THE AFFINE GROUP OF A SMOOTH MANIFOLD

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1. Introduction

The goal of these notes is to explore various results concerning the group of affine transformations of a smooth manifold endowed with a linear connection. Many of those results involve the isometry group of a Riemannian manifold and its relation to the group of affine transformations for the Levi-Civita connection.

Let M be an n-dimensional smooth manifold. For any $x \in M$, we call a frame on M at x any linear isomorphism $\mathbb{R}^n \xrightarrow{\simeq} T_xM$, the set of such frames will be denoted $\mathrm{L}(M)_x$. Clearly, the general linear group $\mathrm{GL}(n,\mathbb{R})$ acts naturally on $\mathrm{L}(M)_x$ via the map:

$$L(M)_x \times \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathrm{L}(M)_x, \quad (z,g) \mapsto z \circ g,$$

and it is not hard to see that this action is simply transitive. Now define

$$\mathrm{L}(M) := \amalg_{x\in M} \mathrm{L}(M)_x$$

and consider the projection $\pi : \mathrm{L}(M) \longrightarrow M$ given by $\pi(\mathrm{L}(M)_x) := x$.

Proposition 1.0.1. *Let M be an n-dimensional manifold. Then $\mathrm{L}(M) \xrightarrow{\pi} M$ is a smooth principal $\mathrm{GL}(n,\mathbb{R})$-bundle over M called the frame bundle of TM such that for any local frame $\{E_1,\dots,E_n\}$ of TM defined on an open subset $U \subset M$,the map:*

$$\sigma : U \longrightarrow \mathrm{L}(M), \quad x \mapsto \{E_{1|x},\dots,E_{n|x}\}, \tag{1}$$

is a local (smooth) section of $\mathrm{L}(M) \xrightarrow{\pi} M$. Conversely, if $\sigma : U \longrightarrow \mathrm{L}(M)$ is any smooth section, then there exists a local frame $\{E_1,\dots,E_n\}$ of TM over U such that σ is of the form (1).

In a similar way one defines on a Riemannian manifold $(M,\langle\ ,\ \rangle)$ the bundle of orthonormal frames $\mathrm{O}(M) := \amalg_{x\in M}\mathrm{O}(M)_x$ where each $\mathrm{O}(M)_x$ consists of linear isometries $(\mathbb{R}^n,\langle\ ,\ \rangle_0) \xrightarrow{\simeq} (T_xM,\langle\ ,\ \rangle_x)$. It is clear that $\mathrm{O}(M) \subset \mathrm{L}(M)$, on the other hand the orthogonal group $\mathrm{O}(n)$ acts simply transitively on $\mathrm{O}(M)$.

Proposition 1.0.2. *Let $(M,\langle\ ,\ \rangle)$ be an n-dimensional Riemannian manifold. Then $\mathrm{O}(M) \xrightarrow{\pi} M$ is a smooth principal $\mathrm{O}(n)$-subbundle of $\mathrm{L}(M)$. If $\{E_1,\dots,E_n\}$ is any local, orthonormal frame of TM defined on an open subset $U \subset M$ then the map:*

$$\sigma : U \longrightarrow \mathrm{L}(M), \quad x \mapsto \{E_{1|x},\dots,E_{n|x}\}, \tag{2}$$

is a local (smooth) section of $\mathrm{O}(M) \xrightarrow{\pi} M$. Conversely, if $\sigma : U \longrightarrow \mathrm{O}(M)$ is any smooth section, then there exists a local orthonormal frame $\{E_1,\dots,E_n\}$ of TM over U such that σ is of the form (2).

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Any diffeomorphism $f : M \longrightarrow M$ induces a principal bundle automorphism $f_* : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathrm{L}(M) \arrow[r, "f_*"] \arrow[d, "\pi"'] & \mathrm{L}(M) \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

Explicitly, for any $z \in \mathrm{L}(M)$ we have $f_*(z) := T_{\pi(z)}f \circ z$. Define on $L(M)$ the $\mathbb{R}^n$-valued 1-form $\theta \in \Omega^1(L(M), \mathbb{R}^n)$ given by $\theta_z(v) := z^{-1}(T_z\pi(v))$, we call it the *canonical form* of $\mathrm{L}(M)$. We have the following result:

Proposition 1.0.3. *Let M be a smooth manifold and let θ denote the canonical form of the frame bundle $\mathrm{L}(M)$. If $f : M \longrightarrow M$ is any diffeomorphism of M then f_* preserves θ. Conversely, if $A : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ is any fiber-preserving transformation leaving θ invariant, then $A = f_*$ for some $f \in \mathrm{Diff}(M)$.*

Proof. Let $z \in \mathrm{L}(M)$ and $v \in T_z\mathrm{L}(M)$, then:

$$\begin{aligned} (f^*\theta)_z(v) &= \theta_{f_*(z)}(T_z f_*(v)) \\ &= (f_*(z))^{-1}(T_{f_*(z)}\pi \circ T_z f_*(v)) \\ &= (f_*z)^{-1}(T_z(\pi \circ f_*)(v)) \\ &= (f_*z)^{-1}(T_z(f \circ \pi)(v)) \\ &= (f_*z)^{-1} \circ T_{\pi(z)}f \circ T_z\pi(v) \\ &= z^{-1} \circ T_z\pi(v) \\ &= \theta_z(v). \end{aligned}$$

Conversely, we first notice that since $A : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ is fiber-preserving, the map:

$$f : M \longrightarrow M, \quad f(x) = \pi(A(z)), \quad z \in \pi^{-1}(x),$$

is a well-defined diffeomorphism of M. Now:

$$(A^*\theta)_z(v) = \theta_{A(z)}(T_z A(v)) = A(z)^{-1} \circ T_{\pi(z)}f \circ T_z\pi(v),$$

so A will preserve θ if and only if $A(z)^{-1} \circ T_{\pi(z)}f = z^{-1}$ for any $z \in P$ which is exactly what $A = f_*$ means. ∎

Proposition 1.0.3 states that the morphism $\mathrm{Diff}(M) \overset{\Psi}{\to} \mathrm{Aut}(\mathrm{L}(M))$, $f \mapsto f_*$ sends the group of diffeomorphisms of M isomorphically onto the subgroup of automorphisms of $\mathrm{L}(M)$ preserving the canonical form θ.

Definition 1.0.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. A connection form on a principal G-bundle $P \overset{\pi}{\longrightarrow} M$ is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying:*

1- *For any $z \in P$ and any $A \in \mathfrak{g}$, $\omega_z(\widetilde{A}_z) = A$ where $\widetilde{A}$ is the fundamental vector field on P corresponding to A, i.e*

$$\widetilde{A}_z := \frac{d}{dt}_{t=0} z \cdot \exp(-tA).$$

2- *For any* $g \in G$ *and any* $z \in P$, $v \in T_z(p)$,

$$(R_g^* \omega)_z(v) = \mathrm{Ad}_{g^{-1}}(\omega_z(v)),$$

with $R_g : P \longrightarrow P$ *being the map* $z \mapsto z \cdot g$.

Let now ∇ be a linear connection on M. A diffeomorphism $f : M \longrightarrow M$ is called an *affine transformation* with respect to ∇ if it satisfies $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$ for any $X, Y \in \mathfrak{X}(M)$ where $f_* X$ is the vector field on M given by:

$$(f_* X)_{f^{-1}(x)} := (T_x f)^{-1}(X_x).$$

The group of such transformations will be denoted $\mathrm{Aff}(M, \nabla)$. On the other hand, we say that $X \in \mathfrak{X}(M)$ is an *affine vector field* if it generates a local 1-parameter group of affine transformations. We are going to prove in this paragraph that $\mathrm{Aff}(M, \nabla)$ is a Lie group of dimension $\leq n^2 + n$ when given the compact-open topology. The idea is to reinterpret the group of affine transformations as a subgroup of $\mathrm{Aut}(\mathrm{L}(M))$ via the identification provided by Proposition 1.0.3, but in order to do this we need an artifact that represents the linear connection ∇ on the frame bundle $\mathrm{L}(M)$. It turns out that the notion of connection form is the adequate solution for this task, more precisely we have the following:

Proposition 1.0.4. *Let* M *be a smooth manifold and* ∇ *a linear connection on* M. *Then there exists a unique connection form* ω *on* $\mathrm{L}(M)$ *such that for any local section* $\sigma := \{E_1, \dots, E_n\}$ *of* $\mathrm{L}(M)$ *defined on* U, $\sigma^* \omega = \Gamma$ *where* $\Gamma \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$ *is given by:*

$$\nabla E_i = \sum_{j=1}^{n} \Gamma_{ij} E_j.$$

Conversely, any connection form ω *on* $\mathrm{L}(M)$ *gives rise to a linear connection* ∇ *on* M *by means of the previous expression.*

Proposition 1.0.5. *Let* M *be a smooth manifold,* ∇ *a linear connection on* M *and* ω *the connection form on* $\mathrm{L}(M)$ *corresponding to* ∇. *Let* $f : M \longrightarrow M$ *be a diffeomorphism, then:*

1- $f \in \mathrm{Aff}(M, \nabla)$ *if and only if* f_* *preserves the connection form* ω.
2- *Conversely, any fiber-preserving tranformation* $A : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ *leaving both* θ *and* ω *invariant is of the form* $A = f_*$ *for some* $f \in \mathrm{Aff}(M, \nabla)$.

