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Introduction et développements de la géométrie de Koszul-Vinberg

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Abstract

In this thesis, we study differential geometry of Koszul-Vinberg manifolds. A Koszul-Vinberg manifold is a manifold M endowed with a pair (∇, h) where ∇ is a flat torsionless connection and h is a symmetric bivector field satisfying a generalized Codazzi equation. When h is invertible, we recover the known notion of pseudo-Hessian manifold. Koszul-Vinberg manifolds have properties similar to Poisson manifolds and, in fact, to any Koszul-Vinberg manifold (M, ∇, h) we associate naturally a Poisson tensor on TM. We investigate these properties and we study in detail many classes of such structures in order to highlight the richness of the geometry of these manifolds. We introduce a notion of Koszul-Vinberg submanifolds and we study their properties by taking into account some developments in the theory of Poisson submanifolds.

In the second part of the thesis, we study invariant Koszul-Vinberg structures on homogeneous spaces. More precisely, we give an algebraic characterization of these structures in the same spirit of Nomizu's theorem on invariant connections on homogeneous spaces. We give many classes of examples mainly for reductive or symmetric pairs (G, H). We establish many properties of pseudo-Hessian homogeneous manifolds.

Dedication

This work is dedicated to the spirit of my father and my mother.
May God have mercy on them and forgive them.

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I thank my family and friends for supporting me throughout my education and reminding me that it is good to take a break from time to time.

I especially want to thank my wife who always cheered me up in difficult times.

Contents

Introduction

A *pseudo-Hessian manifold* is a pseudo-Riemannian manifold (M, g) together with a flat torsionless connection ∇ satisfying the *Codazzi equation*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z). \tag{0.0.1}$$

Typically, these structures occur when investigating exponential families of probability measures (For more details about pseudo-Hessian structures one can see the book of Shima [49]). A *Koszul-Vinberg manifold* is a triple (M, ∇, h) of a manifold M equipped with a flat torsionless connection ∇ and h is a section of $S^2(TM)$ (rather than a section g of $S^2(T^*M)$) which does not have to be non-degenerate, but satisfies the *generalized Codazzi equation*

$$(\nabla_{\alpha^\#} h)(\beta, \gamma) = (\nabla_{\beta^\#} h)(\alpha, \gamma), \tag{0.0.2}$$

for all $\alpha, \beta, \gamma \in \Omega^1(M)$ and where $\alpha^\# := h_\#(\alpha)$ is defined by contraction with h. In case h is non-degenerate, identifying T^*M and TM by $h_\#$ yields a pseudo-Riemannian metric, and (0.0.2) is then equivalent to (0.0.1), so that this structure is indeed more general than pseudo-Hessian structures. This relation is analogous to the one between Poisson geometry and symplectic geometry. The Koszul-Vinberg theory is designed to make a symmetric analog of Poisson geometry where h replaces the Poisson bivector field. Whereas in Poisson geometry the antisymmetric nature of the Poisson tensor helps to kill second order derivatives, making a lot of things natural, in this Koszul-Vinberg theory the strong condition of affine manifolds is needed to get similar conclusions.

In this context, several natural questions arise. First of all, there is a general question: which properties of the usual, pseudo-Hessian manifolds hold for Koszul-Vinberg manifolds? Which properties of Poisson manifolds would remain valid for Koszul-Vinberg manifolds?

These questions lead to this thesis, whose results can be summarized as follows. We start by proving an analog of the Darboux-Weinstein Theorem in some cases. We show that each Koszul-Vinberg manifold gives rise to a Lie algebroid structure on the cotangent bundle. We show that the associated foliation of this Lie algebroid has affine leaves carrying a pseudo-Hessian structure in analogy with the symplectic leaves in the Poisson situation. We show that each Koszul-Vinberg manifold gives rise to a Poisson structure on the tangent bundle whose symplectic leaves are the tangent bundles of the integral manifolds. We show that the tangent functor $M \to TM$ sends the category of Koszul-Vinberg manifolds into the category of Poisson manifolds. We give special classes of examples. Koszul-Vinberg given by affine or quadratic coefficients as well as right invariant ones on Lie groups, relating the latter to well known structures. Analogously to the Poisson case, we show that the dual of any commutative, associative algebra naturally admits a linear Koszul-Vinberg structure. Moreover, we give some classification results in the two-dimensional case.

Then we proceed to introduce morphisms of Koszul-Vinbeg manifolds which are smooth affine maps between affine manifolds relating Koszul-Vinberg bivector fields. This allows us to speak of the category of Koszul-Vinberg manifolds. In this spirit, we define Koszul-Vinberg submanifolds as affine submanifolds with tangent bivector field making the immersion a Koszul-Vinberg map. We show that the tangent bundle of a Koszul-Vinberg submanifold is a Poisson submanifold of the tangent bundle of the ambient Koszul-Vinberg manifold. We introduce a coisotropic submanifold of a Koszul-Vinberg manifold analogously to a coisotropic submanifold in Poisson geometry. We show that the tangent functor sends these submanifolds to coisotropic submanifolds of the tangent bundle.

The last part of the thesis is devoted to the study of invariant Koszul-Vinberg structures on homogeneous spaces. We prove an algebraic characterization of G-invariant Koszul-Vinberg structures on G-homogeneous manifolds in the spirit of Nomizu's theorem on invariant connections (cf. [41, 28]). This leads us to build many classes of invariant Koszul-Vinberg structures on reductive pairs or symmetric spaces. We give also many classes of examples when the Koszul-Vinberg structure is actually a pseudo-Hessian structure. Among the results obtained, we will show that if the Lie group G is semi-simple then G/H does not admit any non trivial G-invariant pseudo-Hessian structure. Then, as an immediate consequence, we obtain a new proof of a result of Shima [49, Theorem 9.2]. We completely describe the regular affine foliation of an invariant Koszul-Vinberg structure.

These results are the subject of our three articles (two published and one submitted):

A. ABOUQATEB, M. BOUCETTA AND C. BOURZIK. *Contravariant Pseudo-Hessian manifolds and their associated Poisson structures*. Differential Geometry and its Applications 70 (2020): 101630.

A. ABOUQATEB, M. BOUCETTA AND C. BOURZIK. *Submanifolds in Koszul–Vinberg Geometry*. Results Math 77, 19 (2022).

A. ABOUQATEB, M. BOUCETTA AND C. BOURZIK. *Homogeneous spaces with invariant Koszul–Vinberg structures*. Submitted. (2022)

Notation. For simplicity, we use K-V notation instead of Koszul-Vinberg.

Chapter 1

K-V manifolds

In this chapter, we explore the basic properties of K-V structures and we will see how we can study them by exploiting their similarities with Poisson structures.

1.1 Affine manifolds

A connection ∇ on a manifold M is a mapping

$$\nabla : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \to \nabla_X Y \in \mathfrak{X}(M),$$

satisfying the following conditions,

1. $\nabla_{X+fY} Z = \nabla_X Z + f \nabla_Y Z$,
2. $\nabla_X (Y + fZ) = \nabla_X Y + X(f) Z + f \nabla_X Z$,

where $X, Y, Z \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

The torsion tensor of ∇ is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The connection ∇ is said to be torsionless if the tensor T vanish identically.

The curvature tensor R of ∇ is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is said to be flat if the tensor R vanish identically.

Definition 1.1.1. *A manifold (M, ∇) endowed with a flat torsionless connection ∇ is called an affine manifold.*

An affine manifold (M, ∇) is called complete if ∇ is complete. Equivalently, if its universal covering is homeomorphic to $\mathbb{R}^n$.

The following result is well known.

Proposition 1.1.2 ([49]).

1. *Suppose that (M, ∇) is an affine manifold. Then there exist a local coordinate systems $(x_1, \ldots, x_n)$ on M such that $\nabla \partial_{x_i} = 0$. The changes between such coordinate systems are affine transformations of $\mathbb{R}^n$.*

2. *Conversely, if M admits a local coordinate systems such that the changes of the local coordinate systems are affine transformations of $\mathbb{R}^n$, then there exists a flat torsionless connection ∇ on M.*

For a flat torsionless connection ∇, a local coordinate system $(x_1, \ldots, x_n)$ satisfying $\nabla \partial_{x_i} = 0$ is called an *affine local coordinate system* with respect ∇.

The flat torsionless connection ∇ on $\mathbb{R}^n$ defined by $\nabla \partial_{x_i} = 0$, where $(x_1, \ldots, x_n)$ is the canonical affine coordinate system on $\mathbb{R}^n$, is called the *canonical flat torsionless connection* on $\mathbb{R}^n$.

1.2 K-V manifolds

Let (M, ∇) be an affine manifold, h a symmetric bivector field on M, $h_\# : T^*M \to TM$ the associated contraction map given by $\prec \beta, h_\#(\alpha) \succ := h(\alpha, \beta)$, and for any $\alpha \in \Omega^1(M)$, $\alpha^\# := h_\#(\alpha)$.

Definition 1.2.1. *The triple (M, ∇, h) is called a K-V manifold if h satisfies the generalized Codazzi equation*

$$(\nabla_{\alpha^\#} h)(\beta, \gamma) = (\nabla_{\beta^\#} h)(\alpha, \gamma). \tag{1.2.1}$$

We call such h a K-V bivector field.

The local expression of the equation (1.2.1) in affine charts is given by the following proposition.

Proposition 1.2.2. *Let (M, ∇) be an affine manifold. Let h be a symmetric bivector field on M. For any $f \in C^\infty(M)$, put $X_f = (df)^\#$. The following assertions are equivalent*

1. *h is a K-V bivector field.*
2. *For any $f, g \in C^\infty(M)$ and any $\alpha \in \Omega^1(M)$,*
$$d\alpha\,(X_f, X_g) = \nabla_{X_f} \alpha^\#(g) - \nabla_{X_g} \alpha^\#(f).$$
3. *For any $f, g, \mu \in C^\infty(M)$,*
$$\nabla_{X_f} X_\mu(g) = \nabla_{X_g} X_\mu(f).$$
4. *for any $m \in M$, there exists a coordinate system $(x_1, \ldots, x_n)$ around m such that for any $1 \le i < j \le n$, $k = 1, \ldots, n$, we have*
$$\nabla_{X_{x_i}} X_{x_k}\,(x_j) = \nabla_{X_{x_j}} X_{x_k}\,(x_i)\,.$$
5. *for any $m \in M$, there exists an affine coordinate system $(x_1, \ldots, x_n)$ around m such that for any $1 \le i < j \le n$ and any $k = 1, \ldots, n$*
$$\sum_{l=1}^{n} [h_{il} \partial_{x_l}(h_{jk}) - h_{jl} \partial_{x_l}(h_{ik})] = 0, \tag{1.2.2}$$
where $h_{ij} = h(dx_i, dx_j)$.

Proof. Let h be a symmetric bivector field on (M, ∇) and $m \in M$. We choose an affine coordinate system $(x_1, \ldots, x_n)$ around m. Let $\alpha = dx_i$, $\beta = dx_j$ and $\gamma = dx_k$.

Then we have

$$\begin{aligned}(\nabla_{\alpha^\#}h)(\beta,\gamma) &= \alpha^\#.h(\beta,\gamma) - h(\nabla_{\alpha^\#}\beta,\gamma) - h(\beta,\nabla_{\alpha^\#}\gamma)\\ &= -\alpha^\#.h(\beta,\gamma) + \prec \beta, \nabla_{\alpha^\#}\gamma^\# \succ + \prec \gamma, \nabla_{\alpha^\#}\beta^\# \succ\\ &= -\sum_{l=1}^n h_{il}\partial_{x_l}(h_{jk}) + \sum_{l=1}^n h_{il}\partial_{x_l}(h_{kj}) + \sum_{l=1}^n h_{il}\partial_{x_l}(h_{jk})\\ &= \sum_{l=1}^n h_{il}\partial_{x_l}(h_{jk}).\end{aligned}$$

Hence we get that

$$(\nabla_{\alpha^\#}h)(\beta,\gamma) = (\nabla_{\beta^\#}h)(\alpha,\gamma),$$

if and only if,

$$\sum_{l=1}^n h_{il}\partial_{x_l}(h_{jk}) = \sum_{l=1}^n h_{jl}\partial_{x_l}(h_{ik}).$$

■

Example 1.2.3.

1. *Take $M = \mathbb{R}^n$ endowed with its canonical flat torsionless connection and consider*

$$h = \sum_{i=1}^n f_i(x_i)\partial_{x_i} \otimes \partial_{x_i},$$

where $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, \dots, n$. Then one can see easily that h satisfies (1.2.2) and hence defines a K-V structure on $\mathbb{R}^n$.

2. *Take $M = \mathbb{R}^n$ endowed with its canonical flat torsionless connection and consider*

$$h = \sum_{i,j=1}^n x_i x_j \partial_{x_i} \otimes \partial_{x_j}.$$

Then one can easily see that h satisfies (1.2.2) and hence defines a K-V structure on $\mathbb{R}^n$.

3. *Let (M, ∇) be an affine manifold, $(X_1, \dots, X_r)$ a family of parallel vector fields and $(a_{i,j})_{1\leq i,j\leq n}$ a symmetric n-matrix. Then*

$$h = \sum_{i,j} a_{i,j} X_i \otimes X_j$$

defines a K-V structure on M.

4. *Consider $\mathbb{R}^n$ endowed with its canonical flat torsionless connection ∇ and denote by $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ its canonical linear coordinates. Let $f \in C^\infty(M)$ such that the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is invertible and put*

$$h = \sum_{i,j=1}^r h_{ij}\partial_{x_i} \otimes \partial_{x_j},$$

where (h_{ij}) is the inverse of the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. Then $(\mathbb{R}^n, \nabla, h)$ is a K-V manifold.

1.3 The Lie algebroid of a K-V manifold

Let (M, ∇, h) be an affine manifold endowed with a symmetric bivector field. We associate to this triple a bracket on $\Omega^1(M)$ by putting

$$[\alpha, \beta]_h := \nabla_{\alpha^\#}\beta - \nabla_{\beta^\#}\alpha, \tag{1.3.1}$$

and a map $\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ given by

$$\prec \mathcal{D}_\alpha \beta, X \succ := (\nabla_X h)(\alpha, \beta) + \prec \nabla_{\alpha^\#}\beta, X \succ, \tag{1.3.2}$$

for any $\alpha, \beta \in \Omega^1(M)$ and $X \in \Gamma(TM)$. This bracket is skew-symmetric and

Lemma 1.3.1. *For any $f \in C^\infty(M), \alpha, \beta \in \Omega^1(M)$,*

$$[\alpha, \beta]_h = \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha \text{ and } [\alpha, f\beta]_h = f[\alpha, \beta]_h + \alpha^\#(f)\beta.$$

Proof. This follows from the fact that

$$\prec [\alpha, \beta]_h, X \succ = \prec \nabla_{\alpha^\#}\beta - \nabla_{\beta^\#}\alpha, X \succ = \prec \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha, X \succ,$$

and

$$[\alpha, f\beta]_h = \nabla_{\alpha^\#} f\beta - \nabla_{(f\beta)^\#}\alpha = \alpha^\#(f)\beta + f\nabla_{\alpha^\#}\beta - f\nabla_{\beta^\#}\alpha = \alpha^\#(f)\beta + f[\alpha, \beta]_h.$$

∎

Theorem 1.3.2. *With the hypothesis and notations above, the following assertions are equivalent:*

(i) *h is a K-V bivector field.*

(ii) *$(T^*M, h_\#, [\ ,\ ]_h)$ is a Lie algebroid.*

In this case, $\mathcal{D}$ is a flat T^*M-connection on the vector bundle $T^*M \to M$ satisfying

$$(\mathcal{D}_\alpha\beta)^\# = \nabla_{\alpha^\#}\beta^\#,$$

for any $\alpha, \beta \in \Omega^1(M)$.

Proof. According to [9, Proposition 2.1], $(T^*M, h_\#, [\ ,\ ]_h)$ is a Lie algebroid if and only if, for any affine coordinate system $(x_1, \ldots, x_n)$,

$$([dx_i, dx_j]_h)^\# = [(dx_i)^\#, (dx_j)^\#] \quad \text{and} \quad \oint_{i,j,k} [dx_i, [dx_j, dx_k]_h]_h = 0,$$

for $1 \le i < j < k \le n$. Since $[dx_i, dx_j]_h = 0$ this is equivalent to $[(dx_i)^\#, (dx_j)^\#] = 0$ for any $1 \le i < j \le n$ which is equivalent to (1.2.2).

Suppose now that (i) or (ii) holds. For any $\alpha, \beta, \gamma \in \Omega^1(M)$,

$$\begin{aligned} \prec \mathcal{D}_\alpha\beta, \gamma^\# \succ &= \nabla_{\gamma^\#} h(\alpha, \beta) + h(\nabla^*_{\alpha^\#}\beta, \gamma) \\ &= \nabla_{\alpha^\#} h(\gamma, \beta) + h(\nabla^*_{\alpha^\#}\beta, \gamma) \\ &= \alpha^\#.h(\beta, \gamma) - h(\nabla^*_{\alpha^\#}\gamma, \beta) \\ &= \prec \gamma, \nabla_{\alpha^\#}\beta^\# \succ . \end{aligned}$$

This shows that $(\mathcal{D}_\alpha\beta)^\# = \nabla_{\alpha^\#}\beta^\#$.

Let us now show that the curvature of $\mathcal{D}$ vanishes. Since $[dx_i, dx_j]_h = 0$, it suffices to show that, for any $i, j, k \in \{1, \ldots, n\}$ with $i < j$, $\mathcal{D}_{dx_i}\mathcal{D}_{dx_j}dx_k = \mathcal{D}_{dx_j}\mathcal{D}_{dx_i}dx_k$. We have

$$\prec \mathcal{D}_{dx_i}dx_k, \frac{\partial}{\partial x_l} \succ = \frac{\partial h_{ik}}{\partial x_l},$$

and hence

$$\mathcal{D}_{dx_i}dx_k = \sum_{l=1}^{n} \frac{\partial h_{ik}}{\partial x_l} dx_l,$$

and then

$$\begin{aligned}
\mathcal{D}_{dx_j}\mathcal{D}_{dx_i}dx_k &= \sum_{l=1}^{n} \mathcal{D}_{dx_j}\left(\frac{\partial h_{ik}}{\partial x_l}dx_l\right) \\
&= \sum_{l=1}^{n}\left((dx_j)^\#\left(\frac{\partial h_{ik}}{\partial x_l}\right)dx_l + \frac{\partial h_{ik}}{\partial x_l}\left(\sum_{s=1}^{n}\frac{\partial h_{jl}}{\partial x_s}dx_s\right)\right) \\
&= \sum_{l,r=1}^{n} h_{jr}\left(\frac{\partial^2 h_{ik}}{\partial x_r \partial x_l}\right)dx_l + \sum_{s,l=1}^{n}\frac{\partial h_{ik}}{\partial x_l}\frac{\partial h_{jl}}{\partial x_s}dx_s \\
&= \sum_{l,r=1}^{n} h_{jr}\left(\frac{\partial^2 h_{ik}}{\partial x_r \partial x_l}\right)dx_l + \sum_{l,r=1}^{n}\frac{\partial h_{ik}}{\partial x_r}\frac{\partial h_{jr}}{\partial x_l}dx_l \\
&= \sum_{l,r=1}^{n}\left(h_{jr}\left(\frac{\partial^2 h_{ik}}{\partial x_r \partial x_l}\right) + \frac{\partial h_{ik}}{\partial x_r}\frac{\partial h_{jr}}{\partial x_l}\right)dx_l \\
&= \sum_{l=1}^{n}\frac{\partial}{\partial x_l}\left(\sum_r h_{jr}\frac{\partial h_{ik}}{\partial x_r}\right)dx_l.
\end{aligned}$$

So

$$\mathcal{D}_{dx_i}\mathcal{D}_{dx_j}dx_k - \mathcal{D}_{dx_j}\mathcal{D}_{dx_i}dx_k = d\left(\sum_{r=1}^{n}\left(h_{jr}\frac{\partial h_{ik}}{\partial x_r} - h_{ir}\frac{\partial h_{jk}}{\partial x_r}\right)\right) \overset{(1.2.2)}{=} d(0) = 0.$$

∎

The following result is an important consequence of Theorem 1.3.2.

Proposition 1.3.3. *Let (M, ∇, h) be a K-V manifold. Then,*

1. *The distribution $\mathrm{Im}h_\#$ is integrable and defines a singular foliation $\mathcal{L}$ on M.*
2. *For any leaf L of $\mathcal{L}$, $(L, \nabla_{|L}, g_L)$ is a pseudo-Hessian manifold where g_L is given by $g_L(\alpha^\#, \beta^\#) = h(\alpha, \beta)$.*

We will call the foliation defined by $\mathrm{Im}h_\#$ the affine foliation associated to (M, ∇, h).

Remark 1.3.4. *This Proposition 1.3.3 shows that K-V bivector fields can be used either to build examples of affine foliations on affine manifolds or to build examples of pseudo-Hessian manifolds.*

Let (M, ∇, h) be a K-V manifold and $\mathcal{D}$ the connection given in (1.3.2). Let $x \in M$ and $\mathfrak{g}_x = \ker h_\#(x)$. Since $(T^*M, h_\#, [\ ,\ ]_h)$ is a Lie algebroid then $(\mathfrak{g}_x, [\ ,\ ])$ is a Lie algebra where the bracket [,] is given by

$$[a, b] = [\alpha, \beta]_h(x),$$

where $\alpha, \beta \in \Omega^1(M)$ are any 1-forms satisfying $\alpha(x) = a$ and $\beta(x) = b$. Now for any $a, b \in \mathfrak{g}_x$ put

$$a \bullet b = (\mathcal{D}_\alpha \beta)(x),$$

where $\alpha, \beta \in \Omega^1(M)$ are any 1-forms satisfying $\alpha(x) = a$ and $\beta(x) = b$.

Proposition 1.3.5. $(\mathfrak{g}_x, \bullet)$ *is a commutative associative algebra.*

Proof. For any $\alpha, \beta \in \Omega^1(M)$,

$$(\mathcal{D}_\alpha \beta)^\# = \nabla_{\alpha^\#} \beta^\#.$$

This shows that if $\alpha_x^\# = 0$ then $(\mathcal{D}_\alpha \beta)_x^\# = 0$. Moreover,

$$\mathcal{D}_\alpha \beta - \mathcal{D}_\beta \alpha = \nabla_{\alpha^\#} \beta - \nabla_{\beta^\#} \alpha.$$

This implies that if $\alpha_x^\# = \beta_x^\# = 0$ then

$$(\mathcal{D}_\alpha \beta)(x) = (\mathcal{D}_\beta \alpha)(x).$$

This implies that $\bullet$ defines a commutative product on $\mathfrak{g}_x$ and moreover, by using the vanishing of the curvature of $\mathcal{D}$, we get the associativity of $\bullet$. ■

Near a point where h vanishes, the algebra structure of $\mathfrak{g}_x$ can be made explicit.

Proposition 1.3.6. *We consider* $\mathbb{R}^n$ *endowed with its canonical affine connection,* h *a symmetric bivector field on* $\mathbb{R}^n$ *such that* $h(0) = 0$ *and* $(\mathbb{R}^n, \nabla, h)$ *is a K-V manifold. Then the product on* $(\mathbb{R}^n)^*$ *given by*

$$e_i^* \bullet e_j^* = \sum_{k=1}^n \frac{\partial h_{ij}}{\partial x_k}(0) e_k^*,$$

is associative and commutative.

