Mesh Parameterization and Applications

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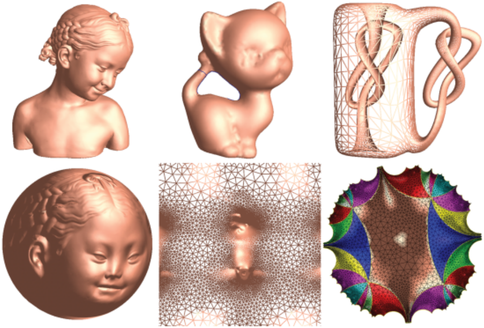
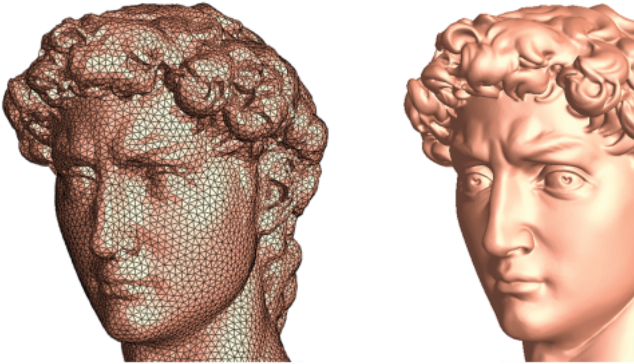
Motivation

Uniformization Theorem (Poincaré- Köbe). Every simply connected Riemann surface is conformally diffeomorphic to the 2-sphere $\mathbb{S}^2$, the plane $\mathbb{R}^2$ or the Poincaré disc $\mathbb{H}^2$



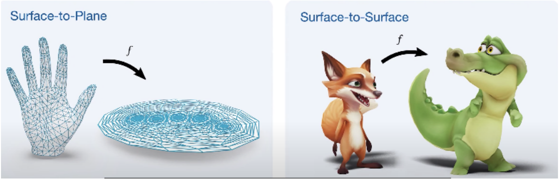
Motivation

Pb : What is the discrete counterpart of the uniformization theorem ?

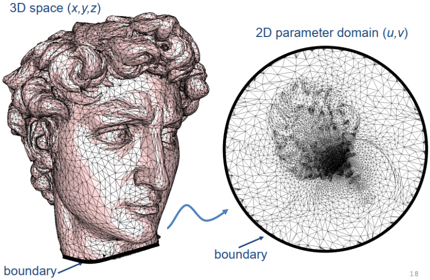


Motivation

Pb : For two surfaces with similar topology, is there a bijective mapping between them ?

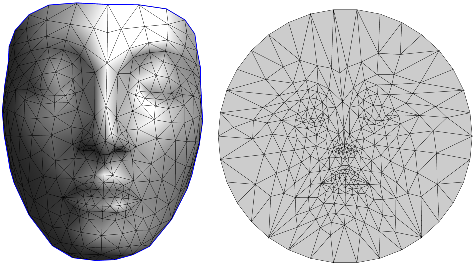


Pb : For two surfaces with similar topology, is there a bijective mapping between them
That maps triangle triangle to triangle and boundary to boundary ?

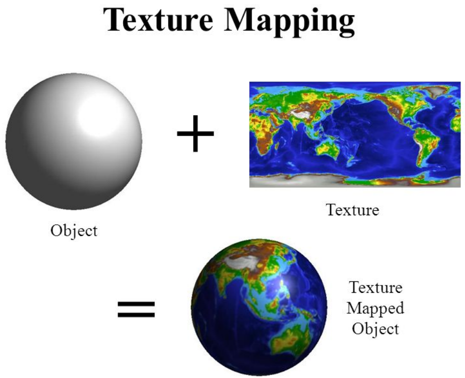


Parameterization Problem

Given a 3D surface-mesh S and a domain Ω (plane, spherical or hyperbolic) : Find a **bijective map** $\Phi : S \longrightarrow \Omega$ that minimize the distortion.