Proof. Define on M the linear connection $\widetilde{\nabla}$ by the expression:

$$\widetilde{\nabla}_X Y = (f_*)^{-1}(\nabla_{f_* X} f_* Y)$$

Fix an open subset $U \subset M$ and let $\{E_1, \dots, E_n\}$ be a local frame of TM defined on U which correponds to a local section $\sigma : U \longrightarrow \mathrm{L}(M)$. Clearly $\{f_* E_1, \dots, f_* E_n\}$ defines a local frame of TM over $V := f^{-1}(U)$ and the corresponding local section is exactly $f_*^{-1} \circ \sigma \circ f : V \longrightarrow \mathrm{L}(M)$. For convenience put $g := f^{-1}$ and write:

$$\widetilde{\nabla} E_i = \sum_{j=1}^{n} \widetilde{\Gamma}_{ij} E_j, \quad \nabla(f_* E_i) = \sum_{j=1}^{n} \Gamma_{ij} f_* E_j,$$

where $\widetilde{\Gamma} := (\widetilde{\Gamma}_{ij})_{i,j} \in \Omega^1(U, \mathfrak{gl}(n,\mathbb{R}))$ and $\Gamma := (\Gamma_{ij})_{i,j} \in \Omega^1(V, \mathfrak{gl}(n,\mathbb{R}))$. On the other hand, a direct computation shows that for any $x \in U$:

$$\begin{aligned}
(\widetilde{\nabla}_{E_k} E_i)_x = g_*(\nabla_{f_*E_k} f_*E_i)_x &= T_{g(x)} f((\nabla_{f_*E_k} f_*E_i)_{g(x)}) \\
&= \sum_{j=1}^{n} \Gamma_{ij}((f_*E_k)_{g(x)}) E_{i|x} \\
&= \sum_{j=1}^{n} (g^*\Gamma_{ij})_x (E_{k|x}) E_{i|x}.
\end{aligned}$$

Therefore $g^*\Gamma = \widetilde{\Gamma}$ and thus $\sigma^*(g^*\omega) = \widetilde{\Gamma}$. In summary we proved that $g^*\omega$ is the (unique) connection form on $\mathrm{L}(M)$ defining $\widetilde{\nabla}$. So $f : M \longrightarrow M$ is an affine transformation i.e $\widetilde{\nabla} = \nabla$ if and only if $\omega = f^*\omega$. ∎

Theorem 1.0.1. *Let M be an n-dimensional smooth manifold with a global trivialization $\{X_1, \dots, X_n\}$ of TM. Denote G the group of transformations preserving this trivialization, i.e diffeomorphisms $f : M \longrightarrow M$ satisfying $T_x f(X_{i|x}) = X_{i|f(x)}$. Then G possesses a unique Lie group structure for the compact-open topology such that $\dim G \leq \dim M$. More precisely for any $p \in M$, the map:*

$$G \longrightarrow M, \quad f \mapsto f(p),$$

is an imbedding of G onto a closed submanifold of M, and the submanifold structure on the image is what makes G a Lie transformation group. Moreover the Lie algebra of G consists of complete vector fields whose 1-parameter subgroups are in G.

Theorem 1.0.2. *Let M be a smooth n-dimensional manifold and ∇ an affine connection on M, then $\mathrm{Aff}(M, \nabla)$ is a Lie group for the compact-topology of dimension $\leq n^2 + n$. More precisely for any $z \in \mathrm{L}(M)$ the map:*

$$\mathrm{Aff}(M, \nabla) \longrightarrow \mathrm{L}(M), \quad f \mapsto f_*(z),$$

is injective and its image is a closed submanifold of $\mathrm{L}(M)$. The submanifold structure on its image makes $\mathrm{Aff}(M, \nabla)$ a Lie transformation group. Its Lie algebra consists of complete affine vector fields on M.

Proof. First some preparation. For any $x \in M$, Recall that since $\mathrm{GL}(n,\mathbb{R})$ acts simply transitively on $P_x := \pi^{-1}(x)$, the map $\mathrm{GL}(n,\mathbb{R}) \longrightarrow P_x$, $g \mapsto z \cdot g$ and its differential $\mathfrak{gl}(n,\mathbb{R}) \longrightarrow T_z P_x$, $A \mapsto \frac{d}{dt}_{t=0} z \cdot \exp(tA)$ is therefore an isomorphism. In other words for any $v \in T_z P_x$ there exists a unique $A \in \mathfrak{gl}(n,\mathbb{R})$ such that $v = \widetilde{A}_z$. On the other hand it is clear that $T_z\pi(T_z P_x) = 0$ and since $T_z\pi : T_z P \longrightarrow T_x M$ is surjective we get that $\ker(T_z\pi) = T_z P_x$.

Denote ω the connection form on $P := \mathrm{L}(M)$ corresponding to ∇ and consider the map $TP \longrightarrow P \times (\mathfrak{gl}(n,\mathbb{R}) \times \mathbb{R}^n)$, $\quad Z \mapsto (p(Z), \omega(Z), \theta(Z))$, then one easily checks that the following diagram is commutative:

$$\begin{tikzcd}
TP \arrow[r, "\phi"] \arrow[d, "p"'] & P \times (\mathfrak{gl}(n,\mathbb{R}) \times \mathbb{R}^n) \arrow[dl, "\mathrm{pr}_1"] \\
P &
\end{tikzcd}$$

i.e the previous map is a vector bundle homomorphism. In fact, this map defines a global trivialization of TP, to see this we first notice that this map is surjective given that $\pi : P \longrightarrow M$ is a submersion so any $v \in \mathbb{R}^n$ is of the form $v = \theta_z(w)$ and that $A = \omega(\widetilde{A})$ for any $A \in \mathfrak{gl}(n, \mathbb{R})$. Let $v \in T_zP$ such that $\theta_z(v) = 0$ and $\omega_z(v) = 0$, then:

$$0 = \theta_z(v) = z^{-1}(T_z\pi(v)),$$

hence $v \in \ker(T_z\pi) := T_zP_{\pi(z)}$, on the other hand write $v = \widetilde{A}_z$ for some A in $\mathfrak{gl}(n, \mathbb{R})$ then we get that

$$0 = \omega_z(v) = \omega_z(\widetilde{A}_z) = A,$$

and so $v = 0$, consequently ϕ is injective. On the other hand if $\sigma : U \longrightarrow P$ is any local section one can define the map:

$$P_{|U} \times (\mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n) \xrightarrow{\psi} P_{|U}, \quad \widetilde{A}_{\sigma(\pi(z))} + (T_{\pi(z)}\sigma)(\sigma_{\pi(z)}(v)),$$

it is clear that ψ is a smooth map and one checks without difficulty that $\phi \circ \psi = \mathrm{Id}$ and thus ϕ is a local diffeomorphism, hence we conclude that ϕ is a vector bundle isomorphism. Finally, let $F : P \longrightarrow P$ be any fiber preserving transformation leaving θ and ω invariant and choose $v \in \mathbb{R}^n$ and $A \in \mathfrak{gl}(n, \mathbb{R})$ such that $(z, A, v) = \phi(w)$. Then:

$$\phi \circ T_zF(w) = (F(z), (F^*\omega)_z(w), (F^*\theta)_z(w)) = (F(z), \omega_z(w), \theta_z(w)),$$

which means that $T_zF(\phi^{-1}(z, A, v)) = (F(z), A, v)$ and so F preserves any global frame of TP defined by ϕ. By Theorem 1.0.1 we get the desired result. ∎

The remaining part of this section will be dedicated to prove that the isometry group of a Riemannian manifold is also a Lie group by following the same footsteps of the previous paragraph.

Assume now that $(M, \langle\ ,\ \rangle)$ is a Riemannian manifold and let ∇ be a metric connection on M, i.e for any $X, Y, Z \in \mathfrak{X}(M)$, $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$. Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame of TM defined on an open subset $U \subset M$ and write

$$\nabla E_i = \sum_{j=1}^{n} \Gamma_{ij} E_j.$$

Then we get that $\Gamma_{ij} = -\Gamma_{ji}$ or in other terms $\Gamma \in \Omega^1(U, \mathfrak{so}(n))$. Thus if $\widetilde{\omega}$ is the connection form on $\mathrm{L}(M)$ corresponding to ∇ then its restriction ω to the orthogonal frame bundle $\mathrm{O}(M)$ is $\mathfrak{so}(n)$-valued and defines therefore a connection form on $\mathrm{O}(M)$ and it is in fact the only connection form on $\mathrm{O}(M)$ representing ∇. Conversely any connection form on $\mathrm{O}(M)$ admits a unique extension to $\mathrm{L}(M)$ and defines therefore a metric connection ∇ on M.

Denote $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ the isometry group of the Riemannian manifold $(M, \langle\ ,\ \rangle)$.

Proposition 1.0.6. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, ∇ its Levi-Civita connection and ω the connection form on $\mathrm{O}(M)$ representing ∇.*

1- *A diffeomorphism $f : M \longrightarrow M$ is an isometry if and only if $f_*(\mathrm{O}(M)) = \mathrm{O}(M)$.*

2- *If* $A : \mathrm{O}(M) \longrightarrow \mathrm{O}(M)$ *is a fiber-preserving transformation leaving invariant the canonical form* θ *of* $\mathrm{O}(M)$*, then there exists a unique isometry* $f : M \longrightarrow M$ *such that* $A = f_*$*.*
3- *Any (principal) bundle automorphism* $\mathrm{O}(M) \longrightarrow \mathrm{O}(M)$ *leaving* θ *invariant, leaves* ω *invariant.*

Proof. The first point is the definition of a Riemannian isometry, the argument for the second point is the same as in Proposition 1.0.5. For the third point, observe that since ∇ is torsion-free, then ω is torsion-free as well i.e the 2-form $T \in \Omega^2(\mathrm{O}(M), \mathbb{R}^n)$, called torsion form of ω, given by:

$$T := \omega \wedge \theta + d\theta, \tag{3}$$

vanishes. Let $A : \mathrm{O}(M) \longrightarrow \mathrm{O}(M)$ be a bundle automorphism, then $A^*\omega$ is a connection form on $\mathrm{O}(M)$. Since A preserves the canonical form θ, we get from expression (3) that $A^*\omega$ is torsion-free as well, so by uniqueness of the Levi-Civita connection we conclude that $A^*\omega = \omega$. ∎