Proof. It is a consequence of the relation $\mathcal{D}_{dx_i} dx_j = dh_{ij}$ true by virtue of (1.3.2). ■

1.4 The product of K-V manifolds and the splitting theorem

As the product of two Poisson manifolds is a Poisson manifold [56], the product of two K-V manifolds is a K-V manifold.

Let (M_1, ∇^1, h^1) and (M_2, ∇^2, h^2) be two K-V manifolds. We denote by $p_i : M := M_1 \times M_2 \to M_i, i = 1, 2$ the canonical projections. For any $X \in \Gamma(TM_1)$ and $Y \in \Gamma(TM_2)$, we denote by $X + Y$ the vector field on M given by $(X + Y)(m_1, m_2) = (X(m_1), Y(m_2))$. The product of the affine atlases on M_1 and M_2 is an affine atlas on M and the corresponding flat torsionless connection is the unique flat torsionless connection ∇ on M satisfying

$$\nabla_{X_1+Y_1}(X_2 + Y_2) = \nabla^1_{X_1} X_2 + \nabla^2_{Y_1} Y_2,$$

for any $X_1, X_2 \in \Gamma(TM_1)$ and $Y_1, Y_2 \in \Gamma(TM_2)$.

Moreover, the product of h_1 and h_2 is the unique symmetric bivector field h satisfying,

$$h(p_1^*\alpha_1, p_1^*\beta_1) = h^1(\alpha_1, \beta_1)\circ p_1,\ h(p_2^*\alpha_2, p_2^*\beta_2) = h^2(\alpha_2, \beta_2)\circ p_2 \quad \text{and} \quad h(p_1^*\alpha_1, p_2^*\alpha_2) = 0,$$

for any $\alpha_1, \beta_1 \in \Omega^1(M_1)$, $\alpha_2, \beta_2 \in \Omega^1(M_2)$,

Proposition 1.4.1. *(M, ∇, h) is a K-V manifold.*

Proof. Let $(m_1, m_2) \in M$. Choose an affine coordinate system $(x_1, \ldots, x_{n_1})$ near m_1 and an affine coordinate system $(y_1, \ldots, y_{n_2})$ near m_2. Then

$$h = \sum_{i,j} h^1_{ij} \circ p_1 \partial_{x_i} \otimes \partial_{x_j} + \sum_{l,k} h^2_{lk} \circ p_2 \partial_{y_l} \otimes \partial_{y_k}$$

and one can check easily that h satisfies (1.2.2). ∎

If we continue our investigation of the analogies between Poisson manifolds and K-V manifolds, we can naturally ask if there is an analog of the Darboux-Weinstein theorem (see [56]) in the context of K-V manifolds. To be more specific, assume that (M, ∇, h) is an affine manifold endowed with a non degenerate bivector field h. Then h is a K-V bivector field if and only if, for any $m \in M$ there exits an affine coordinates system $(x_1, \ldots, x_n)$ around m such that $(h_{ij})_{1\leq i,j\leq n}$ is invertible and is the inverse of $\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq n}$ for some smooth function ϕ around m (ϕ is called the potential). More generally, if we assume just h is a K-V bivector field and $m \in M$ such that $\mathrm{rank} h_\#(m) = r$. One can inquire whether there exists an affine coordinate system $(x_1, \ldots, x_r, y_1, \ldots, y_{n-r})$ such that

$$h = \sum_{i,j=1}^{r} h_{ij}(x_1, \ldots, x_r)\partial_{x_i} \otimes \partial_{x_j} + \sum_{i,j=1}^{n-r} f_{ij}(y_1, \ldots, y_{n-r})\partial_{y_i} \otimes \partial_{y_j},$$

where $(h_{ij})_{1\leq i,j\leq r}$ is invertible and is the inverse of $\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq r}$ and $f_{ij}(m) = 0$ for any i, j. Moreover, if the rank of $h_\#$ is constant near m then the functions f_{ij} vanish.

The answer is no in general for a geometric reason. Suppose that m is regular, i.e., the rank of h is constant near m and suppose that there exists an affine coordinate system $(x_1, \ldots, x_r, y_1, \ldots, y_{n-r})$ such that

$$h = \sum_{i,j=1}^{r} h_{ij}(x_1, \ldots, x_r)\partial_{x_i} \otimes \partial_{x_j}.$$

This will have a strong geometric consequence, namely that $\mathrm{Im} h_\# = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_r})$ and the associated affine foliation is parallel, i.e., if X is a local vector field and Y is tangent to the foliation then $\nabla_X Y$ is tangent to the foliation. We give now an example of a regular K-V manifold whose associated affine foliation is not parallel, which shows that the analog of Darboux-Weinstein is not true in general.

Example 1.4.2. *We consider $M = \mathbb{R}^4$ endowed with its canonical affine connection ∇, denote by (x, y, z, t) its canonical coordinates and consider*

$$X = \cos(t)\partial_x + \sin(t)\partial_y + \partial_z,\ Y = -\sin(t)\partial_x + \cos(t)\partial_y \quad \textit{and} \quad h = X \otimes Y + Y \otimes X.$$

We have $\nabla_X X = \nabla_Y X = \nabla_X Y = \nabla_Y Y = 0$ *and hence* h *is a K-V bivector field,* $\mathrm{Im} h_\# = \mathrm{span}\{X, Y\}$ *and the rank of* h *is constant equal to 2. However, the foliation associated to* $\mathrm{Im} h_\#$ *is not parallel since* $\nabla_{\partial_t} Y = -X + \partial_z \notin \mathrm{Im} h_\#$.

However, when h has constant rank equal to $\dim M - 1$, we have the following result and its important corollary.

Theorem 1.4.3. *Let* (M, ∇, h) *be a K-V manifold and* $m \in M$ *such that* m *is a regular point and the rank of* $h_\#(m)$ *is equal to* $n-1$. *Then there exists an affine coordinate system* $(x_1, \ldots, x_n)$ *around* m *and a function* $f(x_1, \ldots, x_n)$ *such that*

$$h = \sum_{i,j=1}^{n-1} h_{ij} \partial_{x_i} \otimes \partial_{x_j},$$

and the matrix $(h_{ij})_{1 \leq i,j \leq n-1}$ *is invertible and its inverse is the matrix* $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq n-1}$.

Corollary 1.4.4. *Let* (M, ∇, h) *be a K-V manifold with* h *of constant rank equal to* $\dim M - 1$. *Then the affine foliation associated to* $\mathrm{Im} h_\#$ *is* ∇*-parallel.*

In order to prove Theorem 1.4.3, we need the following lemma.

Lemma 1.4.5. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a differentiable function such that* $\partial_x(f) + f\partial_y(f) = 0$. *Then* f *is a constant.*

Proof. Let f be a solution of the equation above. We consider the vector field $X_f = \partial_x + f\partial_y$. The integral curve $(x(t), y(t))$ of X_f passing through $(a, b) \in \mathbb{R}^2$ satisfies

$$x'(t) = 1, \quad y'(t) = f(x(t), y(t)) \quad \text{and} \quad (x(0), y(0)) = (a, b).$$

Now

$$y''(t) = \partial_x(f)(x(t), y(t)) + y'(t)\partial_y(f)(x(t), y(t)) = 0,$$

and hence, the flow of X_f is given by $\phi(t, (x, y)) = (t + x, f(x, y)t + y)$. The relation $\phi(t + s, (x, y)) = \phi(t, \phi(s, (x, y)))$ implies that the map $F(x, y) = (1, f(x, y))$ satisfies

$$F(u + tF(u)) = F(u), \ u \in \mathbb{R}^2, t \in \mathbb{R}.$$

Let $u, v \in \mathbb{R}^2$ such that $F(u)$ and $F(v)$ are linearly independent. Then there exist $s, t \in \mathbb{R}$ such that $u - v = tF(u) + sF(v)$ and hence $F(u) = F(v)$ which is a contradiction. So $F(x, y) = \alpha(x, y)(a, b)$, i.e., $(1, f(x, y)) = (\alpha(x, y)a, \alpha(x, y)b)$ and α must be constant and hence f is constant. ∎

Proof of Theorem 1.4.3. Let $(x_1, \ldots, x_n)$ be an affine coordinate system near m such that $(X_1, \ldots, X_{n-1})$ are linearly independent in a neighborhood of m, where $X_i = (dx_i)^\#$, $X_n = \sum_{j=1}^{n-1} f_j X_j$ and, by virtue of the proof of Theorem 1.3.2, for any $1 \leq i < j \leq n$, $[X_i, X_j] = 0$. For any $i = 1, \ldots, n-1$, the relation $[X_i, X_n] = 0$ is equivalent to

$$X_i(f_j) = h_{in}\partial_{x_n}(f_j) + \sum_{l=1}^{n-1} h_{il}\partial_{x_l}(f_j) = 0, \qquad j = 1, \ldots, n-1.$$

But $h_{in} = X_n(x_i) = \sum_{l=1}^{n-1} f_l h_{il}$ and hence, for any $i, j = 1, \dots n-1$,

$$\sum_{l=1}^{n-1} h_{il}(f_l \partial_{x_n}(f_j) + \partial_{x_l}(f_j)) = 0.$$

Or the matrix $(h_{ij})_{1 \leq i,j \leq n-1}$ is invertible so we get

$$f_l \partial_{x_n}(f_j) + \partial_{x_l}(f_j) = 0, \quad l, j = 1, \dots, n-1. \tag{1.4.1}$$

For $l = j$ we get that f_j satisfies $f_j \partial_{x_n}(f_j) + \partial_{x_j}(f_j) = 0$ so, according to Lemma 1.4.5, $\partial_{x_n}(f_j) = \partial_{x_j}(f_j) = 0$ and from (1.4.1), f_j =constant. We consider $y = f_1 x_1 + \dots + f_{n-1} x_{n-1} - x_n$, we have $(dy)^\# = 0$ and $(x_1, \dots, x_{n-1}, y)$ is an affine coordinate system around m.

On the other hand, there exists a coordinate system $(z_1, \dots, z_n)$ such that

$$(dx_i)^\# = \partial_{z_i}, \ i = 1, \dots, n-1.$$

We deduce that

$$\partial_{x_i} = \sum_{j=1}^{n-1} h^{ij} \partial_{z_j}, \ i = 1, \dots, n-1,$$

with $h^{ij} = \frac{\partial z_j}{\partial x_i}$. We consider $\sigma = \sum_{j=1}^{n-1} z_j dx_j$. We have $d\sigma = 0$ so, according to the foliated Poincaré Lemma (see [13, p.56]) there exists a function f such that $h^{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. ■

1.5 The divergence and the modular class of a K-V manifold

We define now the divergence of a K-V structure. We first recall the definition of the divergence of multivector fields associated to a connection on a manifold.

Let (M, ∇) be a manifold endowed with a connection. We define $\mathrm{div}_\nabla : \Gamma(\otimes^p TM) \to \Gamma(\otimes^{p-1} TM)$ by

$$\mathrm{div}_\nabla(T)(\alpha_1, \dots, \alpha_{p-1}) = \sum_{i=1}^{n} \nabla_{e_i}(T)(e_i^*, \alpha_1, \dots, \alpha_{p-1}),$$

where $\alpha_1, \dots, \alpha_{p-1} \in T_x^* M$, $(e_1, \dots, e_n)$ a basis of $T_x M$ and $(e_1^*, \dots, e_n^*)$ its dual basis. This operator respects the symmetries of tensor fields.

Suppose now that (M, ∇, h) is a K-V manifold. The divergence of this structure is the vector field $\mathrm{div}_\nabla(h)$. This vector field is an invariant of the K-V structure and has an important property. Indeed, let $\mathbf{d}_h : \Gamma(\wedge^\bullet TM) \longrightarrow \Gamma(\wedge^{\bullet+1} TM)$ be the differential associated to the Lie algebroid structure $(T^*M, h_\#, [\ ,\ ]_h)$ and given by

$$\begin{aligned}\mathbf{d}_h Q(\alpha_1,\ldots,\alpha_p) &= \sum_{j=1}^{p}(-1)^{j+1}\alpha_j^\#.Q(\alpha_1,\ldots,\hat{\alpha}_j,\ldots\alpha_p)\\ &+\sum_{1\leq i<j\leq p}(-1)^{i+j}Q([\alpha_i,\alpha_j]_h,\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots,\alpha_p).\end{aligned}$$

Proposition 1.5.1. $\mathbf{d}_h(\mathrm{div}_\nabla(h)) = 0$.

Proof. Let $(x_1,\ldots,x_n)$ be an affine coordinate system. We have

$$\begin{aligned}\mathbf{d}_h\mathrm{div}_\nabla(h)(\alpha,\beta) &= \sum_{i=1}^{n}\alpha^\#.\nabla_{\partial_{x_i}}(h)(dx_i,\beta)-\beta^\#.\nabla_{\partial_{x_i}}(h)(dx_i,\alpha)-\nabla_{\partial_{x_i}}(h)(dx_i,\nabla_{\alpha^\#}\beta)\\ &\quad+\nabla_{\partial_{x_i}}(h)(dx_i,\nabla_{\beta^\#}\alpha)\\ &= \sum_{i=1}^{n}\left(\nabla_{\alpha^\#}\nabla_{\partial_{x_i}}(h)(dx_i,\beta)-\nabla_{\beta^\#}\nabla_{\partial_{x_i}}(h)(dx_i,\alpha)\right)\\ &\overset{(1.2.1)}{=} \sum_{i=1}^{n}\left(\nabla_{[\alpha^\#,\partial_{x_i}]}(h)(dx_i,\beta)-\nabla_{[\beta^\#,\partial_{x_i}]}(h)(dx_i,\alpha)\right).\end{aligned}$$

If we take $\alpha = dx_l$ and $\beta = dx_k$, we have

$$[\partial_{x_i},(dx_l)^\#] = \sum_{m=1}^{n}\partial_{x_i}(h_{ml})\partial_{x_m},$$

and hence

$$\mathbf{d}_h\mathrm{div}_\nabla(h)(\alpha,\beta) = \sum_{i,m=1}^{n}\left(\partial_{x_i}(h_{ml})\partial_{x_m}(h_{ik})-\partial_{x_i}(h_{mk})\partial_{x_m}(h_{il})\right)=0.$$

∎

Let (M,∇,h) be an orientable K-V manifold and Ω a volume form on M. For any f we denote by $X_f = h_\#(df)$ and we define $\mathbf{M}_\Omega : C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ by putting for any $f \in C^\infty(M,\mathbb{R})$,

$$\nabla_{X_f}\Omega = \mathbf{M}_\Omega(f)\Omega.$$

It is obvious that $\mathbf{M}_\Omega$ is a derivation, hence a vector field, and that

$$\mathbf{M}_{e^f\Omega} = X_f + \mathbf{M}_\Omega.$$

Moreover, if $(x_1,\ldots,x_n)$ is an affine coordinate system and $\mu = \Omega(\partial_{x_1},\ldots,\partial_{x_n})$ then

$$\nabla_{X_f}\Omega(\partial_{x_1},\ldots,\partial_{x_n}) = X_f(\mu) = X_{\ln|\mu|}(f)\mu.$$

So in the coordinate system $(x_1,\ldots,x_n)$, we have $\mathbf{M}_\Omega = X_{\ln|\mu|}$. This implies $\mathbf{d}_h\mathbf{M}_\Omega = 0$. The cohomology class of $\mathbf{M}_\Omega$ doesn't depend on Ω and we call it the *modular class* of (M,∇,h).

Proposition 1.5.2. *The modular class of (M,∇,h) vanishes if and only if there exists a volume form Ω such that $\nabla_{X_f}\Omega = 0$ for any $f \in C^\infty(M,\mathbb{R})$.*

By analogy with the case of Poisson manifolds, one can ask if it is possible to find a volume form Ω such that $\mathcal{L}_{X_f}\Omega = 0$ for any $f \in C^\infty(M,\mathbb{R})$. The following proposition gives a negative answer to this question unless $h = 0$.

Proposition 1.5.3. *Let (M, ∇, h) be an orientable K-V manifold. Then,*

1. *For any volume form Ω and any $f \in C^\infty(M,\mathbb{R})$,*

$$\mathcal{L}_{X_f}\Omega = [\mathbf{M}_\Omega(f) + \mathrm{div}_\nabla(h)(f) + \prec h, \mathrm{Hess}(f) \succ]\,\Omega,$$

   where $\mathrm{Hess}(f)(X,Y) = \nabla_X(df)(Y)$ and $\prec h, \mathrm{Hess}(f) \succ$ is the pairing between the bivector field h and the 2-form $\mathrm{Hess}(f) = \nabla df$.

2. *There exists a volume form Ω such that $\mathcal{L}_{X_f}\Omega = 0$ for any $f \in C^\infty(M,\mathbb{R})$ if and only if, $h = 0$.*

Proof.

1. Let $(x_1, \ldots, x_n)$ be an affine coordinate system. Then,

$$\begin{aligned}
[X_f, \partial_{x_i}] &= \sum_{l,j=1}^{n} [\partial_{x_j}(f) h_{jl} \partial_{x_l}, \partial_{x_i}] \\
&= -\sum_{l,j=1}^{n} \left(h_{jl} \partial_{x_i} \partial_{x_j}(f) + \partial_{x_j}(f) \partial_{x_i}(h_{jl}) \right) \partial_{x_l}, \\
\mathcal{L}_{X_f}\Omega(\partial_{x_1}, \ldots, \partial_{x_n}) &= (\nabla_{X_f}\Omega)(\partial_{x_1}, \ldots, \partial_{x_n}) - \sum_{i=1}^{n} \Omega((\partial_{x_1}, \ldots, [X_f, \partial_{x_i}], \ldots, \partial_{x_n})) \\
&= (\nabla_{X_f}\Omega)(\partial_{x_1}, \ldots, \partial_{x_n}) \\
&\quad + \sum_{i,j=1}^{n} \left(h_{ji} \partial_{x_i} \partial_{x_j}(f) + \partial_{x_j}(f) \partial_{x_i}(h_{ji}) \right) \Omega(\partial_{x_1}, \ldots, \partial_{x_n}),
\end{aligned}$$

   and the formula follows since $\mathrm{div}_\nabla(h) = \sum_{i,j=1}^{n} \partial_{x_i}(h_{ji}) \partial_{x_j}$.

2. This is a consequence of the fact that $\mathbf{M}_\Omega$ and $\mathrm{div}_\nabla(h)$ are derivation and

$$\prec h, \mathrm{Hess}(fg) \succ = f \prec h, \mathrm{Hess}(g) \succ + g \prec h, \mathrm{Hess}(f) \succ + 2 \prec h, df \odot dg \succ .$$

∎

1.6 The tangent bundle of a K-V manifold

In this section, we define and study the associated Poisson tensor on the tangent bundle of a K-V manifold. We will start this paragraph by recalling some valuable results concerning the geometry of the tangent bundle.

Let (M, ∇) be an n-dimensional smooth manifold endowed with a connection and denote by $p : TM \to M$ the canonical projection of the tangent bundle. It is a well known fact that one can define on TM the so called *Sasaki connection* $\overline{\nabla}$ associated to ∇, and also the *Sasaki almost complex structure* $J : T(TM) \to T(TM)$.

For more details, one can see [23, 58, 21]. Indeed, associated to ∇ there exists a splitting

$$T(TM) = V(M) \oplus H(M),$$

such that for any $u \in TM$, $T_u p : H_u(M) \longrightarrow T_{p(x)}M$ is an isomorphism. For any vector field $X \in \Gamma(TM)$ we denote by $X^v \in \Gamma(V(M))$ its vertical lift and by $X^h \in \Gamma(H(M))$ its horizontal lift. These are given, for any $u \in TM$, by

$$X_u^v = \frac{d}{dt}_{|t=0}(u + tX_{p(u)}), \quad \text{and} \quad Tp(X_u^h) = X_{p(u)}.$$

The vector X_u^h can also be defined using parallel transport. Indeed, let γ be a smooth curve on M starting at $x \in M$ and $\gamma'(0) = X_x \in T_xM$. Then

$$X_u^h = \frac{d}{dt}_{|t=0}\tau_{0t}^{\gamma}(u), \tag{1.6.1}$$

where $\tau_{0t}^{\gamma} : T_xM \xrightarrow{\cong} T_{\gamma(t)}M$ is the parallel transport map along γ.

The *Sasaki almost complex structure* $J : T(TM) \longrightarrow T(TM)$ determined by ∇ is defined by

$$J(X^h) = X^v \quad \text{and} \quad J(X^v) = -X^h.$$

It is integrable to a complex structure on TM if and only if ∇ is flat.

Suppose now that (M, ∇) is an affine manifold. Since the curvature of ∇ vanishes, for any $X, Y \in \Gamma(TM)$,

$$[X^h, Y^h] = [X, Y]^h, \; [X^h, Y^v] = (\nabla_X Y)^v \quad \text{and} \quad [X^v, Y^v] = 0. \tag{1.6.2}$$

As for the vector fields, for any $\alpha \in \Omega^1(M)$, we define $\alpha^v, \alpha^h \in \Omega^1(TM)$ by

$$\begin{cases} \alpha^v(X^v) = \alpha(X) \circ p, \\ \alpha^v(X^h) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \alpha^h(X^h) = \alpha(X) \circ p, \\ \alpha^h(X^v) = 0. \end{cases}$$

The *Sasaki connection* $\overline{\nabla}$ on TM is defined by

$$\overline{\nabla}_{X^h} Y^h = (\nabla_X Y)^h, \; \overline{\nabla}_{X^h} Y^v = (\nabla_X Y)^v \quad \text{and} \quad \overline{\nabla}_{X^v} Y^h = \overline{\nabla}_{X^v} Y^v = 0, \tag{1.6.3}$$

where $X, Y \in \Gamma(TM)$. This connection is torsionless and flat and hence $(TM, \overline{\nabla})$ is an affine manifold. Moreover, J is parallel with respect to $\overline{\nabla}$.

Remark 1.6.1. *All the above geometrical structures on TM could be described locally in an easy way. In fact, let $(x_1, \cdots, x_n)$ be an affine coordinates system on an open set $U \subset M$. Then we can see easily that the connection $\overline{\nabla}$ is the canonical one for which the associated canonical coordinate system $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ on TU is affine (where $x_i(u) := x_i(p(u))$ and $y_j(u) := dx_i(u)$ for any $u \in TU$). The complex structure is given by $J(\partial_{x_i}) = \partial_{y_i}$, $J(\partial_{y_i}) = -\partial_{x_i}$.*

Now, let h be a symmetric bivector field on M. We associate to h a skew-symmetric bivector field Π on TM by putting

$$\Pi(\alpha^v, \beta^v) = \Pi(\alpha^h, \beta^h) = 0 \quad \text{and} \quad \Pi(\alpha^h, \beta^v) = -\Pi(\beta^v, \alpha^h) = h(\alpha, \beta) \circ p,$$

for any $\alpha, \beta \in \Omega^1(M)$. For any $\alpha \in \Omega^1(M)$,

$$\Pi_\#(\alpha^v) = -(\alpha^\#)^h \quad \text{and} \quad \Pi_\#(\alpha^h) = (\alpha^\#)^v. \tag{1.6.4}$$

To prove the main result of this chapter, we need the following proposition which is a part of the folklore.