Texture



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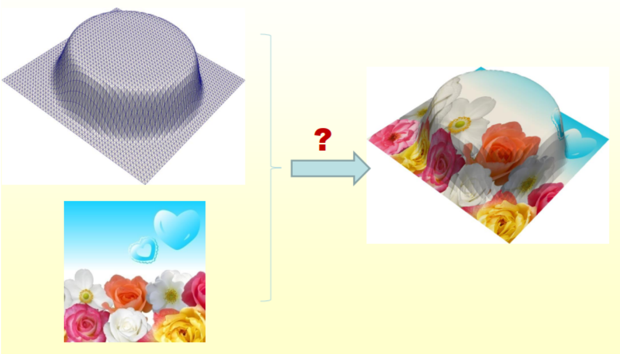
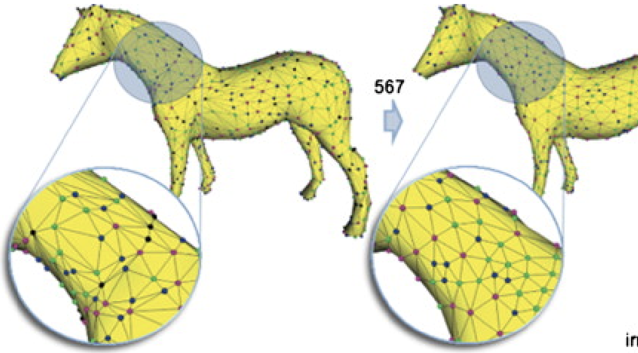
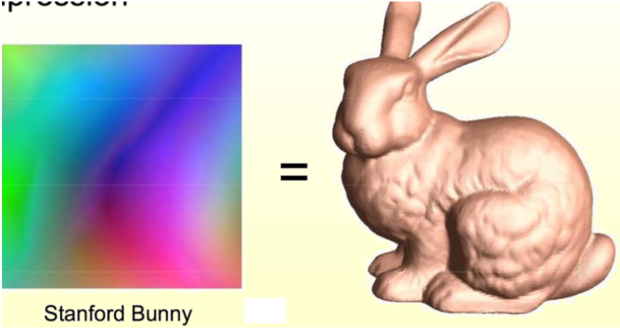


Remeshing

Remesh surfaces, or replace one triangulation by another, to obtain a parameterization that minimize the distortion.



Mesh Compression



Morphing

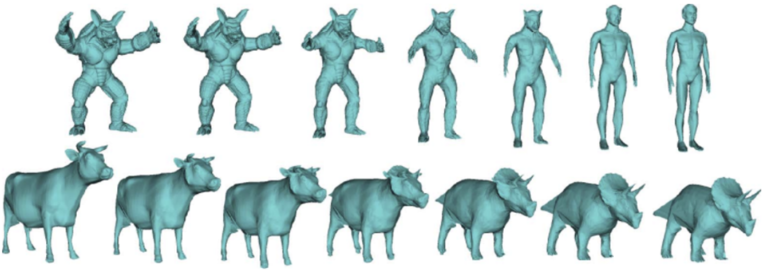


Fig. 6. 3D mesh morphing examples

Smooth setting

Let $f : \Omega \to \mathbb{R}^3$ be a parameterization of $S = f(\Omega)$ over the parameter domain $\Omega \subset \mathbb{R}^2$.
the first fundamental form

$$\mathbf{I}_f = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$\mathbf{I}_f = J_f^T J_f = \begin{pmatrix} f_u^T \\ f_v^T \end{pmatrix} (f_u f_v) = \mathbf{I}_f$$

where $J_f = (f_u f_v)$ is the Jacobian of f, i.e. the 3×2 matrix with the partial derivatives of f as column vectors.

Smooth setting

The two eigenvalues λ_1 and λ_2 of $\mathbf{I}_f$:

$$\lambda_{1,2} = \frac{1}{2}\left((E+G) \pm \sqrt{4F^2 + (E-G)^2}\right)$$

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f is equiareal equiareal or area-preserving $\Longleftrightarrow \quad \lambda_1 \lambda_2 = 1.$

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isometric $\iff$ conformal + equiareal.

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Cylinder

Parameter domain : $\Omega = \{(u, v) \in \mathbb{R}^2 : u \in [0, 2\pi), v \in [0, 1]\}$
Surface : $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0, 1]\}$

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Parameterization : $f(u,v) = (\cos u, \sin u, v)$

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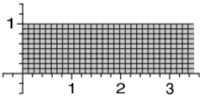
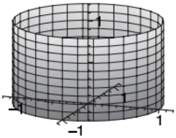
Parameterization : $f(u,v) = (\cos u, \sin u, v)$
Inverse : $f^{-1}(x,y,z) = (\arccos x, z)$.

$$\mathbf{I}_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

eigenvalues : $\lambda_1 = 1, \quad \lambda_2 = 1$.
Then f is an isometry.





hemisphere (stereographic)

Parameter domain : $\Omega = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$
Surface : hemisphere.

