The Group of affine transformations of a Smooth Manifold

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Preliminaries

Let M be an n-dimensional smooth manifold. For any $x \in M$, we call a frame on M at x any linear isomorphism $\mathbb{R}^n \xrightarrow{\simeq} T_x M$, the set of such frames will be denoted $\mathrm{L}(M)_x$. Clearly, the general linear group $\mathrm{GL}(n, \mathbb{R})$ acts naturally on $\mathrm{L}(M)_x$ via the map :

$$L(M)_x \times \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{L}(M)_x, \quad (z, g) \mapsto z \circ g,$$

and it is not hard to see that this action is simply transitive. Now define

$$\mathrm{L}(M) := \amalg_{x \in M} \mathrm{L}(M)_x$$

and consider the projection $\pi : \mathrm{L}(M) \longrightarrow M$ given by $\pi(\mathrm{L}(M)_x) := x$.

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Proposition 1.1

Let M be an n-dimensional manifold. Then $\mathrm{L}(M) \xrightarrow{\pi} M$ has a unique structure of a smooth principal $\mathrm{GL}(n, \mathbb{R})$-bundle over M called the frame bundle of TM such that for any local frame $\{E_1, \ldots, E_n\}$ of TM defined on an open subset $U \subset M$, the map :

$$\sigma : U \longrightarrow \mathrm{L}(M), \quad x \mapsto \{E_{1|x}, \ldots, E_{n|x}\}, \tag{1}$$

is a local (smooth) section of $\mathrm{L}(M)$. Conversely, if $\sigma : U \longrightarrow \mathrm{L}(M)$ is any smooth section, then there exists a local frame $\{E_1, \ldots, E_n\}$ of TM over U such that σ is of the form (1).

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In a similar way one defines on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ the bundle of orthonormal frames $\mathrm{O}(M) := \amalg_{x \in M} \mathrm{O}(M)_x$ where each $\mathrm{O}(M)_x$ consists of linear isometries

$$(\mathbb{R}^n, \langle\ ,\ \rangle_0) \xrightarrow{\simeq} (T_x M, \langle\ ,\ \rangle_x).$$

It is clear that $\mathrm{O}(M) \subset \mathrm{L}(M)$, on the other hand the orthogonal group $\mathrm{O}(n)$ acts simply transitively on $\mathrm{O}(M)$.

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Proposition 1.2

Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional Riemannian manifold. Then $\mathrm{O}(M) \overset{\pi}{\longrightarrow} M$ is a smooth principal $\mathrm{O}(n)$-subbundle of $\mathrm{L}(M)$. Futhermore if $\{E_1, \ldots, E_n\}$ is any local, orthonormal frame of TM defined on an open subset $U \subset M$ then the map :

$$\sigma : U \longrightarrow \mathrm{L}(M), \quad x \mapsto \{E_{1|x}, \ldots, E_{n|x}\}, \tag{2}$$

is a local (smooth) section of $\mathrm{O}(M)$. Conversely, if $\sigma : U \longrightarrow \mathrm{O}(M)$ is any smooth section, then there exists a local orthonormal frame $\{E_1, \ldots, E_n\}$ of TM over U such that σ is of the form (2).

Preliminaries

Any diffeomorphism $f : M \longrightarrow M$ induces a principal bundle automorphism $f_* : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ such that the following diagram is commutative :

$$\begin{array}{ccc} \mathrm{L}(M) & \xrightarrow{f_*} & \mathrm{L}(M) \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[f]{} & M \end{array}$$

Explicitly, for any $z \in \mathrm{L}(M)$ we have $f_*(z) := T_{\pi(z)} f \circ z$.

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Define on $L(M)$ the $\mathbb{R}^n$-valued 1-form $\theta \in \Omega^1(L(M), \mathbb{R}^n)$ given by

$$\theta_z(v) := z^{-1}(T_z\pi(v)),$$

we call it the *canonical form* of $\mathrm{L}(M)$. We have the following result :

Proposition 1.3

Let M be a smooth manifold and let θ denote the canonical form of the frame bundle $\mathrm{L}(M)$. If $f : M \longrightarrow M$ is any diffeomorphism of M then f_* preserves θ. Conversely, if $A : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ is any fiber-preserving transformation leaving θ invariant, then $A = f_*$ for some $f \in \mathrm{Diff}(M)$.

Proof.

The first point is a straightforward computation. Conversely, we first notice that since $A : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ *is fiber-preserving, the map :*

$$f : M \longrightarrow M, \quad f(x) = \pi(A(z)), \ \ z \in \pi^{-1}(x),$$

is a well-defined diffeomorphism of M*. Now :*

$$(A^*\theta)_z(v) = \theta_{A(z)}(T_z A(v)) = A(z)^{-1} \circ T_{\pi(z)} f \circ T_z \pi(v),$$

so A *will preserve* θ *if and only if* $A(z)^{-1} \circ T_{\pi(z)} f = z^{-1}$ *for any* $z \in P$ *which is exactly what* $A = f_*$ *means.* □

Preliminaries

Proposition 1.3 states that the morphism $\mathrm{Diff}(M) \xrightarrow{\Psi} \mathsf{Aut}(\mathrm{L}(M))$, $f \mapsto f_*$ sends the group of diffeomorphisms of M isomorphically onto the subgroup of automorphisms of $\mathrm{L}(M)$ preserving the canonical form θ.

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Definition 1.1

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A connection form on a principal G-bundle $P \stackrel{\pi}{\longrightarrow} M$ is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying :

1- *For any $z \in P$ and any $A \in \mathfrak{g}$, $\omega_z(\widetilde{A}_z) = A$ where $\widetilde{A}$ is the fundamental vector field on P corresponding to A, i.e*

$$\widetilde{A}_z := \frac{d}{dt}_{t=0} z \cdot \exp(-tA).$$

2- *For any $g \in G$ and any $z \in P$, $v \in T_z(p)$,*

$$(R_g^* \omega)_z(v) = \mathrm{Ad}_{g^{-1}}(\omega_z(v)),$$

with $R_g : P \longrightarrow P$ being the map $z \mapsto z \cdot g$.

The Affine Group as a Lie group

Let now ∇ be a linear connection on M. A diffeomorphism $f : M \longrightarrow M$ is called an *affine transformation* with respect to ∇ if it satisfies

$$f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$$

for any $X, Y \in \mathcal{X}(M)$ where $f_* X$ is the vector field on M given by :

$$(f_* X)_{f^{-1}(x)} := (T_x f)^{-1}(X_x).$$

The group of such transformations will be denoted $\mathrm{Aff}(M, \nabla)$. On the other hand, we say that $X \in \mathcal{X}(M)$ is an *affine vector field* if it generates a local 1-parameter group of affine transformations.

The Affine Group as a Lie group

Proposition 2.1

Let M be a smooth manifold and ∇ a linear connection on M. Then there exists a unique connection form ω on $\mathrm{L}(M)$ such that for any local section $\sigma := \{E_1, \ldots, E_n\}$ of $\mathrm{L}(M)$ defined on U, $\sigma^*\omega = \Gamma$ where $\Gamma \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$ is given by :

$$\nabla E_i = \sum_{j=1}^{n} \Gamma_{ij} E_j.$$

Conversely, any connection form ω on $\mathrm{L}(M)$ gives rise to a linear connection ∇ on M by means of the previous expression.

The Affine Group as a Lie group

Proposition 2.2

Let M be a smooth manifold, ∇ a linear connection on M and ω the connection form on $\mathrm{L}(M)$ corresponding to ∇. Let $f : M \longrightarrow M$ be a diffeomorphism, then :

1- $f \in \mathrm{Aff}(M, \nabla)$ *if and only if* f_* *preserves the connection form* ω.