Proposition 1.6.2. *Let (P, ∇) be a manifold endowed with a torsionless connection and π is a bivector field on P. Then the Nijenhuis-Schouten bracket $[\pi, \pi]$ is given by*

$$[\pi, \pi](\alpha, \beta, \gamma) \;=\; 2\left(\nabla_{\pi_\#(\alpha)}\pi(\beta, \gamma) + \nabla_{\pi_\#(\beta)}\pi(\gamma, \alpha) + \nabla_{\pi_\#(\gamma)}\pi(\alpha, \beta)\right).$$

Theorem 1.6.3. *The following assertions are equivalent:*

(i) *(M, ∇, h) is a K-V manifold.*

(ii) *(TM, Π) is a Poisson manifold.*

In this case, if L is a leaf of $\mathrm{Im}h_\#$ then $TL \subset TM$ is a symplectic leaf of Π which is also a complex submanifold of TM. Moreover, if ω_L is the symplectic form of TL induced by Π and g_L is the pseudo-Riemannian metric given by $g_L(U, V) = \omega(JU, V)$ then (TL, g_L, ω_L, J) is a pseudo-Kähler manifold.

Proof. We will use Proposition 1.6.2 to prove the equivalence. Indeed, by a direct computation one can establish easily, for any $\alpha, \beta, \gamma \in \Omega^1(M)$, the following relations

$$\begin{aligned}
\overline{\nabla}_{\Pi_\#(\alpha^v)}\Pi(\beta^v, \gamma^v) &= \overline{\nabla}_{\Pi_\#(\alpha^v)}\Pi(\beta^h, \gamma^h) = 0, \\
\overline{\nabla}_{\Pi_\#(\alpha^h)}\Pi(\beta^v, \gamma^v) &= \overline{\nabla}_{\Pi_\#(\alpha^h)}\Pi(\beta^h, \gamma^h) = \overline{\nabla}_{\Pi_\#(\alpha^h)}\Pi(\beta^h, \gamma^v) = 0, \\
\overline{\nabla}_{\Pi_\#(\alpha^v)}\Pi(\beta^h, \gamma^v) &= \nabla_{\alpha^\#}(h)(\beta, \gamma) \circ p,
\end{aligned}$$

and the equivalence follows. The second part of the theorem is obvious and the only point which needs to be checked is that g_L is non-degenerate. ∎

Remark 1.6.4.

1. *The total space of the dual of a Lie algebroid carries a Poisson tensor (see appendix B). If (M, ∇, h) is a K-V manifold then, according to Theorem 1.3.2, T^*M carries a Lie algebroid structure and one can see easily that Π is the corresponding Poisson tensor on TM.*

2. *The equivalence of (i) and (ii) in Theorem 1.6.3 deserves to be stated explicitly in the case of $\mathbb{R}^n$ endowed with its canonical affine structure ∇. Indeed, let $(h_{ij})_{1 \leq i,j \leq n}$ be a symmetric matrix where $h_{ij} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and h the associated symmetric bivector field on $\mathbb{R}^n$. The associated bivector field Π_h on $T\mathbb{R}^n = \mathbb{C}^n$ is*

$$\Pi_h = \sum_{i,j=1}^{n} h_{ij}(x)\partial_{x_i} \wedge \partial_{y_j},$$

where $(x_1 + iy_1, \ldots, x_n + iy_n)$ are the canonical coordinates of $\mathbb{C}^n$. Then, according to Theorem 1.3.2, $(\mathbb{R}^n, \nabla, h)$ is a K-V manifold if and only if $(\mathbb{C}^n, \Pi_h)$ is a Poisson manifold.

We explore now some relations between some invariants of (M, ∇, h) and some invariants of (TM, Π).

Proposition 1.6.5. *Let (M,∇,h) be a K-V manifold. Then* $(\mathrm{div}_\nabla h)^v = \mathrm{div}_{\overline{\nabla}}\Pi$.

Proof. Fix $(x,u)\in TM$ and choose a basis $(e_1,\ldots,e_n)$ of T_xM. Then $(e_1^v,\ldots,e_n^v,e_1^h,\ldots,e_n^h)$ is a basis of $T_{(x,u)}TM$ with $((e_1^*)^v,\ldots,(e_n^*)^v,(e_1^*)^h,\ldots,(e_n^*)^h)$ as a dual basis. For any $\alpha\in T_x^*M$, we have

$$\begin{aligned}\prec \alpha^v, \mathrm{div}_{\overline{\nabla}}\Pi \succ &= \sum_{i=1}^n \left(\overline{\nabla}_{e_i^v}(\Pi)((e_i^*)^v,\alpha^v) + \overline{\nabla}_{e_i^h}(\Pi)((e_i^*)^h,\alpha^v)\right)\\ &\overset{(1.6.3)}{=} \prec \alpha, \mathrm{div}_\nabla(h)\succ \circ p = \prec \alpha^v, (\mathrm{div}_\nabla(h))^v \succ .\end{aligned}$$

In the same way we get that $\prec \alpha^h, \mathrm{div}_{\overline{\nabla}}\Pi \succ = 0$ and the result follows. ∎

Let (M,∇,h) be a K-V manifold. For any multivector field Q on M we define its vertical lift Q^v on TM by

$$i_{\alpha^h}Q^v = 0 \quad \text{and} \quad Q^v(\alpha_1^v,\ldots,\alpha_q^v) = Q(\alpha_1,\ldots,\alpha_q)\circ p.$$

Recall that h defines a Lie algebroid structure on T^*M whose anchor is $h_\#$ and the Lie bracket is given by (1.3.1). The Poisson tensor Π defines a Lie algebroid structure onT^*TM whose anchor is $\Pi_\#$ and the Lie bracket is the Koszul bracket

$$[\phi_1,\phi_2]_\Pi = \mathcal{L}_{\Pi_\#(\phi_1)}\phi_2 - \mathcal{L}_{\Pi_\#(\phi_2)}\phi_1 - d\Pi(\phi_1,\phi_2), \quad \phi_1,\phi_2\in\Omega^1(TM).$$

We denote by $\mathbf{d}_h$ (resp. $\mathbf{d}_\Pi$) the differential associated to the Lie algebroid structure on T^*M (resp. T^*TM) defined by h (resp. Π).

Proposition 1.6.6.

(i) *For any $\alpha,\beta\in\Omega^1(M)$ and $X\in\Gamma(TM)$,*

$$\begin{cases}\mathcal{L}_{X^h}\alpha^h = (\mathcal{L}_X\alpha)^h,\ \mathcal{L}_{X^h}\alpha^v = (\nabla_X\alpha)^v,\ \mathcal{L}_{X^v}\alpha^h = 0 \quad \textit{and} \quad \mathcal{L}_{X^v}\alpha^v = (\mathcal{L}_X\alpha)^h - (\nabla_X\alpha)^h,\\ [\alpha^h,\beta^h]_\Pi = 0,\ [\alpha^v,\beta^v]_\Pi = -[\alpha,\beta]_h^v \quad \textit{and} \quad [\alpha^h,\beta^v]_\Pi = (\mathcal{D}_\beta\alpha)^h,\end{cases}$$

where $\mathcal{D}$ is the connection given by (1.3.2).

(ii) $(\mathbf{d}_hQ)^v = -\mathbf{d}_\Pi(Q^v)$.

Proof. The relations in (i) can be established by a straightforward computation. From these relations and the fact that $\Pi_\#(\alpha^h) = (\alpha^\#)^v$ one can deduce easily that $i_{\alpha^h}\mathbf{d}_\Pi(Q^v) = 0$. On the other hand, since $\Pi_\#(\alpha^v) = -(\alpha^\#)^h$ and $[\alpha^v,\beta^v]_\Pi = -[\alpha,\beta]^v$ we can conclude. ∎

Remark 1.6.7. *From Propositions 1.5.1 and Proposition 1.6.6, we can deduce that* $\mathbf{d}_\Pi(\mathrm{div}_{\overline{\nabla}}\Pi) = 0$. *This is not a surprising result because $\overline{\nabla}$ is flat and $\mathrm{div}_{\overline{\nabla}}\Pi$ is a representative of the modular class of Π.*

As a consequence of Proposition 1.6.6 we can define a linear map from the cohomology of $(T^*M, h_\#, [\ ,\ ]_h)$ to the cohomology of $(T^*TM, \Pi_\#, [\ ,\ ]_\Pi)$ by

$$V : H^*(M,h) \longrightarrow H^*(TM,\Pi),\ [Q]\mapsto [Q^v].$$

Proposition 1.6.8. *V is injective.*

Proof. An element $P \in \Gamma(\wedge^d TTM)$ is of type $(r, d-r)$ if for any $q \neq r$

$$P(\alpha_1^v, \ldots, \alpha_q^v, \beta_1^h, \ldots, \beta_{d-q}^h) = 0,$$

for any $\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_{d-q} \in \Omega^1(M)$. We have

$$\begin{cases} \Gamma(\wedge^d TTM) = \bigoplus_{r=0}^d \Gamma_{(r,d-r)}(\wedge^d TTM), \\ \mathbf{d}_\Pi(\Gamma_{(r,d-r)}(\wedge^d TTM)) \subset \Gamma_{(r+1,d-r)}(\wedge^{d+1} TTM) \oplus \Gamma_{(r,d+1-r)}(\wedge^{d+1} TTM). \end{cases}$$

Let $Q \in \Gamma(\wedge^d TM)$ such that $\mathbf{d}_h Q = 0$ and there exists $P \in \Gamma(\wedge^{d-1} TTM)$ such that $\mathbf{d}_\Pi P = Q^v$. Since $Q^v \in \Gamma_{(d,0)}(\wedge^d TTM)$ then $P \in \Gamma_{(d-1,0)}(\wedge^{d-1} TTM)$. Let us show that $P = T^v$. For $\alpha_1, \ldots, \alpha_{d-1}, \beta \in \Omega^1(M)$, we have

$$0 \;=\; \mathbf{d}_\Pi P(\beta^h, \alpha_1^v, \ldots, \alpha_{d-1}^v) = (\beta^\#)^v.P(\alpha_1^v, \ldots, \alpha_{d-1}^v).$$

So the function $P(\alpha_1^v, \ldots, \alpha_{d-1}^v)$ is constant on the fibers of TM and hence there exists $T \in \Gamma(\wedge^{d-1} TM)$ such that $P(\alpha_1^v, \ldots, \alpha_{d-1}^v) = T(\alpha_1, \ldots, \alpha_{d-1}) \circ p$. So $[Q] = 0$ which completes the proof. ∎

1.7 Linear and affine K-V structures

As in the Poisson geometry context, we have the notions of linear and affine K-V structures. One can see [32] for the notion of cocycle in associative algebras.

Let (V, ∇) be a finite dimensional real vector space endowed with its canonical affine structure. A symmetric bivector field h on V is called affine if there exists a commutative product $\bullet$ on V^* and a symmetric bilinear form B on V^* such that, for any $\alpha, \beta \in V^* \subset \Omega^1(V)$ and $u \in V$,

$$h(\alpha, \beta)(u) = \prec \alpha \bullet \beta, u \succ + B(\alpha, \beta).$$

One can see easily that if $\alpha, \beta \in \Omega^1(V) = C^\infty(V, V^*)$ then

$$h(\alpha, \beta)(u) = \prec \alpha(u) \bullet \beta(u), u \succ + B(\alpha(u), \beta(u)).$$

If $B = 0$, h is called linear.

If $(x_1, \ldots, x_n)$ is a linear coordinate system on V^* associated to a basis $(e_1, \ldots, e_n)$ then

$$h(dx_i, dx_j) = b_{ij} + \sum_{k=1}^n C_{ij}^k x_k,$$

where $e_i \bullet e_j = \sum_{k=1}^n C_{ij}^k e_k$ and $b_{ij} = B(e_i, e_j)$.

Proposition 1.7.1. *(V, ∇, h) is a K-V manifold if and only if $\bullet$ is associative and B is a scalar 2-cocycle of $(V^*, \bullet)$, i.e.,*

$$B(\alpha \bullet \beta, \gamma) = B(\alpha, \beta \bullet \gamma)$$

for any $\alpha, \beta, \gamma \in V^*$.

Proof. For any $\alpha \in V^*$ and $u \in V$, $\alpha^\#(u) = \mathrm{L}_\alpha^* u + i_\alpha B$ where $\mathrm{L}_\alpha(\beta) = \alpha \bullet \beta$ and $i_\alpha B \in V^{**} = V$. We denote by $\phi^{\alpha^\#}$ the flow of the vector field $\alpha^\#$. Then, for any

$\alpha, \beta, \gamma \in V^*$,

$$\begin{aligned}\nabla_{\alpha^\#}(h)(\beta,\gamma)(u) &= \frac{d}{dt}_{|t=0}\Big(\prec \beta\bullet\gamma, \phi^{\alpha^\#}(t,u)\succ + B(\beta,\gamma)\Big)\\ &= \prec \beta\bullet\gamma, \mathrm{L}^*_\alpha u + i_\alpha B\succ\\ &= \prec \alpha\bullet(\beta\bullet\gamma), u\succ + B(\alpha, \beta\bullet\gamma)\end{aligned}$$

and the result follows. ■

Conversely, we have the following result.

Proposition 1.7.2. *Let $(\mathcal{A}, \bullet, B)$ be a commutative and associative algebra endowed with a symmetric scalar 2-cocycle. Then,*

1. *$\mathcal{A}^*$ carries a K-V structure (∇, h) where ∇ is the canonical affine structure of $\mathcal{A}^*$ and h is given by*

$$h(u,v)(\alpha) = \prec \alpha, u(\alpha)\bullet v(\alpha)\succ + B(u(\alpha), v(\alpha)).$$

   where $\alpha \in \mathcal{A}^*, u, v \in \Omega^1(\mathcal{A}^*)$.

2. *When $B = 0$, the leaves of the affine foliation associated to $\mathrm{Im}h_\#$ are the orbits of the action Φ of $(\mathcal{A}, +)$ on $\mathcal{A}^*$ given by $\Phi(u,\alpha) = \exp(\mathrm{L}^*_u)(\alpha)$. In particular, if $\mathcal{A}^4 = 0$ they are affine special real manifolds.*

3. *The associated Poisson tensor Π on $T\mathcal{A}^* = \mathcal{A}^*\times\mathcal{A}^*$ is the affine Poisson tensor dual associated to the Lie algebra $(\mathcal{A}\times\mathcal{A}, [\ ,\ ])$ endowed with the 2-cocycle B_0 where*

$$[(a,b),(c,d)] = (a\bullet d - b\bullet c, 0) \quad \textit{and} \quad B_0((a,b),(c,d)) = B(a,d) - B(c,b).$$

Proof. It is only the third point which needs to be checked. One can see easily that $[\ ,\ ]$ is a Lie bracket on $\mathcal{A}\times\mathcal{A}$ and B_0 is a scalar 2-cocycle for this Lie bracket. For any $a \in \mathcal{A} \subset \Omega^1(\mathcal{A}^*)$, $a^v = (0,a) \in \mathcal{A}\times\mathcal{A} \subset \Omega^1(\mathcal{A}^*\times\mathcal{A}^*)$ and $a^h = (a,0)$. So

$$\Pi(a^h, b^v)(\alpha,\beta) = h(a,b)(\alpha) = \prec \alpha, a\bullet b\succ + B(a,b).$$

On the other hand, if Π^* is the Poisson tensor dual, then

$$\begin{aligned}\Pi^*(a^h, b^v)(\alpha,\beta) &= \Pi^*((a,0),(0,b))(\alpha,\beta)\\ &= \prec (\alpha,\beta), [(a,0),(0,b)]\succ + B_0((a,0),(0,b))\\ &= \prec \alpha, a\bullet b\succ + B(a,b)\\ &= \Pi(a^h, b^v)(\alpha,\beta).\end{aligned}$$

In the same way one can check the others equalities. ■

This Proposition 1.7.2 can be used as machinery to build examples of pseudo-Hessian manifolds. Indeed, by virtue of Proposition 1.3.3, any orbit L of the action Φ has an affine structure ∇^L and a pseudo-Riemannian metric g_L such that (L, ∇^L, g_L) is a pseudo-Hessian manifold.

Example 1.7.3 ([9]). *All the algebras bellow are identified with $\mathbb{R}^n$ with its canonical basis $(e_i)_{i=1}^n$ and $(e_i^*)_{i=1}^n$ is the dual basis. The action Φ of $\mathcal{A}$ on $\mathcal{A}^*$ is given by $\Phi(a,\mu) = \exp(L_a^*)(\mu)$ and for any $a \in \mathcal{A}, X_a$ is the vector fields on $\mathcal{A}^*$ given by $X_a = L_a^*$, where L_a is the left multiplication by a. We denote by ∇ the canonical flat torsionless connection on $\mathcal{A}^*$.*

1. *We take $\mathcal{A} = \mathbb{R}^n$ as a product of n copies of the associative, commutative algebra $\mathbb{R}$. The non vanishing product is given by $e_ie_i = e_i$ for $i = 1, \ldots, n$. We denote by $(a_i)_{i=1}^n$ the linear coordinates of $\mathcal{A}$ and $(x_i)_{i=1}^n$ the dual coordinates on $\mathcal{A}^*$. We have*

$$\Phi\left(\sum_{i=1}^n a_ie_i, \sum_{i=1}^n x_ie_i^*\right) = \sum_{i=1}^n e^{a_i}x_ie_i^*.$$

Moreover, for any $i = 1, \ldots, n, X_{e_i} = x_i\partial_{x_i}$. The orbit of a point $x \in \mathcal{A}^*$ is $M_x = \{\sum_{i=1}^n e^{a_i}x_ie_i^*, a_i \in \mathbb{R}\}$. It is a convex cone and one can see easily that if $\phi : \mathcal{A}^* \to \mathbb{R}$ is the function given by

$$\phi(u) = \sum_{i=1}^n u_i \ln|u_i|.$$

Then the restriction of $\nabla d\phi$ to M_x together with the restriction of ∇ to M_x defines the pseudo-Hessian structure on M_x. Note here that the signature of the pseudo-Hessian metric on M_x is exactly (p, q) where p is the number of x_i such that $x_i > 0$ and q is the number of x_i such that $x_i < 0$. Note that if $x_i > 0$ for $i = 1, \ldots, n$ then the metric on M_x is definite positive and we recover the example given in [49, pp. 17].

2. *We take $\mathcal{A} = \mathbb{C}$ endowed with its canonical structure of commutative and associative algebra. The non vanishing products are*

$$e_1e_1 = e_1,\ e_1e_2 = e_2e_1 = e_2,\ e_2e_2 = -e_1.$$

We denote here by (x, y) the linear coordinates on $\mathcal{A}$ associated to (e_1, e_2) and (α, β) the dual coordinates on $\mathcal{A}^*$. We have

$$X_{e_1} = \alpha\partial_\alpha + \beta\partial_\beta \quad \text{and} \quad X_{e_2} = \beta\partial_\alpha - \alpha\partial_\beta,$$

and it is easy to check that

$$\Phi\left(xe_1 + ye_2, \alpha e_1^* + \beta e_2^*\right) = e^x\left((\alpha\cos(y) + \beta\sin(y))e_1^* + (-\alpha\sin(y) + \beta\cos(y))e_2^*\right).$$

We deduce that we have two orbits the origin and $\mathcal{A}^*\backslash\{0\}$. Let's describe the pseudo-Hessian structure of $M := \mathcal{A}^*\backslash\{0\}$. The pseudo-Hessian metric g satisfies

$$g\left(X_{e_1}, X_{e_1}\right) = \alpha, g\left(X_{e_1}, X_{e_2}\right) = \beta, g\left(X_{e_2}, X_{e_2}\right) = -\alpha.$$

and hence

$$g = \frac{1}{\alpha^2 + \beta^2}\left(\alpha d\alpha^2 + 2\beta d\alpha d\beta - \alpha d\beta^2\right).$$

Thus (M, ∇, g) is a Lorentzian pseudo-Hessian manifold. Moreover, the metric g is flat. Now we look for a function f on M such that $g = \nabla df$, i.e.,

$$\frac{\partial^2 f}{\partial\alpha^2} = \frac{\alpha}{\alpha^2 + \beta^2}, \frac{\partial^2 f}{\partial\beta^2} = \frac{-\alpha}{\alpha^2 + \beta^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial\alpha\partial\beta} = \frac{\beta}{\alpha^2 + \beta^2}.$$

The function f given by

$$f(\alpha, \beta) = \frac{1}{2}\alpha\ln\left(\alpha^2 + \beta^2\right) + \beta\arctan\left(\frac{\alpha}{\beta}\right),$$

satisfies these equations on the open set $\{\beta \neq 0\}$. Note that this function is harmonic.

3. *We take $\mathcal{A} = \mathbb{R}^3$ with the commutative, associative product given by $e_1e_1 = e_2$ and $e_1e_2 = e_3$ and the others products are zero. We have $\mathcal{A}^3 \neq 0$ and $\mathcal{A}^4 = 0$. We denote by (a, b, c) the linear coordinates of $\mathcal{A}$ and (x, y, z) the dual coordinates of $\mathcal{A}^*$. We have*
$$X_{e_1} = y\partial_x + z\partial_y, X_{e_2} = z\partial_x \quad \textit{and} \quad X_{e_3} = 0,$$
and
$$\Phi\left(ae_1 + be_2 + ce_3, xe_1^* + ye_2^* + ze_3^*\right) = \left(x + ay + \left(\frac{1}{2}a^2 + b\right)z, y + az, z\right).$$
The orbits of this action are the plans $\{z = c, c \neq 0\}$, the lines $\{z = 0, y = c, c \neq 0\}$ and the points $\{(c, 0, 0)\}$. The pseudo-Riemannian metric on $M_c = \{z = c, c \neq 0\}$ is given by
$$g_c\left(X_{e_1}, X_{e_1}\right) = y, g_c\left(X_{e_1}, X_{e_2}\right) = c \quad \textit{and} \quad g_c\left(X_{e_2}, X_{e_2}\right) = 0.$$
This is a Lorentzian metric and one can check easily that, if $\phi(x, y, z) = -\frac{y^3}{6z^2} + \frac{xy}{z}$ then g_c is the restriction of $\nabla d\phi$ to M_c. Note that since $\mathcal{A}^4 = 0$ then $\left(M_c, \nabla_{|M_c}, g_c\right)$ is an affine special real manifold. However, the pseudo-Hessian metric on the line $L_c = \{z = 0, y = c, c \neq 0\}$ is given by the restriction of $\nabla d\phi_1$, where $\phi_1(x, y, z) = \frac{x^2}{2y}$.