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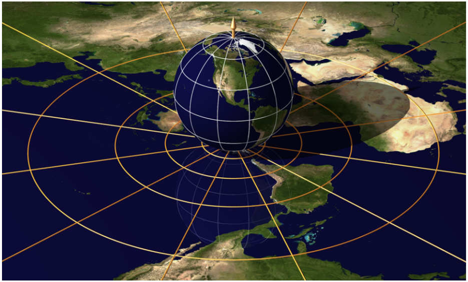
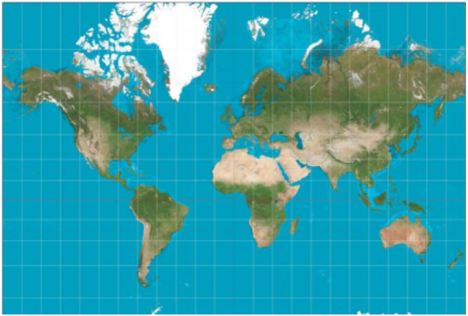
Surface : hemisphere.
Parameterization : $f(u, v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$
eigenvalues : $\lambda_1 = 4d^2$, $\lambda_2 = 4d^2$, with $d = \frac{1}{1+u^2+v^2}$.
This mapping is always conformal but not equiareal.



Mercator



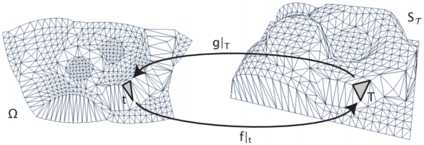
Maps loxodromes to lines. And it is conformal.

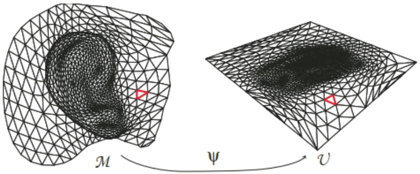
Theorem [1827]

Gauss [1827] : A globally isometric parameterization exists only for developable surfaces like planes, cones, and cylinders with vanishing Gaussian curvature.

Parameterization by Affine Combinations

$S_{\mathcal{T}}$: a 3D-triangle mesh surface. Ω a 2D-triangle mesh surface. $f : \Omega \longrightarrow S_{\mathcal{T}}$ and $g := f^{-1} : S_{\mathcal{T}} \longrightarrow \Omega$ is defined vertex by vertex and is continious and linear on each triangle T. g is called a parameterization of the mesh-surface $S_{\mathcal{T}}$.





Parameterization by Affine Combinations

We first specify the parameter points :
$\mathbf{u}_i = (u_i, v_i), i = n+1, ..., n+b$ for the boundary vertices $\mathbf{p}_i \in B = \partial\Omega$. Then we minimize the overall spring energy

$$E = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\frac{1}{2}D_{ij}\left\|\boldsymbol{u}_i - \boldsymbol{u}_j\right\|^2$$

where $D_{ij} = D_{ji}$ is the spring constant of the spring between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$, with respect to the unknown parameter positions $\boldsymbol{u}_i = (u_i, v_i)$ for the interior points.

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$$\frac{\partial E}{\partial \boldsymbol{u}_i} = \sum_{j \in N_i} D_{ij} \left(\boldsymbol{u}_i - \boldsymbol{u}_j \right)$$

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$$\frac{\partial E}{\partial \boldsymbol{u}_i} = \sum_{j\in N_i} D_{ij}\left(\boldsymbol{u}_i - \boldsymbol{u}_j\right)$$

the minimum of E is obtained if

$$\sum_{j\in N_i} D_{ij}\boldsymbol{u}_i = \sum_{j\in N_i} D_{ij}\boldsymbol{u}_j$$