2- *Conversely, any fiber-preserving tranformation* $A : \mathrm{L}(M) \longrightarrow \mathrm{L}(M)$ *leaving both* θ *and* ω *invariant is of the form* $A = f_*$ *for some* $f \in \mathrm{Aff}(M, \nabla)$.

The Affine Group as a Lie group

Proof.

Define on M the linear connection $\widetilde{\nabla}$ by the expression :

$$\widetilde{\nabla}_X Y = (f_*)^{-1}(\nabla_{f_* X} f_* Y)$$

The idea is to prove that $(f^{-1})^*\omega$ is the (unique) connection form on $\mathrm{L}(M)$ defining $\widetilde{\nabla}$. So f is an affine transformation i.e $\widetilde{\nabla} = \nabla$ if and only if $\omega = f^*\omega$. □

The Affine Group as a Lie group

Theorem 1

Let M be an n-dimensional smooth manifold with a global trivialization $\{X_1, \ldots, X_n\}$ of TM. Denote G the group of transformations preserving this trivialization, i.e diffeomorphisms $f : M \longrightarrow M$ satisfying $T_x f(X_{i|x}) = X_{i|f(x)}$. Then G possesses a unique Lie group structure for the compact-open topology such that $\dim G \leq \dim M$. More precisely for any $p \in M$, the map :

$$G \longrightarrow M, \quad f \mapsto f(p),$$

is an imbedding of G onto a closed submanifold of M, and the submanifold structure on the image is what makes G a Lie transformation group. Moreover the Lie algebra of G consists of complete vector fields whose 1-parameter subgroups are in G.

The Affine Group as a Lie group

Theorem 2

Let M be a smooth n-dimensional manifold and ∇ an affine connection on M, then $\mathrm{Aff}(M,\nabla)$ is a Lie group for the compact-topology of dimension $\leq n^2+n$. More precisely for any $z \in \mathrm{L}(M)$ the map :

$$\mathrm{Aff}(M,\nabla) \longrightarrow \mathrm{L}(M), \quad f \mapsto f_*(z),$$

is injective and its image is a closed submanifold of $\mathrm{L}(M)$. The submanifold structure on its image makes $\mathrm{Aff}(M,\nabla)$ a Lie transformation group. Its Lie algebra consists of complete affine vector fields on M.

Proof.

For any $x \in M$, Recall that since $\mathrm{GL}(n, \mathbb{R})$ acts simply transitively on $P_x := \pi^{-1}(x)$, the map $\mathrm{GL}(n, \mathbb{R}) \longrightarrow P_x$, $g \mapsto z \cdot g$ is a diffeomorphism and its differential

$$\mathfrak{gl}(n, \mathbb{R}) \longrightarrow T_z P_x, \quad A \mapsto \frac{d}{dt}_{t=0} z \cdot \exp(tA),$$

is an isomorphism. Thus for any $v \in T_z P_x$ there is a unique $A \in \mathfrak{gl}(n, \mathbb{R})$ such that $v = \widetilde{A}_z$. On the other hand $T_z P_x = \ker(T_z \pi)$.

Denote ω the connection form on $P := \mathrm{L}(M)$ corresponding to ∇. We prove that the map

$$TP \xrightarrow{\phi} P \times (\mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n), \quad Z \mapsto (p(Z), \omega(Z), \theta(Z)),$$

defines a trivialization of TP. First notice that ϕ is surjective given that $\pi : P \longrightarrow M$ is a submersion and $A = \omega(\widetilde{A})$ for any $A \in \mathfrak{gl}(n, \mathbb{R})$.

Let $v \in T_zP$ such that $\theta_z(v) = 0$ and $\omega_z(v) = 0$, then :

$$0 = \theta_z(v) = z^{-1}(T_z\pi(v)),$$

hence $v \in \ker(T_z\pi) := T_zP_{\pi(z)}$, on the other hand write $v = \widetilde{A}_z$ for some A in $\mathfrak{gl}(n, \mathbb{R})$ then we get that

$$0 = \omega_z(v) = \omega_z(\widetilde{A}_z) = A,$$

and so $v = 0$, i.e ϕ is injective. On the other hand if $\sigma : U \longrightarrow P$ is any local section one can define the map :

$$P_{|U} \times (\mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n) \xrightarrow{\psi} P_{|U}, \quad \widetilde{A}_{\sigma(\pi(z))} + (T_{\pi(z)}\sigma)(\sigma_{\pi(z)}(v)),$$

it is clear that ψ is a smooth map and $\phi \circ \psi = \mathrm{Id}$, thus ϕ is a local diffeomorphism and we conclude that it is a vector bundle isomorphism. Finally, one checks that any fiber preserving transformation $F : P \longrightarrow P$ leaving θ and ω invariant leaves ϕ invariant. By Theorem 1 we get the desired result. $\square$

The Isometry Group as a Lie group

Assume now that $(M, \langle\ ,\ \rangle)$ is a Riemannian manifold and let ∇ be a metric connection on M, i.e $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.
Let $\{E_1, \dots, E_n\}$ be a local orthonormal frame of TM defined on an open subset $U \subset M$ and write

$$\nabla E_i = \sum_{j=1}^{n} \Gamma_{ij} E_j.$$

Then we get that $\Gamma_{ij} = -\Gamma_{ji}$ or in other terms $\Gamma \in \Omega^1(U, \mathfrak{so}(n))$.

The Isometry Group as a Lie group

Thus if $\widetilde{\omega}$ is the connection form on $\mathrm{L}(M)$ corresponding to ∇ then its restriction ω to the orthogonal frame bundle $\mathrm{O}(M)$ is $\mathfrak{so}(n)$-valued and defines therefore a connection form on $\mathrm{O}(M)$ and it is in fact the only connection form on $\mathrm{O}(M)$ representing ∇. Conversely any connection form on $\mathrm{O}(M)$ admits a unique extension to $\mathrm{L}(M)$ and defines therefore a metric connection ∇ on M.

The Isometry Group as a Lie group

Proposition 3.1

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, ∇ its Levi-Civita connection and ω the connection form on $\mathrm{O}(M)$ representing ∇.

1- *A diffeomorphism $f : M \longrightarrow M$ is an isometry if and only if $f_*(\mathrm{O}(M)) = \mathrm{O}(M)$.*

2- *If $A : \mathrm{O}(M) \longrightarrow \mathrm{O}(M)$ is a fiber-preserving transformation leaving invariant the canonical form θ of $\mathrm{O}(M)$, then there exists a unique isometry $f : M \longrightarrow M$ such that $A = f_*$.*

3- *Any (principal) bundle automorphism $\mathrm{O}(M) \longrightarrow \mathrm{O}(M)$ leaving θ invariant, leaves ω invariant.*

The Isometry Group as a Lie group

Proof.

The first point is the definition of a Riemannian isometry, the argument for the second point is the same as in Proposition 2.2. For the third point, observe that since ∇ is torsion-free, then ω is torsion-free as well i.e the 2-form $T \in \Omega^2(\mathrm{O}(M), \mathbb{R}^n)$, called torsion form of ω, given by :

$$T := \omega \wedge \theta + d\theta, \tag{3}$$

vanishes. Let $A : \mathrm{O}(M) \longrightarrow \mathrm{O}(M)$ be a bundle automorphism, then $A^*\omega$ is a connection form on $\mathrm{O}(M)$. Since A preserves the canonical form θ, we get from expression (3) that $A^*\omega$ is torsion-free as well, so by uniqueness of the Levi-Civita connection we conclude that $A^*\omega = \omega$.