4. *We take $\mathcal{A} = \mathbb{R}^3$ with the commutative, associative product given by*
$$e_1e_1 = e_2,\ e_1e_3 = e_1,\ e_2e_3 = e_2,\ e_3e_3 = e_3,$$
the others products are zero. We denote by (a, b, c) the linear coordinates on $\mathcal{A}$ and (x, y, z) the dual coordinates on $\mathcal{A}^*$. We have
$$\Phi\left(ae_1 + be_2 + ce_3, xe_1^* + ye_2^* + ze_3^*\right) = e^c\left(x + ay, y, ax + \frac{1}{2}\left(a^2 + 2b\right)y + z\right).$$
The orbits have dimension $3, 2, 1$ or 0. The three dimensional orbits are $\{y > 0\}$ and $\{y < 0\}$. The two dimensional orbits are $\{y = 0, x > 0\}$ and $\{y = 0, x < 0\}$. The one dimensional orbit are $\{y = x = 0, z > 0\}$ and $\{y = x = 0, z < 0\}$. The origin is the only zero dimensional orbit. Let describe the pseudo-Hessian structure on $M = \{y > 0\}$ or $M = \{y < 0\}$. We have
$$X_{e_1} = y\partial_x + x\partial_z, X_{e_2} = y\partial_z \quad \textit{and} \quad X_{e_3} = x\partial_x + y\partial_y + z\partial_z,$$
and the pseudo-Hessian metric g on M satisfies
$$g\left(X_{e_1}, X_{e_1}\right) = y, g\left(X_{e_1}, X_{e_2}\right) = 0$$
$$g\left(X_{e_1}, X_{e_3}\right) = x, g\left(X_{e_2}, X_{e_2}\right) = 0, g\left(X_{e_2}, X_{e_3}\right) = y \quad \textit{and} \quad g\left(X_{e_3}, X_{e_3}\right) = z.$$
Note that the matrix of g in $(X_{e_1}, X_{e_2}, X_{e_3})$ is just the passage matrix P from $(X_{e_1}, X_{e_2}, X_{e_3})$ to $(\partial_x, \partial_y, \partial_z)$ and hence the matrix of g in $(\partial_x, \partial_y, \partial_z)$ is P^{-1}. Thus, in the coordinates (x, y, z), we have
$$g = \frac{1}{y}\left(dx^2 + \frac{x^2 - yz}{y}dy_2 + 2dydz - \frac{2x}{y}dxdy\right).$$

One can check easily that g is the restriction of $\nabla d\phi$ where $\phi(x,y,z) = z\ln|y| + \frac{x^2}{2y}$. This metric is of signature $(+,+,-)$ in $\{y>0\}$ and $(+,-,-)$ in $\{y<0\}$.

5. *We take $\mathcal{A} = \mathbb{R}^4$ with the commutative, associative product given by*

$$e_1e_1 = e_2,\ e_1e_2 = e_3,\ e_1e_3 = e_2e_2 = e_4,$$

the others products are zero. We have $\mathcal{A}^3 \neq 0$ and $\mathcal{A}^4 = 0$. We denote by (a,b,c,d) the linear coordinates on $\mathcal{A}$ and (x,y,z,t) the dual coordinates on $\mathcal{A}^*$. We have

$$\begin{aligned}&\Phi\left(ae_1+be_2+ce_3+de_4, xe_1^*+ye_2^*+ze_3^*+te_4^*\right)\\ &= \left(x+ay+\left(\frac{1}{2}a^2+b\right)z+\left(\frac{1}{6}a^3+ab+c\right)t, y+az+\left(\frac{1}{2}a^2+b\right)t, z+at, t\right),\end{aligned}$$

and

$$X_{e_1} = y\partial_x + z\partial_y + t\partial_z, X_{e_2} = z\partial_x + t\partial_y, X_{e_3} = t\partial_x \quad \text{and} \quad X_{e_4} = 0.$$

Let's describe the pseudo-Hessian structure of the hyperplane $M_c = \{t = c, c \neq 0\}$ endowed with the coordinates (x,y,z). Since the matrix of g_c in $(X_{e_1}, X_{e_2}, X_{e_3})$ is the passage matrix P from $(X_{e_1}, X_{e_2}, X_{e_3})$ to $(\partial_x, \partial_y, \partial_z)$, we get

$$g_c = \frac{1}{c}\left(2dxdz + dy^2 - \frac{2z}{c}dydz + \frac{(z^2-yc)}{c^2}dz^2\right).$$

The signature of this metric is $(+,+,-)$ if $c>0$ and $(+,-,-)$ if $c<0$. One can check easily that g_c is the restriction of $\nabla d\phi$ to M_c, where

$$\phi(x,y,z,t) = \frac{z^4}{12t^3} + \frac{y^2}{2t} - \frac{z^2y}{2t} + \frac{xz}{t}.$$

Since $\mathcal{A}^4 = 0$, M_c is an affine special real manifold.

6. *We take $\mathcal{A} = \mathbb{R}^4$ with the commutative, associative product given by*

$$e_1e_1 = e_1,\ e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_1e_4 = e_4,\ e_2e_2 = e_3,\ e_2e_3 = e_4.$$

We have

$$X_{e_1} = x\partial_x + y\partial_y + z\partial_z + t\partial_t, X_{e_2} = y\partial_x + z\partial_y + t\partial_z, X_{e_3} = z\partial_x + t\partial_y \quad \text{and} \quad X_{e_4} = t\partial_x.$$

Thus $\{t>0\}$ and $\{t<0\}$ are orbits and hence carry a pseudo-Hessian structures. Let us determine the pseudo-Hessian metric. The same argument as above gives that the metric is given by the inverse of the passage matrix from $(X_{e_1}, \ldots, X_{e_4})$ to $(\partial_x, \partial_y, \partial_z, \partial_t)$. Thus

$$g = \frac{1}{t}\left(2dxdt + 2dydz - \frac{2z}{t}dydt - \frac{z}{t}dz^2 + \frac{2\left(z^2-yt\right)}{t^2}dzdt + \frac{2zyt - xt^2 - z^3}{t^3}dt^2\right).$$

The signature of this metric is $(+,+,-,-)$. One can check easily that g is the restriction of $\nabla d\phi$ to M, where

$$\phi(x,y,z,t) = -\frac{z^3}{6t^2} + \frac{yz}{t} + x\ln|t|.$$

1.8 Multiplicative K-V structures

A K-V structure (∇, h) on a Lie group G is called multiplicative if the multiplication $m : (G \times G, \nabla \oplus \nabla, h \oplus h) \longrightarrow (G, \nabla, h)$ preserves the connections and sends $h \oplus h$ to h.

Lemma 1.8.1. *Let G be a connected Lie group and ∇ a connection on G such that the multiplication $m : (G \times G, \nabla \oplus \nabla) \longrightarrow (G, \nabla)$ preserves the connections. Then G is abelian and ∇ is bi-invariant.*

Proof. We will denote by $\chi^r(G)$ (resp. $\chi^l(G)$) the space of right invariant vector fields (resp. the left invariant vector fields) on G. It is clear that for any $X \in \chi^r(G)$ and $Y \in \chi^l(G)$, the vector field (X, Y) on $G \times G$ is m-related to the vector field $X + Y$ on G:

$$Tm(X_a, Y_b) = X_a.b + a.Y_b = X_{ab} + Y_{ab} = (X + Y)_{ab}$$

It follows that for any $X_1, X_2 \in \chi^r(G)$ and $Y_1, Y_2 \in \chi^l(G)$, the vector field $(\nabla \oplus \nabla)_{(X_1,Y_1)}(X_2, Y_2)$ is m-related to $\nabla_{X_1+Y_1}(X_2 + Y_2)$, hence

$$Tm((\nabla_{X_1}X_2)_a, (\nabla_{Y_1}Y_2)_b) = (\nabla_{(X_1+Y_1)}(X_2 + Y_2))_{ab}$$

So we get

$$(\nabla_{X_1}X_2)_a.b + a.(\nabla_{Y_1}Y_2)_b = (\nabla_{X_1}X_2 + \nabla_{X_1}Y_2 + \nabla_{Y_1}X_2 + \nabla_{Y_1}Y_2)_{ab} \tag{1.8.1}$$

If we take $Y_1 = 0 = Y_2$ we obtain that ∇ is right invariant. In the same way, we get that ∇ is left invariant. Now, if we return back to the equation (1.8.1) we obtain that for any $X \in \chi^r(G)$ and $Y \in \chi^l(G)$ we have $\nabla X = 0 = \nabla Y$. This implies that any left invariant vector field is also right invariant ; indeed, if $Y = \sum_{i=1}^n f_i X_i$ with $Y \in \chi^l(G)$ and $X_i \in \chi^r(G)$ then $X_j f_i = 0$ for all $i, j = 1, \cdot, n$. Hence the adjoint representation is trivial and hence G must be abelian. ∎

Another proof of this Lemma based on parallel transport is as follows,

Proof. For any $\gamma : [0, 1] \longrightarrow G \times G$, $t \mapsto (\gamma_1(t), \gamma_2(t))$ with $\gamma(0) = (a, b)$ and $\gamma(1) = (c, d)$,

$$\tau_{m(\gamma)}(T_{(a,b)}m(u, v)) = T_{(c,d)}m(\tau_\gamma(u, v)),$$

where $\tau_\gamma : T_{(a,b)}(G \times G) \longrightarrow T_{(c,d)}(G \times G)$ and $\tau_{m(\gamma)} : T_{ab}G \longrightarrow T_{cd}G$ are the parallel transports. But

$$T_{(a,b)}m(u, v) = T_a R_b(u) + T_b L_a(v) \quad \text{and} \quad \tau_\gamma(u, v) = (\tau_{\gamma_1}(u), \tau_{\gamma_2}(v)).$$

So we get

$$\tau_{\gamma_1\gamma_2}(T_a R_b(u)) + \tau_{\gamma_1\gamma_2}(T_b L_a(v)) = T_c R_d(\tau_{\gamma_1}(u)) + T_d L_c(\tau_{\gamma_2}(v)).$$

If we take $v = 0$ and $\gamma_2(t) = b = d$. We get

$$\tau_{\gamma_1 b}(T_a R_b(u)) = T_c R_b(\tau_{\gamma_1}(u)),$$

and hence ∇ is right invariant. In the same way, we get that ∇ is left invariant. And finally

$$\tau_{\gamma_1\gamma_2}(T_a R_b(u)) = T_c R_d(\tau_{\gamma_1}(u)) \quad \text{and} \quad \tau_{\gamma_1\gamma_2}(T_b L_a(v)) = T_d L_c(\tau_{\gamma_2}(v)).$$

If we take $\gamma_2 = \gamma_1^{-1}$ we get that

$$\tau_{\gamma_1}(u) = T_a R_{a^{-1}c}(u) = T_a L_{ca^{-1}}(u).$$

This implies that the adjoint representation is trivial and hence G must be abelian. ∎

Corollary 1.8.2. *Let (∇, h) be a multiplicative K-V structure on a simply connected Lie group G. Then G is a vector space, ∇ its canonical affine connection and h is linear.*

Example 1.8.3. *Based on the classification of complex associative, commutative algebras given in [46], we can give a list of examples of affine K-V structures up to dimension 4.*

1. *On* $\mathbb{R}^2$

$$h_1 = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} x_1 & x_2 \\ x_2 & 0 \end{pmatrix} \quad \textit{and} \quad h_3 = \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. *On* $\mathbb{R}^3$

$$h_1 = \begin{pmatrix} a & 0 & x_2 \\ 0 & 0 & 0 \\ x_2 & 0 & b \end{pmatrix}, \; h_2 = \begin{pmatrix} x_2 & x_3 & a \\ x_3 & a & 0 \\ a & 0 & 0 \end{pmatrix}, \; h_3 = \begin{pmatrix} a & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$
$$h_4 = \begin{pmatrix} x_2 & 0 & x_2 \\ 0 & 0 & x_2 + a \\ x_2 & x_2 + a & x_3 \end{pmatrix} \quad \textit{and} \quad h_5 = \begin{pmatrix} x_2 & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

3. *On* $\mathbb{R}^4$

$$h_1 = \begin{pmatrix} x_3 & a & x_4 + b & 0 \\ a & -x_4 + c & 0 & 0 \\ x_4 + b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} x_2 & x_3 & x_4 & a \\ x_3 & x_4 & a & 0 \\ x_4 & a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix},$$
$$h_3 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{pmatrix}, \; h_4 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_4 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad h_5 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & 0 \\ x_3 & x_4 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{pmatrix}.$$

1.9 Quadratic K-V structures

Let V be a vector space of dimension n. Denote by ∇ its canonical affine connection. A symmetric bivector field h on V is quadratic if there exists a basis $\mathbb{B}$ of V such that, for any $i, j = 1, \ldots, n$,

$$h(dx_i, dx_j) = \sum_{l,k=1}^{n} a_{l,k}^{i,j} x_l x_k,$$

where the $a_{k,l}^{i,j}$ are real constants and $(x_1, \ldots, x_n)$ are the linear coordinates associated to $\mathbb{B}$.

For any linear endomorphism A of V we denote by $\widetilde{A}$ the associated linear vector field on V.

The key point is that if h is a quadratic K-V bivector field on V then its divergence is a linear vector field, i.e., $\mathrm{div}_\nabla(h) = \widetilde{L^h}$ where L^h is a linear endomorphism of V.

Moreover, if $F = (A, u)$ is an affine transformation of V then $\mathrm{div}_\nabla(F_* h) = \widetilde{A^{-1} L^h A}$. So the Jordan form of L_h is an invariant of the quadratic K-V structure. By using Maple we can classify quadratic K-V structures on $\mathbb{R}^2$. The same approach have been used in [27] to classify quadratic Poisson structures on $\mathbb{R}^4$. Note that if h is a quadratic K-V bivector field on $\mathbb{R}^n$ then its associated Poisson tensor on $\mathbb{C}^n$ is also quadratic.

Theorem 1.9.1.

1. *Up to an affine isomorphism, there are two quadratic K-V structures on $\mathbb{R}^2$ which are divergence free*

$$h_1 = \begin{pmatrix} 0 & 0 \\ 0 & ux^2 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} \frac{r^2x^2}{c} - 2rxy + cy^2 & \frac{r^3x^2}{c^2} - \frac{2r^2xy}{c} + ry^2 \\ \frac{r^3x^2}{c^2} - \frac{2r^2xy}{c} + ry^2 & -\frac{2r^3xy}{c^2} + \frac{r^4x^2}{c^3} + \frac{r^2y^2}{c} \end{pmatrix}.$$

2. *Up to an affine isomorphism, there are two K-V structures on $\mathbb{R}^2$ with the divergence equivalent to the Jordan form* $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$,

$$h_1 = \begin{pmatrix} cy^2 + xy & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} \frac{1}{2}xy + cy^2 & \frac{y^2}{4} \\ \frac{y^2}{4} & 0 \end{pmatrix}.$$

3. *Up to an affine isomorphism, there are five quadratic K-V structures on $\mathbb{R}^2$ with diagonalizable divergence*

$$h_1 = \begin{pmatrix} ax^2 & 0 \\ 0 & by^2 \end{pmatrix}, \; h_2 = \begin{pmatrix} ax^2 + by^2 & 0 \\ 0 & 0 \end{pmatrix}, \; h_3 = \begin{pmatrix} ax^2 & axy \\ axy & ay^2 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} \frac{2r^2x^2}{c} - 2rxy + cy^2 & ry^2 \\ ry^2 & \frac{2r^2y^2}{c} \end{pmatrix}$$

and

$$h_5 = \begin{pmatrix} (\frac{2p^2}{u} + \frac{q}{2})x^2 + \frac{pqxy}{u} + \frac{q^2y^2}{4u} & px^2 + qxy - \frac{pqy^2}{2u} \\ px^2 + qxy - \frac{pqy^2}{2u} & (\frac{2p^2}{u} + \frac{q}{2})y^2 + ux^2 - 2pxy \end{pmatrix}.$$

4. *Up to an affine isomorphism, there is a unique quadratic pseudo-Hessian structure on $\mathbb{R}^2$ with the divergence having non real eigenvalues*

$$h = \begin{pmatrix} -2pxy - ux^2 + uy^2 & px^2 - py^2 - 2uxy \\ px^2 - py^2 - 2uxy & 2pxy + ux^2 - uy^2 \end{pmatrix}.$$

Example 1.9.2. *The study of quadratic K-V structures on $\mathbb{R}^3$ is more complicated and we give here a class of quadratic pseudo-Hessian structures on $\mathbb{R}^3$ of the form $\widetilde{A} \odot \widetilde{\mathrm{I}}_3$ where $\widetilde{A}$ is linear.*

1. *A is diagonal:*

$$h_1 = \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} x^2 & xy & 0 \\ xy & y^2 & 0 \\ 0 & 0 & -z^2 \end{pmatrix}.$$

2. $A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$:

$$h_3 = \begin{pmatrix} 2x(y-px) & (y-px)y+pyx & pxz+(y-px)z \\ (y-px)y+pyx & 2py^2 & 2pyz \\ pxz+(y-px)z & 2pyz & 2pz^2 \end{pmatrix},$$

$$\text{and} \quad h_4 = \begin{pmatrix} 2x(y+px) & (y+px)y-pyx & pxz+(y+px)z \\ (y+px)y+pyx & -2py^2 & 0 \\ pxz+(y+px)z & 0 & 2pz^2 \end{pmatrix}.$$

1.10 Right-invariant K-V structures on Lie groups

Let $(\mathfrak{g}, \bullet)$ be a left symmetric algebra, such that, for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{ass}(u,v,w) = \mathrm{ass}(v,u,w) \text{ where } \mathrm{ass}(u,v,w) = (u \bullet v) \bullet w - u \bullet (v \bullet w).$$

Then $[u,v] = u \bullet v - v \bullet u$ is a Lie bracket on $\mathfrak{g}$ and $\mathrm{L} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $u \mapsto \mathrm{L}_u$ is a representation of the Lie algebra $(\mathfrak{g}, [\ ,\ ])$, where L_u denotes left multiplication by u.

We consider a connected Lie group G whose Lie algebra is $(\mathfrak{g}, [\ ,\ ])$ and we define on G a right invariant connection by

$$\nabla_{u^-} v^- = -(u \bullet v)^-, \tag{1.10.1}$$

where u^- is the right vector field associated to $u \in \mathfrak{g}$. This connection is flat torsionless hence (G, ∇) is an affine manifold. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a symmetric bivector and let r^- be the associated right invariant symmetric bivector field.

Proposition 1.10.1. *(G, ∇, r^-) is a K-V manifold if and only if, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,*

$$[[r,r]](\alpha,\beta,\gamma) := \prec \gamma, [\alpha,\beta]_r^\# - [\alpha^\#, \beta^\#] \succ = 0, \tag{1.10.2}$$

where

$$[\alpha,\beta]_r = \mathrm{L}^*_{\alpha^\#}\beta - \mathrm{L}^*_{\beta^\#}\alpha \quad \textit{and} \quad \prec \mathrm{L}^*_u \alpha, v \succ = - \prec \alpha, u \bullet v \succ .$$

In this case, the product on $\mathfrak{g}^*$ given by $\alpha.\beta = \mathrm{L}^*_{\alpha^\#}\beta$ is left symmetric, $[\ ,\ ]_r$ is a Lie bracket and $r_\#$ is a morphism of Lie algebras.

Proof. Note first that for any $\alpha \in \mathfrak{g}^*$, $(\alpha^-)^\# = (\alpha^\#)^-$ and $\nabla_{u^-}\alpha^- = -(\mathrm{L}^*_u\alpha)^-$ and hence, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,

$$\nabla_{(\alpha^-)^\#}(r^-)(\beta^-, \gamma^-) = r(\mathrm{L}^*_{\alpha^\#}\beta, \gamma) + r(\beta, \mathrm{L}^*_{\alpha^\#}\gamma).$$

So, (G, ∇, r^-) is a K-V manifold if and only if, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,

$$\begin{aligned} 0 &= r(\mathrm{L}^*_{\alpha^\#}\beta, \gamma) + r(\beta, \mathrm{L}^*_{\alpha^\#}\gamma) - r(\mathrm{L}^*_{\beta^\#}\alpha, \gamma) - r(\alpha, \mathrm{L}^*_{\beta^\#}\gamma) \\ &= \prec \gamma, [\alpha,\beta]_r^\# - \alpha^\# \bullet \beta^\# + \beta^\# \bullet \alpha^\# \succ \\ &= \prec \gamma, [\alpha,\beta]_r^\# - [\alpha^\#, \beta^\#] \succ . \end{aligned}$$

Hence the first part of the proposition follows. Suppose now that $[\alpha,\beta]_r^\# = [\alpha^\#, \beta^\#]$ for any $\alpha, \beta \in \mathfrak{g}^*$. Then, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,

$$\mathrm{ass}(\alpha,\beta,\gamma) - \mathrm{ass}(\beta,\alpha,\gamma) = \mathrm{L}^*_{[\alpha,\beta]_r^\#}\gamma - \mathrm{L}^*_{\alpha^\#}\mathrm{L}^*_{\beta^\#}\gamma + \mathrm{L}^*_{\beta^\#}\mathrm{L}^*_{\alpha^\#}\gamma = 0.$$

This completes the proof. ∎

Definition 1.10.2.

1. *Let* $(\mathfrak{g}, \bullet)$ *be a left symmetric algebra. A symmetric bivector* $r \in \mathfrak{g} \otimes \mathfrak{g}$ *satisfying* $[[r, r]] = 0$ *is called a* S*-matrix.*

2. *A left symmetric algebra* $(\mathfrak{g}, \bullet, r)$ *endowed with a* S*-matrix is called a K-V algebra.*

Let $(\mathfrak{g}, \bullet, r)$ be a K-V algebra, $[u, v] = u \bullet v - v \bullet u$ and G a connected Lie group with $(\mathfrak{g}, [\ ,\ ])$ as a Lie algebra. We have shown that G carries a right invariant K-V structure (∇, r^-). On the other hand, in Section 1.6, we have associated to (∇, r^-) a flat connection $\overline{\nabla}$, a complex structure J and a Poisson tensor Π on TG. Now we will show that TG carries a structure of Lie group and the triple $(\overline{\nabla}, J, \Pi)$ is right invariant. This structure of Lie group on TG is different from the usual one defined by the adjoint action of G on $\mathfrak{g}$.

Let us start with a general algebraic construction which is interesting on its own. Let $(\mathfrak{g}, \bullet)$ be a left symmetric algebra, put $\Phi(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$ and define a product $\star$ and a bracket on $\Phi(\mathfrak{g})$ by

$$(a, b) \star (c, d) = (a \bullet c, a \bullet d) \quad \text{and} \quad [(a, b), (c, d)] = ([a, c], a \bullet d - c \bullet b),$$

for any $(a, b), (c, d) \in \Phi(\mathfrak{g})$. It is easy to check that $\star$ is left symmetric, $[\ ,\ ]$ is the commutator of $\star$ and hence is a Lie bracket. We define also $J_0 : \Phi(\mathfrak{g}) \longrightarrow \Phi(\mathfrak{g})$ by $J_0(a, b) = (b, -a)$. It is also a straightforward computation to check that

$$N_{J_0}((a, b), (c, d)) = [J_0(a, b), J_0(c, d)] - J_0[(a, b), J_0(c, d)] - J_0[J_0(a, b), (c, d)] - [(a, b), (c, d)] = 0.$$

For $r \in \otimes^2\mathfrak{g}$ symmetric, we define $R \in \otimes^2\Phi(\mathfrak{g})$ by

$$R((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = r(\alpha_1, \beta_2) - r(\alpha_2, \beta_1), \tag{1.10.3}$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathfrak{g}^*$. We have obviously that $R_\#(\alpha_1, \beta_1) = (-\beta_1^\#, \alpha_1^\#)$.