Theorem 1.0.3. *Let* $(M, \langle\ ,\ \rangle)$ *be a Riemannian manifold, then* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *with the compact-open topology is a Lie group of dimension* $\leq \frac{n(n+1)}{2}$*. In fact for any* $z \in \mathrm{O}(M)$*, the map:*

$$\mathrm{Isom}(M, \langle\ ,\ \rangle) \longrightarrow \mathrm{O}(M), \quad f \mapsto f_*(z),$$

is an imbedding and its image is a closed submanifold of $\mathrm{O}(M)$*. If* ∇ *is the Levi-Civita connection of* $\langle\ ,\ \rangle$ *then* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *is a closed subgroup of* $\mathrm{Aff}(M, \nabla)$*. Its Lie algebra consists of complete Killing vector fields on* M*.*

Proposition 1.0.7. *Let* $(M, \langle\ ,\ \rangle)$ *be a Riemannian manifold, then the natural action of* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *on* M *is proper. In particular if* M *is compact then* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *is compact.*

Proof. Choose a compact $K \subset M$ and put $G_K = \{f \in \mathrm{Isom}(M),\ f(K) \cap K \neq \emptyset\}$. Let $(f_n)_n$ be an arbitrary sequence of G_K, then for any $n \in \mathbb{N}$ we can find $p_n \in K$ such that $f_n(p_n) \in K$. Since K is compact we can assume without loss of generality that $(f_n(p_n))_n$ is convergent and we denote $q \in K$ its limit, similarly there exists a subsequence $(p_{\varphi(n)})_n$ of $(p_n)_n$ converging to some $p \in K$ and therefore $(f_{\varphi(n)}(p_{\varphi(n)}))_n$ is also convergent.

Denote $d : M \times M \longrightarrow \mathbb{R}^+$ the geodesic distance, which is preserved by elements of $\mathrm{Isom}(M, \langle\ ,\ \rangle)$, hence:

$$\begin{aligned} d(f_{\varphi(n)}(p), q) &\leq d(f_{\varphi(n)}(p), f_{\varphi(n)}(p_{\varphi(n)})) + d(f_{\varphi(n)}(p_{\varphi(n)}), q) \\ &\leq d(p, p_{\varphi(n)}) + d(f_{\varphi(n)}(p_{\varphi(n)}), q) \underset{n \to +\infty}{\longrightarrow} 0. \end{aligned}$$

Next put $p = \pi(z)$ with $z \in \mathrm{O}(M)$ and $C = \{f_{\varphi(n)}(p),\ n \in \mathbb{N}\} \cup \{q\}$, then C is a compact subset of M and since $\mathrm{O}(n)$ is compact we get that $\pi^{-1}(C)$ is a compact subset of $\mathrm{O}(M)$ containing the sequence $((f_{\varphi(n)})_*(z))_n$ and so it has a convergent subsequence $((f_{\psi(n)})_*(z))_n$. Theorem 1.0.3 implies that $\{f_*(z),\ f \in \mathrm{Isom}(M)\}$ is closed submanifold in $\mathrm{O}(M)$ so $((f_{\psi(n)})_*(z))_n$ converges to $f_*(z)$ for some $f \in \mathrm{Isom}(M, \langle\ ,\ \rangle)$ and by the same result we get that $(f_{\psi(n)})_n$ converges to f in $\mathrm{Isom}(M, \langle\ ,\ \rangle)$. We conclude that G_K is compact and so $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ acts properly on M. ∎

2. Results on the dimension of the affine group of a manifold

In this section we state some results that illustrates how the dimension of the group of affine transformations affects the global structure of the manifold. The main paragraph of this section relies on the notion of parallel transport along curves and its how it is perceived from the perspective of the frame bundle, and this will be our starting point.

Let M be an n-dimensional smooth manifold and ∇ a linear connection on M. Let $\gamma : [0,1] \longrightarrow M$ be a smooth a curve, then ∇ provides a unique linear operator denoted $D_{\dot\gamma} : \Gamma(\gamma^{-1}TM) \longrightarrow \Gamma(\gamma^{-1}TM)$ satisfying:

$$D_{\dot\gamma}(fV) = f'V + fD_{\dot\gamma}V, \quad f \in \mathcal{C}^\infty([0,1],\mathbb{R}),\ V \in \Gamma(\gamma^{-1}TM),$$

we call it the *covariant derivation along* γ, here $\Gamma(\gamma^{-1}TM)$ is the space of vector fields along γ i.e smooth maps $V : [0,1] \longrightarrow TM$ such that $V(t) \in T_{\gamma(t)}M$. In a local frame $\{E_1,\dots,E_n\}$ where $\nabla E_i = \sum_{j=1}^n \Gamma_{ij}E_j$ we get that:

$$D_{\dot\gamma}V(t) = \sum_{k=1}^{n}\Big(f_k'(t) + \sum_{i,j=1}^{n} f_i(t)g_j(t)\Gamma_{ji}^k(\gamma(t))\Big)E_{k\,|\gamma(t)}, \tag{4}$$

where $V(t) = \sum f_i(t)E_{i\,|\gamma(t)}$ and $\dot\gamma(t) = \sum g_i(t)E_{i\,|\gamma(t)}$. We say that V *is parallel along* γ if $D_{\dot\gamma}V = 0$, using that equation (4) is a linear system of ordinary differential equations one gets the following important result:

Theorem 2.0.1. *Let M be an n-dimensional manifold with a linear connection ∇ and $\gamma : [0,1] \longrightarrow M$ a smooth curve. For any $v \in T_{\gamma(0)}M$, there exists a unique parallel vector field V along γ satisfying $V(0) = v$. We call V the parallel transport of v along γ.*

Let ω be the connection form on $\mathrm{L}(M)$ representing ∇, for any basis $\{e_1,\dots,e_n\}$ of $T_{\gamma(0)}M$ one obtain a *parallel frame* $\{E_1,\dots,E_n\}$ along γ i.e E_i is the parallel vector field along γ satisfying $E_i(0) = e_i$, and it is an easy matter to see that $\{E_1(t),\dots,E_n(t)\}$ is a basis of $T_{\gamma(t)}M$, thus one obtains a smooth curve $\widetilde{\gamma} : [0,1] \longrightarrow \mathrm{L}(M)$ given by:

$$\widetilde{\gamma}(t) = (E_1(t),\dots,E_n(t)),$$

such a curve is called *the horizontal lift* of γ to $\mathrm{L}(M)$ through $\{e_1,\dots,e_n\}$.

Proposition 2.0.1. *Let M be an n-dimensional manifold with a linear connection ∇ and let $\gamma : [0,1] \longrightarrow M$ and $\alpha : [0,1] \longrightarrow \mathrm{L}(M)$ be smooth curves. Then α is a horizontal lift of γ if and only if $\pi \circ \alpha = \gamma$ and $\omega_{\alpha(t)}(\dot\alpha(t)) = 0$, where ω is the connection form corresponding to ∇.*

Proof. Since $\pi(\alpha(s)) = \gamma(s)$ for any $0 \le s \le 1$, we can write $\alpha(s) = (V_1(s),\dots,V_n(s))$ where $V_i \in \Gamma(\gamma^{-1}TM)$. Next choose a local frame $\{E_1,\dots,E_n\}$ of M defined on an open subset $U \subset M$ and let $\epsilon > 0$ such that $\gamma(]t-\epsilon, t+\epsilon[) \subset U$ then write:

$$V_i(s) = \sum_{j=1}^{n} g_{ij}(s)E_{j\,|\gamma(s)}, \quad \nabla E_i = \sum_{j=1}^{n}\Gamma_{ij}E_j, \tag{5}$$

for $s \in]t-\epsilon, t+\epsilon[$. Then it is straightforward to prove that:

$$D_{\dot\gamma}V_i(t) = \Big(g'(t) + g(t)\cdot\Gamma_{\gamma(t)}(\dot\gamma(t))\Big)\cdot E_{j\,|\gamma(t)},$$

where $g := (g_{ij})_{i,j} \in \mathcal{C}^\infty([0,1], \mathrm{GL}(n,\mathbb{R}))$ and $\Gamma := (\Gamma_{ij})_{i,j}$. On the other hand if $\sigma : U \longrightarrow \mathrm{L}(M)$ is the local section corresponding to $\{E_1, \dots, E_n\}$ then (5) means exactly that $\alpha(s) = \sigma \circ \gamma(s) \cdot g(s)^{-1}$, for convenience denote $\hat{g}(s) := g(s)^{-1}$. Thus:

$$\begin{aligned}\dot{\alpha}(t) &= T_{(\sigma\circ\gamma)(t)} R_{g(t)}((\sigma\circ\gamma)'(t)) + \frac{d}{ds}_{|s=t} (\sigma\circ\gamma)(t)\cdot \hat{g}(s) \\ &= T_{(\sigma\circ\gamma)(t)} R_{g(t)}((\sigma\circ\gamma)'(t)) + \frac{d}{ds}_{|s=t} (\sigma\circ\gamma)(t)\cdot \hat{g}(t)\cdot \underbrace{(g(t)\hat{g}(s))}_{h(s)}.\end{aligned}$$

Now $s \mapsto h(s)$ is a curve in G satisfying $h(t) = e$ and therefore:

$$\frac{d}{ds}_{|s=t} (\sigma\circ\gamma)(t)\cdot \hat{g}(t)\cdot h(s) = -\widetilde{h'(t)}_{(\sigma\circ\gamma)(t)\cdot\hat{g}(t)},$$

where $\widetilde{h'(t)}$ is the fundamental vector field on $\mathrm{L}(M)$ corresponding to $-h'(t) = -g(t)\hat{g}'(t)$. Thus using that $R_g^*\omega = \mathrm{Ad}_{g^{-1}}(\omega)$ and $\omega(\widetilde{A}) = A$ we get from the previous computation:

$$\begin{aligned}\omega_{\alpha(t)}(\dot{\alpha}(t)) &= g(t)\cdot(\sigma^*\omega)_{\gamma(t)}(\dot{\gamma}(t))\cdot g(t)^{-1} - g(t)\hat{g}'(t) \\ &= g(t)\cdot \Gamma_{\gamma(t)}(\dot{\gamma}(t))\cdot g(t)^{-1} - g(t)\hat{g}'(t) \\ &= g(t)\cdot \Gamma_{\gamma(t)}(\dot{\gamma}(t))\cdot g(t)^{-1} - g'(t)\hat{g}(t)^{-1} \quad (g'(s)\hat{g}(s) = \mathrm{I}_n) \\ &= \Big(g(t)\cdot \Gamma_{\gamma(t)}(\dot{\gamma}(t)) - g'(t)\Big)\cdot \hat{g}(t)^{-1}\end{aligned}$$

So we conclude that $\omega_{\alpha(t)}(\dot{\alpha}(t)) = 0$ if and only if $D_{\dot{\gamma}} V_i(t) = 0$ for all $1 \leq i \leq n$. ∎

Recall that a vector field Z on $\mathrm{L}(M)$ is called *horizontal* if $\omega(Z) = 0$ and *standard* if $\theta(Z)$ is a constant function.