The Isometry Group as a Lie group

Theorem 3

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, then $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ with the compact-open topology is a Lie group of dimension $\leq \frac{n(n+1)}{2}$. In fact for any $z \in \mathrm{O}(M)$, the map :

$$\mathrm{Isom}(M, \langle\ ,\ \rangle) \longrightarrow \mathrm{O}(M), \quad f \mapsto f_*(z),$$

is an imbedding and its image is a closed submanifold of $\mathrm{O}(M)$. If ∇ is the Levi-Civita connection of $\langle\ ,\ \rangle$ then $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ is a closed subgroup of $\mathrm{Aff}(M, \nabla)$. Its Lie algebra consists of complete Killing vector fields on M.

The Isometry Group as a Lie group

Proposition 3.2

The natural action of $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *on* M *is proper. In particular if* M *is compact then* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *is compact.*

Proof.

Choose a compact $K \subset M$ *and put* $G_K = \{f \in \mathrm{Isom}(M),\ f(K) \cap K \neq \emptyset\}$. *Let* $(f_n)_n$ *be an arbitrary sequence of* G_K*, then for any* $n \in \mathbb{N}$ *we can find* $p_n \in K$ *such that* $f_n(p_n) \in K$. *Since* K *is compact, one can show that there exists a subsequence* $(p_{\varphi(n)})_n$ *of* $(p_n)_n$ *converging to some* $p \in K$ *such that* $(f_{\varphi(n)}(p_{\varphi(n)}))_n$ *is also convergent, denote* q *its limit.*

Let $d : M \times M \longrightarrow \mathbb{R}^+$ *be the geodesic distance, then :*

$$\begin{aligned} d(f_{\varphi(n)}(p), q) &\leq d(f_{\varphi(n)}(p), f_{\varphi(n)}(p_{\varphi(n)})) + d(f_{\varphi(n)}(p_{\varphi(n)}), q) \\ &\leq d(p, p_{\varphi(n)}) + d(f_{\varphi(n)}(p_{\varphi(n)}), q) \underset{n \to +\infty}{\longrightarrow} 0. \end{aligned}$$

Next put $p = \pi(z)$ *with* $z \in \mathrm{O}(M)$ *and* $C = \{f_{\varphi(n)}(p),\ n \in \mathbb{N}\} \cup \{q\}$, *since* $\mathrm{O}(n)$ *is compact we get that* $\pi^{-1}(C)$ *is a compact subset of* $\mathrm{O}(M)$, *therefore* $((f_{\varphi(n)})_*(z))_n$ *and so it has a convergent subsequence.*
By Theorem 3, $\{f_*(z),\ f \in \mathrm{Isom}(M)\}$ *is closed submanifold in* $\mathrm{O}(M)$ *so* $((f_{\psi(n)})_*(z))_n$ *converges to* $f_*(z)$ *for some isometry* f *and by the same result we get that* $(f_{\psi(n)})_n$ *converges to* f *in* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$. *We conclude that* G_K *is compact and so* $\mathrm{Isom}(M, \langle\ ,\ \rangle)$ *acts properly on* M. □

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Results on the dimension of the group of affine transformations

Theorem 4

Let M be an n-dimensional manifold with a linear connection ∇. Then $\dim(\mathrm{Aff}(M, \nabla)) = n(n+1)$ *if and only if M is an ordinary affine space with the natural flat affine connection.*

Parallel Transport and horizontal lift

Let M be an n-dimensional smooth manifold and ∇ a linear connection on M. Let $\gamma : [0,1] \longrightarrow M$ be a smooth a curve, then ∇ provides a unique linear operator denoted $D_{\dot\gamma} : \Gamma(\gamma^{-1}TM) \longrightarrow \Gamma(\gamma^{-1}TM)$ satisfying :

$$D_{\dot\gamma}(fV) = f'V + fD_{\dot\gamma}V, \quad f \in \mathcal{C}^\infty([0,1],\mathbb{R}),\ V \in \Gamma(\gamma^{-1}TM),$$

we call it the *covariant derivation along* γ, here $\Gamma(\gamma^{-1}TM)$ is the space of vector fields along γ i.e smooth maps $V : [0,1] \longrightarrow TM$ such that $V(t) \in T_{\gamma(t)}M$.

Finally V is said to be *parallel along* γ if $D_{\dot\gamma}V = 0$.

Parallel Transport and horizontal lift

Theorem 5

Let M be an n-dimensional manifold with a linear connection ∇ and $\gamma : [0, 1] \longrightarrow M$ a smooth curve. For any $v \in T_{\gamma(0)}M$, there exists a unique parallel vector field V along γ satisfying $V(0) = v$. We call V the parallel transport of v along γ.

Parallel Transport and horizontal lift

Let ω be the connection form on $\mathrm{L}(M)$ representing ∇, for any basis $\{e_1, \dots, e_n\}$ of $T_{\gamma(0)}M$ one obtain a *parallel frame* $\{E_1, \dots, E_n\}$ along γ i.e E_i is the parallel vector field along γ satisfying $E_i(0) = e_i$.

It is an easy matter to see that $\{E_1(t), \dots, E_n(t)\}$ is a basis of $T_{\gamma(t)}M$, thus one obtains a smooth curve $\widetilde{\gamma} : [0, 1] \longrightarrow \mathrm{L}(M)$ given by :

$$\widetilde{\gamma}(t) = (E_1(t), \dots, E_n(t)),$$

this curve is called *the horizontal lift* of γ to $\mathrm{L}(M)$ through $\{e_1, \dots, e_n\}$.

Parallel Transport and horizontal lift

Proposition 1.1

Let M be an n-dimensional manifold with a linear connection ∇ and let $\gamma : [0,1] \longrightarrow M$ and $\alpha : [0,1] \longrightarrow \mathrm{L}(M)$ be smooth curves. Then α is a horizontal lift of γ if and only if $\pi \circ \alpha = \gamma$ and $\omega_{\alpha(t)}(\dot{\alpha}(t)) = 0$, where ω is the connection form corresponding to ∇.

Parallel Transport and horizontal lift

Recall that a vector field Z on $\mathrm{L}(M)$ is called *horizontal* if $\omega(Z) = 0$ and *standard* if $\theta(Z)$ is a constant function.

Proposition 1.2

Let M be an n-dimensional manifold with a linear connection ∇ and ω the connection form of ∇ on $\mathrm{L}(M)$.

1. *Let Z be a standard horizontal vector field on $\mathrm{L}(M)$. For any $z \in \mathrm{L}(M)$, the curve defined by $\gamma(t) := \pi(\varphi_t^Z(z))$ is a geodesic on M.*
2. *Conversely, given a geodesic $\gamma : [-a, a] \longrightarrow M$, there exists a local standard horizontal vector field Z on $\mathrm{L}(M)$ and $\epsilon > 0$ such that $\gamma(t) = \varphi_t^Z(z)$ for any $-\epsilon < t < \epsilon$.*

Parallel Transport and horizontal lift

Proof.

We only prove the first point. Let $z \in \mathrm{L}(M)$ *then there exists* $\epsilon > 0$ *such that the curve* $\alpha :]-\epsilon, \epsilon[\longrightarrow \mathrm{L}(M)$ *given by* $\alpha(t) = \varphi_t^Z(z)$ *is well-defined and smooth. Let* $\gamma(t) = \pi(\alpha(t))$*, since* $\omega_{\alpha(t)}(\alpha'(t)) = \omega_{\alpha(t)}(Z_{\alpha(t)}) = 0$ *then Proposition 1.1 gives that* α *is a horizontal lift of* γ *on* $\mathrm{L}(M)$*, therefore if we write :*

$$\alpha(t) = (V_1(t), \dots, V_n(t)), \quad V_i \in \Gamma(\gamma^{-1}TM),$$

we get that $\{V_1, \dots, V_n\}$ *is a parallel frame along* γ*. On the other hand if we write* $\theta_z(Z_z) = (a_1, \dots, a_n)$ *then we get that :*

$$\gamma'(t) = T_{\alpha(t)}\pi(Z_{\alpha(t)}) = \alpha(t)(\theta_{\alpha(t)}(Z_{\alpha(t)})) = \alpha(t)(\theta_z(Z_z)) = \sum_{i=1}^{n} a_i V_i(t),$$

which shows that γ' *is parallel along* γ *i.e* γ *is a geodesic.* □

Parallel Transport and horizontal lift

Denote $\mathfrak{aff}(M, \nabla) := \mathrm{Lie}(\mathrm{Aff}(M, \nabla))$, any $X \in \mathfrak{aff}(M, \nabla)$ defines a smooth vector field $\hat{X}$ of $\mathrm{L}(M)$ given by :

$$\hat{X}_z := \frac{d}{dt}_{t=0} \exp(tX)_*(z).$$

It is clear that $\hat{X}$ is a complete vector field on $\mathrm{L}(M)$.