Proposition 1.10.3. $[[r, r]] = 0$ *if and only if,* $[R, R] = 0$*, where* $[R, R]$ *is the Schouten bracket associated to the Lie algebra structure of* $\Phi(\mathfrak{g})$ *given by*

$$[R, R](\alpha, \beta, \gamma) = \oint_{\alpha,\beta,\gamma} \prec \gamma, [R_\#(\alpha), R_\#(\beta)] \succ, \quad \alpha, \beta, \gamma \in \Phi^*(\mathfrak{g}).$$

Proof. For any $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in \Phi(\mathfrak{g})^*$,

$$\begin{aligned}
\prec \gamma, [R_\#(\alpha), R_\#(\beta)] &= \prec (\gamma_1, \gamma_2), [(-\alpha_2^\#, \alpha_1^\#), (-\beta_2^\#, \beta_1^\#)] \succ \\
&= \prec \gamma_1, [\alpha_2^\#, \beta_2^\#] \succ - \prec \gamma_2, \alpha_2^\# \bullet \beta_1^\# \succ + \prec \gamma_2, \beta_2^\# \bullet \alpha_1^\# \succ \\
&= \prec \gamma_1, [\alpha_2^\#, \beta_2^\#] \succ + \prec \beta_1, (\mathrm{L}^*_{\alpha_2^\#}\gamma_2)^\# \succ - \prec \alpha_1, (\mathrm{L}^*_{\beta_2^\#}\gamma_2)^\# \succ, \\
\prec \beta, [R_\#(\gamma), R_\#(\alpha)] \succ &= \prec \beta_1, [\gamma_2^\#, \alpha_2^\#] \succ + \prec \alpha_1, (\mathrm{L}^*_{\gamma_2^\#}\beta_2)^\# \succ - \prec \gamma_1, (\mathrm{L}^*_{\alpha_2^\#}\beta_2)^\# \succ \\
\prec \alpha, [R_\#(\beta), R_\#(\gamma)] \succ &= \prec \alpha_1, [\beta_2^\#, \gamma_2^\#] \succ + \prec \gamma_1, (\mathrm{L}^*_{\beta_2^\#}\alpha_2)^\# \succ - \prec \beta_1, (\mathrm{L}^*_{\gamma_2^\#}\alpha_2)^\# \succ .
\end{aligned}$$

So

$$[R, R](\alpha, \beta, \gamma) = -[[r, r]](\beta_2, \gamma_2, \alpha_1) - [[r, r]](\gamma_2, \alpha_2, \beta_1) - [[r, r]](\alpha_2, \beta_2, \gamma_1),$$

and the result follows. ∎

Let G be a Lie group whose Lie algebra is $(\mathfrak{g}, [\ ,\ ])$ and let $\rho : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ be the homomorphism of groups such that $d_e\rho = \mathrm{L}$ where $\mathrm{L} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is the representation associated to $\bullet$. Then the product

$$(g, u).(h, v) = (gh, u + \rho(g)(v)), \quad g, h \in G, u, v \in \mathfrak{g}$$

induces a Lie group structure on $G \times \mathfrak{g}$ whose Lie algebra is $(\Phi(\mathfrak{g}), [\ ,\ ])$. The complex endomorphism J_0 and the left symmetric product $\star$ induce a right invariant complex tensor J_0^- and a right invariant connection $\widetilde{\nabla}$ given by

$$J_0^-(a, b)^- = (b, -a)^- \quad \text{and} \quad \widetilde{\nabla}_{(a,b)^-}(c, d)^- = -((a, b) \star (c, d))^-.$$

Let $r \in \otimes^2\mathfrak{g}$ be a symmetric bivector such that $[[r, r]] = 0$, r^- the associated right invariant symmetric bivector field and ∇ the flat torsionless connection given by (1.10.1). Then (G, ∇, r^-) is a K-V manifold and let $\overline{\nabla}$, J and Π be the associated structure on TG defined in Section 1.4.

Theorem 1.10.4. *If we identify TG with $G \times \mathfrak{g}$ by $u_g \longrightarrow (g, T_gR_{g^{-1}}u_g)$, and denote by Π, $\overline{\nabla}$ and J the images of Π, $\overline{\nabla}$ and J under this identification then $\Pi = R^-$, $\overline{\nabla} = \widetilde{\nabla}$ and $J = J_0^-$.*

To prove this theorem, we need some preparation.

Proposition 1.10.5. *Let (G, ∇) be a Lie group endowed with a right invariant connection and $\gamma : [0, 1] \longrightarrow G$ a curve. Let $V : [0, 1] \longrightarrow TG$ be a vector field along γ. We define $\mu : [0, 1] \longrightarrow \mathfrak{g}$ and $W : [0, 1] \longrightarrow \mathfrak{g}$ by*

$$\mu(t) = T_{\gamma(t)}R_{\gamma(t)^{-1}}(\gamma'(t)) \quad \textit{and} \quad W(t) = T_{\gamma(t)}R_{\gamma(t)^{-1}}(V(t)).$$

Then V is parallel along γ with respect ∇ if and only if

$$W'(t) - \mu(t) \bullet W(t) = 0,$$

where $u \bullet v = -(\nabla_{u^-}v^-)(e)$.

Proof. We consider $(u_1, \ldots, u_n)$ a basis of $\mathfrak{g}$ and $(X_1, \ldots, X_n)$ the corresponding right invariant vector fields. Then

$$\begin{cases} \mu(t) = \sum_{i=1}^n \mu_i(t)u_i, \ W(t) = \sum_{i=1}^n W_i(t)u_i, \\ \gamma'(t) = \sum_{i=1}^n \mu_i(t)X_i, \ V(t) = \sum_{i=1}^n W_i(t)X_i. \end{cases}$$

Then

$$\begin{aligned} \nabla_t V(t) &= \sum_{i=1}^n W_i'(t)X_i + \sum_{i=1}^n W_i(t)\nabla_{\gamma'(t)}X_i \\ &= \sum_{i=1}^n W_i'(t)X_i + \sum_{i,j=1}^n W_i(t)\mu_j(t)\nabla_{X_j}X_i \\ &= \sum_{i=1}^n W_i'(t)X_i - \sum_{i,j=1}^n W_i(t)\mu_j(t)(u_j \bullet u_i)^- \\ &= (W'(t) - \mu(t) \bullet W(t))^- \end{aligned}$$

and the result follows having in mind that u^- is the right invariant vector field associated to $u \in \mathfrak{g}$. ∎

Let (G, ∇) be a Lie group endowed with a right invariant connection. Then ∇ induces a splitting of $TTG = \ker dp \oplus \mathcal{H}$. For any tangent vector $X \in T_gG$, we denote by $X^v, X^h \in T_{(g,u)}TG$ the vertical and the horizontal lift of X.

Proposition 1.10.6. *If we identify TG to $G \times \mathfrak{g}$ by $X_g \mapsto (g, T_gR_{g^{-1}}(X_g))$ then for any $X \in T_gG$,*

$$X^v(g,u) = (0, T_gR_{g^{-1}}(X)) \quad \textit{and} \quad X^h(g,u) = (X, T_gR_{g^{-1}}(X) \bullet u).$$

Proof. The first relation is obvious. Recall that the horizontal lift of X at $u_g \in TG$ is given by:

$$X^h(u_g) = \frac{d}{dt}_{|t=0} V(t),$$

where $V : [0,1] \longrightarrow TG$ is the parallel vector field along $\gamma : [0,1] \longrightarrow G$ a curve such that $\gamma(0) = g$ and $\gamma'(0) = X$. If we denote by $\Theta_R : TG \longrightarrow G \times \mathfrak{g}$ the identification $u_g \mapsto (g, T_gR_{g^{-1}}(u_g))$ then, by virtue of Proposition 1.10.5,

$$T_{u_g}\Theta_R(X^h) = \frac{d}{dt}_{|t=0}(\gamma(t), W(t)) = (X, T_gR_{g^{-1}}(X) \bullet u).$$

∎

We consider now a left symmetric algebra $(\mathfrak{g}, \bullet)$, G a connected Lie group associated to $(\mathfrak{g}, [\ ,\ ])$, ∇ the right invariant affine connection associated to $\bullet$. We have seen that $G \times \mathfrak{g}$ has a structure of Lie group. We identify TG to $G \times \mathfrak{g}$ and, for any vector field X on G, we denote by X^v and X^h the vector fields on $G \times \mathfrak{g}$ obtained from the identification of the horizontal and the vertical lift of X. For $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$, a^- (resp. α^-) is the right invariant vector field (resp. 1-form) on G associated to a (resp. α), $(a,b)^-$ (resp. $(\alpha,\beta)^-$) the right invariant vector field (resp. 1-form) on $G \times \mathfrak{g}$ associated to (a,b) (resp. (α,β)).

Proposition 1.10.7. *For any $(a,b) \in \mathfrak{g} \times \mathfrak{g}$ and $(\alpha,\beta) \in \mathfrak{g}^* \times \mathfrak{g}^*$,*

$$(a,b)^- = (a^-)^h + (b^-)^v \quad \textit{and} \quad (\alpha,\beta)^- = (\alpha^-)^h + (\beta^-)^v.$$

Proof. We have

$$\begin{aligned}
(a,b)^-(g,u) &= T_{(e,0)}R_{(g,u)}(a,b) \\
&= \frac{d}{dt}_{|t=0}(\exp(ta), tb)(g,u) \\
&= \frac{d}{dt}_{|t=0}(\exp(ta)g, tb + \rho(\exp(ta))(u)) \\
&= (a^-(g), b + a \bullet u) \\
&= (a^-(g), T_gR_{g^{-1}}(a^-(g)) \bullet u) + (0, T_gR_{g^{-1}}(b^-(g)) \\
&= (a^-)^h(g,u) + (b^-)^v(g,u). \qquad \text{(Proposition 1.10.5)}
\end{aligned}$$

The. second relation can be easily deduced from the first one. ∎

Proof of Theorem 3.2.7. Let Π be the Poisson tensor on $G \times \mathfrak{g}$ associated to r^-. Then, by using Proposition 1.10.7,

$$\begin{aligned}\Pi((\alpha_1,\beta_1)^-,(\alpha_2,\beta_2)^-) &= \Pi((\alpha_1^-)^h + (\beta_1^-)^v, (\alpha_2^-)^h + (\beta_2^-)^v)\\ &= r^-(\alpha_1^-,\beta_2^-) - r^-(\alpha_2^-,\beta_1^-)\\ &= r(\alpha_1,\beta_2) - r(\alpha_2,\beta_1)\\ &= R^-((\alpha_1,\beta_1)^-,(\alpha_2,\beta_2)^-).\end{aligned}$$

In the same way,

$$\begin{aligned}J_0^-(a,b)^- &= (b,-a)^- = (b^-)^h - (a^-)^v,\\ J(a,b)^- &= (b^-)^h - (a^-)^v,\\ \overline{\nabla}_{(a,b)^-}(c,d)^- &= (\nabla_{a^-}c^-)^h + (\nabla_{a^-}d^-)^v\\ &= -((a\bullet c)^-)^h - ((a\bullet d)^-)^v\\ &= -((a,b).(c,d))^-\\ &= \widetilde{\nabla}_{(a,b)^-}(c,d)^-.\end{aligned}$$

■

Let $(\mathfrak{g},\bullet)$ be a left symmetric algebra, (M,∇) an affine manifold and $\rho : \mathfrak{g} \longrightarrow \Gamma(TM)$ a linear map such that $\rho(u\bullet v) = \nabla_{\rho(u)}\rho(v)$. Then ρ defines an action on M of the Lie algebra $(\mathfrak{g},[\ ,\ ])$. We consider $\rho^l : \Phi(\mathfrak{g}) \longrightarrow \Gamma(TTM)$, $(u,v) \longrightarrow \rho(u)^h + \rho^v(v)$. It is easy to check that

$$\rho^l([a,b]) = [\rho^l(a),\rho^l(b)].$$

Let $r \in \otimes^2\mathfrak{g}$ satisfying $[[r,r]] = 0$ and $R \in \otimes^2\Phi(\mathfrak{g})$ given by (1.10.3).

Theorem 1.10.8. *The bivector field on TM associated to $\rho(r)$ is $\rho^l(R)$ which is a Poisson tensor and $(M,\nabla,\rho(r))$ is a K-V manifold.*

Proof. Let $(e_1,\ldots,e_n)$ a basis of $\mathfrak{g}$ and $E_i = (e_i,0)$ and $F_i = (0,e_i)$. Then $(E_1,\ldots,E_n,F_1,\ldots,F_n)$ is a basis of $\Phi(\mathfrak{g})$. Then

$$r = \sum_{i,j} r_{i,j}e_i\otimes e_j \quad \text{and} \quad R = \sum_{i,j} r_{i,j}\left(E_i\otimes F_j - F_i\otimes E_j\right).$$

So

$$\rho(r) = \sum_{i,j=1}^n r_{i,j}\rho(e_i)\otimes\rho(e_j) \quad \text{and} \quad \rho^l(R) = \sum_{i,j=1}^n r_{i,j}\left(\rho(e_i)^h\otimes\rho(e_j)^v - \rho(e_i)^v\otimes\rho(e_j)^h\right).$$

Then for any $\alpha,\beta\in\Omega^1(M)$

$$\rho^l(R)(\alpha^v,\beta^v) = \rho^l(R)(\alpha^h,\beta^h) = 0 \quad \text{and} \quad \rho^l(R)(\alpha^h,\beta^v) = \rho(r)(\alpha,\beta)\circ p.$$

According to Proposition 1.10.3, R is a solution of the classical Yang-Baxer equation and hence $\rho^l(R)$ is a Poisson tensor. By using Theorem 1.6.3, we get that $(M,\nabla,\rho(r))$ is a K-V manifold. ■

Example 1.10.9.

1. *Let $\mathfrak{g} = \mathrm{gl}(n,\mathbb{R})$ be the Lie algebra of n-square matrices. It has a structure of left symmetric algebra given by $A\bullet B = BA$. Let $\rho : \mathfrak{g} \longrightarrow \Gamma(T\mathbb{R}^n)$ given by $\rho(A) = A$.*

Then $\rho(A \bullet B) = \nabla_A B$, where ∇ is the canonical connection of $\mathbb{R}^n$. According to Theorem 1.10.8, any S-matrix on $\mathfrak{g}$ gives rise to a quadratic K-V structure on $\mathbb{R}^n$.

2. *More generally, let (M, ∇) be an affine manifold and $\mathfrak{g}$ the finite dimensional Lie algebra of affine vector fields. Recall that $X \in \mathfrak{g}$ if for any $Y, Z \in \Gamma(TM)$,*

$$[X, \nabla_Y Z] = \nabla_{[X,Y]} Z + \nabla_Y [X, Z].$$

Since the curvature and the torsion of ∇ vanish this is equivalent to

$$\nabla_{\nabla_Y Z} X = \nabla_Y \nabla_Z X.$$

From this relation, one can see easily that, for any $X, Y \in \mathfrak{g}$, $X \bullet Y := \nabla_X Y \in \mathfrak{g}$ and $(\mathfrak{g}, \bullet)$ is an associative finite dimensional Lie algebra that acts on M by $\rho(X) = X$. Moreover, $\rho(X \bullet Y) = \nabla_X Y$. According to Theorem 1.10.8, any S-matrix on $\mathfrak{g}$ gives rise to a K-V structure on M.

1.11 Classification of two-dimensional K-V algebras

Using the classification of two-dimensional non-abelian left symmetric algebras (see [12]) and the classification of abelian left symmetric algebras (see [46]), we give a classification (over the field $\mathbb{R}$) of 2-dimensional K-V algebras. We proceed as follows

1. For any left symmetric 2-dimensional algebra $\mathfrak{g}$, we determine its automorphism group $\mathrm{Aut}(\mathfrak{g})$ and the space of S-matrices on $\mathfrak{g}$, which we denote by $\mathcal{A}(\mathfrak{g})$.

2. We give the quotient $\mathcal{A}(\mathfrak{g})/\sim$ where $\sim$ is the equivalence relation

   $$r^1 \sim r^2 \iff \exists\ A \in \mathrm{Aut}(\mathfrak{g}) \text{ such that } r^2_\sharp = A \circ r^1_\sharp \circ A^t \text{ or } \exists\ \lambda \in \mathbb{R} \text{ such that } r^2 = \lambda r^1.$$

$(\mathfrak{g}, .)$	$\mathrm{Aut}(\mathfrak{g})$	$\mathcal{A}(\mathfrak{g})/\sim$
$b_{1,\alpha\neq-1,1}$ $e_2.e_1{=}e_1;\ e_2.e_2{=}\alpha e_2$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r^1_\sharp{=}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^3_\sharp{=}0$
$b_{1,\alpha=-1}$ $e_2.e_1{=}e_1;\ e_2.e_2{=}-e_2$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r^1_\sharp = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix};\ r^2_\sharp = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$ $r^3_\sharp = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^4_\sharp = 0$
$b_{1,\alpha=1}$ $e_2.e_1{=}e_1;\ e_2.e_2{=}e_2$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, ab \neq 0$	$r^1_\sharp{=}\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^3_\sharp = 0$
$b_{2,\beta\neq0,1,2}$ $e_2.e_1{=}(\beta{-}1)e_1;$ $e_1.e_2{=}\beta e_1;\ e_2.e_2{=}\beta e_2$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r^1_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^3_\sharp = 0$
$b_{2,\beta=1}$ $e_1.e_2{=}e_1;\ e_2.e_2{=}e_2$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r^1_\sharp{=}\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^3_\sharp = 0$
$b_{2,\beta=2}$ $e_1.e_2{=}2e_1;$ $e_2.e_1{=}e_1;\ e_2.e_2{=}2e_2$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r^1_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^3_\sharp{=}0$
b_3 $e_2.e_1{=}e_1;\ e_2.e_2{=}e_1+e_2$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$r^1_\sharp{=}\begin{pmatrix} 1/2 & 1 \\ 1 & 1 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ r^3_\sharp{=}0$
b_4 $e_1.e_1{=}2e_1;\ e_1.e_2{=}e_2;\ e_2.e_2{=}e_1$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$r^1_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ r^3_\sharp = 0$
b_5 $e_1.e_2{=}e_1;\ e_2.e_2{=}e_1+e_2$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$r^1_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ r^2_\sharp{=}0$
As^1_2 $e_1.e_1{=}e_2$	$\begin{pmatrix} a & 0 \\ b & a^2 \end{pmatrix}, a \neq 0$	$r^1_\sharp{=}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ r^3_\sharp{=}0$
As^4_2 $e_1.e_1{=}e_1;\ e_1.e_2{=}e_2;\ e_2.e_2{=}e_2$	$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, a \neq 0$	$r^1_\sharp{=}\begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix};\ r^2_\sharp{=}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$ $r^3_\sharp{=}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ r^4_\sharp{=}0$

Chapter 2

Submanifolds in K-V Geometry

Keeping in mind the analogies given in the previous chapter between K-V structures and Poisson structures, we introduce in this chapter *K-V Hamiltonian vector fields* and we explore their properties. We also introduce many classes of submanifolds in K-V geometry.

2.1 K-V Hamiltonian vector fields

In Poisson geometry, Hamiltonian vector fields play an important role (see [52]) and it is therefore natural to examine some properties of their analogue in K-V geometry.

Let (M, ∇, h) be a K-V manifold. For any smooth function $f \in C^\infty(M)$ we associate a vector field $X_f := (df)^\#$, which will be called the *K-V Hamiltonian vector field* associated to f. The symmetry of the bivector field h leads to the following

$$X_{f_1}(f_2) = X_{f_2}(f_1) = \prec df_2, X_{f_1} \succ = \prec df_1, X_{f_2} \succ,$$

where $f_1, f_2 \in C^\infty(M)$. Contrary to what happens in Poisson geometry the flow of the vector field X_f does not generally preserve the K-V bivector field h, this can be seen through the following example.

Example 2.1.1. *Consider the K-V manifold $M = (\mathbb{R}^2, \nabla, h)$ where ∇ is the canonical affine connection on $\mathbb{R}^2$ and $h = x\partial_x \otimes \partial_x + y\partial_y \otimes \partial_y$. Let $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x$, by a direct computation, we get that $L_{X_f}(h)(dx, dx) = -f$.*

More generally,

Proposition 2.1.2. *For any $f \in C^\infty(M)$ and any $\alpha, \beta \in \Omega^1(M)$, we have*

$$\mathcal{L}_{X_f}(h)(\alpha, \beta) = -\nabla_{X_f}(h)(\alpha, \beta) + 2 \prec \nabla_{\alpha^\#} df, \beta^\# \succ .$$

Proof. Let $(x_1, \ldots, x_n)$ be an affine local coordinates system on M. Denote by $X_i = (dx_i)^\#$. Then

$$\begin{aligned}
\mathcal{L}_{X_f}(h)(dx_i, dx_j) &= X_f.h_{ij} - h\left(\mathcal{L}_{X_f}dx_i, dx_j\right) - h\left(dx_i, \mathcal{L}_{X_f}dx_j\right) \\
&= -X_f.h_{ij} + \prec dx_i, [X_f, X_j] \succ + \prec dx_j, [X_f, X_i] \succ \\
&= -X_f.h_{ij} + \prec dx_i, [df, dx_j]^\# \succ + \prec dx_j, [df, dx_i]^\# \succ \\
&= -X_f.h_{ij} + \prec [df, dx_j], X_i \succ + \prec [df, dx_i], X_j \succ \\
&= -X_f.h_{ij} + \prec \nabla_{X_j} df, X_i \succ - \prec \nabla_{X_f} dx_j, X_i \succ \\
&\quad + \prec \nabla_{X_i} df, X_j \succ - \prec \nabla_{X_f} dx_i, X_j \succ .
\end{aligned}$$

Now, since $\nabla dx_i = 0$ and $\prec \nabla_X df, Y \succ = \prec \nabla_Y df, X \succ$, we get

$$\mathcal{L}_{X_f}(h)(dx_i, dx_j) = -\nabla_{X_f}(h)(dx_i, dx_j) + 2 \prec \nabla_{X_i} df, X_j \succ .$$

■

As a reminder, a smooth function $f \in C^\infty(M)$ on an affine manifold (M, ∇) is called affine if it is affine in local affine coordinates. In the following, we introduce the following space.

$$\mathcal{E} := \{f \in C^\infty(M) \,/\, \prec \nabla_{\alpha^\#} df, \beta^\# \succ = 0, \forall \alpha, \beta \in \Omega^1(M)\},$$

which is the space of smooth functions $f : M \to \mathbb{R}$ that are affine along the leaves of the affine foliation, i.e., the restriction of f to any leaf $L \subset M$ is affine.

Example 2.1.3. *Let $(\mathcal{A}, \bullet)$ be a n-dimensional commutative associative algebra. We have seen in Proposition 1.7.2 that the dual $\mathcal{A}^*$ carries a linear K-V structure (∇, h) where ∇ is the canonical affine connection on $\mathcal{A}^*$ and h is the linear symmetric bivector field on $\mathcal{A}^*$ given by*

$$h(u, v)(\alpha) = \prec \alpha, u(\alpha) \bullet v(\alpha) \succ,$$

where $\alpha \in \mathcal{A}^*$ and $u, v \in \Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*, \mathcal{A})$. Now we take $\mathcal{A} = (\mathbb{R}^2, \bullet)$ where $\bullet$ is the commutative, associative product given by $e_1 \bullet e_1 = e_1$ and the others products are zero. Denote by (x, y) the canonical dual coordinates of $\mathcal{A}^*$, so we have

$$(dx)^\# = x\partial_x \text{ and } (dy)^\# = 0.$$

According to Proposition 2.1.2 we get that for any $f \in C^\infty(\mathcal{A}^*)$, which depend only on the y-variable

$$\mathcal{L}_{X_f}(h) = 0.$$

We know that for any K-V manifold (M, ∇, h) the tangent bundle (TM, Π) is a Poisson manifold. So we can ask the following question. When the vertical lift X_f^v and the horizontal lift X_f^h are Poisson vector fields on (TM, Π)?