Proposition 2.0.2. *Let M be an n-dimensional manifold with a linear connection ∇ and ω the connection form of ∇ on $\mathrm{L}(M)$.*

1. *Let Z be a standard horizontal vector field on $\mathrm{L}(M)$. For any $z \in \mathrm{L}(M)$, the curve defined by $\gamma(t) := \pi(\varphi_t^Z(z))$ is a geodesic on M.*
2. *Conversely, given a geodesic $\gamma : [-a,a] \longrightarrow M$, there exists a local standard horizontal vector field Z on $\mathrm{L}(M)$ and $\epsilon > 0$ such that $\gamma(t) = \varphi_t^Z(z)$ for any $-\epsilon < t < \epsilon$.*

Proof. Let $z \in \mathrm{L}(M)$ then there exists $\epsilon > 0$ such that the curve $\alpha :]-\epsilon, \epsilon[\longrightarrow \mathrm{L}(M)$ given by $\alpha(t) = \varphi_t^Z(z)$ is well-defined and smooth. Define $\gamma(t) = \pi(\alpha(t))$, since $\omega_{\alpha(t)}(\alpha'(t)) = \omega_{\alpha(t)}(Z_{\alpha(t)}) = 0$ then Proposition 2.0.1 gives that α is a horizontal lift of γ on $\mathrm{L}(M)$, therefore if we write:

$$\alpha(t) = (V_1(t), \dots, V_n(t)), \quad V_i \in \Gamma(\gamma^{-1}TM),$$

we get that $\{V_1, \dots, V_n\}$ is a parallel frame along γ. On the other hand if we write $\theta_z(Z_z) = (a_1, \dots, a_n)$ then we get that:

$$\gamma'(t) = T_{\alpha(t)}\pi(Z_{\alpha(t)}) = \alpha(t)(\theta_{\alpha(t)}(Z_{\alpha(t)})) = \alpha(t)(\theta_z(Z_z)) = \sum_{i=1}^n a_i E_i(t),$$

which shows that γ' is parallel along γ i.e γ is a geodesic.
Conversely, let $\gamma : [-a,a] \longrightarrow M$ be a geodesic and let U be a normal neighborhood of $p := \gamma(0)$ in M, i.e there exists an open neighborhood V of 0 in T_pM such that $\exp_p : V \longrightarrow U$ is a diffeomorphism. For any $x \in U$, there exists a unique normal

geodesic $t \mapsto (\exp_p(t \exp_p^{-1}(x)))$ joining p to x, denote $\alpha_z : [0,1] \longrightarrow \mathrm{L}(M)$ its (unique) horizontal lift through $z \in \mathrm{L}(M)_p$ and define:

$$Z_{\alpha_z(1)} := \alpha_z'(1).$$

It is clear from this definition that $\omega_z(Z_z) = 0$ and by uniqueness of the horizontal lift, one proves that $\alpha_z'(t) = Z_{\alpha_z(t)}$. Furthermore be shown that Z defines a smooth vector field on a neighborhood of any $z \in \mathrm{L}(M)_{\gamma(0)}$ in which case $\alpha_z(t) = \varphi_t^Z(z)$. Since γ is a normal geodesic in a small neighborhood $]-\epsilon, \epsilon[$, we get the desired result. ∎

Denote $\mathfrak{aff}(M,\nabla) := \mathrm{Lie}(\mathrm{Aff}(M,\nabla))$, any $X \in \mathfrak{aff}(M,\nabla)$ defines a smooth vector field $\hat{X}$ of $\mathrm{L}(M)$ given by:

$$\hat{X}_z := \frac{d}{dt}_{t=0} \exp(tX)_*(z).$$

It is clear that $\hat{X}$ is a complete vector field on $\mathrm{L}(M)$.

Proposition 2.0.3. *Let M be an n-dimensional manifold, ∇ a linear connection on M, and let $X \in \mathfrak{aff}(M,\nabla)$. Suppose that $\omega_z(\hat{X}_z) = 0$ for some $z \in \mathrm{L}(M)$. Then the curve $\gamma : \mathbb{R} \longrightarrow M$, $\gamma(t) = \exp(tX) \cdot x$ with $x = \pi(z)$ is a geodesic and its horizontal lift at z is the curve $\hat{\gamma}(t) := \exp(tX)_*(z)$, $t \in \mathbb{R}$.*

Proof. Put $\hat{\gamma}(t) = \exp(tX)_*(z)$, then clearly $\pi(\hat{\gamma}(t)) = \gamma(t)$, moreover from the relation $\exp(tX)^*\omega = \omega$ we get that $\omega_{\exp(tX)_* z}(\hat{X}_{\exp(tX)_* z}) = \omega_z(Z_z) = 0$ which means that $\hat{\gamma}$ is the horizontal lift of γ through z, in particular if $z = (e_1, \ldots, e_n)$ then:

$$\hat{\gamma}(t) = (E_1(t), \ldots, E_n(t)),$$

E_i being the parallel transport of e_i along γ. Moreover $\exp(tX)^*\theta = \theta$ gives that $\theta_{\exp(tX)_* z}(\hat{X}_{\exp(tX)_* z}) = \theta_z(\hat{X}_z)$ so if we put $\theta_z(X_z) = (a_1, \ldots, a_n)$ we get that:

$$\gamma'(t) = T_{\hat{\gamma}(t)}\pi(\hat{X}_{\hat{\gamma}(t)}) = \hat{\gamma}(t)(\theta_{\hat{\gamma}(t)}(\hat{X}_{\hat{\gamma}(t)})) = \hat{\gamma}(t)(\theta_z(\hat{X}_z)) = \sum_{i=1}^n a_i E_i(t).$$

Hence γ' is parallel along γ, i.e γ is a geodesic. ∎

Theorem 2.0.2. *Let M be an n-dimensional manifold with a linear connection ∇. Then $\dim(\mathrm{Aff}(M,\nabla)) = n(n+1)$ if and only if M is an ordinary affine space with the natural flat affine connection.*

Proof. Denote $G := \mathrm{Aff}(M,\nabla)$ and for any $x \in M$ denote G_x the isotropy at x for the natural action of G on M. First note that the map:

$$G_x \longrightarrow \mathrm{GL}(T_x M), \quad f \mapsto T_x f \tag{6}$$

is an injective Lie group homomorphism. Assume now that $\dim G = n(n+1)$, then let $x \in M$ and $z \in \mathrm{L}(M)$ such that $\pi(z) = x$. Since the map:

$$G \xrightarrow{\Psi} \mathrm{L}(M), \quad f \mapsto f_*(z)$$

is an imbedding of G onto a closed submanifold of $\mathrm{L}(M)$ and $\dim \mathrm{L}(M) = n(n+1)$, then either $\Psi(G) = \mathrm{L}(M)$ or $\Psi(G)$ is a connected component of $\mathrm{L}(M)$ and in any case we get that $M = G \cdot x \simeq G/G_x$, therefore:

$$\dim G_x = \dim G - \dim M = n^2.$$

This gives in particular that $G_x^0 = \mathrm{GL}^+(T_xM)$ under the identification (6). Now let $t > 0$ and consider the transformation $A_t \in \mathrm{GL}^+(T_xM)$ given by $A_t(u) = tu$. From the previous remark there exists $f_t \in G_x^0$ such that $T_x f_t = A_t$, hence for any $u, v, w \in T_xM$ we get that:

$$A_t(R_x^\nabla(u,v)w) = R_x^\nabla(A_tu, A_tv)A_tw, \quad A_t(T_x^\nabla(u,v)) = T_x^\nabla(A_tu, A_tv),$$

therefore $R_x^\nabla(u,v)w = t^{-2}R_x^\nabla(u,v)w$ and $T_x^\nabla(u,v) = t^{-1}T_x^\nabla(u,v)$ for all $t > 0$, and so we conclude that $R^\nabla = 0$ and $T^\nabla = 0$.

On the other hand, let Z be a standard horizontal vector field on $\mathrm{L}(M)$ i.e $\omega(Z) = 0$ and $\theta(Z)$ is constant where ω is the connection form representing ∇ and θ is the canonical form of $\mathrm{L}(M)$. If $\mathfrak{g} := \mathrm{Lie}(G)$ then there exists a unique $X \in \mathfrak{g}$ such that $Z_z = \hat{X}_z$ where:

$$\hat{X}_{\widetilde{z}} := \frac{d}{dt}_{t=0} \exp(tX)_*\widetilde{z}, \quad \widetilde{z} \in \mathrm{L}(M).$$

From Proposition 2.0.3 we get that $\gamma(t) = \exp(tX) \cdot x$ is a geodesic with horizontal lift at z the curve $\hat{\gamma}(t) = \exp(tX)_*(z)$ defined for any $t \in \mathbb{R}$. Now $\gamma : \mathbb{R} \longrightarrow M$ is the geodesic with initial conditions $\gamma(0) = x$ and $\gamma'(0) = T_z\pi(Z_z)$ and therefore its horizontal lift at z is exactly

$$\alpha :]-\epsilon, \epsilon[\longrightarrow \mathrm{L}(M), \quad t \mapsto \varphi_t^Z(z),$$

which proves that α can be extended to all of $\mathbb{R}$. Since $z \in \mathrm{L}(M)$ was arbitrary we get that Z is complete, and since we know by Proposition 2.0.2 that geodesics of M are exactly the projections of integral curves of standard horizontal vector fields, we conclude that M is (geodesically) complete.