Parallel Transport and horizontal lift

Proposition 1.3

Let M be an n-dimensional manifold, ∇ a linear connection on M, and let $X \in \mathfrak{aff}(M, \nabla)$. Suppose that $\omega_z(\hat{X}_z) = 0$ for some $z \in \mathrm{L}(M)$. Then the curve $\gamma : \mathbb{R} \longrightarrow M$, $\gamma(t) = \exp(tX) \cdot x$ with $x = \pi(z)$ is a geodesic and its horizontal lift at z is the curve $\hat{\gamma}(t) := \exp(tX)_*(z)$, $t \in \mathbb{R}$.

Parallel Transport and horizontal lift

Proof.

Put $\hat{\gamma}(t) = \exp(tX)_*(z)$, *then clearly* $\pi(\hat{\gamma}(t)) = \gamma(t)$, *moreover from the relation* $\exp(tX)^*\omega = \omega$ *we get that* $\omega_{\exp(tX)_* z}(\hat{X}_{\exp(tX)_* z}) = 0$ *which means that* $\hat{\gamma}$ *is the horizontal lift of* γ *through* z*, in particular if* $z = (e_1, \ldots, e_n)$ *then :*

$$\hat{\gamma}(t) = (E_1(t), \ldots, E_n(t)),$$

E_i *being the parallel transport of* e_i *along* γ*. Moreover* $\exp(tX)^*\theta = \theta$ *gives that* $\theta_{\exp(tX)_* z}(\hat{X}_{\exp(tX)_* z}) = \theta_z(\hat{X}_z)$ *so if* $\theta_z(X_z) = (a_1, \ldots, a_n)$ *we get that :*

$$\gamma'(t) = T_{\hat{\gamma}(t)}\pi(\hat{X}_{\hat{\gamma}(t)}) = \hat{\gamma}(t)(\theta_{\hat{\gamma}(t)}(\hat{X}_{\hat{\gamma}(t)})) = \hat{\gamma}(t)(\theta_z(\hat{X}_z)) = \sum_{i=1}^{n} a_i E_i(t).$$

Hence γ' *is parallel along* γ*, i.e* γ *is a geodesic.* □

Main Results

Theorem 6

Let M be an n-dimensional manifold with a linear connection ∇. Then $\dim(\mathrm{Aff}(M, \nabla)) = n(n+1)$ *if and only if M is an ordinary affine space with the natural flat affine connection.*

Proof.

Denote $G := \mathrm{Aff}(M, \nabla)$ and for any $x \in M$ denote G_x the isotropy at x for the natural action of G on M. First note that the map :

$$G_x \longrightarrow \mathrm{GL}(T_x M), \quad f \mapsto T_x f \tag{4}$$

is an injective Lie group homomorphism. Assume now that $\dim G = n(n+1)$, then let $x \in M$ and $z \in \mathrm{L}(M)$ such that $\pi(z) = x$. Since the map :

$$G \xrightarrow{\Psi} \mathrm{L}(M), \quad f \mapsto f_*(z)$$

is an imbedding of G onto a closed submanifold of $\mathrm{L}(M)$ and $\dim \mathrm{L}(M) = n(n+1)$, then either $\Psi(G) = \mathrm{L}(M)$ or $\Psi(G)$ is a connected component of $\mathrm{L}(M)$ and in any case we get that $M = G \cdot x \simeq G/G_x$, therefore :

$$\dim G_x = \dim G - \dim M = n^2.$$

This gives that $G_x^0 = \mathrm{GL}^+(T_x M)$ under the identification (4).

Main Results

Now let $t > 0$ and consider the transformation $A_t \in \mathrm{GL}^+(T_x M)$ given by $A_t(u) = tu$. From the previous remark there exists $f_t \in G_x^0$ such that $T_x f_t = A_t$, hence for any $u, v, w \in T_x M$ we get that :

$$A_t(R_x^\nabla(u,v)w) = R_x^\nabla(A_t u, A_t v)A_t w, \quad A_t(T_x^\nabla(u,v)) = T_x^\nabla(A_t u, A_t v),$$

therefore $R_x^\nabla(u,v)w = t^{-2} R_x^\nabla(u,v)w$ and $T_x^\nabla(u,v) = t^{-1} T_x^\nabla(u,v)$ for all $t > 0$, and so we conclude that $R^\nabla = 0$ and $T^\nabla = 0$.

Main Results

On the other hand, let Z be a standard horizontal vector field on $\mathrm{L}(M)$. If $\mathfrak{g} := \mathrm{Lie}(G)$ then there exists a unique $X \in \mathfrak{g}$ such that $Z_z = \hat{X}_z$ where :

$$\hat{X}_{\widetilde{z}} := \frac{d}{dt}_{t=0} \exp(tX)_* \widetilde{z}, \quad \widetilde{z} \in \mathrm{L}(M).$$

From Proposition 1.3 we get that $\gamma(t) = \exp(tX) \cdot x$ is a geodesic with horizontal lift at z the curve $\hat{\gamma}(t) = \exp(tX)_*(z)$ defined for any $t \in \mathbb{R}$. Now $\gamma : \mathbb{R} \longrightarrow M$ is the geodesic with initial conditions $\gamma(0) = x$ and $\gamma'(0) = T_z\pi(Z_z)$ and therefore its horizontal lift at z is exactly

$$\alpha :]-\epsilon, \epsilon[\longrightarrow \mathrm{L}(M), \quad t \mapsto \varphi_t^Z(z),$$

which proves that α can be extended to all of $\mathbb{R}$. Since $z \in \mathrm{L}(M)$ was arbitrary we get that Z is complete, and since we know by Proposition 1.2 that geodesics of M are exactly the projections of integral curves of standard horizontal vector fields, we conclude that M is (geodesically) complete.

Main Results

Consider now the universal cover $\widetilde{M}$ of (M, ∇) with its induced induced linear connection, then there exists an affine transformation $\widetilde{M} \xrightarrow{\simeq} \mathbb{R}^n$. Next $M = \widetilde{M}/\Gamma$ where Γ is a discrete subgroup of

$$\mathrm{Aff}(\widetilde{M}, \nabla) \simeq \mathrm{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$$

hence commuting with $\mathrm{Aff}(\widetilde{M}, \nabla)^0 \simeq \mathrm{GL}^+(n, \mathbb{R}) \rtimes \mathbb{R}^n$. But one can show that only the trivial element commutes with connected component of $\mathrm{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$, hence Γ is trivial and M is itself simply connected i.e $\widetilde{M} = M$, this completes the proof. □

Main Results

Theorem 7

Let M be an n-dimensional manifold with an affine connection and assume that $\dim \mathrm{Aff}(M, \nabla) > n^2$. *Then ∇ is torsion-free.*

This result is a consequence of the following algebraic Lemma :

Lemma 8

Let V ba an n-dimensional vector space and $T : V \times V \longrightarrow V$ a non-trivial skew-symmetric bilinear map i.e $T \in V \otimes \Lambda^2 V^*$. Denote H the subgroup of linear transformation preserving T, then $\dim H \leq n^2 - n$.