Proposition 2.1.4. *For any $f \in C^\infty(M)$, we have*

1. *X_f^v is the Hamiltonian vector field on (TM, Π) associated to the function $f \circ p \in C^\infty(TM)$.*

2. *X_f^h is a Poisson vector field on (TM, Π) if and only if $f \in \mathcal{E}$.*

Proof. From (1.6.4) we get $X_f^v = \Pi_\#((df)^h) = \Pi_\#(d(f \circ p))$ which leads to 1. For the second assertion we need to compute $\mathcal{L}_{X_f^h}(\Pi)$. Let $\alpha, \beta \in \Omega^1(M)$, by using the formulas proved in Proposition 1.6.6, we get that

$$\mathcal{L}_{X_f^h}(\Pi)(\alpha^v, \beta^v) = \mathcal{L}_{X_f^h}(\Pi)(\alpha^h, \beta^h) = 0.$$

Furthermore we have

$$\begin{aligned}
\mathcal{L}_{X_f^h}(\Pi)(\alpha^v,\beta^h) &= X_f^h.\Pi(\alpha^v,\beta^h)-\Pi(\mathcal{L}_{X_f^h}\alpha^v,\beta^h)-\Pi(\alpha^v,\mathcal{L}_{X_f^h}\beta^h)\\
&= -X_f^h.(h(\alpha,\beta)\circ p)-\Pi((\nabla_{X_f}\alpha)^v,\beta^h)-\Pi(\alpha^v,(\mathcal{L}_{X_f}\beta)^h)\\
&= -(X_f.h(\alpha,\beta))\circ p+h(\nabla_{X_f}\alpha,\beta)\circ p+h(\alpha,\mathcal{L}_{X_f}\beta)\circ p\\
&= \big(-X_f.h(\alpha,\beta)+\prec\nabla_{X_f}\alpha,\beta^{\#}\succ+\prec\mathcal{L}_{X_f}\beta,\alpha^{\#}\succ\big)\circ p\\
&= \big(\prec\mathcal{L}_{X_f}\beta,\alpha^{\#}\succ-\prec\alpha,\nabla_{X_f}\beta^{\#}\succ\big)\circ p\\
&= \big(X_f.h(\alpha,\beta)-\prec\beta,[X_f,\alpha^{\#}]\succ-\prec\alpha,\nabla_{X_f}\beta^{\#}\succ\big)\circ p\\
&= \big(\prec\nabla_{X_f}\alpha,\beta^{\#}\succ-\prec\beta,[X_f,\alpha^{\#}]\succ\big)\circ p.
\end{aligned}$$

Now, since $[\alpha^{\#},\beta^{\#}]=[\alpha,\beta]_h^{\#}$ and $\nabla_{\alpha^{\#}}\beta-\nabla_{\beta^{\#}}\alpha=[\alpha,\beta]_h$, we get

$$\mathcal{L}_{X_f^h}(\Pi)(\alpha^v,\beta^h)=\prec\nabla_{\alpha^{\#}}df,\beta^{\#}\succ\circ p.$$

∎

Now we consider the two vector spaces

$$\mathcal{V}_h:=\{X_f\in\Gamma(TM)|\ f\in\mathcal{E}\}\quad\text{and}\quad\mathcal{V}_\Pi:=\{X_f^h\in\Gamma(T(TM))|\ f\in\mathcal{E}\}.$$

Proposition 2.1.5. *$\mathcal{V}_h$ and $\mathcal{V}_\Pi$ are two abelian Lie subalgebras of vector fields.*

Proof. For all $f_1,f_2\in E$, we have

$$[X_{f_1},X_{f_2}]=([df_1,df_2]_h)^{\#}=\left(\nabla_{X_{f_1}}df_2-\nabla_{X_{f_2}}df_{f_1}\right)^{\#}=0.$$

The other assumption follows from $[X_{f_1}^h,X_{f_2}^h]=[X_{f_1},X_{f_2}]^h$. ∎

It is well known that for any affine manifold (M,∇), the space of vector fields $\Gamma(TM)$ endowed with the product $X\bullet Y=\nabla_XY$ is a left symmetric algebra. It is therefore natural to look at the behavior of the subspaces $\mathcal{V}_h$ and $\mathcal{V}_\Pi$ with respect to the left symmetric structures in $\Gamma(TM)$ and in $\Gamma(TTM)$.

Theorem 2.1.6. *Let (M,∇,h) be a K-V manifold satisfying the following condition:*

$$f_1,f_2\in\mathcal{E}\Longrightarrow h(df_1,df_2)\in\mathcal{E}. \tag{2.1.1}$$

Then $(\mathcal{V}_h,\bullet)$ and $(\mathcal{V}_\Pi,\bullet)$ are two commutative, associative subalgebras of $(\Gamma(TM),\bullet)$ and $(\Gamma(TTM),\bullet)$ respectively. Moreover, the map $X_f\mapsto X_f^h$ is an isomorphism from $\mathcal{V}_h$ to $\mathcal{V}_\Pi$.

Proof. Let $f_1,f_2\in\mathcal{E}$. For any $\alpha\in\Omega^1(M)$, we have

$$\begin{aligned}
\prec\alpha,X_{f_1}\bullet X_{f_2}\succ &= \prec\alpha,(\mathcal{D}_{df_1}df_2)^{\#}\succ\\
&= \prec\mathcal{D}_{df_1}df_2,\alpha^{\#}\succ\\
&= \alpha^{\#}.h(df_1,df_2)\\
&= \prec\alpha,X_{h(df_1,df_2)}\succ.
\end{aligned}$$

Hence $X_{f_1}\bullet X_{f_2}=X_{h(df_1,df_2)}$ which is in $\mathcal{V}_h$ by the condition (2.1.1). The remainder of the theorem is obvious. ∎

Remark 2.1.7. *The class of K-V manifolds satisfying condition (2.1.1) deserves a special study.*

In the following, we provide a class of K-V manifolds satisfying condition (2.1.1).

Example 2.1.8. *Let* $(\mathcal{A}, \bullet, B)$ *be a finite dimensional commutative and associative algebra endowed with a symmetric scalar* 2*-cycle. We have seen in Proposition 1.7.2 that the dual* $\mathcal{A}^*$ *carries a K-V structure* (∇, h) *where* ∇ *is the canonical affine structure on* $\mathcal{A}^*$ *and* h *is the affine symmetric bivector field on* $\mathcal{A}^*$ *given by*

$$h(u,v)(\alpha) = \prec \alpha, u(\alpha).v(\alpha) \succ + B(u(\alpha), v(\alpha))$$

where $\alpha \in \mathcal{A}^*$ *and* $u, v \in \Omega^1(\mathcal{A}^*)$*. We consider the canonical coordinates system* $(x_1, \ldots, x_n)$ *on* $\mathcal{A}^*$ *associated to a basis* $(e_1, \ldots, e_n)$*. Then*

$$h(dx_i, dx_j) = b_{ij} + \sum_{k=1}^{n} C_{ij}^k x_k,$$

where $C_{ij}^k = \prec e_k^*, e_i.e_j \succ$ *and* $b_{ij} = B(e_i, e_j)$*. Hence a function* f *belongs to* $\mathcal{E}$ *if and only if, for all* $i, j = 1, \cdots, n$*,*

$$\sum_{l,k} h_{il} h_{jk} \frac{\partial^2 f}{\partial x_l \partial x_k} = 0. \tag{2.1.2}$$

Let $f_1, f_2 \in \mathcal{E}$*, then*

$$h(df_1, df_2) = \sum_{i,j} h_{ij} \frac{\partial f_1}{\partial x_i} \frac{\partial f_2}{\partial x_j}.$$

A straightforward computation shows that

$$\begin{aligned}
\sum_{k,l} h_{il} h_{jk} \frac{\partial^2 h(df_1, df_2)}{\partial x_k \partial x_l} &= \sum_{k,l,m,s} h_{il} h_{jk} h_{ms} \frac{\partial^3 f_1}{\partial x_k \partial x_l \partial x_m} \frac{\partial f_2}{\partial x_s} + h_{il} h_{jk} h_{ms} \frac{\partial^2 f_1}{\partial x_l \partial x_m} \frac{\partial^2 f_2}{\partial x_k \partial x_s} \\
&\quad + h_{il} h_{jk} C_{ms}^k \frac{\partial^2 f_1}{\partial x_l \partial x_m} \frac{\partial f_2}{\partial x_s} + h_{il} h_{jk} h_{ms} \frac{\partial^2 f_1}{\partial x_k \partial x_m} \frac{\partial^2 f_2}{\partial x_l \partial x_s} \\
&\quad + h_{il} h_{jk} h_{ms} \frac{\partial f_1}{\partial x_m} \frac{\partial^3 f_2}{\partial x_k \partial x_l \partial x_s} + h_{il} h_{jk} C_{ms}^k \frac{\partial f_1}{\partial x_m} \frac{\partial^2 f_2}{\partial x_l \partial x_s} \\
&\quad + h_{il} h_{jk} C_{ms}^l \frac{\partial^2 f_1}{\partial x_k \partial x_m} \frac{\partial f_2}{\partial x_s} + h_{il} h_{jk} C_{ms}^l \frac{\partial f_1}{\partial x_m} \frac{\partial^2 f_2}{\partial x_k \partial x_s}.
\end{aligned}$$

Based on the fact that $\sum_k h_{ik} C_{jm}^k = \sum_k h_{jk} C_{im}^k$ *together with equation (2.1.2) we get*

$$\begin{aligned}
\sum_{k,l,m,s} h_{il} h_{jk} h_{ms} \frac{\partial^3 f_1}{\partial x_k \partial x_l \partial x_m} \frac{\partial f_2}{\partial x_s} &= \sum_{m,s} h_{ms} \frac{\partial f_2}{\partial x_s} \frac{\partial}{\partial x_m} \left(\sum_{k,l} h_{il} h_{jk} \frac{\partial^2 f_1}{\partial x_k \partial x_l} \right) - \sum_{k,l,s} h_{jk} \frac{\partial^2 f_1}{\partial x_k \partial x_l} \frac{\partial f_2}{\partial x_s} \left(\sum_m h_{ms} C_{il}^m \right) \\
&\quad - \sum_{k,l,s} h_{il} \frac{\partial^2 f_1}{\partial x_k \partial x_l} \frac{\partial f_2}{\partial x_s} \left(\sum_m h_{ms} C_{jk}^m \right) \\
&= -\sum_{m,s} C_{is}^m \frac{\partial f_2}{\partial x_s} \left(\sum_{k,l} h_{ml} h_{jk} \frac{\partial^2 f_1}{\partial x_k \partial x_l} \right) - \sum_{m,s} C_{js}^m \frac{\partial f_2}{\partial x_s} \left(\sum_{k,l} h_{mk} h_{il} \frac{\partial^2 f_1}{\partial x_k \partial x_l} \right) \\
&= 0.
\end{aligned}$$

Hence

$$\sum_{k,l,m,s} h_{il}h_{jk}h_{ms}\frac{\partial^2 f_1}{\partial x_l \partial x_m}\frac{\partial^2 f_2}{\partial x_k \partial x_s} = \sum_{k,s} h_{jk}\frac{\partial^2 f_2}{\partial x_k \partial x_s}\left(\sum_{l,m} h_{il}h_{sm}\frac{\partial^2 f_1}{\partial x_l \partial x_m}\right) = 0.$$

This proves that $h(df_1, df_2) \in \mathcal{E}$. Hence the K-V manifold $(\mathcal{A}^*, \nabla, h)$ satisfies condition (2.1.1).

It is natural to ask whether all K-V manifolds satisfy the condition (2.1.1). The response is no, as the following example shows.

Example 2.1.9. *Consider the K-V manifold $M = (\mathbb{R}^2, \nabla, h)$ where ∇ is the canonical affine structure on $\mathbb{R}^2$ and $h = x^2\partial_x \otimes \partial_x$. Let $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x$, obviously $f \in \mathcal{E}$ but $h(df, df) = f^2$ does not belong to the space $\mathcal{E}$ because*

$$\prec \nabla_{(dx)^\#} df^2, (dx)^\# \succ = f^4 \neq 0.$$

2.2 K-V maps

In this section, we will study smooth maps between K-V manifolds that preserve these structures.

Let (M^1, ∇^1) and (M^2, ∇^2) be two affine manifolds. Denote by $p_i : TM^i \to M^i$, the canonical projections. Let $F : M^1 \to M^2$ be a smooth map.

Definition 2.2.1. *F is said to be affine if it is affine in the changes of coordinates.*

Recall that two vector fields $X \in \Gamma(TM^1)$ and $Y \in \Gamma(TM^2)$ are said to be F-related if and only if, $T_xF(X_x) = Y_{F(x)}$ for all $x \in M^1$.

Proposition 2.2.2. *The map F is affine if and only if for any related vector fields $X^1, X^2 \in \Gamma(TM^1)$ and $Y^1, Y^2 \in \Gamma(TM^2)$ respectively, the two vector fields $\nabla^1_{X^1}X^2 \in \Gamma(TM^1)$ and $\nabla^2_{Y^1}Y^2 \in \Gamma(TM^2)$ are F-related.*

In a more geometric terms, F is affine if and only if for every parallel vector field X along a curve γ on M^1 the image F_*X is also parallel along the curve $F \circ \gamma$, or equivalently F is totally geodesic, i.e., the image $F \circ \gamma$ of each geodesic γ of M^1 is a geodesic of M^2 (see [33]).

Let (M^i, ∇^i, h^i) for $i = 1, 2$ be two K-V manifolds. Denote by Π^i the induced Poisson bivector fields on TM^i.

Theorem 2.2.3. *Let $F : (M^1, \nabla^1, h^1) \to (M^2, \nabla^2, h^2)$ be an affine map. Then the following assertions are equivalent*

(i) *F is a K-V map, i.e., for any $\alpha, \beta \in \Omega^1(M^2)$, we have*

$$h^1(F^*\alpha, F^*\beta) = h^2(\alpha, \beta) \circ F. \quad (2.2.1)$$

(ii) *The tangent map $TF : (TM^1, \Pi^1) \to (TM^2, \Pi^2)$ is a Poisson map.*

(iii) *For any one form $\alpha \in \Omega^1(M^2)$, the vector fields $(F^*\alpha)^{\#_1} \in \Gamma(TM^1)$ and $\alpha^{\#_2} \in \Gamma(TM^2)$ are F-related.*

(iv) *For any $f \in C^\infty(M^2)$, the Koszul-Vinberg Hamiltonian vector fields $X_f \in \Gamma(TM^2)$ and $X_{f\circ F} \in \Gamma(TM^1)$ are F-related.*

In order to show this theorem, we need the following lemma.

Lemma 2.2.4. *Let $F : (M^1, \nabla^1) \to (M^2, \nabla^2)$ be an affine map. Then we have*

1. *For any F-related vector fields $X \in \Gamma(TM^1)$ and $Y \in \Gamma(TM^2)$, their respective vertical lifts X^v and Y^v are TF-related. The same thing happens to their horizontal lifts X^h and Y^h.*

2. *For any $\alpha \in \Omega^1(M^2)$, we have*

$$(F^*\alpha)^v = (TF)^*(\alpha^v) \quad \text{and} \quad (F^*\alpha)^h = (TF)^*(\alpha^h).$$

Proof. 1. Let $x \in M^1$ and $u \in T_xM^1$,

$$\begin{aligned} T_u(TF)(X_u^v) &= \frac{d}{dt}_{|t=0} T_xF(u + tX_x) \\ &= \frac{d}{dt}_{|t=0} T_xF(u) + tT_xF(X_x) \\ &= \frac{d}{dt}_{|t=0} T_xF(u) + tY_{F(x)} \\ &= Y^v_{TF(u)}. \end{aligned}$$

Hence X^v and Y^v are TF-related. Now let $\gamma : I \to M^1$ be a curve with $\gamma(0) = x$ and $\gamma'(0) = X_x$. According to (1.6.1) we get that

$$T_u(TF)(X_u^h) = \frac{d}{dt}_{|t=0} T_{\gamma(t)}F(\tau_{0t}^{\gamma}(u)).$$

From the commutativity of the parallel transport maps, we obtain

$$T_u(TF)(X_u^h) = \frac{d}{dt}_{|t=0} \tau_{0t}^{F\circ\gamma}(TF(u)) = Y^h_{TF(u)}.$$

2. Now from this first point, we deduce that for any $X \in \Gamma(TM^1)$ and $\alpha \in \Omega^1(M^2)$, we have

$$\begin{aligned} \prec (TF)^*\alpha^v, X^v \succ &= \prec \alpha^v, Y^v \succ \circ TF \\ &= \prec \alpha, Y \succ \circ p_2 \circ TF \\ &= \prec \alpha, Y \succ \circ F \circ p_1 \\ &= \prec F^*\alpha, X \succ \circ p_1 \\ &= \prec (F^*(\alpha))^v, X^v \succ . \end{aligned}$$

Furthermore,

$$\prec (TF)^*(\alpha^v), X^h \succ = \prec \alpha^v, Y^h \succ \circ TF = 0 = \prec (F^*(\alpha))^v, X^h \succ .$$

This proves that $(TF)^*(\alpha^v) = (F^*(\alpha))^v$. Similarly, we get $(TF)^*(\alpha^h) = (F^*(\alpha))^h$. ∎

Proof of Theorem 2.2.3. Suppose that (i) is satisfied. According to the Lemma 2.2.4 we get that for any $\alpha, \beta \in \Omega^1(M^2)$,

$$\begin{aligned} \Pi^1((TF)^*(\alpha^h), (TF)^*(\beta^h)) &= \Pi^1((F^*\alpha)^h, (F^*\beta)^h) \\ &= 0 \\ &= \Pi^2(\alpha^h, \beta^h) \circ TF. \end{aligned}$$

Similarly, we get $\Pi^1((TF)^*(\alpha^v), (TF)^*(\beta^v)) = \Pi^2(\alpha^v, \beta^v) \circ TF$. Furthermore, we have

$$\begin{aligned} \Pi^1((TF)^*(\alpha^h), (TF)^*(\beta^v)) &= \Pi^1((F^*\alpha)^h, (F^*\beta)^v) \\ &= h^1(F^*\alpha, F^*\beta) \circ p_1 \\ &= h^2(\alpha, \beta) \circ F \circ p_1 \\ &= h^2(\alpha, \beta) \circ p_2 \circ TF \\ &= \Pi^2(\alpha^h, \beta^v) \circ TF. \end{aligned}$$

Hence we get (ii).

Conversely suppose that (ii) is satisfied. Because TF is Poisson map we get that for any $\alpha, \beta \in \Omega^1(M^2)$,

$$\Pi^2(\alpha^h, \beta^v) \circ TF = \Pi^1((TF)^*(\alpha^h), (TF)^*(\beta^v)).$$

According to Lemma 2.2.4 we have

$$h^2(\alpha, \beta) \circ p_2 \circ TF = h^1(F^*\alpha, F^*\beta) \circ p_1.$$

Since $p_2 \circ TF = F \circ p_1$, we get

$$h^2(\alpha, \beta) \circ F \circ p_1 = h^1(F^*\alpha, F^*\beta) \circ p_1.$$

Then from the surjectivity of p_1, the equality (2.2.1) follows.

To prove that (i) and (iii) are equivalent, it suffices to see that that (2.2.1) becomes $(F^*\beta)(X) = \beta(Y) \circ F$. Which is equivalent to say that the two vector fields X and Y are F-related.

The equivalence between (iii) and (iv) is obvious. ∎

Remark 2.2.5. *Obviously one can see that if F is a K-V map. Then*

$$TF : (TM^1, J^1) \to (TM^2, J^2),$$

is a pseudo-holomorphic map, i.e., $J^2 \circ T(TF) = T(TF) \circ J^1$.

Example 2.2.6.

1. *Let (M^i, ∇^i, h^i) for $i = 1, 2$ be two K-V manifolds. We consider the K-V structure (∇, h) on the product manifold $M^1 \times M^2$ (as described in Proposition 1.4.1). Then the canonical projections*

$$p_i : (M^1 \times M^2, \nabla, h) \to (M^i, \nabla^i, h^i),$$

are a K-V map.

2. *Let* (G, ∇, h) *be a simply connected Lie group equipped with a K-V structure. Then the multiplication map*

$$m : (G \times G, \nabla \oplus \nabla, h \oplus h) \longrightarrow (G, \nabla, h)$$

is a K-V map if and only if, G *is a vector space,* ∇ *its canonical affine connection and* h *is linear (As seen in Corollary 1.8.2).*

3. *Let* $(\mathcal{A}_i, \bullet_i, B_i)$ *for* $i = 1, 2$ *be two commutative, associative algebras endowed with two symmetric scalar 2-cocycle. Denote by* $(\mathcal{A}^*, \nabla, h^i)$ *its associated K-V manifolds. We consider an affine map* $F : \mathcal{A}_1 \to \mathcal{A}_2$*, that satisfy*

$$F(u \bullet_1 v) = F(u) \bullet_2 F(v) \text{ and } B_1(u, v) = B_2(F(u), F(v)),$$

where $u, v \in \mathcal{A}_1$*. Then*

$$F^* : (\mathcal{A}_2^*, \nabla, h^2) \to (\mathcal{A}_1^*, \nabla, h^1),$$

is a K-V map.

4. *Let* $(\mathfrak{g}, \bullet, r^1)$ *be a K-V algebra. Denote by* (G, ∇) *the associated affine manifold to the left symmetric algebra* $(\mathfrak{g}, \bullet)$*. Let* $F : \mathfrak{g} \to \mathfrak{g}$ *be a morphism of left symmetric algebra. Define a symmetric bivector* $r^2 \in \otimes^2 \mathfrak{g}$ *by setting*

$$r^2(\alpha, \beta) := r^1(F^*\alpha, F^*\beta),$$

for any $\alpha, \beta \in \mathfrak{g}^*$*. Let* $\tilde{F} : G \to G$ *be the morphism of Lie group which integrates* F*, and* r^{i-} *be the right invariant symmetric bivector field associated to* r^i*. Then* (G, ∇, r^{2-}) *is a K-V manifold. Moreover*

$$\tilde{F} : (G, \nabla, r^{1-}) \to (G, \nabla, r^{2-}),$$

is a K-V map.

As a consequence of the first point in Example 2.2.6 and Theorem 2.2.3 we obtain

Corollary 2.2.7. *The canonical diffeomorphism*

$$\psi := Tp_1 \times Tp_2 : (T(M^1 \times M^2), \Pi) \to (TM^1 \times TM^2, \Pi^1 \oplus \Pi^2),$$

is a Poisson map, where Π *is the Poisson bivector associated to the K-V bivector field* $h = h^1 \oplus h^2$*.*

Proof. We know that the skew-symmetric bivector field $\Pi^1 \oplus \Pi^2$ is the unique Poisson structure on the product manifold $TM^1 \times TM^2$ such that the canonical projections

$$\widetilde{p}_i : (TM^1 \times TM^2, \Pi^1 \oplus \Pi^2) \to (TM^i, \Pi^i),$$

are Poisson maps. Since the one-forms given by $\widetilde{p}_i^* \alpha_i^h$ and $\widetilde{p}_i^* \alpha_i^v$, where $\alpha_i \in \Omega^1(M^i)$, span the vector space of one-forms on $TM^1 \times TM^2$. Then for any $\alpha_1, \beta_1 \in \Gamma(T^*M^1)$, we have

$$\begin{aligned} \Pi^1 \oplus \Pi^2 \left(\widetilde{p}_1^* \alpha_1^h, \widetilde{p}_1^* \beta_1^v\right) \circ \psi &= \Pi^1(\alpha_1^h, \beta_1^v) \circ \widetilde{p}_1 \circ \psi \\ &= \Pi^1(\alpha_1^h, \beta_1^v) \circ Tp_1. \end{aligned}$$

According to Theorem 2.2.3 it follows that $Tp_i : (T(M^1 \times M^2), \Pi) \to (TM^i, \Pi^i)$ are Poisson maps. Therefore, we have

$$\begin{aligned}\Pi^1 \oplus \Pi^2 \left(\widetilde{p}_1^*\alpha_1^h, \widetilde{p}_1^*\beta_1^v\right) \circ \psi &= \Pi\left(Tp_1^*\alpha_1^h, Tp_1^*\beta_1^h\right) \\ &= \Pi\left(\psi^*(\widetilde{p}_1^*\alpha_1^h), \psi^*(\widetilde{p}_1^*\beta_1^v)\right).\end{aligned}$$

Similarly, we verify that we have the same last equality for all types of one-forms that span the vector space of one-forms on $TM^1 \times TM^2$. This implies that ψ is a Poisson map. ■

We finish this section by giving some proprieties about K-V maps.