Consider now the universal cover $\widetilde{M}$ of (M, ∇) with its induced induced linear connection, then there exists an affine transformation $\widetilde{M} \xrightarrow{\simeq} \mathbb{R}^n$. Next $M = \widetilde{M}/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Aff}(\widetilde{M}, \nabla) \simeq \mathrm{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$ hence commuting with $\mathrm{Aff}(\widetilde{M}, \nabla)^0 \simeq \mathrm{GL}^+(n, \mathbb{R}) \rtimes \mathbb{R}^n$. But one can show with ease that only the trivial element commutes with connected component of $\mathrm{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$, hence Γ is trivial and M is itself simply connected i.e $\widetilde{M} = M$, this completes the proof. ∎

Theorem 2.0.3. *Let M be an n-dimensional manifold with an affine connection and assume that* $\dim \mathrm{Aff}(M, \nabla) > n^2$. *Then ∇ is torsion-free.*

This result is a consequence of the following algebraic Lemma:

Lemma 2.0.1. *Let V ba an n-dimensional vector space and $T : V \times V \longrightarrow V$ a non-trivial skew-symmetric bilinear map i.e $T \in V \otimes \Lambda^2 V^*$. Denote H the subgroup of linear transformation preserving T, then* $\dim H \leq n^2 - n$.

Proof of the Theorem. Denote $G := \mathrm{Aff}(M, \nabla)$ then let $x \in M$ and denote G_x the isotropy at x for the natural action of G on M, then from $G/G_x \simeq G \cdot x$ we get:

$$\dim(G_x) \geq \dim(G) - \dim(M) > n^2 - n \tag{7}$$

On the other hand denote T^∇ the torsion tensor of ∇, then for every $f \in G_x$ we get that:

$$T_x^\nabla(T_xf(u), T_xf(v)) = T_xf(T_x^\nabla(u,v)), \quad u, v \in T_xM.$$

Therefore the group $\{T_x f,\ f \in G_x\} \simeq G_x$ preserves T_x^{∇}, but according to the previous Lemma and (7) we conclude that $T_x^{\nabla} = 0$ for any $x \in M$, i.e ∇ is torsion-free. ∎

Another result in the same spirit is the following Theorem due to Egorov [3] and can be proved by essentially the same procedure:

Theorem 2.0.4. *Let M be an n-dimensional manifold and ∇ a linear connection on M such that $\dim \mathrm{Aff}(M, \nabla) > n^2$. Then ∇ has neither torsion nor curvature provided that $n \geq 4$.*

3. Relation between the isometry group and the affine group of a Riemannian manifold

In this section, we go through a number of results that have treated the relation between the group of affine transformations and the isometry group in the case of a Riemannian manifold. These results exploit the holonomy Riemannian manifolds along with the De Rham decomposition theorem in order to get to conclusions, therefore we begin by introducing the notion of holonomy.

Let M be an n-dimensional manifold and ∇ a linear connection on M. For any smooth curve $\gamma : [a, b] \longrightarrow M$ one can define a map $\tau^{\gamma}_{a,b} : T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M$ by the formula $\tau^{\gamma}_{a,b}(v) = V(b)$ where $V \in \Gamma(\gamma^{-1}TM)$ is the parallel transport of v along γ (relative to ∇). It is clear that this map is linear since V is a solution of a system of linear ordinary differential equations, furthermore:

Proposition 3.0.1.

(1) *$\tau^{\gamma}_{a,b}$ does not depend on the orientation-preserving parametrization of the curve γ.*
(2) *Denote $\gamma_1 := \gamma_{|[a,t_0]}$ and $\gamma_2 := \gamma_{|[t_0,b]}$ i.e $\gamma = \gamma_1 * \gamma_2$, then:*

$$\tau^{\gamma}_{a,b} = \tau^{\gamma_2}_{t_0,b} \circ \tau^{\gamma_1}_{a,t_0}.$$

(3) *For any smooth curve γ, $\tau^{\gamma}_{a,b}$ is an isomorphism and its inverse is exactly the linear operator $\tau^{\gamma^-}_{a,b} : T_{\gamma(b)}M \longrightarrow \tau_{\gamma(a)}M$ with $\gamma^-(t) = \gamma(a + b - t)$.*

Proof.
(1) & (3)- Choose a diffeomorphism $\varphi : [c, d] \longrightarrow [a, b]$ and put $\alpha = \gamma \circ \varphi$. Let V be the parallel transport of $v \in T_{\gamma(a)}M$ along γ and $W := V \circ \varphi$. Next, write:

$$V(\varphi(s)) = \sum_{i=1}^{n} g_i(\varphi(s)) E_{i|\alpha(s)}, \quad \dot{\gamma}(\varphi(s)) = \sum_{i=1}^{n} f_i(\varphi(s)) E_{i|\alpha(s)}$$

for any $s \in]t-\epsilon, t+\epsilon[$, where $\{E_1, \ldots, E_n\}$ is a local frame of M defined on an open subset U and $\alpha(]t-\epsilon, t+\epsilon[) \subset U$. Since $\dot{\alpha}(s) = \varphi'(s)\dot{\gamma}(\varphi(s))$, then:

$$\begin{aligned} D_{\dot{\alpha}}W(s) &= \sum_{i=1}^{n} \varphi'(s) g_i'(\varphi(s)) E_{i|\alpha(s)} + \sum_{i=1}^{n} g_i(\varphi(s))(\nabla_{\dot{\alpha}} E_i)_{|\alpha(s)} \\ &= \sum_{i=1}^{n} \varphi'(s) g_i'(\varphi(s)) E_{i|\alpha(s)} + \sum_{i,j=1}^{n} \varphi'(s) g_i(\varphi(s)) f_j(\varphi(s))(\nabla_{E_j} E_i)_{|\alpha(s)} \\ &= \varphi'(s)(D_{\dot{\gamma}}V)(\varphi(s)) \\ &= 0 \end{aligned}$$

Thus $D_{\dot{\alpha}}W(t) = 0$ for any $t \in [c,d]$ i.e W is parallel along α, moreover if φ is orientation-preserving we get that W is the parallel transport of v along α, thus:

$$\tau^{\alpha}_{c,d}(v) = W(d) = V(b) = \tau^{\gamma}_{a,b}(v).$$

In the case where $[c,d] = [a,b]$ and $\varphi(t) = a+b-t$ i.e $\alpha = \gamma^-$ then we get that:

$$\tau^{\gamma^-}_{a,b} \circ \tau^{\gamma}_{a,b}(v) = \tau^{\gamma^-}_{a,b}(V(b)) = \tau^{\gamma^-}_{a,b}(W(a)) = W(b) = V(a) = v.$$

(2)$-$ Let V be the parallel transport of $v \in T_{\gamma(a)}M$ along γ and denote V_1 the parallel vector field along γ_1 satisfying $V_1(a) = v$ and V_2 the parallel vector field along γ_2 with initial condition $V_2(t_0) = V_1(t_0)$. Then by uniqueness of parallel transport and the expressions of γ_1 and γ_2, we get $V_1 = V_{|[a,t_0]}$ and $V_2 := V_{[t_0,b]}$. In particular:

$$\tau^{\gamma_2}_{t_0,b} \circ \tau^{\gamma_1}_{a,t_0}(v) = \tau^{\gamma_2}_{t_0,b}(V(t_0)) = V(b) = \tau^{\gamma}_{a,b}(v).$$

This ends the proof. ∎

These properties allows to extend the definition of $\tau^{\gamma}_{a,b}$ for piecewise smooth curves $\gamma : [a,b] \longrightarrow M$ by setting:

$$\tau^{\gamma}_{a,b} := \tau^{\gamma}_{t_k,b} \circ \tau^{\gamma}_{t_{k-1},t_k} \circ \cdots \circ \tau^{\gamma}_{t_1,t_2} \circ \tau^{\gamma}_{a,t_1},$$

where $a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$ is any subdivision of $[a,b]$ such that the curve $\gamma_{[t_i,t_{i+1}]}$ is smooth. The previous properties extend to this situation as well:

Proposition 3.0.2. *Let M be a smooth manifold with a linear connection ∇. Then:*

(1) *$\tau^{\gamma}_{a,b}$ does not depend on the orientation-preserving parametrization of the piecewise smooth curve $\gamma : [a,b] \longrightarrow M$.*
(2) *Given two piecewise smooth curves $\gamma_1 : [a,b] \longrightarrow M$ and $\gamma_2 : [b,c] \longrightarrow M$ such that $\gamma_2(b) = \gamma_1(b)$ then $\tau^{\gamma_1 * \gamma_2}_{a,c} = \tau^{\gamma_2}_{b,c} \circ \tau^{\gamma_1}_{a,b}$.*
(3) *For any piecewise smooth curve $\gamma : [a,b] \longrightarrow M$, $\tau^{\gamma}_{a,b}$ is invertible with inverse $\tau^{\gamma^-}_{a,b}$.*

It is more convenient therefore to denote $\tau^{\gamma}_{a,b}$ by $\tau^{\gamma}_{\gamma(a),\gamma(b)}$ instead or just $\tau^{\gamma}_{\gamma(a)}$ when γ is a loop. Fix $x_0 \in M$ and define:

$$\mathrm{Hol}_{x_0}(M,\nabla) := \{\tau^{\gamma}_{x_0} : T_{x_0}M \xrightarrow{\simeq} T_{x_0}M,\ \gamma \text{ is a loop based at } x_0\}.$$

Using the above observations, it is clear that $\mathrm{Hol}_{x_0}(M,\nabla)$ is a subgroup of $\mathrm{GL}(T_{x_0}M)$ called the *holonomy group of (M,∇) at x_0*. We also define the *restricted holonomy* *of* (M,∇) *at* x_0 to be:

$$\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla) := \{\tau^{\gamma}_{x_0} : T_{x_0}M \xrightarrow{\simeq} T_{x_0}M,\ \gamma \text{ is a contractible loop based at } x_0\},$$

which is obviously a subgroup of the holonomy group since concatenation and inverse of contractible loops remains contractible. It is also straightforward to see that $\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$ is normal in $\mathrm{Hol}_{x_0}(M,\nabla)$.