Main Results

Proof of the Theorem.

Denote $G := \mathrm{Aff}(M, \nabla)$ then let $x \in M$ and denote G_x the isotropy at x for the natural action of G on M, then from $G/G_x \simeq G \cdot x$ we get :

$$\dim(G_x) \geq \dim(G) - \dim(M) > n^2 - n \tag{5}$$

On the other hand denote T^∇ the torsion tensor of ∇, then for every $f \in G_x$ we get that :

$$T_x^\nabla(T_x f(u), T_x f(v)) = T_x f(T_x^\nabla(u, v)), \quad u, v \in T_x M.$$

Therefore the group $\{T_x f,\ f \in G_x\} \simeq G_x$ preserves T_x^∇, but according to the previous Lemma and (5) we conclude that $T_x^\nabla = 0$ for any $x \in M$, i.e ∇ is torsion-free. □

Main Results

Another result in the same spirit is the following Theorem due to Egorov and can be proved by essentially the same procedure :

Theorem 9

Let M be an n-dimensional manifold and ∇ a linear connection on M such that $\dim \mathrm{Aff}(M, \nabla) > n^2$. Then ∇ has neither torsion nor curvature provided that $n \geq 4$.

Table of Contents

The affine group of a Riemannian manifold

The goal of this part it to prove the following Result :

Yano's Theorem

Let $(M, \langle\ ,\ \rangle)$ be a compact Riemannian manifold with Levi-Civita connection ∇. Then $\mathrm{Aff}(M, \nabla)^0 = \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$.

Holonomy

Let M be an n-dimensional manifold and ∇ a linear connection on M. For any smooth curve $\gamma : [a, b] \longrightarrow M$ one can define the linear map $\tau^{\gamma}_{a,b} : T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M$ by the formula $\tau^{\gamma}_{a,b}(v) = V(b)$ where $V \in \Gamma(\gamma^{-1}TM)$ is the parallel transport of v along γ (relative to ∇).

Holonomy

Proposition 1.1

1. $\tau^{\gamma}_{a,b}$ *does not depend on the orientation-preserving parametrization of the curve* γ.
2. *Denote* $\gamma_1 := \gamma_{|[a,t_0]}$ *and* $\gamma_2 := \gamma_{|[t_0,b]}$ *i.e* $\gamma = \gamma_1 * \gamma_2$, *then :*

$$\tau^{\gamma}_{a,b} = \tau^{\gamma_2}_{t_0,b} \circ \tau^{\gamma_1}_{a,t_0}.$$

3. *For any smooth curve* γ, $\tau^{\gamma}_{a,b}$ *is an isomorphism and its inverse is exactly the linear operator*

$$\tau^{\gamma^-}_{a,b} : T_{\gamma(b)}M \longrightarrow \tau_{\gamma(a)}M$$

with $\gamma^-(t) = \gamma(a+b-t)$.

Holonomy

These properties allows to extend the definition of $\tau^{\gamma}_{a,b}$ for piecewise smooth curves $\gamma : [a, b] \longrightarrow M$ by setting :

$$\tau^{\gamma}_{a,b} := \tau^{\gamma}_{t_k,b} \circ \tau^{\gamma}_{t_{k-1},t_k} \circ \cdots \circ \tau^{\gamma}_{t_1,t_2} \circ \tau^{\gamma}_{a,t_1},$$

where $a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$ is any subdivision of $[a, b]$ such that the curve $\gamma_{[t_i,t_{i+1}]}$ is smooth. The previous properties extend to this situation as well :

Holonomy

Proposition 1.2

Let M be a smooth manifold with a linear connection ∇. Then :

1. $\tau_{a,b}^{\gamma}$ *does not depend on the orientation-preserving parametrization of the piecewise smooth curve* $\gamma : [a,b] \longrightarrow M$.
2. *Given two piecewise smooth curves* $\gamma_1 : [a,b] \longrightarrow M$ *and* $\gamma_2 : [b,c] \longrightarrow M$ *such that* $\gamma_2(b) = \gamma_1(b)$ *then* $\tau_{a,c}^{\gamma_1 * \gamma_2} = \tau_{b,c}^{\gamma_2} \circ \tau_{a,b}^{\gamma_1}$.
3. *For any piecewise smooth curve* $\gamma : [a,b] \longrightarrow M$, $\tau_{a,b}^{\gamma}$ *is invertible with inverse* $\tau_{a,b}^{\gamma^{-}}$.

Holonomy

It is more convenient therefore to denote $\tau^{\gamma}_{a,b}$ by $\tau^{\gamma}_{\gamma(a),\gamma(b)}$ instead or just $\tau^{\gamma}_{\gamma(a)}$ when γ is a loop. Fix $x_0 \in M$ and define :

$$\mathrm{Hol}_{x_0}(M,\nabla) := \{\tau^{\gamma}_{x_0} : T_{x_0}M \xrightarrow{\simeq} T_{x_0}M,\ \gamma \text{ is a loop based at } x_0\}.$$

Using the above observations, it is clear that $\mathrm{Hol}_{x_0}(M,\nabla)$ is a subgroup of $\mathrm{GL}(T_{x_0}M)$ called the *holonomy group of* (M,∇) *at* x_0. We also define the *restricted holonomy of* (M,∇) *at* x_0 to be :

$$\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla) := \{\tau^{\gamma}_{x_0} : T_{x_0}M \xrightarrow{\simeq} T_{x_0}M,\ \gamma \text{ is a contractible loop based at } x_0\},$$

which is obviously a subgroup of the holonomy group since concatenation and inverse of contractible loops remains contractible. It is also straightforward to see that $\widetilde{\mathrm{Hol}}_{x_0}(M,\nabla)$ is normal in $\mathrm{Hol}_{x_0}(M,\nabla)$.

Holonomy

Theorem 10

Let M be a smooth manifold and ∇ a linear connection on M. The holonomy group $\mathrm{Hol}_{x_0}(M, \nabla)$ possesses the structure of an (immersed) Lie subgroup of $\mathrm{GL}(T_{x_0}M)$ and $\widetilde{\mathrm{Hol}}_{x_0}(M, \nabla) = \mathrm{Hol}_{x_0}(M, \nabla)^0$.

Holonomy

Proposition 1.3

Let M be a connected manifold with a linear connection ∇ and let $x, y \in M$. Then for any piecewise smooth curve $\gamma : [a, b] \longrightarrow M$ joining x to y, the map :

$$\mathrm{Hol}_x(M, \nabla) \longrightarrow \mathrm{Hol}_y(M, \nabla), \quad g \mapsto \tau^{\gamma}_{x,y} \circ g \circ (\tau^{\gamma}_{x,y})^{-1},$$

is an isomorphism.

De Rham decomposition Theorem

Let $(M, \langle \ , \ \rangle)$ be a connected Riemannian manifold with Levi-Civita connection ∇. Then following that $\nabla \langle \ , \ \rangle = 0$ we get that for any smooth curve $\gamma : [a, b] \longrightarrow M$ and any $V, W \in \Gamma(\gamma^{-1}TM)$:

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle D_{\dot\gamma} V(t), W(t)\rangle + \langle V(t), D_{\dot\gamma} W(t)\rangle.$$

In particular if V and W are parallel along γ then $t \mapsto \langle V(t), W(t)\rangle$ is a constant map and therefore $\langle \tau^{\gamma}_{a,b}(v), \tau^{\gamma}_{a,b}(w)\rangle = \langle v, w\rangle$. This leads to the following result :

De Rham decomposition Theorem

Proposition 2.1

Let $(M, \langle\ ,\ \rangle)$ *be a Riemannian manifold with Levi-Civita connection* ∇*. Then* $\mathrm{Hol}_x(M, \nabla) \subset \mathrm{O}(T_xM, \langle\ ,\ \rangle)$ *for any* $x \in M$*.*

De Rham decomposition Theorem

For any $x \in M$, we will say that $T_x M$ is irreducibe if it does not admit any proper, non-trival subspace that is invariant by the action of the holonomy group at x.