Proposition 2.2.8. *Let (M^i, ∇^i, h^i) for $i = 1, 2, 3$ be a K-V manifolds, $\phi : (M^1, \nabla^1, h^1) \to (M^2, \nabla^2, h^2)$ a Koszul-Vinberg map and $\psi : (M^2, \nabla^2, h^2) \to (M^3, \nabla^3, h^3)$ an affine map. Then*

1. *For any $x \in M^1$, $\mathrm{rank}(h^1_\#(x)) \geq \mathrm{rank}(h^2_\#(\phi(x)))$.*
2. *If (M^2, ∇^2, h^2) is a pseudo-Hessian manifold and ϕ is surjective, then ϕ is a submersion map.*
3. *If ψ is a K-V map, then $\psi \circ \phi$ is a K-V map.*
4. *If $\psi \circ \phi$ is a K-V map, and ϕ is surjective, then ψ is a K-V map.*
5. *If ϕ is a diffeomorphism, then ϕ^{-1} is a K-V map.*

2.3 K-V submanifolds

By similarity to the Poisson submanifolds studied in [47, 35, 55], we shall introduce in this section a class of submanifolds in the context of K-V manifolds.

Let (M, ∇, h) be a K-V manifold. We say that an immersed submanifold $\iota : N \hookrightarrow M$ is a *K-V submanifold* of (M, ∇, h), if it can be equipped with a flat torsionless connection ∇^N and a K-V bivector field h^N such that the immersion $\iota : N \hookrightarrow M$ becomes a K-V map. In particular, (N, ∇^N) is an affine submanifold as studied in [45].

For any vector subspace V of a finite-dimensional vector space E we denote by V° the annihilator of V in E^*.

Proposition 2.3.1. *Given an immersed affine submanifold $\iota : (N, \nabla^N) \hookrightarrow (M, \nabla)$, there is at most one K-V bivector field h^N on N that makes (N, ∇, h^N) into a K-V submanifold. This happens if and only if any of the following equivalent conditions holds.*

1. *For any $x \in N$, $\mathrm{Im}(h_\#(\iota(x))) \subset T_x\iota(T_xN)$.*
2. *For any $f \in C^\infty(M)$, the K-V Hamiltonian vector field X_f is tangent to $\iota(N)$.*
3. *For any $x \in N$, $h_\#((T_x\iota(T_xN))^\circ) = 0$.*

If N is a closed submanifold. Then these conditions are also equivalent to

4. *For any $f \in C^\infty(M)$ and $g \in \mathcal{J}(N) := \{g \in C^\infty(M) |\ g_{|_N} = 0\}$, $h(df, dg) \in \mathcal{J}(N)$.*

Proof. If $\iota : (N, \nabla^N, h^N) \hookrightarrow (M, \nabla, h)$ is a K-V map, then the two bivector fields h and h^N are ι-related, i.e., for any $x \in N$,

$$T_x\iota \circ h^N_\#(x) \circ (T_x\iota)^* = h_\#(\iota(x)).$$

Since $T_x\iota$ is injective, this shows that h^N is unique. It also shows that 1 is satisfied if (N, ∇^N, h^N) is a K-V submanifold.

Next, suppose that $\iota:(N, \nabla^N) \hookrightarrow (M, \nabla)$ is an affine immersion that satisfies $\mathrm{Im}(h_\#(i(x))) \subset T_x(i)(T_xN)$. We claim that there exists a unique smooth bivector field h^N in N such that $h_\#(\iota(x))$ factors as

$$\begin{array}{ccc} T^*_{\iota(x)}M & \xrightarrow{h_\#(\iota(x))} & T_{\iota(x)}M \\ {\scriptstyle (T_x\iota)^*}\downarrow & & \uparrow{\scriptstyle T_x\iota} \\ T^*_xN & \dashrightarrow_{h^N_\#(x)} & T_xN. \end{array}$$

Since we already know that $\mathrm{Im}(h_\#(\iota(x))) \subset T_x(\iota)(T_xN)$, it is enough to check that for any $\alpha \in (T_x\iota(T_xN))^\circ$ we have $\alpha^\# = 0$. In fact, we find for any $\beta \in T^*_{\iota(x)}M$,

$$\langle \beta, \alpha^\# \rangle = \langle \alpha, \beta^\# \rangle = 0,$$

which proves the claim (the smoothness of h^N is automatic).

Now observe that the skew-symmetric bivector field Π^N on TN associated to (∇^N, h^N) and the Poisson bivector field Π on TM associated to (∇, h) are $T\iota$-related, this implies that the Schouten brackets $[\Pi^N, \Pi^N]$ and $[\Pi, \Pi]$ are also $T\iota$-related. Hence

$$[\Pi^N, \Pi^N] = 0.$$

According to Theorem 1.3.2 (N, ∇^N, h^N) is a K-V manifold. This shows that if 1 hold, then N has a unique K-V structure (∇^N, h^N), such that the immersion $\iota : (N, \nabla^N, h^N) \hookrightarrow (M, \nabla, h)$ is a K-V map.

The equivalence between 1 and 2 follows from the fact that

$$\mathrm{Im}(h_\#(\iota(x))) = \mathrm{span}\{h_\#(\iota(x))(df) | f \in C^\infty(M)\}.$$

The equivalence between 2 and 3 follows from observing that for any $\alpha \in (T_x\iota(T_xN))^\circ$ and $\beta \in T^*_{\iota(x)}M$, we have $\langle \beta, \alpha^\# \rangle = \langle \alpha, \beta^\# \rangle$. So $h_\#((T_x\iota(T_xN))^\circ) = 0$ if and only if, $h_\#(\iota(x))(T^*_{\iota(x)}M) \subset T_x\iota(T_xN)$. Finally, notice that if N is a closed submanifold, a vector field $X \in \Gamma(TM)$ is tangent to N if and only if, for any $f \in \mathcal{J}(N)$, we have $X(f) \in \mathcal{J}(N)$. Hence, the result follows from the first part and the fact that $h(df, dg) = X_g(f)$. ■

Now let (N, ∇) be an affine submanifold (M, ∇, h) and h^N be a symmetric bivector field on N.

Corollary 2.3.2. *The following statements are equivalent.*

1. (N, ∇^N, h^N) *is a K-V submanifold.*

2. (TN, Π^N) *is a Poisson submanifold of* (TM, Π).

Example 2.3.3.

1. *Take* $M = (\mathbb{R}^2, \nabla, h^2)$ *endowed with its canonical affine structure and the K-V bivector* $h^2 = x^2\partial_x \otimes \partial_x + y^2\partial_y \otimes \partial_y$. *We consider the following affine immersion*

$$F : \mathbb{R} \to \mathbb{R}^2, x \mapsto (\lambda x, \mu x),$$

where $(\lambda, \mu) \neq (0,0)$ *and* $\mathbb{R}$ *is endowed with its canonical affine structure. Then the only way to make* F *as a K-V map is to take* $\lambda = 0$ *or* $\mu = 0$ *and* $h^1 = x^2\partial_x \otimes \partial_x$.

2. *Take* $M = (\mathbb{R}^m, \nabla, h^m)$ *and* $N = (\mathbb{R}^{m-k} \times \{0_k\}, \nabla, h^{m-k})$ *endowed with its canonical affine structures and the K-V bivectors*

$$h^m = \sum_{i,j=1}^{m} x_i x_j \partial_{x_i} \otimes \partial_{x_j} \text{ and } h^{m-k} = \sum_{i,j=1}^{m-k} x_i x_j \partial_{x_i} \otimes \partial_{x_j}.$$

Then N *is a K-V submanifold of* M.

3. *Consider* $(\mathbb{R}^n, \nabla, h)$ *endowed with its canonical affine connection and*

$$h = \sum_{i,i=1}^{m} f_i(x_i)\partial_{x_i} \otimes \partial_{x_i},$$

where $f_i \in C^\infty(\mathbb{R})$. *Then* $N = \mathbb{R}^{m-k} \times \{0_k\}$ *as an affine submanifold* $(\mathbb{R}^n, \nabla, h)$ *is a K-V submanifold if and only if,* $f_i(0) = 0$, *for* $i = k+1, \ldots, n$.

4. *Let* (M, ∇, h) *be a K-V manifold and* $f \in C^\infty(M)$ *an affine function, i.e.,* $\nabla df = 0$, *such that* $X_f(g) = 0$, *for all* $g \in C^\infty(M)$. *Then all the smooth level sets of* f *are K-V submanifolds. Indeed, since* $X_g(f) = X_f(g) = 0$, *shows that all K-V Hamiltonian vector fields are tangent to the level sets of* f.

5. *Let* $(\mathcal{A}^*, \nabla, h)$ *be a linear K-V manifold and* $I \subset (\mathcal{A}, \bullet)$ *be an ideal. Then* $I^\circ \subset \mathcal{A}^*$ *is a K-V submanifold.*

6. *Let* $(\mathfrak{g}, \bullet, r)$ *be K-V algebra and* $\mathfrak{h} \subset (\mathfrak{g}, \bullet)$ *be a left symmetric subalgebra such that* $\mathrm{Im}(r_\#) \subset \mathfrak{h}$. *Denote by* (G, ∇, h) *the associated K-V manifold to* $(\mathfrak{g}, \bullet, r)$ *and by* (H, ∇^H) *be the associated affine Lie subgroup to* $(\mathfrak{h}, \bullet)$. *Then* H *is a K-V submanifold.*

In what follows, we show that a K-V structure can be defined by its affine foliation instead of the K-V bivector.

Theorem 2.3.4. *Let* (M, ∇) *be an affine manifold, and* $\mathcal{F}$ *a general foliation such that*

1. *Each leaf* L *of* $\mathcal{F}$ *is endowed with a pseudo-Hessian structure* (∇^L, g_L) *and* (L, ∇^L) *is an affine submanifold of* (M, ∇).

2. *If* $f \in C^\infty(M)$, *the vector field* X_f *defined by* $X_f(x) =$ *the gradient vector field of* $f|_L$ *on* (L, ∇^L, g_L) *at* x *is a smooth vector field on* M *where* L *is the leaf passing through* x.

Then (M, ∇) *has a unique K-V bivector field* h *whose affine foliation is* $\mathcal{F}$. *Moreover, each leaf* (L, ∇^L, h^L) *is a K-V submanifold of* (M, ∇, h) *where* $h^L = g_L^{-1}$.

Proof. We define a symmetric bivector on M by putting for all $\alpha, \beta \in \Omega^1(M)$ and $x \in M$,

$$h(\alpha, \beta)(\iota(x)) = h^L(\iota^*\alpha, \iota^*\beta)(x),$$

where L is the affine leaf passing through $x \in M$ and $\iota : L \hookrightarrow M$ is the canonical injection. The smoothness of h follows automatically from 2. From 1 and 2 one can deduce that for any one forms $\alpha, \beta, \gamma \in \Omega^1(M)$, we have

$$\begin{aligned} \nabla_{\alpha^\#}(h)\,(\beta,\gamma)\,(\iota(x)) &= \nabla^L_{(\iota^*\alpha)^{\#_L}}(h^L)\,(\iota^*\beta,\iota^*\gamma)\,(x) \\ &= \nabla^L_{(\iota^*\beta)^{\#_L}}(h^L)\,(\iota^*\alpha,\iota^*\gamma)\,(x) \\ &= \nabla_{\beta^\#}(h)\,(\alpha,\gamma)\,(\iota(x)). \end{aligned}$$

■

Let us now examine the relation between the concept of K-V submanifold and affine foliation.

Proposition 2.3.5. *Let (M, ∇, h) be a K-V manifold with affine foliation $\mathcal{F}$. An affine submanifold $(N, \nabla^N) \subset (M, \nabla)$ is a K-V submanifold if and only if, for each leaf $L \in \mathcal{L}$ the intersection $L \cap N$ is an open subset of L. Hence, the affine foliation of (N, ∇^N, h^N) consists of the connected components of the intersection $L \cap N$.*

Proof. An affine submanifold $(N, \nabla^N) \subset (M, \nabla)$ is a K-V submanifold if and only if,

$$\mathrm{Im}(h_\#(x)) \subset T_xN, \ \forall x \in N.$$

It follows that for a K-V submanifold $N \subset M$, each affine leaf of (N, ∇^N, h^N) is also an integral submanifold of (M, ∇, h). Hence, each affine leaf of (N, ∇^N, h^N) is an open subset of an affine leaf of $\mathcal{F}$.

Conversely, if for each affine leaf $L \in \mathcal{F}$ the intersection $L \cap N$ is an open subset of L. Then for any $x \in N$, we have $\mathrm{Im}(h_\#(x)) = T_xL \subset T_xN$, where $L \in \mathcal{F}$ is the affine leaf passing through x. This shows that N is a K-V submanifold. ■

Remark 2.3.6. *Let (M, ∇, g) be a pseudo-Hessian manifold. Then the only K-V submanifolds are the open subsets of M.*

Finally, we give the relation between K-V maps and the affine foliation.

Proposition 2.3.7. *Let $F : (M^1, \nabla^1, h^1) \to (M^2, \nabla^2, h^2)$ be a K-V map. Then, for each pseudo-Hessian leaf L of (M^2, ∇^2, h^2), the set $L \cap \mathrm{Im}(F)$ is open in L. In particular, if $\mathrm{Im}(F)$ is an affine submanifold, then it is a K-V submanifold of (M^2, ∇^2, h^2).*

Proof. Let $x \in M^1$ and set $y = F(x)$. Denote by L the affine leaf of (M^2, ∇^2, h^2) containing y. Any point in L can be reached from y by piecewise smooth curves consisting of integral curves of K-V Hamiltonian vector field X_f.

Given $f \in C^\infty(M)$, according to Theorem 2.2.3, the vector fields X_f and $X_{f\circ F}$ are F-related. Hence, if $\gamma_2(t) \in M^2$ and $\gamma_1(t) \in M^1$ are the integral curves of X_f and $X_{f\circ F}$ satisfying $\gamma_2(0) = y$ and $\gamma_1(0) = x$, we have $\gamma_2(t) = F(\gamma_1(t))$, for all small enough t. It follows that a neighborhood of y in L is contained in the image of F. ■

Corollary 2.3.8. *Let (M, ∇, h) be a K-V manifold. If $N^1, N^2 \subset M$ are two K-V submanifolds which intersect transversely then $N^1 \cap N^2 \subset M$ is also a K-V submanifold.*

Proof. Let $\gamma : I \to N^1 \cap N^2$ be a curve with $\gamma(0) = x$ and $\gamma(1) = y$. We take $u \in T_x(N^1 \cap N^2) = T_xN^1 \cap T_xN^2$. Since N^1 and N^2 are affine submanifolds of M then $\tau^\gamma(u) \in T_yN^1 \cap T_yN^2 = T_y(N^1 \cap N^2)$ where $\tau^\gamma : T_xM \to T_yM$ is the parallel transport along γ. This show that $N^1 \cap N^2$ is an affine submanifold of M. Applying the first assertion of Proposition 2.3.1 to N^1 and N^2, we get that for all $x \in N^1 \cap N^2$, $\mathrm{Im}(h_\#(x)) \subset T_xN^1 \cap T_xN^2 = T_x(N^1 \cap N^2)$. Hence $N^1 \cap N^2$ is a K-V submanifold of (M, ∇, h). ∎

In general, K-V submanifolds don't have functoriality under K-V maps which are explained by the following example.

Example 2.3.9. *Consider the K-V map*

$$\begin{aligned} F : (\mathbb{R}^3, \nabla, h^1) &\to (\mathbb{R}^2, \nabla, h^2) \\ (x, y, z) &\longmapsto (x + y - \sqrt{2}z, x + y - \sqrt{2}z), \end{aligned}$$

where ∇ is the canonical affine structure of $\mathbb{R}^n$,

$$h^1 = \partial_x \otimes \partial_x + \partial_y \otimes \partial_y - \partial_z \otimes \partial_z \text{ and } h^2 = 0.$$

The affine submanifold $N := \{(x, -x) |\, x \in \mathbb{R}\} \subset \mathbb{R}^2$ is a transversal to the map F. Moreover, $F^{-1}(N) = \{(x, y, z) |\, x + y - \sqrt{2}z = 0\} \subset \mathbb{R}^3$ is a hyperplane, hence $F^{-1}(N)$ cannot be a K-V submanifold of $\mathbb{R}^3$, since the only one are the open subset of $\mathbb{R}^3$.

2.4 K-V transversals

We recall from [47] that a Poisson transversal of a Poisson manifold (P, π) is a submanifold $N \subset P$ such that, at every point $x \in N$, we have

$$T_xP = T_xN + \pi_\#(T_xN^\circ). \tag{2.4.1}$$

Definition 2.4.1. *A K-V transversal of a K-V manifold (M, ∇, h) is an affine submanifold $N \subset M$ such that, at every point $x \in N$, we have*

$$T_xM = T_xN + h_\#(T_xN^\circ). \tag{2.4.2}$$

Note that the equality $\mathrm{rank}(TN^\circ) = \mathrm{rank}(T_NM) - \mathrm{rank}(TN)$ implies that condition (2.4.2) is equivalent to the direct sum decomposition

$$T_NM = TN \oplus h_\#(TN^\circ). \tag{2.4.3}$$

The main reason to consider K-V transversals is that they have naturally induced K-V structures. Indeed, let N be a K-V tranversal of (M, ∇, h). Then the decomposition for T_NM and the dual decomposition for T^*_NM gives a sequence of bundle maps

$$T^*N \xrightarrow{p^*} T^*_NM \xrightarrow{h_\#} T_NM \xrightarrow{p} TN. \tag{2.4.4}$$

The resulting bundle map $T^*N \to TN$ is symmetric and so it is of the form $h^N_\#$ for a unique symmetric bivector field h^N on N. Hence we get that

Proposition 2.4.2. *(N, ∇, h^N) is a K-.V manifold.*

To show this proposition we need the following lemma.

Lemma 2.4.3. *Let N be a K-V tranversal of (M,∇,h). Then TN is a Poisson transversal submanifold of (TM,Π).*

Proof. Let $x\in M$, $\alpha,\beta\in\Gamma(T^*M)$ and $X\in\Gamma(TM)$ such that $\alpha_x\in T_xN^\circ$. Let $u\in T_xN$, for all $Z\in\Gamma(TN)$,

$$\prec(\alpha^h)_u,(Z^h)_u\succ=\prec\alpha_x,Z_x\succ=0,$$

and

$$\prec(\alpha^v)_u,(Z^v)_u\succ=\prec\alpha_x,Z_x\succ=0,$$

Hence $(\alpha^h)_u,(\alpha^v)_u\in T_u(TN)^\circ$. On the other hand, there exist $Y\in\Gamma(TN)$ and $\gamma\in\Gamma(T^*M)$ such that $\gamma_x\in T_xN^\circ$ and $X_x=Y_x+\gamma_x^\#$. Hence, we have

$$\begin{aligned}\prec(\beta^v)_u,(X^v)_u\succ &= \prec\beta_x,X_x\succ\\ &= \prec\beta_x,Y_x+\gamma_x^\#\succ\\ &= \prec(\beta)_u^v,(Y^v)_u+(\gamma_x^\#)_u^v\succ\\ &= \prec(\beta)_u^v,(Y^v)_u+\Pi_\#(\gamma_u^h)\succ.\end{aligned}$$

This implies that $(X^v)_u=(Y^v)_u+\Pi_\#(\gamma_u^h)$. Similarly, we get that $(X^h)_u=(Y^h)_u-\Pi_\#(\gamma_u^v)$. Hence the equality $T_u(TM)=T_u(TN)+\Pi_\#(T_u(TN)^\circ)$ follows from the fact that

$$T_u(TM)=\operatorname{span}\left\{(X^v)_u,(X^h)_u|\,X\in\Gamma(TM)\right\}.$$

■

Proof of Proposition 2.4.2. Let Π^N be the skew-symmetric bivector field associated to the pair (∇,h^N), one can see also that Π^N coincide with the bundle map given by the following composition maps

$$T^*(TN)\xrightarrow{Tp^*}T^*_{TN}(TM)\xrightarrow{\Pi_\#}T_{TN}(TM)\xrightarrow{Tp}T(TN).$$

According to the Lemma 2.4.3 and [47, Proposition 2.13], Π^N is a Poisson bivector field on TN, hence we get that (∇,h^N) is a K-V structure on N. ■

It is important to note that for a K-V transversal N in (M,∇,h), with induced K-V structure h^N, the inclusion map $\iota:(N,\nabla,h^N)\hookrightarrow(M,\nabla,h)$ is not in general a K-V map (Unless is an open set in M). This will be clear in the next example.

Example 2.4.4. *Take $M=(\mathbb{R}^3,\nabla,h)$ endowed with its canonical affine structure and $h=x\partial_x\otimes\partial_x+y\partial_y\otimes\partial_y$. Obviously (M,∇,h) is a K-V manifold and*

$$N:=\{(0,0,z)|\,z\in\mathbb{R}\}\subset(M,\nabla,h),$$

is a K-V transversal. However, the induced K-V bivector field on N vanish identically.

K-V transversal behave functorially under pullbacks by K-V maps. This turns out to be a very useful property.

Proposition 2.4.5. *Let $F:(M^1,\nabla^1,h^1)\to(M^2,\nabla^2,h^2)$ be a K-V map and let $N^2\subset M^2$ be a K-V transversal. Then F is transverse to N^2 and $N^1:=F^{-1}(N^2)$ is a K-V transversal in M^1. Moreover, F restricts to a K-V map between the induced K-V structure on N^1 and N^2.*

Proof. Let $x\in N^1$ and $y=F(x)\in N^2$. Because F is a K-V map we get that for all $\alpha\in T_y^*M^2$,

$$\alpha^{\#_2}=T_xF((T_xF^*\alpha)^{\#_1}).\qquad(2.4.5)$$

Therefore, $h^2_{\#}(T^*_y M^2) \subset T_x F(T_x M^1)$. Since N^2 is a K-V transversal this implies that

$$T_y M^2 = T_y N^2 + h^2_{\#}(T^*_y M^2) = T_y N^2 + T_x F(T_x M^1).$$

This shows that F is transverse to N^2. In particular, N^1 is a submanifold of M^1.

The affinity of N^1 follows from [37, Theorem 2]. Let $v \in T_x M^1$, and decompose $T_x F(v) = u + \alpha^{\#_2}$, with $u \in T_y N^2$ and $\alpha \in (T_y N^2)^\circ$. Then, $F^*\alpha \in (T_x N^1)^\circ$, and from (2.4.5) we get that the vector $w := v - (T_x F^*\alpha)^{\#_1}$ is mapped by $T_x F$ to u. Hence $w \in T_x N^1$. This shows that

$$v = w + (T_x F^*\alpha)^{\#_1} \in T_x N^1 + h^1_\sharp((T_x N^1)^\circ).$$

Therefore N^1 is a K-V transversal.