Theorem 3.0.1. *Let* M *be a smooth manifold and* ∇ *a linear connection on* M*. The holonomy group* $\mathrm{Hol}_{x_0}(M,\nabla)$ *possesses the structure of an (immersed) Lie subgroup of* $\mathrm{GL}(T_{x_0}M)$ *and* $\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla) = \mathrm{Hol}_{x_0}(M,\nabla)^0$.

Proof.
We start by proving that $\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$ is arcwise connected. Let $g \in \widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$, then there exists a piecewise smooth loop γ based at x_0 homotopic to the constant loop x_0 such that $g = \tau^{\gamma}_{x_0}$. Denote $F : [0,1]\times[a,b] \longrightarrow M$ this homotopy, i.e:

$$F(0,t) = \gamma(t),\ F(1,t) = x_0, \quad F(s,a) = F(s,b) = x_0.$$

Put $F(s,t) := \gamma_s(t)$, clearly each loop γ_s is contractible as well and one can assume without loss of generality that γ_s is piecewise smooth for any $0 \le s \le 1$, so we get a correspondence $\alpha : [0,1] \longrightarrow \widetilde{\mathrm{Hol}}_{x_0}(M,\nabla),\ s \mapsto \tau^{\gamma_s}_{a,b}$ with $\alpha(0) = g$ and $\alpha(1) = \mathrm{Id}$. The claim amounts to proving that α is continuous.

Fix a system of local coordinates $(U, x^1, \ldots, x^n)$ centered at x_0. For every $0 \le s \le 1$, $F(s,1) = x_0 \in U$ therefore we can find an open neighborhood U_s of s in $[0,1]$ and $0 \le \alpha_s < 1$ such that $F(U_s \times [\alpha_s, 1]) \subset U$. Since $[0,1]$ is compact, it is the union of finitely many U_{s_i} with $1 \le i \le r$, so if we put $\alpha = \max\limits_{1\le i\le r} \alpha_{s_i} < 1$ we obtain that:

$$F([0,1]\times[\alpha,1]) = F\Big(\Big(\bigcup_{i=1}^{r} U_{s_i}\Big)\times[\alpha,1]\Big) = \bigcup_{i=1}^{r} F(U_{s_i}\times[\alpha,1]) \subset U.$$

A similar argument allows to choose $\alpha < 1$ such that every γ_s is smooth on $[\alpha,1]$. Now let $v \in T_{x_0}M$ and denote V_s the parallel transport of v along γ_s. Put $E_i := \frac{\partial}{\partial x^i}$ and write for any $\alpha \le t \le 1$:

$$V_s(t) = \sum_{i=1}^{n} f_i(s,t) E_{i|F(s,t)}, \quad \frac{\partial F}{\partial t}(s,t) = \sum_{i=1}^{n} g_i(s,t) E_{i|F(s,t)}, \quad \nabla E_i = \sum_{j=1}^{n} \Gamma_{ij} E_j.$$

Then $V_{s|[\alpha,1]}$ is the solution of the following system of ordinary linear differential equations:

$$\sum_{k=1}^{n}\Big(\frac{\partial f_k}{\partial t}(s,t) + \sum_{i,j=1}^{n} g_j(s,t) f_i(s,t)\Gamma^k_{ji}(F(s,t))\Big) E_{k|F(s,t)}$$

It is clear that the coefficients of this system are all continuous (in fact uniformly continuous) with respect to the parameter s, therefore we get that the solution $[0,1]\times[\alpha,1] \longrightarrow \mathbb{R}^n,\ (s,t) \mapsto (f_1(s,t),\ldots,f_n(s,t))$ is continuous as well, in particular one obtains that the map $[0,1] \longrightarrow T_{x_0}M,\ s \mapsto V_s(1) = \alpha(s)(v)$ is also continuous for every $v \in T_{x_0}M$. Since $T_{x_0}M$ is a finite-dimensional vector space we conclude that $\alpha : [0,1] \longrightarrow \widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$ is continuous.

By Yamabe's Theorem we get that the restricted holonomy group $\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$

has the structure of an (immersed) Lie subgroup of $\mathrm{GL}(T_{x_0}M)$. Next, observe that there exists a well-defined group homomorphism:

$$\pi_1(M,x_0) \longrightarrow \mathrm{Hol}_{x_0}(M,\nabla)/\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla), \quad [\gamma] \mapsto [\tau^{\gamma}_{x_0}],$$

proving that the quotient $\mathrm{Hol}_{x_0}(M,\nabla)/\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$ is at most countable. Using this we can show that $\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$ is the identity component of $\mathrm{Hol}_{x_0}(M,\nabla)$, and since $\mathrm{Hol}_{x_0}(M,\nabla)$ is second-countable with countably many connected components, it has the structure of an (immersed) Lie subgroup of $\mathrm{GL}(T_{x_0}M)$. ∎

If M is connected, the holonomy group at any point is essentially the same, in the sense of the following result:

Proposition 3.0.3. *Let M be a connected manifold with a linear connection ∇ and let $x, y \in M$. Then for any piecewise smooth curve $\gamma : [a,b] \longrightarrow M$ joining x to y, the map:*

$$\mathrm{Hol}_x(M,\nabla) \longrightarrow \mathrm{Hol}_y(M,\nabla), \quad g \mapsto \tau^{\gamma}_{x,y} \circ g \circ (\tau^{\gamma}_{x,y})^{-1},$$

is an isomorphism.

Proof. Straightforward ∎

In what follows $(M, \langle\ ,\ \rangle)$ will always be a connected Riemannian manifold with Levi-Civita connection ∇. Then following that $\nabla\langle\ ,\ \rangle = 0$ we get that for any smooth curve $\gamma : [a,b] \longrightarrow M$ and any $V, W \in \Gamma(\gamma^{-1}TM)$:

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle D_{\dot\gamma}V(t), W(t)\rangle + \langle V(t), D_{\dot\gamma}W(t)\rangle.$$

In particular if V and W are parallel along γ then $t \mapsto \langle V(t), W(t)\rangle$ is a constant map and therefore $\langle \tau^{\gamma}_{a,b}(v), \tau^{\gamma}_{a,b}(w)\rangle = \langle v, w\rangle$. This leads to the following result:

Proposition 3.0.4. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold with Levi-Civita connection ∇. Then $\mathrm{Hol}_x(M,\nabla) \subset \mathrm{O}(T_xM, \langle\ ,\ \rangle)$ for any $x \in M$.*

We use the symbol $\mathrm{Hol}_x(M, \langle\ ,\ \rangle)$ instead of $\mathrm{Hol}_x(M,\nabla)$ in what follows. For any $x \in M$, we will say that T_xM is irreducibe if it is irreducible as a $\mathrm{Hol}_x(M, \langle\ ,\ \rangle)$-module i.e does not admit any proper, non-trival subspace that is invariant by the action of $\mathrm{Hol}_x(M, \langle\ ,\ \rangle)$. In view of Proposition 3.0.3 we see that if T_xM is irreducible then T_yM is also irreducible for any $y \in M$. This suggests the following definition:

Definition 3.0.1. *A Riemannian manifold $(M, \langle\ ,\ \rangle)$ is said to be irreducible if T_xM is irreducible for some (hence every) $x \in M$.*

The next result is a classical Theorem in riemannian geometry and will be essential for the next part:

Theorem 3.0.2 (De Rham decomposition theorem)**.** *A simply connected, complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ is isometric to the direct product $M_0 \times \ldots, \times M_k$ where M_0 is a Euclidean space and $M_1, \ldots, M_k$ are all simply connected, irreducible Riemannian manifolds. Such a decomposition is a unique up to an order of the factors involved.*

Corollary 3.0.1. *Let $(M, \langle\ ,\ \rangle)$ be a simply connected, complete Riemannian manifold and $M = M_0 \times \cdots \times M_k$ its de Rham decomposition. Let $x = (x_0, \ldots, x_k)$, then:*

(1) *The identification* $\mathrm{Hol}_{x_1}(M_1, \langle\ ,\ \rangle) \times \ldots \mathrm{Hol}_{x_k}(M_k, \langle\ ,\ \rangle) \mapsto \mathrm{Hol}_x(M, \langle\ ,\ \rangle)$ *given by* $(\tau_{x_1}^{\gamma_1}, \ldots, \tau_{x_k}^{\gamma_k}) \mapsto \tau_x^{\alpha_1} \circ \cdots \circ \tau_x^{\alpha_k}$ *is an isomorphism, where* α_i *is the loop given by:*

$$\alpha_i(t) = (x_1, \ldots, \gamma_i(t), \ldots, x_k).$$

(2) *Under the previous identification,* $\mathrm{Hol}_{x_i}(M_i, \langle\ ,\ \rangle)$ *is a normal subgroup of* $\mathrm{Hol}_x(M, \langle\ ,\ \rangle)$ *acting trivially on* $T_{x_j}M_j$ *for* $j \neq i$ *and irreducibly on* $T_{x_i}M_i$.