Parameterization by Affine Combinations

This is equivalent to saying that each interior parameter point $\boldsymbol{u}_i$ is an affine combination of its neighbours,

$$\boldsymbol{u}_i = \sum_{j \in N_i} \lambda_{ij} \boldsymbol{u}_j$$

with normalized coefficients :

$$\lambda_{ij} = D_{ij} / \sum_{k \in N_i} D_{ik}$$

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$$\iff \quad \boldsymbol{u}_i - \sum_{j \in N_i, j \leq n} \lambda_{ij} \boldsymbol{u}_j = \sum_{j \in N_i, j > n} \lambda_{ij} \boldsymbol{u}_j$$

Parameterization by Affine Combinations

To find the solution in practice :

1. Fix the boundary points $\mathbf{b}_i \in B$

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$$\begin{aligned} &\mathbf{u}_i = \mathbf{b}_i, && \text{if } i \in B \\ &\boldsymbol{u}_i - \sum_{j \in N_i, j \leq n} \lambda_{ij} . \boldsymbol{u}_j = \sum_{j \in N_i, j > n} \lambda_{ij} \boldsymbol{u}_j, && \text{if } i \notin B \end{aligned}$$

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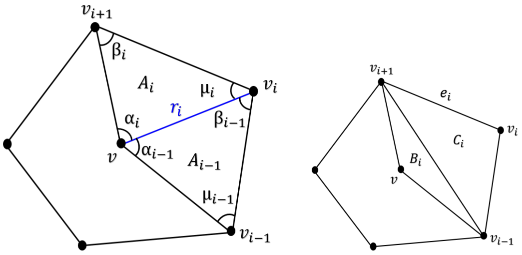
3. Assemble into two linear systems (one for each coordinate) :

$$LU = \bar{U}, \quad LV = \bar{V} \quad L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\lambda_{ij} & \text{if } j \in \mathcal{N}_i, i \notin B \\ 0 & \text{otherwise} \end{cases}$$

Planar Barycentric Coordinates

Let $P = [v_1, \ldots, v_n] \subset \mathbb{R}^2$ be a convex polygon.
Any functions $\lambda_i : P \longrightarrow \mathbb{R}$, $i = 1, \ldots, n$, will be called Generalized Barycentric coordinates (GBC) if, for all v within P :

$$\sum_{i=1}^{n} \lambda_i(v) = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i(v) v_i = v.$$

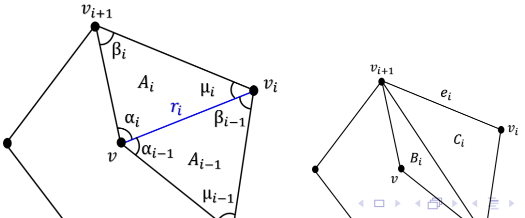


Wachspress Coordinates

Wachspress Coordinates (WC) :

$$\lambda_i = \frac{\omega_i}{\sum_{j=1}^{n} \omega_j}, \quad \text{where} \quad \omega_i = \frac{A_i}{C_{i-1} C_i},$$ and A_i, C_i are respectively the areas of the triangles $[v, v_i, v_{i+1}]$, $[v_{i-1}, v_i, v_{i+1}]$. Or

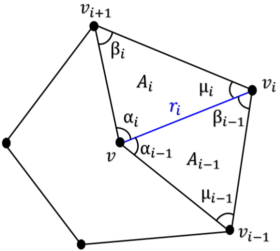
$$\lambda_i = \frac{\omega_i}{\sum_{j=1}^{n} \omega_j}, \quad \text{where} \quad \omega_i = \frac{\cot(\beta_{i-1}) + \cot(\mu_i)}{r_i^2},$$



Mean value coordinates

Mean Value Coordinates (MVC) :

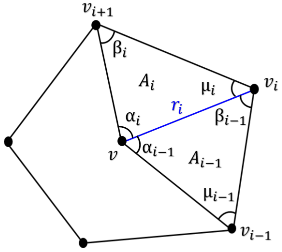
$$\lambda_i = \frac{\omega_i}{\sum_{j=1}^{n} \omega_j}, \qquad \omega_i = \frac{\tan(\frac{\alpha_{i-1}}{2}) + \tan(\frac{\alpha_i}{2})}{r_i},$$



Discrete harmonic coordinates

Harmonic Coordinates (HC) on convex polygons :

$$\lambda_i = \frac{\omega_i}{\sum_{j=1}^{n} \omega_j}, \qquad \omega_i = \cot(\mu_{i-1}) + \cot(\beta_i),$$