In view of Proposition 1.3 we see that if $T_x M$ is irreducible then $T_y M$ is also irreducible for any $y \in M$. This suggests the following definition :

De Rham decomposition Theorem

Definition 2.1

A Riemannian manifold $(M, \langle\ ,\ \rangle)$ is said to be irreducible if T_xM is irreducible for some (hence every) $x \in M$.

De Rham decomposition Theorem

Theorem 11 (De Rham decomposition theorem)

A simply connected, complete Riemannian manifold $(M, \langle \ , \ \rangle)$ *is isometric to the direct product* $M_0 \times \ldots, \times M_k$ *where* M_0 *is a Euclidean space and* $M_1, \ldots, M_k$ *are all simply connected, irreducible Riemannian manifolds. Such a decomposition is a unique up to the order of the factors involved.*

The affine group of a Riemannian manifold

Corollary 3.1

Let $(M, \langle\ ,\ \rangle)$ be a simply connected, complete Riemannian manifold and $M = M_0 \times \cdots \times M_k$ its de Rham decomposition. Let $x = (x_0, \ldots, x_k)$.

1. *The identification*

   $$\mathrm{Hol}_{x_1}(M_1, \langle\ ,\ \rangle) \times \cdots \times \mathrm{Hol}_{x_k}(M_k, \langle\ ,\ \rangle) \mapsto \mathrm{Hol}_x(M, \langle\ ,\ \rangle)$$

   given by $(\tau_{x_1}^{\gamma_1}, \ldots, \tau_{x_k}^{\gamma_k}) \mapsto \tau_x^{\alpha_1} \circ \cdots \circ \tau_x^{\alpha_k}$ is an isomorphism, where α_i is the loop given by $\alpha_i(t) = (x_1, \ldots, \gamma_i(t), \ldots, x_k)$.
2. *Under the previous identification, $\mathrm{Hol}_{x_i}(M_i, \langle\ ,\ \rangle)$ is a normal subgroup of $\mathrm{Hol}_x(M, \langle\ ,\ \rangle)$ acting trivially on $T_{x_j}M_j$ for $j \neq i$ and irreducibly on $T_{x_i}M_i$.*
3. *For any $f \in \mathrm{Aff}(M, \nabla)$ and any $i = 1, \ldots, k$,*

   $$T_x f(T_{x_0}M_0) = T_{f(x)_0}M_0, \quad \text{and} \quad T_x f(T_{x_i}M_i) = T_{f(x)_j}M_j,$$

   for some $j = 1, \ldots, k$. If $f \in \mathrm{Aff}(M, \nabla)^0$, $T_x f(T_{x_i}M_i) = T_{f(x)_i}M_i$.

Proof.
We only prove the third point. Let $f \in \mathrm{Aff}(M, \nabla)$ and choose a piecewise smooth loop $\gamma : [0,1] \longrightarrow M$ based at $f(x)$. Then for any $v \in T_{x_0}M_0$:

$$\tau_{f(x)}^{\gamma}(T_x f(v)) = T_x f(\tau_x^{f^{-1}\circ\gamma}(v)) = T_x f(v),$$

so $T_x f(v)$ is invariant by $\mathrm{Hol}_{f(x)}(M, \nabla)$ thus $T_x f(T_{x_0}M_0) = T_{f(x)_0}M_0$. On the other hand, if $w \in T_{x_i}M_i$ for $i \neq 0$ then :

$$\tau_{f(x)}^{\gamma}(T_x f(w)) = T_x f(\tau_x^{f^{-1}\circ\gamma}(w)) \in T_x f(T_{x_i}M_i),$$

thus $T_x f(T_{x_i}M_i)$ is invariant, furthermore if $V \subset T_x f(T_{x_i}M_i)$ is any invariant subspace then in the same way $(T_x f)^{-1}(V)$ is an invariant subspace of $T_{x_i}M_i$ thus it is either trivial or equal to $T_{x_i}M_i$ proving that $T_x f(T_{x_i}M_i)$ is irreducible, in particular one gets the decomposition of $T_{f(x)}M$ into the sum of irreducible subspaces :

$$T_{f(x)}M = T_x f(T_{x_0}M_0) \oplus \cdots \oplus T_x f(T_{x_k}M_k),$$

and by uniqueness of such decomposition we conclude that $T_x f(T_{x_i}M_i) = T_{f(x)_j}M_j$ for some $j = 1, \ldots, k$.

The affine group of a Riemannian manifold

Proof. (Continued)
Next let X be a complete affine vector field on M i.e $X \in \mathfrak{aff}(M, \nabla)$, and consider the curve $\gamma(t) = \exp(tX) \cdot x$. Let $v_i \in T_{x_i} M_i$, and consider $u : \mathbb{R} \longrightarrow \mathbb{R}$ given by :

$$u(t) = \langle T_x \exp(tX)(v_i), \tau^{\gamma}_{0,t}(v_i) \rangle_{\gamma(t)}.$$

Then u is a smooth function satisfying $u(0) = \langle v_i, v_i \rangle_x \neq 0$ and therefore $u(t) \neq 0$ for $-\delta < t < \delta$, which shows that $T_x \exp(tX)(v_i) \in T_{\gamma(t)_i} M_i$ for all $-\delta < t < \delta$. In fact since $T_{x_i} M_i$ is finite-dimensional, one can choose $\delta > 0$ small enough so that

$$T_x \exp(tX)(T_{x_i} M_i) \in T_{\gamma(t)_i} M_i,$$

for any $-\delta < t < \delta$. The result follows from the fact that $\mathrm{Aff}(M, \nabla)^0$ is generated by 1-parameter subgroups. □

The affine group of a Riemannian manifold

Theorem 12

Let $M = M_0 \times \cdots \times M_k$ be the de Rham decomposition of a complete, simply connected Riemannian manifold $(M, \langle\ ,\ \rangle)$. Then :

$$\mathrm{Isom}(M, \langle\ ,\ \rangle)^0 = \mathrm{Isom}(M_0, \langle\ ,\ \rangle)^0 \times \cdots \times \mathrm{Isom}(M_k, \langle\ ,\ \rangle)^0,$$

$$\mathrm{Aff}(M, \nabla)^0 = \mathrm{Aff}(M_0, \nabla)^0 \times \cdots \times \mathrm{Aff}(M_k, \nabla)^0,$$

where ∇ is the Levi-Civita connection of M.