(3) *For any* $f \in \mathrm{Aff}(M, \nabla)$ *and any* $i = 1, \ldots, k$,

$$T_x f(T_{x_0}M_0) = T_{f(x)_0}M_0, \quad \text{and} \quad T_x f(T_{x_i}M_i) = T_{f(x)_j}M_j,$$

for some $j = 1, \ldots, k$. *If* $f \in \mathrm{Aff}(M, \nabla)^0$ *then* $T_x f(T_{x_i}M_i) = T_{f(x)_i}M_i$.

Proof. We only prove the third point. Let $f : M \longrightarrow M$ be an affine transformation and choose a piecewise smooth loop $\gamma : [0, 1] \longrightarrow M$ based at $f(x)$. Then for any $v \in T_{x_0}M_0$:

$$\tau_{f(x)}^{\gamma}(T_x f(v)) = T_x f(\tau_x^{f^{-1}\circ\gamma}(v)) = T_x f(v),$$

thus $T_x f(v)$ is invariant by $\mathrm{Hol}_{f(x)}(M, \nabla)$ which gives that $T_x f(T_{x_0}M_0) = T_{f(x)_0}M_0$. On the other hand, if $w \in T_{x_i}M_i$ for $i \neq 0$ then:

$$\tau_{f(x)}^{\gamma}(T_x f(w)) = T_x f(\tau_x^{f^{-1}\circ\gamma}(w)) \in T_x f(T_{x_i}M_i),$$

which shows that $T_x f(T_{x_i}M_i)$ is invariant, furthermore if $V \subset T_x f(T_{x_i}M_i)$ is any invariant subspace then in the same way $(T_x f)^{-1}(V)$ is an invariant subspace of $T_{x_i}M_i$ thus it is either trivial or equal to $T_{x_i}M_i$ proving that $T_x f(T_{x_i}M_i)$ is irreducible, in particular one gets the decomposition of $T_{f(x)}M$ into the sum of irreducible subspaces:

$$T_{f(x)}M = T_x f(T_{x_0}M_0) \oplus \cdots \oplus T_x f(T_{x_k}M_k),$$

and by uniqueness of such decomposition we conclude that $T_x f(T_{x_i}M_i) = T_{f(x)_j}M_j$ for some $j = 1, \ldots, k$.

Next let X be a complete affine vector field on M i.e $X \in \mathfrak{aff}(M, \nabla)$, and consider the curve $\gamma(t) = \exp(tX) \cdot x$. Let $v_i \in T_{x_i}M_i$, and consider $u : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

$$u(t) = \langle T_x \exp(tX)(v_i), \tau_{0,t}^{\gamma}(v_i)\rangle_{\gamma(t)}.$$

Then u is a smooth function satisfying $u(0) = \langle v_i, v_i\rangle_x \neq 0$ and therefore $u(t) \neq 0$ for $-\delta < t < \delta$, which shows that $T_x \exp(tX)(v_i) \in T_{\gamma(t)_i}M_i$ for all $-\delta < t < \delta$. In fact since $T_{x_i}M_i$ is finite-dimensional, one can choose $\delta > 0$ small enough so that

$$T_x \exp(tX)(T_{x_i}M_i) \in T_{\gamma(t)_i}M_i,$$

for any $-\delta < t < \delta$. The result follows from the fact that $\mathrm{Aff}(M, \nabla)^0$ is generated by 1-parameter subgroups. ∎

Theorem 3.0.3. *Let* $M = M_0 \times \cdots \times M_k$ *be the de Rham decomposition of a complete, simply connected Riemannian manifold* $(M, \langle\ ,\ \rangle)$*. Then:*

$$\mathrm{Isom}(M, \langle\ ,\ \rangle)^0 = \mathrm{Isom}(M_0, \langle\ ,\ \rangle)^0 \times \cdots \times \mathrm{Isom}(M_k, \langle\ ,\ \rangle)^0,$$

$$\mathrm{Aff}(M, \nabla)^0 = \mathrm{Aff}(M_0, \nabla)^0 \times \cdots \times \mathrm{Aff}(M_k, \nabla)^0,$$

where ∇ *is the Levi-Civita connection of* M.

Proof. Consider the homomorphism $\Psi : \mathrm{Diff}(M_0) \times \cdots \times \mathrm{Diff}(M_k) \longrightarrow \mathrm{Diff}(M)$ which corresponds to any k-tuple of diffeomorphisms $(f_0, \ldots, f_k)$ the transformation $f : M \longrightarrow M$ given by:

$$f(x_0, \ldots, x_k) = (f_0(x_0), \ldots, f_k(x_k)).$$

Clearly Ψ is continuous and injective. We claim that $f = \Psi(f_0, \ldots, f_k)$ is an affine transformation if and only if f_i is an affine transformation for any $0 \leq i \leq k$. Indeed let $\gamma : [0,1] \longrightarrow M$ be any piecewise smooth curve and write $\gamma := (\gamma_0, \ldots, \gamma_k)$ then choose $v = v_0 \oplus \cdots \oplus v_k \in T_{\gamma(0)}M$ with $v_i \in T_{\gamma_i(0)}M_i$, then:

$$T_{\gamma(1)}f \circ \tau^{\gamma}_{0,1}(v) = \sum_{i=1}^{k} T_{y_i} f_i(\tau^{\gamma_i}_{0,1}(v_i)), \quad \tau^{f\circ\gamma}_{0,1}(T_{\gamma(0)}f(v)) = \sum_{i=1}^{k} \tau^{f_i\circ\gamma_i}_{0,1}(T_{x_i} f_i(v_i)),$$

which shows that f preserves parallel transports on M if and only if each f_i does so on M_i proving the claim. One can also prove in a similar way that f is an isometry if and only if every f_i is an isometry. In particular:

$$\Psi(\mathrm{Aff}(M_0)\times\cdots\times\mathrm{Aff}(M_k)) \subset \mathrm{Aff}(M), \quad \Psi(\mathrm{Isom}(M_0)\times\cdots\times\mathrm{Isom}(M_k)) \subset \mathrm{Isom}(M).$$

Let $f \in \mathrm{Aff}(M, \nabla)^0$ and $\mathrm{pr}_i : M \longrightarrow M_i$ be the projection on the i-th component then denote $g_i := \mathrm{pr}_i \circ f$, we will show that $g_i(x_0, \ldots, x_k)$ only depends on x_i. Indeed let $x = (x_0, \ldots, x_k) \in M$, $j \neq i$ and $v_j \in T_{x_j}M_j$ then by (3) of Corollary 3.0.1:

$$T_x g_i(v_j) = T_{f(x)}\mathrm{pr}_i(\underbrace{T_x f(v_j)}_{\in M_j}) = 0$$

Therefore if we fix $(a_0, \ldots, a_k) \in M$ and define $f_i : M_i \longrightarrow M_i$ by the expession:

$$f_i(y) := g_i(a_0, \ldots, y, \ldots, a_k),$$

then f_i is a well-defined diffeomorphism of M_i and $f = \Psi(f_0, \ldots, f_k)$. It also follows that if $f \in \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$ then each f_i is an isometry. ∎

Theorem 3.0.4. *Let $(M, \langle\ ,\ \rangle)$ be a complete, irreducible Riemannian manifold, then $\mathrm{Aff}(M, \nabla) = \mathrm{Isom}(M, \langle\ ,\ \rangle)$ except when M is a 1-dimensional Euclidean space.*

The proof of this Theorem will be done in two steps: First one proves that on any such manifold, any affine transformation is homothetic and if furthermore $(M, \langle\ ,\ \rangle)$ is not Euclidean then homothetic transformations are isometries, the result follows then by observing that only 1-dimensional Euclidean spaces can be irreducible.

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, and recall that $f \in \mathrm{Diff}(M)$ is said to be a *homothetic transformation* if there exists a positive constant $c > 0$ such that $\langle T_x f(v), T_x f(w)\rangle = c^2\langle v, w\rangle$ for all $x \in M$ and $v, w \in T_xM$, i.e $f^*\langle\ ,\ \rangle = c^2\langle\ ,\ \rangle$. It is a well-known fact that if ∇ is the Levi-Civita of $(M, \langle\ ,\ \rangle)$, then the Levi-Civita connection $\widetilde{\nabla}$ for $f^*\langle\ ,\ \rangle$ is given by:

$$\widetilde{\nabla}_X Y := f_*^{-1}(\nabla_{f_*X} f_*Y), \tag{8}$$

When $f : M \longrightarrow M$ is a homothetic transformation we get from its definition that $\langle\ ,\ \rangle$ and $f^*\langle\ ,\ \rangle$ share the same Levi-Civita connection i.e $\widetilde{\nabla} = \nabla$, which is to say that *any homothetic transformation is an affine transformation.* Conversely:

Lemma 3.0.1. *If $(M, \langle\ ,\ \rangle)$ is an irreducible Riemannian manifold, then every affine transformation $f : M \longrightarrow M$ is homothetic.*

Proof. According to (8), $\langle\ ,\ \rangle$ and $f^*\langle\ ,\ \rangle$ define the same Levi-Civita ∇. Next recall that if $G \subset \mathrm{O}(V, \langle\ ,\ \rangle)$ acts irreducibly on a Euclidean vector space $(V, \langle\ ,\ \rangle)$ and preserves a symmetric bilinear form $B : V \times V \longrightarrow \mathbb{R}$, then $B = c^2\langle\ ,\ \rangle$ for some constant $c > 0$. Applying this result to

$$(V, \langle\ ,\ \rangle) = (T_xM, \langle\ ,\ \rangle_x), \quad G = \mathrm{Hol}_x(M, \langle\ ,\ \rangle) \quad \text{and} \quad B = (f^*\langle\ ,\ \rangle)_x,$$

we obtain that for any $x \in M$, $(f^*\langle\ ,\ \rangle)_x = c_x^2\langle\ ,\ \rangle_x$ for some $c_x > 0$. Finally, if $y \in M$ is another point and $\gamma : [0, 1] \longrightarrow M$ is a piecewise smooth curve joining x to y, then for every $v \in T_xM$:

$$\begin{aligned} c_y^2\langle\tau_x^\gamma(v), \tau_x^\gamma(v)\rangle_y &= \langle T_yf(\tau_x^\gamma(v)), T_yf(\tau_x^\gamma(v))\rangle_{f(y)} \\ &= \langle\tau_{f(x)}^{f\circ\gamma}(T_xf(v)), \tau_{f(x)}^{f\circ\gamma}(T_xf(v))\rangle_{f(y)} \\ &= \langle T_xf(v), T_xf(v)\rangle_{f(x)} \\ &= c_x^2\langle v, v\rangle_x. \end{aligned}$$

Since $\langle v, v\rangle_x = \langle\tau_x^\gamma(v), \tau_x^\gamma(v)\rangle_y$ for any $v \in T_xM$ we get that $c_x = c_y$, completing the proof. ∎

Lemma 3.0.2. *If $(M, \langle\ ,\ \rangle)$ is a complete Riemannian manifold which is not locally Euclidean, then every homothetic transformation is an isometry.*

Proof. Suppose that $(M, \langle\ ,\ \rangle)$ admits a homothetic transformation $f : M \longrightarrow M$ that isn't an isometry, and write $f^*\langle\ ,\ \rangle = c^2\langle\ ,\ \rangle$ with $c > 0$. Next notice that f^{-1} is homothetic as well with ration $1/c$, therefore we suppose without loss of generality that $0 < c < 1$.