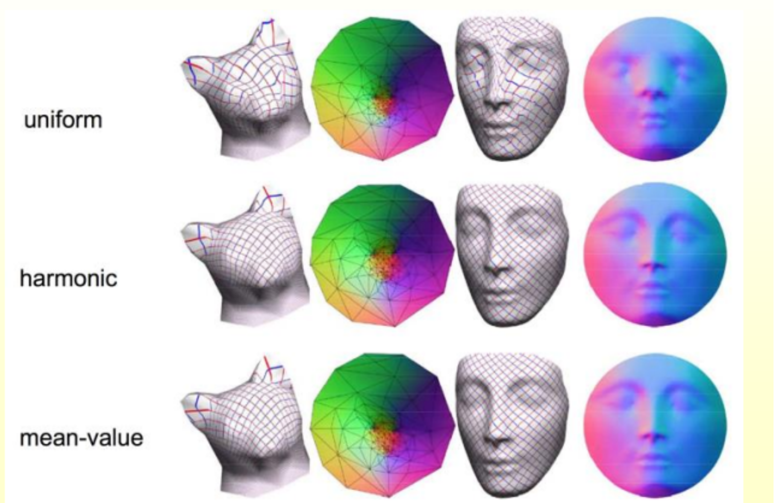
Tutte-Floater Theorem

If we take in the equation $AU = \bar{U}$, $AV = \bar{V}$, where $A = (a_{ij})$

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\lambda_{ij} & \text{if } j \in N_i \\ 0 & \text{otherwise} \end{cases}$$

and λ_{ij} are ones of the BC of a point v_i in the 3D mesh we found an approximating conformal mapping that is bijective.
Remark : For Harmonic coordinates, we impose a mesh that verifies
$\mu_{i-1} + \beta_i < \pi$. i.e. Delaunay Triangulation.

Practice : Tutte Embedding



Parameterization by minimizing the conformal energy

Let $\mathbf{M} = (V, E, T)$ be an oriented 3 -connected disk-type triangular surface-mesh. $V = (v_i)$ set of vertices. ω_{ij} a barycentric coordinates associated to v_i.
Discrete Dirichlet energy associated to $\mathbf{M}$ is

$$E(\Phi) := \frac{1}{2} \sum_{e_{ij} \in E} \omega_{ij} \|\Phi_i - \Phi_j\|^2,$$

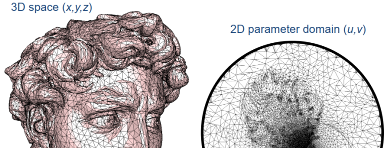
where $\Phi_j = \Phi(v_j)$.

Parameterization by minimizing the conformal energy

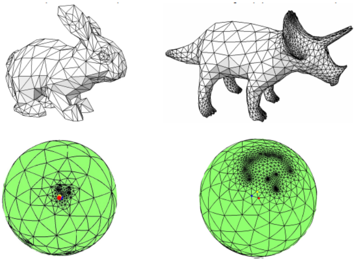
Planar Dirichlet Problem (DP) :

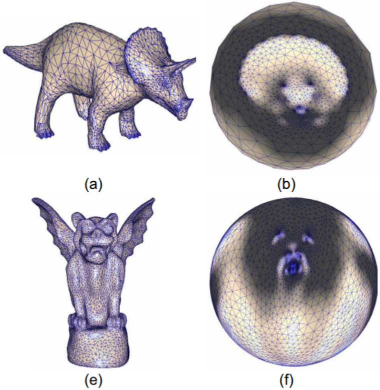
$$\begin{aligned} \min \quad & E(\Phi) \\ s.t. \quad & \Phi_i = p_i \quad \forall v_i \in \partial \mathbf{M} \end{aligned}$$

where $p_i \in B$ are vertices of a convex polygon B on the plane, and the assignment $\Phi_i \to p_i$ defines a homeomorphic boundary map $\partial \mathbf{M} \to \partial B$.
(i) there exists a critical points of DP contained in B ; and
(ii) every critical points of (DP) contained in B defines a bijection between $\mathbf{M}$ and B.



Spherical Embedding





Hyperbolic Embedding