Proof.
Consider the homomorphism $\Psi : \mathrm{Diff}(M_0) \times \cdots \times \mathrm{Diff}(M_k) \longrightarrow \mathrm{Diff}(M)$ which corresponds to any k-tuple of diffeomorphisms $(f_0, \ldots, f_k)$ the transformation $f : M \longrightarrow M$ given by :

$$f(x_0, \ldots, x_k) = (f_0(x_0), \ldots, f_k(x_k)).$$

Clearly Ψ is continuous and injective. We claim that $f = \Psi(f_0, \ldots, f_k)$ is an affine transformation if and only if f_i is an affine transformation for any $0 \leq i \leq k$. Indeed let $\gamma : [0,1] \longrightarrow M$ be any piecewise smooth curve and write $\gamma := (\gamma_0, \ldots, \gamma_k)$ then choose $v = v_0 \oplus \cdots \oplus v_k \in T_{\gamma(0)}M$ with $v_i \in T_{\gamma_i(0)}M_i$, then :

$$T_{\gamma(1)}f \circ \tau_{0,1}^{\gamma}(v) = \sum_{i=1}^{k} T_{y_i}f_i(\tau_{0,1}^{\gamma_i}(v_i)), \quad \tau_{0,1}^{f\circ\gamma}(T_{\gamma(0)}f(v)) = \sum_{i=1}^{k} \tau_{0,1}^{f_i\circ\gamma_i}(T_{x_i}f_i(v_i)),$$

which shows that f preserves parallel transports on M if and only if each f_i does so on M_i proving the claim. One can also prove in a similar way that f is an isometry if and only if every f_i is an isometry. In particular :

$$\Psi(\mathrm{Aff}(M_0) \times \cdots \times \mathrm{Aff}(M_k)) \subset \mathrm{Aff}(M)$$

$$, \quad \Psi(\mathrm{Isom}(M_0) \times \cdots \times \mathrm{Isom}(M_k)) \subset \mathrm{Isom}(M).$$

Proof. (Continued)
Let $f \in \mathrm{Aff}(M, \nabla)^0$ and $\mathrm{pr}_i : M \longrightarrow M_i$ be the projection on the i-th component then denote $g_i := \mathrm{pr}_i \circ f$, we will show that $g_i(x_0, \ldots, x_k)$ only depends on x_i. Indeed let $x = (x_0, \ldots, x_k) \in M$, $j \neq i$ and $v_j \in T_{x_j} M_j$ then by (3) of Corollary 3.1 :

$$T_x g_i(v_j) = T_{f(x)} \mathrm{pr}_i(\underbrace{T_x f(v_j)}_{\in M_j}) = 0$$

Therefore if we fix $(a_0, \ldots, a_k) \in M$ and define $f_i : M_i \longrightarrow M_i$ by the expession :

$$f_i(y) := g_i(a_0, \ldots, y, \ldots, a_k),$$

then f_i is a well-defined diffeomorphism of M_i and $f = \Psi(f_0, \ldots, f_k)$. It also follows that if $f \in \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$ then each f_i is an isometry. □

The affine group of a Riemannian manifold

Theorem 13

Let $(M, \langle\ ,\ \rangle)$ be a complete, irreducible Riemannian manifold, then $\mathrm{Aff}(M, \nabla) = \mathrm{Isom}(M, \langle\ ,\ \rangle)$ except when M is a 1-dimensional Euclidean space.

The affine group of a Riemannian manifold

The proof of this Theorem will be done in two steps : First one proves that on any such manifold, any affine transformation is homothetic and if furthermore $(M, \langle\ ,\ \rangle)$ is not Euclidean then homothetic transformations are isometries, the result follows then by observing that only 1-dimensional Euclidean spaces can be irreducible.

The affine group of a Riemannian manifold

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, and recall that $f \in \mathrm{Diff}(M)$ is said to be a *homothetic transformation* if there exists a positive constant $c > 0$ such that

$$\langle T_x f(v), T_x f(w) \rangle = c^2 \langle v, w \rangle$$

for all $x \in M$ and $v, w \in T_x M$, i.e $f^* \langle\ ,\ \rangle = c^2 \langle\ ,\ \rangle$.

The affine group of a Riemannian manifold

If ∇ is the Levi-Civita of $(M, \langle\ ,\ \rangle)$, then the Levi-Civita connection $\widetilde{\nabla}$ for $f^*\langle\ ,\ \rangle$ is given by :

$$\widetilde{\nabla}_X Y := f_*^{-1}(\nabla_{f_*X} f_* Y), \tag{6}$$

When $f : M \longrightarrow M$ is a homothetic transformation then $\langle\ ,\ \rangle$ and $f^*\langle\ ,\ \rangle$ share the same Levi-Civita connection i.e $\widetilde{\nabla} = \nabla$, hence *any homothetic transformation is an affine transformation*. Conversely :

The affine group of a Riemannian manifold

Lemma 14

If $(M, \langle \ , \ \rangle)$ *is an irreducible Riemannian manifold, then every affine transformation* $f : M \longrightarrow M$ *is homothetic.*

Proof. Clearly $\langle\ ,\ \rangle$ and $f^*\langle\ ,\ \rangle$ define the same Levi-Civita ∇.

Recall that if $G \subset \mathrm{O}(V, \langle\ ,\ \rangle)$ acts irreducibly on a Euclidean vector space $(V, \langle\ ,\ \rangle)$ and preserves a symmetric bilinear form B, then we can find $c > 0$ such that $B = c^2\langle\ ,\ \rangle$. Applying this fact to

$$(V, \langle\ ,\ \rangle) = (T_x M, \langle\ ,\ \rangle_x), \quad G = \mathrm{Hol}_x(M, \langle\ ,\ \rangle) \ \text{ and } \ B = (f^*\langle\ ,\ \rangle)_x,$$

we obtain that for any $x \in M$, $(f^*\langle\ ,\ \rangle)_x = c_x^2\langle\ ,\ \rangle_x$ for some $c_x > 0$. Finally, if $y \in M$ is another point and $\gamma : [0,1] \longrightarrow M$ is a piecewise smooth curve joining x to y, then for every $v \in T_x M$:

$$\begin{aligned} c_y^2\langle\tau_x^\gamma(v), \tau_x^\gamma(v)\rangle_y &= \langle T_y f(\tau_x^\gamma(v)), T_y f(\tau_x^\gamma(v))\rangle_f(y) \\ &= \langle\tau_{f(x)}^{f\circ\gamma}(T_x f(v)), \tau_{f(x)}^{f\circ\gamma}(T_x f(v))\rangle_{f(y)} \\ &= \langle T_x f(v), T_x f(v)\rangle_{f(x)} \\ &= c_x^2\langle v, v\rangle_x. \end{aligned}$$

Since $\langle v, v\rangle_x = \langle\tau_x^\gamma(v), \tau_x^\gamma(v)\rangle_y$ for any $v \in T_x M$ we get that $c_x = c_y$, completing the proof. □

The affine group of a Riemannian manifold

Lemma 15

If $(M, \langle\ ,\ \rangle)$ is a complete Riemannian manifold which is not locally Euclidean, then every homothetic transformation is an isometry.

Proof.
Suppose that $(M, \langle\ ,\ \rangle)$ admits a homothetic transformation $f : M \longrightarrow M$ that isn't an isometry, and write $f^*\langle\ ,\ \rangle = c^2\langle\ ,\ \rangle$ with $c > 0$. Next notice that f^{-1} is homothetic as well with ration $1/c$, therefore we suppose without loss of generality that $0 < c < 1$.

We start by proving that f has a fixed point. Denote $d : M \times M \longrightarrow \mathbb{R}^+$ the geodesic distance and take an arbitrary point $x \in M$ then put $\ell := d(x, f(x))$. Let $\gamma : [0,1] \longrightarrow M$ be a minimizing geodesic joining x to $f(x)$, which exists since M is complete, then $f^i \circ \gamma$ is a smooth curve joining $f^i(x)$ and $f^{i+1}(x)$ with length :

$$\ell_i = \int_0^1 \langle (f^i \circ \gamma)'(t), (f^i \circ \gamma)'(t) \rangle^{\frac{1}{2}}_{f^i \circ \gamma(t)} dt = c^i \ell,$$

Therefore if $m, n \in \mathbb{N}$ are such that $m < n$ then :

$$d(f^m(x), f^n(x)) \leq \sum_{i=m}^{n-1} d(f^i(x), f^{i+1}(x)) \leq \sum_{i=m}^{n+1} \ell_i = \sum_{i=m}^{n+1} c^i \ell \leq \frac{c^m \ell}{1-c},$$

and thus $(f^m(x))_m$ is a Cauchy sequence in (M, d) hence converges to some $x^* \in M$ since M is complete.

