^v(t)} X^w,$$

where X^w denotes that radially parallel vector field with $X^w(p_0) = w$. According to Lemma 2 we get

$$z'(t) = \widetilde{R}_u(t)(j(t), w).$$

Since $z(0) = 0 \in V$ (because $J^v(0) = 0$), we obtain from (5) and (6)

$$z(I_u) \subset V,$$

hence in particular

$$\nabla_{J^v(1)} X^w \in V_u = T_{p(u)} N.$$

Proof of Theorem 1

Repeating the argument of the absense of conjugate points, we find that for every vector field X on M tangent to N the vector field $\nabla_X X^w$ is tangent to N.

If now $(w_1, \ldots, w_r)$ is a basis of V, then $(X^{W_1}|N, \ldots, (X^{W_r}|N)$ is a frame field of the tangent bundle TN, and therefore, we finally get for all vector fields X, Y on M which tangent to N the vector field $\nabla_X Y$ is tangent to N.

Notation

From Theorem 1 we derive a criterion on the existence of a local affine map for which at one point the differential is prescribed by a linear map $A : T_{p_1}M^1 \to T_{p_2}M^2$.

• For every $u \in U_1^T$ we define the linear map

$$A_u := \tau^2_{Au} \circ A \circ \tau^1_u : T_{\gamma^1_u(1)}M^1 \to T_{\gamma^2_{Au}(1)}M^2.$$

• If there exists an affine map $F : U_1 \to M^2$ with $T_{p_1}F = A$, then we have $\gamma^2_{Au} = F \circ \gamma_u$ for every $u \in U_1$, hence $A_u = T_{\gamma^1_u(1)}f$; thus in this case the maps A_u preserve the torsion and curvature tensor fields.

Theorem 2

Theorem (2)

If for every $u \in U_1^T$ the linear map A_u preserves the torsion and curavature tensor fields, then

$$F := \exp^2_{p_2} \circ A \circ (\exp^1_{p_1})^{-1} : U_1 \to M^2$$

is an affine map satisfying $F(p_1) = p_2$ and $T_{p_1} F = A$.

Proof of Theorem 2

Proof. Of course, F is a smooth map satisfying $F(p_1) = p_2$ and $T_{p_1}F = A$; its graph is the geodesic umbrella

$$N := \exp^{\times}_{(p_1,p_2)}(V(A) \cap (U_1^T \times U_2^T))$$

in the affine product $M^\times = M^1 \times M^2$.

According to the above Proposition (2) it remains to prove the affinity of N in $M^\times$.

Starting from the linear subspace $V(A) = T_(p_1, p_2)N \subset T_(p_1, p_2)M^\times$ we define the linear subspaces $V(A)_{(u,Au)}$ as above.

It is easy to see that $V(A)_{(u,Au)}$ coincides with $V(A_u)$.

As every map A_u preserves the torsion and curvature tensor fields, we know from Proposition (1) that the subspaces $V(A)_{(u,Au)}$ are torsion and curvature invariant.

Therefore, Theorem 1 implies the affinity of the submanifold N.

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