We start by proving that f has a fixed point. Denote $d : M \times M \longrightarrow \mathbb{R}^+$ the geodesic distance and take an arbitrary point $x \in M$ then put $\ell := d(x, f(x))$. Let $\gamma : [0, 1] \longrightarrow M$ be a minimizing geodesic joining x to $f(x)$, which exists since M is complete, then $f^i \circ \gamma$ is a smooth curve joining $f^i(x)$ and $f^{i+1}(x)$ with length:

$$\ell_i = \int_0^1 \langle(f^i \circ \gamma)'(t), (f^i \circ \gamma)'(t)\rangle_{f^i\circ\gamma(t)}^{\frac{1}{2}} dt = c^i\ell,$$

Therefore if $m, n \in \mathbb{N}$ are such that $m < n$ then:

$$d(f^m(x), f^n(x)) \leq \sum_{i=m}^{n-1} d(f^i(x), f^{i+1}(x)) \leq \sum_{i=m}^{n+1} \ell_i = \sum_{i=m}^{n+1} c^i\ell \leq \frac{c^m\ell}{1-c},$$

and thus $(f^m(x))_m$ is a Cauchy sequence in (M, d) hence converges to some $x^* \in M$ since M is complete. Now x^* is obviously a fixed point of f, furthermore x^* does not depend on the choice of x, indeed given $z \in M$ and α a geodesic joining z to x^*, we get that $f^m \circ \alpha$ is a curve joining $f^m(z)$ to $f^m(x^*) = x^*$ and so:

$$d(f^m(z), x^*) \leq \ell(f^m \circ \alpha) = c^m\ell(\alpha) \underset{m\to+\infty}{\longrightarrow} 0. \tag{9}$$

Now fix a neighborhood U of x^* in M with compact closure. Then there exists a constant $K^* > 0$ such that for any $y \in U$ and any unit vectors $v_1, v_2 \in T_yM$:

$$|\langle R_y(v_1, v_2)v_1, v_2\rangle_y| \leq K^*, \tag{10}$$

where R denotes the curvature tensor of $(M, \langle\ ,\ \rangle)$. Since f is also an affine transformation, then for any $z \in M$ and any orthonormal family $\{v, w\}$ of T_zM:

$$\langle R_{f^m(z)}(f_*^m v, f_*^m w) f_*^m v, f_*^m w\rangle = \langle f_*^m(R_z(v,w)v), f_*^m w\rangle = c^{2m}\langle R_z(v,w)v, w\rangle. \tag{11}$$

On the other hand, according to (9) there exists $N \in \mathbb{N}$ such that $f^m(z) \in U$ for any $m \geq N$, moreover $\|f_*^m v\| = \|f_*^m w\| = c^m$, thus using (10):

$$|\langle R_{f^m(z)}(f_*^m v, f_*^m w) f_*^m v, f_*^m w\rangle| \leq K^* \|f_*^m v\|^2 \|f_*^m w\|^2 = c^{4m} K^*,$$

and finally (11) gives that $|\langle R_z(v,w)v, w\rangle| \leq c^{2m}$ for every $m \leq N$. We conclude that $\langle R_z(v,w)v, w\rangle = 0$ for every $z \in M$ and any orthonormal family $\{v, w\}$ of T_zM i.e $(M, \langle\ ,\ \rangle)$ is locally Euclidean. ∎

Theorems 3.0.3 and 3.0.4 have a number of interesting consequences, before we state them we need to make some remarks:
Let X be an affine vector field on a complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ and denote $\widetilde{M}$ the universal cover of M with the induced metric $p^*\langle\ ,\ \rangle$ where $p : \widetilde{M} \longrightarrow M$ is the natural projection, then let $\widetilde{M} = M_0 \times \cdots \times M_k$ be its de Rham decomposition. Next denote $\widetilde{X}$ the lift of X to $\widetilde{M}$, i.e the unique vector field on $\widetilde{M}$ satisfying $T_zp(\widetilde{X}_z) = X_{p(z)}$, then $\widetilde{X}$ is an affine transformation and since X is complete, $\widetilde{X}$ is also complete hence an element of $\mathfrak{aff}(\widetilde{M}, \nabla)$. Moreover $\widetilde{X}$ is Killing if and only if X is Killing.
By Theorem 3.0.3, we have $\mathfrak{aff}(\widetilde{M}, \nabla) \simeq \mathfrak{aff}(M_0, \nabla) \times \cdots \times \mathfrak{aff}(M_k, \nabla)$ and so $\widetilde{X}$ corresponds to a unique family $(X_0, \ldots, X_k)$ such that $X_i \in \mathfrak{aff}(M_i, \nabla)$. According to Theorem 3.0.4 gives that $X_1, \ldots, X_k$ are all Killing vector fields, therefore X will be Killing if and only if X_0 is.

Corollary 3.0.2. *If $(M, \langle\ ,\ \rangle)$ is a complete whose restricted holonomy group $\widetilde{\mathrm{Hol}}_x(M, \langle\ ,\ \rangle)$ have no nonzero invariant vector, then $\mathrm{Aff}(M, \nabla)^0 = \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$, where ∇ is the Levi-Civita connection of $(M, \langle\ ,\ \rangle)$.*

Proof. The restricted holonomy group $\widetilde{\mathrm{Hol}}_x(M, \langle\ ,\ \rangle)$ is isomorphic to the restricted holonomy group of its universal cover $\widetilde{M}$, this is due to the fact that contractible loops on M lift to (contractible) loops on $\widetilde{M}$, this gives that the map:

$$\widetilde{\mathrm{Hol}}_z(\widetilde{M}, \langle\ ,\ \rangle) \longrightarrow \widetilde{\mathrm{Hol}}_{p(z)}(M, \langle\ ,\ \rangle), \quad \tau_x^\gamma \mapsto T_zp \circ \tau_z^\gamma \circ (T_zp)^{-1} = \tau_z^{p\circ\gamma},$$

is an isomorphism for any $z \in \widetilde{M}$, moreover the restricted holonomy group of $\widetilde{M}$ coincides with the total holonomy group since $\widetilde{M}$ is simply connected. So our assumption just states that $\widetilde{M}$ has no Euclidean factor i.e M_0 is a point, which gives in view of the previous remarks that every affine vector field on M is a Killing vector field. ∎

Corollary 3.0.3. *If X is an affine vector field of a complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ and if the length of X is bounded, then X is a Killing vector field.*

Proof. Denote $\widetilde{M}$ the universal cover of M with the induced metric $g := p^*\langle\ ,\ \rangle$ where $p : \widetilde{M} \longrightarrow M$ is the natural projection and let $\widetilde{M} = M_0 \times \cdots \times M_k$ be its De Rham decomposition. Next let $\widetilde{X}$ be the lift of X to $\widetilde{M}$ and write $\widetilde{X} = (X_0, \ldots, X_k)$ such that $X_i \in \mathfrak{aff}(M_i, \nabla)$. Then:

$$g(X_0, X_0) \leq g(\widetilde{X}, \widetilde{X}) = \langle X, X\rangle,$$

hence if X has bounded length then so does X_0. Now write $X_0 = \sum \xi^i \partial/\partial x^i$ in some (global) Euclidean coordinate system $x^1, \dots, x^r$ of M_0. Since X_0 is an affine vector field then it satisfies:

$$(L_{X_0} \circ \nabla_Y - \nabla_Y \circ L_{X_0})Z = \nabla_{[X_0,Z]} Y.$$

For $Y = \partial/\partial x^j$ and $Z = \partial/\partial x^k$ we get that $\nabla Y = 0$, $\nabla Z = 0$ hence by the previous expression $\nabla_Y [X_0, Z] = 0$ which is equivalent to:

$$\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0, \quad i, j, k = 1, \dots r.$$

This means that $X_0 = \sum_{i=1}^r (\sum_{j=1}^r a_{ij} x^j + b_i) \partial/\partial x^i$ where a_{ij} and b_i are constants, but since X_0 has bounded length it follows that $a_{ij} = 0$ for all $i, j = 1, \dots, r$ proving that X_0 is a linear combination of $\partial/\partial x^1, \dots, \partial/\partial x^r$ each of which is a Killing vector field on M_0, we thus conclude that X_0 is Killing. By the previous remarks, X is a Killing vector field on M. ∎

If M is a compact Riemannian manifold then the length of any vector field is bounded, therefore:

Corollary 3.0.4 (Yano's Theorem). *Let $(M, \langle\ ,\ \rangle)$ be a compact Riemannian manifold with Levi-Civita connection ∇. Then* $\mathrm{Aff}(M, \nabla)^0 = \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$.

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