Proof. (Continued)
Now x^* is obviously a fixed point of f, furthermore x^* does not depend on the choice of x, indeed given $z \in M$ and α a geodesic joining z to x^*, we get that $f^m \circ \alpha$ is a curve joining $f^m(z)$ to $f^m(x^*) = x^*$ and so :

$$d(f^m(z), x^*) \leq \ell(f^m \circ \alpha) = c^m \ell(\alpha) \underset{m \to +\infty}{\longrightarrow} 0. \tag{7}$$

Now fix a neighborhood U of x^* in M with compact closure. Then there exists a constant $K^* > 0$ such that for any $y \in U$ and any unit vectors $v_1, v_2 \in T_yM$:

$$|\langle R_y(v_1, v_2)v_1, v_2\rangle_y| \leq K^*, \tag{8}$$

where R denotes the curvature tensor of $(M, \langle\ ,\ \rangle)$. Since f is also an affine transformation, then for any $z \in M$ and any orthonormal family $\{v, w\}$ of T_zM :

$$\langle R_{f^m(z)}(f_*^m v, f_*^m w)f_*^m v, f_*^m w\rangle = \langle f_*^m(R_z(v, w)v), f_*^m w\rangle = c^{2m}\langle R_z(v, w)v, w\rangle. \tag{9}$$

Proof. (Continued)
According to (7) there exists $N \in \mathbb{N}$ such that $f^m(z) \in U$ for any $m \geq N$, moreover $\|f_*^m v\| = \|f_*^m w\| = c^m$, thus using (8) :

$$|\langle R_{f^m(z)}(f_*^m v, f_*^m w) f_*^m v, f_*^m w\rangle| \leq K^* \|f_*^m v\|^2 \|f_*^m w\|^2 = c^{4m} K^*,$$

and finally (9) gives that $|\langle R_z(v,w)v,w\rangle| \leq c^{2m} K^*$ for every $m \geq N$. We conclude that $\langle R_z(v,w)v,w\rangle = 0$ for every $z \in M$ and any orthonormal family $\{v,w\}$ of T_zM i.e $(M, \langle\ ,\ \rangle)$ is locally Euclidean. □

The affine group of a Riemannian manifold

Theorems 12 and 13 have a number of interesting consequences, before we state them we need to make some remarks :

The affine group of a Riemannian manifold

Let X be an affine vector field on a complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ and denote $\widetilde{M}$ the universal cover of M with the induced metric $p^*\langle\ ,\ \rangle$ where $p : \widetilde{M} \longrightarrow M$ is the natural projection, then let $\widetilde{M} = M_0 \times \cdots \times M_k$ be its de Rham decomposition.

Next denote $\widetilde{X}$ the lift of X to $\widetilde{M}$, i.e the unique vector field on $\widetilde{M}$ satisfying

$$T_z p(\widetilde{X}_z) = X_{p(z)},$$

then $\widetilde{X}$ is an affine transformation and since X is complete, $\widetilde{X}$ is also complete hence an element of $\mathfrak{aff}(\widetilde{M}, \nabla)$. Moreover $\widetilde{X}$ is Killing if and only if X is Killing.

The affine group of a Riemannian manifold

By Theorem 12, we have $\mathfrak{aff}(\widetilde{M}, \nabla) \simeq \mathfrak{aff}(M_0, \nabla) \times \cdots \times \mathfrak{aff}(M_k, \nabla)$ and so $\widetilde{X}$ corresponds to a unique family $(X_0, \ldots, X_k)$ such that

$$X_i \in \mathfrak{aff}(M_i, \nabla).$$

According to Theorem 13 gives that $X_1, \ldots, X_k$ are all Killing vector fields, therefore X will be Killing if and only if X_0 is.

The affine group of a Riemannian manifold

Corollary 3.2

If M is a complete whose restricted holonomy group $\widetilde{\mathrm{Hol}}_x(M, \langle\ ,\ \rangle)$ have no nonzero invariant vector, then $\mathrm{Aff}(M, \nabla)^0 = \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$, where ∇ is the Levi-Civita connection of $(M, \langle\ ,\ \rangle)$.

Proof.
The restricted holonomy group $\widetilde{\mathrm{Hol}}_x(M, \langle\ ,\ \rangle)$ is isomorphic to the restricted holonomy group of its universal cover $\widetilde{M}$, this is due to the fact that contractible loops on M lift to (contractible) loops on $\widetilde{M}$, this gives that the map :

$$\widetilde{\mathrm{Hol}}_z(\widetilde{M}, \langle\ ,\ \rangle) \longrightarrow \widetilde{\mathrm{Hol}}_{p(z)}(M, \langle\ ,\ \rangle), \quad \tau_x^\gamma \mapsto T_z p \circ \tau_z^\gamma \circ (T_z p)^{-1} = \tau_z^{p\circ\gamma},$$

is an isomorphism for any $z \in \widetilde{M}$, moreover the restricted holonomy group of $\widetilde{M}$ coincides with the total holonomy group since $\widetilde{M}$ is simply connected. So our assumption just states that $\widetilde{M}$ has no Euclidean factor i.e M_0 is a point, which gives in view of the previous remarks that every affine vector field on M is a Killing vector field. □

The affine group of a Riemannian manifold

Corollary 3.3

If X is an affine vector field of a complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ and if the length of X is bounded, then X is a Killing vector field.

The affine group of a Riemannian manifold

Proof.
Denote $\widetilde{M}$ the universal cover of M with the induced metric $g := p^*\langle\ ,\ \rangle$ where $p : \widetilde{M} \longrightarrow M$ is the natural projection and let $\widetilde{M} = M_0 \times \cdots \times M_k$ be its De Rham decomposition. Next let $\widetilde{X}$ be the lift of X to $\widetilde{M}$ and write $\widetilde{X} = (X_0, \ldots, X_k)$ such that $X_i \in \mathfrak{aff}(M_i, \nabla)$. Then :

$$g(X_0, X_0) \leq g(\widetilde{X}, \widetilde{X}) = \langle X, X \rangle,$$

hence if X has bounded length then so does X_0. Write $X_0 = \sum \xi^i \partial/\partial x^i$ in some (global) Euclidean coordinate system $x^1, \ldots, x^r$ of M_0. Since X_0 is an affine vector field then it satisfies :

$$(L_{X_0} \circ \nabla_Y - \nabla_Y \circ L_{X_0})Z = \nabla_{[X_0, Z]} Y, \quad \text{i.e} \quad L_{X_0}\nabla = 0.$$

For $Y = \partial/\partial x^j$ and $Z = \partial/\partial x^k$ we get that $\nabla Y = 0$, $\nabla Z = 0$ hence by the previous expression $\nabla_Y[X_0, Z] = 0$ which is equivalent to :

$$\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0, \quad i, j, k = 1, \ldots r.$$

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Proof. (Continued)

This means that

$$X_0 = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} a_{ij} x^j + b_i \right) \partial/\partial x^i$$

where a_{ij} and b_i are constants, but since X_0 has bounded length it follows that $a_{ij} = 0$ for all $i, j = 1, \ldots, r$ proving that X_0 is a linear combination of $\partial/\partial x^1, \ldots, \partial/\partial x^r$ each of which is a Killing vector field on M_0, we thus conclude that X_0 is Killing. By the previous remarks, X is a Killing vector field on M. □

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If M is a compact Riemannian manifold then the length of any vector field is bounded, therefore :

Corollary 3.4 (Yano's Theorem)

Let $(M, \langle\ ,\ \rangle)$ be a compact Riemannian manifold with Levi-Civita connection ∇. Then $\mathrm{Aff}(M, \nabla)^0 = \mathrm{Isom}(M, \langle\ ,\ \rangle)^0$.

END

Thanks for your attention