According to the Lemma 2.4.3 we get that $TN^2 \subset (TM^2, \Pi^2)$ is a Poisson transversal. And from [47, Proposition 2.20] it follows that

$$TF : (TN^1, \Pi^{N^1}) \to (TN^2, \Pi^{N^2}),$$

is a Poison map. Therefore

$$F : (N^1, \nabla^1, h^{N^1}) \to (N^2, \nabla^2, h^{N^2}),$$

is a K-V map. ∎

Corollary 2.4.6. *Let (M, ∇, h) be a K-V manifold, $(N^1, \nabla^1, h^1) \subset M$ be a K-V transversal and $(N^2, \nabla^2, h^2) \subset M$ be a K-V submanifold. Then*

1. *N^1 and N^2 intersect transversally.*
2. *$N^1 \cap N^2$ is a K-V submanifold of N^1.*
3. *$N^1 \cap N^2$ is a K-V transversal in N^2.*
4. *The two induced K-V structures on $N^1 \cap N^2$ coincide.*

Now we give the relation between K-V transversal and the affine foliation.

Proposition 2.4.7. *Let (M, ∇, h) be a K-V manifold with affine foliation $\mathcal{F}$. An affine submanifold $N \subset M$ is a K-V transversal if and only if, for all $L \in \mathcal{F}$, the intersection $L \cap N$ is a K-V submanifold of L. Hence, the affine foliation of $\left(N, \nabla, h^N\right)$ consists of the connected components of the intersection $L \cap N$.*

Proof. The condition on affine submanifold $N \subset M$ to be a K-V transversal is

$$T_N M = TN \oplus h_{\#}\left(TN^\circ\right).$$

This last condition is equivalent to have both the following conditions satisfied

(i) $T_N M = TN + h_{\#}\left(TN^\circ\right)$.

(ii) $TN \cap h_{\#}\left(TN^\circ\right) = 0$.

Condition (i) says that N is transverse to the affine leaves and condition (ii) (provided (i) is satisfied) says that the kernel of the pullback of g_L to $L \cap N$ is trivial. So $L \cap N$ is a K-V submanifold of L. ∎

We finish this section with the following example.

Example 2.4.8. *Consider $\mathbb{R}^n$ endowed with its canonical affine structure ∇ and the standard K-V bivector field*

$$h = \sum_{i=}^{n} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_i}.$$

Let $F : (\mathbb{R}^n, \nabla, h) \to (\mathbb{R}^m, \nabla, h)$ be an affine map. Obviously, F is a K-V map. Moreover, for every regular value $x \in \mathbb{R}^m$ of F the affine submanifold $F^{-1}(x) \subset \mathbb{R}^n$ is a K-V transversal.

2.5 Coisotropic K-V submanifolds

Let (P, π) be a Poisson manifold. A submanifold $N \subset P$ is called coisotropic submanifold if and only if, $\pi_{\#}(TN^{\circ}) \subset TN$. Now we have a similar statement for K-V manifolds, we consider a K-V manifold (M, ∇, h) and an affine submanifold N of M.

Proposition 2.5.1. *The following assertions are equivalent*

1. $h_{\#}(TN^{\circ}) \subset TN$.
2. TN *is a coisotropic submanifold of* (TM, Π).

Proof. The equivalence follows from the fact that $\alpha_{|_{TN}} = 0$ if and only if,

$$\alpha^h_{|_{T(TN)}} = \alpha^v_{|_{T(TN)}} = 0,$$

for any $\alpha \in \Omega^1(M)$. ∎

Definition 2.5.2. *A coisotropic K-V submanifold $N \subset M$ is an affine submanifold such that $h_{\#}(TN^{\circ}) \subset TN$.*

K-V submanifolds are a subclass of coisotropic Koszul-Vinberg submanifolds.

Proposition 2.5.3. *Let (M, ∇, h) be a K-V manifold. For any affine closed submanifold $N \subset M$, the following conditions are equivalent*

(i) *N is a coisotropic K-V submanifold.*

(ii) *For every $f, g \in \mathcal{J}(N)$, where $\mathcal{J}(N)$ is the vanishing ideal we have $X_f(g) \in \mathcal{J}(N)$.*

(iii) *For every $f \in \mathcal{J}(N)$ the Hamiltonian vector field X_f is tangent to N.*

Proof. Let's prove that $(i) \implies (ii)$. Let $f, g \in \mathcal{J}(N)$. Obviously, for any $x \in N$, we have $d_x f, d_x g \in T_x N^{\circ}$. Since $N \subset M$ is a coisotropic K-V submanifold then $h(x)(d_x f, d_x g) = 0$, so $h(df, dg) \in \mathcal{J}(N)$.

Let's prove that $(ii) \implies (iii)$. Assume that $\mathcal{J}(N)$ is stable under h. Let $f, g \in \mathcal{J}(N)$, for all $x \in N$,

$$X_f(g)(x) = h(x)(d_x f, d_x g) = 0.$$

The closedness of the submanifold N implies that X_f is tangent to N.

Let's prove that $(iii) \implies (i)$. Again the closedness of the submanifold N implies that $T_x N^{\circ}$ is generated by elements $d_x f$ where $f \in \mathcal{J}(N)$. So we conclude that for any $\alpha, \beta \in TN^{\circ}$, $h(\alpha, \beta) = 0$. Therefore, N is coisotropic. ∎

Proposition 2.5.4. *Let $F:(M^1,\nabla^1,h^1)\to(M^2,\nabla^2,h^2)$ be a K-V map and assume that F is transverse to a coisotropic K-V submanifold $N^2\subset M^2$. Then $F^{-1}(N^2)\subset M^1$ is a coisotropic K-V submanifold.*

Proof. Firstly the affinity of $F^{-1}(N^2)$ is guaranteed (this follows from [37, Theorem 2]). Hence the result can be deduced directly by applying [47, Propostion 2.34] to the Poisson map $TF:(TM^1,\Pi^1)\to(TM^2,\Pi^2)$. ■

There is one more important property of coisotropic objects and which shows their relevance in K-V geometry. In order to express it, we introduce the following notation. Let (M^1,∇^1,h^1) and (M^2,∇^2,h^2) be a K-V manifold. Denote by $(M^1\times\overline{M}^2,\nabla,h)$ the K-V manifold such that the canonical projections $p_1:(M^1\times\overline{M}^2,\nabla,h)\to(M^1,\nabla^1,h^1)$ and $p_2:(M^1\times\overline{M}^2,\nabla,h)\to(M^2,\nabla^2,-h^2)$ are K-V maps.

Proposition 2.5.5. *For a smooth map $F:(M^1,\nabla^1,h^1)\to(M^2,\nabla^2,h^2)$, the following conditions are equivalent*

(i) *F is a K-V map.*

(ii) $\mathrm{Graph}(F)\subset M^1\times\overline{M}^2$ *is a coisotropic K-V submanifold.*

This proposition is a consequence of the following lemma which is a generalization of the following fact. A map $F:\mathbb{R}^n\to\mathbb{R}^m$ is affine if and only if, its graph is an affine subspace of $\mathbb{R}^{n+m}$. Now let (M^1,∇^1) and (M^2,∇^2) be two affine manifolds and $F:(M^1,\nabla^1)\to(M^2,\nabla^2)$ is a smooth map.

Lemma 2.5.6. *F is an affine map if and only if its graph is an affine submanifold of $(M^1\times M^2,\nabla^1\oplus\nabla^2)$.*

Proof. Let $\tilde\gamma:[0,1]\to\mathrm{Graph}(F)$, $\tilde\gamma=(\gamma,F(\gamma))$ where γ is a curve on M^1 such that $\gamma(0)=p$ and $\gamma(1)=q$. Hence, for any $u\in T_pM^1$ we have

$$\tau^{\tilde\gamma}(u,T_pF(u))=(\tau^{\gamma}(u),\tau^{F(\gamma)}(T_pF(u))),$$

where $\tau^{\tilde\gamma}$ is the parallel transport along the curve $\tilde\gamma$, seen as a curve on $M^1\times M^2$. Hence, $T_qF(\tau^\gamma(u))=\tau^{F(\gamma)}(T_pF(u))$ if and only if $\tau^{\tilde\gamma}(u,T_pF(u))\in T_{(q,F(q))}\mathrm{Graph}(F)$. This shows that F is an affine map if and only if it commutes with parallel transport if and only if $\mathrm{Graph}(F)$ is an affine submanifold of $(M^1\times M^2,\nabla^1\oplus\nabla^2)$. ■

Now let's explore more proprieties of coisotropic K-V subamanifold. For that we endowed the space $\Gamma(TN^\circ)$ with the product

$$\alpha\bullet\beta=\mathcal{D}_\alpha\beta.$$

Proposition 2.5.7. *Let $N\subset(M,\nabla,h)$ be a coisotropic submanifold. Then $(TN^\circ,N,\bullet,\rho)$ is a left symmetric algebroid, where ρ is the restriction of $h_\#$ to the subbundle $TN^\circ\subset T^*_NM$. Moreover, for any $x\in N$, the vector space $\mathfrak{g}_x=\ker\rho_x$ is a commutative associative algebra.*

Proof. What we need to show is that the product $\bullet$ is well defined and the other assertions are a direct consequence of this fact. Let X be a vector field on M which is tangent to N and $\alpha,\beta\in\Gamma(TN^\circ)$. Then

$$\begin{aligned}\prec\mathcal{D}_\alpha\beta,X\succ\ &=\ -X.(\prec\alpha,\beta^\#\succ)+\prec\alpha,\nabla_X\beta^\#\succ+\prec\beta,\nabla_X\alpha^\#\succ\\&\quad+h_\#(\alpha).(\prec\beta,X\succ)-\prec\beta,\nabla_{\alpha^\#}X\succ.\end{aligned}$$

Using the affinity of N together with the condition $h_{\#}(TN^{\circ}) \subset TN$, we get that $\prec \mathcal{D}_{\alpha}\beta, X \succ = 0$, hence $\mathcal{D}_{\alpha}\beta \in \Gamma(TN^{\circ})$. ■

Example 2.5.8.

1. *N is a K-V submanifold if and only if $\rho = 0$.*
2. *If $x \in M$ is a point at which the K-V structure vanishes, then $\{x\}$ is coisotropic.*
3. *Let $\iota : \mathcal{H} \hookrightarrow \mathcal{A}$ be a subalgebra of the associative, commutative algebra $\mathcal{A}$. Then $(\iota^*)^{-1}(\{0\}) \subset \mathcal{A}^*$ is a coisotropic K-V submanifold, where $\mathcal{A}^*$ is endowed with its canonical linear K-V structure.*

Recall that two submanifolds $N^1, N^2 \subset M$ are said to have a clean intersection if $N^1 \cap N^2$ is a submanifold of M and $T(N^1 \cap N^2) = TN^1 \cap TN^2$. A submanifold has a clean intersection with a foliation if it intersects cleanly every leaf of the foliation.

Proposition 2.5.9. *Let N be an affine submanifold of a K-V manifold (M, ∇, h) which has a clean intersection with its affine foliation $\mathcal{F}$. Then N is a coisotropic submanifold of (M, ∇, h) if and only if, for each affine leaf $L \in \mathcal{F}$ the intersection $L \cap N$ is a coisotropic submanifold of N.*

Proof. Assume that $N \subset M$ is an affine submanifold which is transverse to the affine foliation $\mathcal{F}$. This means that for each $L \in \mathcal{F}$ the inclusion $\iota : L \hookrightarrow M$ is transverse N. Now

(a) If N is coisotropic in M, it follows that $\iota^{-1}(N) = L \cap N$ is coisotropic in N, since the inclusion is a K-V map.

(b) If $L \subset N$ is coisotropic in L, then we have

$$h^L_{\#}(T(L \cap N)^{\circ}) \subset T(L \cap N) = TL \cap TN,$$

where the annihilator is taking in T^*L.

It follows that for any $\alpha \in \Omega^1(M)$ such that $\alpha_{|_{T(L\cap N)}} = 0$, we have $\alpha^{\#} \in \Gamma(TN)$. But $(TL \cap TN)^{\circ} = TL^{\circ} + TN^{\circ}$, so we conclude that $h_{\#}(TN^{\circ}) \subset TN$, which means that N is coisotropic. ■

Chapter 3

Homogeneous spaces with invariant K-V structures

The study of invariant structures (such as pseudo-Riemannian metrics, symplectic, Hessian, or contact structures ...) on homogeneous manifolds is an important step in any geometrical study and this was a motivation for many interesting contributions, one can see for instance [43, 44, 16, 8, 49]. In this chapter, we undertake the study of invariant K-V structures on homogeneous manifolds.

3.1 K-V transformations

3.1.1 Infinitesimal K-V transformations

Let (M, ∇) be an affine manifold. A vector field $X \in \Gamma(TM)$ is called an *infinitesimal affine transformation* if it satisfies

$$[\mathcal{L}_X, \nabla_Y] = \nabla_{[X,Y]},$$

for any vector field $Y \in \Gamma(TM)$. The set $\mathfrak{aff}(M, \nabla)$ of complete infinitesimal affine transformations of M is a Lie subalgebra of $\Gamma(TM)$ of dimension at most $n^2 + n$, where $n = \dim M$. It is the Lie algebra of the Lie group $\mathrm{Aff}(M, \nabla)$ of affine transformations (M, ∇) (see [33, 34]).

Let (M, ∇, h) be a K-V manifold. Denote by $\mathrm{Aff}_h(M)$ the subgroup of $\mathrm{Aff}(M)$ consisting of K-V transformations of (M, ∇, h). This group will be called the *group of K-V transformations*. A *K-V vector field* $X \in \Gamma(TM)$ is an infinitesimal affine transformation satisfying $\mathcal{L}_X h = 0$. Denote by $\mathfrak{aff}_h(M, \nabla)$ the space of complete infinitesimal K-V transformations.

Proposition 3.1.1. *Let (M, ∇, h) be a connected K-V manifold. Then*

1. $\mathrm{Aff}_h(M)$ *is a closed subgroup of* $\mathrm{Aff}(M)$.
2. $\mathrm{Aff}_h(M)$ *is a Lie group and* $\mathfrak{aff}_h(M, \nabla)$ *is its Lie algebra.*

Proof. 1. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{Aff}_h(M)$ which converges to $\varphi \in \mathrm{Aff}(M)$. Since $\mathrm{Aff}(M)$ is a Lie transformation group, we get

$$\varphi^* h = \lim_{n\to\infty} \varphi_n^* h = \lim_{n\to\infty} h = h.$$

2. Clearly $\mathfrak{aff}_h(M, \nabla)$ is a Lie subalgebra of $\mathfrak{aff}(M, \nabla)$ and $X \in \mathfrak{aff}_h(M, \nabla)$ if and only if, $\varphi_t^X \in \mathrm{Aff}_h(M)$ for all $t \in \mathbb{R}$. Hence, according to [34, Theorem 3.1 pp. 13] we obtain the result.

∎

Let (M,∇,h) be a K-V manifold. Denote by $\mathrm{Aut}_\Pi(TM)$ the group of diffeomorphisms of TM that preserves the Poisson bivector Π associated to the pair (∇,h). Denote by $\mathrm{Aut}_\Pi(TM)$ the group of diffeomorphisms $F : M \to M$ such that $TF : TM \to TM$ preserves the Poisson bivector Π associated to the pair (∇,h). Then according to [2, Theorem 3.1], we have

Corollary 3.1.2. $\mathrm{Aff}_h(M) = \mathrm{Aff}(M) \cap \mathrm{Aut}_\Pi(TM)$.

Now we give a class of K-V manifolds for which the Lie group $\mathrm{Aff}_h(M)$ can be computed.

Let $(\mathcal{A},\bullet)$ be a finite dimensional associative, commutative algebra. Denote by (∇,h) the canonical K-V structure on $\mathcal{A}^*$ introduced in [10]. Recall that ∇ is the canonical flat torsionless connection of $\mathcal{A}^*$ and h is the K-V bivector field defined by

$$h(\alpha,\beta)(\mu) :=\prec \mu, \alpha(\mu)\bullet\beta(\mu)\succ,$$

where $\alpha,\beta \in \Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*,\mathcal{A})$ and $\mu \in \mathcal{A}^*$. Denote by $\mathrm{Aut}(\mathcal{A},\bullet)^*$ the subgroup of $\mathrm{GL}(\mathcal{A}^*)$ given by

$$\mathrm{Aut}(\mathcal{A},\bullet)^* := \{g^* |\, g \in \mathrm{Aut}(\mathcal{A},\bullet)\},$$

and $(\mathcal{A}^2)^\circ := \{\mu \in \mathcal{A}^*,\ \mu|_{\mathcal{A}^2} = 0\}$.

Proposition 3.1.3. $\mathrm{Aff}_h(\mathcal{A}^*)$ *is a semi-direct product*

$$\mathrm{Aff}_h(\mathcal{A}^*) = (\mathcal{A}^2)^\circ \rtimes \mathrm{Aut}(\mathcal{A},\bullet)^*.$$

Moreover, if we suppose that $\mathcal{A}$ is unitary. Then $\mathrm{Aff}_h(\mathcal{A}^*) = \mathrm{Aut}(\mathcal{A},\bullet)^*$.

Proof. Let $F \in \mathrm{Aff}_h(\mathcal{A}^*)$ then there exists a unique isomorphism g of $\mathcal{A}$ and a unique $\epsilon \in \mathcal{A}^*$ such that $F = g^* + \epsilon$. Since F preserve h then for any $\mu \in \mathcal{A}^*$ and $\alpha,\beta \in \Omega^1(\mathcal{A}^*)$,

$$\prec \mu, (g\circ\alpha\circ F(\mu))\bullet(g\circ\beta\circ F(\mu))\succ=\prec g^*(\mu)+\epsilon, (\alpha\circ F(\mu))\bullet(\beta\circ F(\mu))\succ .$$

From the bijectivity of g^* and F it follows that, for any $u,v \in \mathcal{A}$,

$$\prec \mu, g(u)\bullet g(v)\succ=\prec g^*(\mu)+\epsilon, u\bullet v\succ .$$

In this last equation we replace μ by $t\mu$, where $t \in \mathbb{R}$, and we take its derivative with respect to t we get

$$\prec \mu, g(u)\bullet g(v)\succ=\prec g^*(\mu), u\bullet v\succ .$$

This shows that $\mathrm{Aff}_h(\mathcal{A}^*)\subset(\mathcal{A}^2)^\circ \rtimes \mathrm{Aut}(\mathcal{A},\bullet)^*$. The other inclusion is obvious.

If $\mathcal{A}$ is unitary then $\mathcal{A}^2=\mathcal{A}$. Hence we have $\mathrm{Aff}_h(\mathcal{A}^*) = \{0\} \rtimes \mathrm{Aut}(\mathcal{A},\bullet)^*$. ■

Let us give some more examples where we can compute $\mathrm{Aff}_h(M)$.

Example 3.1.4. *We consider $M = (\mathbb{R}^2,\nabla)$ endowed with its canonical affine structure and denote by (x,y) its canonical affine coordinates. Then $\mathrm{Aff}(M) = \mathbb{R}^2 \rtimes \mathrm{GL}(\mathbb{R}^2)$. We enumerate $\mathrm{Aff}_h(\mathbb{R}^2)$ for some K-V bivector fields.*

1. *The bivector field $h := y\partial_x \otimes \partial_x$ is K-V and*

$$\mathrm{Aff}_h(M) = \left\{ \left(\begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} \right) \mid a \in \mathbb{R}^*, b,c \in \mathbb{R} \right\}.$$

2. *The bivector field* $h = x\partial_x \otimes \partial_x + y\partial_x \otimes \partial_y$ *is K-V and*

$$\mathrm{Aff}_h(M) = \mathbb{R}\backslash\{0\}.$$

3. *The bivector field* $h = y\partial_x \otimes \partial_x + \partial_x \otimes \partial_y$ *is K-V and*

$$\mathrm{Aff}_h(M) = \left\{ \left(\begin{pmatrix} b \\ 2a \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) \mid a, b \in \mathbb{R} \right\}.$$

4. *The bivector field* $h = x^2\partial_x \otimes \partial_x + y^2\partial_y \otimes \partial_y$ *is K-V and*

$$\mathrm{Aff}_h(M) = \left\{ \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right); \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right) \mid a, b \in \mathbb{R}^* \right\}.$$

3.1.2 K-V actions

Let M be a smooth, connected manifold and $\mathrm{Diff}(M)$ the group of diffeomorphisms of M. A smooth action of a Lie group G on M is a group morphism $\lambda : G \to \mathrm{Diff}(M)$ such that the induced map

$$G \times M \xrightarrow{\lambda} M, \qquad (g, x) \mapsto \lambda(g, x) = \lambda_g(x) = g \cdot x.$$

is smooth. Each smooth action $\lambda : G \times M \to M$ induce an action of G on TM given by

$$G \times TM \longrightarrow TM, \qquad (g, u) \mapsto T(\lambda_g)(u).$$

Let (M, ∇, h) be a K-V manifold and G a Lie group acting smoothly on M. We say that the action is a *K-V action* if $\lambda(G) \subset \mathrm{Aff}_h(M)$.

Let (M, ∇, h) be a K-V manifold and G is a Lie group acting smoothly on (M, ∇) by affine transformations. Then, according to [2, Theorem 3.1], we have

Proposition 3.1.5. *The action of G on (M, ∇, h) is a K-V action if and only if the action of G on TM is made by Poisson transformations.*

Obviously the natural action of $\mathrm{Aff}_h(M)$ on (M, ∇, h) is a K-V action. We can build a naturally K-V action as the following example shows.

Example 3.1.6. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\mathcal{A} := \mathbb{R}[A]$ be the unitary commutative, associative subalgebra of $\mathcal{M}_n(\mathbb{R})$ generated by A. Denote by $G(A)$ the subgroup of $\mathrm{GL}_n(\mathbb{R})$ given by*

$$G(A) := \{g \in \mathrm{GL}_n(\mathbb{R}) |\ gAg^{-1} \in \mathcal{A}\}.$$

Obviously $G(A)$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ hence it is a Lie group, which acts on $\mathcal{A}$ by automorphisms through conjugation

$$G(A) \times \mathcal{A} \to \mathcal{A},\ (g, u) \mapsto \tau_g(u) = g.u := gug^{-1}.$$

On the other hand, consider the dual vector space $\mathcal{A}^*$ endowed with the contragradient action

$$G(A) \times \mathcal{A}^* \to \mathcal{A}^*,\ (g, \mu) \mapsto \lambda_g(\mu) := (\tau_{g^{-1}})^*(\mu),$$

and its canonical K-V structure (∇, h) (see [1]). The action λ induces an action of $G(A)$ on $\Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*, \mathcal{A})$ given by $g.\alpha = \tau_g \circ \alpha \circ (\tau_{g^{-1}})^*$. Hence λ is a K-V action.

3.1.3 Covering K-V transformations

Let Γ be a discrete group acting smoothly, freely, and properly on M. Then the orbit space M/Γ is a smooth manifold and the canonical projection $p : M \to M/\Gamma$ is a covering map (see [36, pp. 91]). A tensor field on M/Γ can be seen as a Γ-invariant tensor field on M. In particular a Γ-invariant vector field X on M gives rise to a vector field on M/Γ denoted by p_*X which is p-related to X.

Now let (∇, h) be a K-V structure on M such that $\Gamma \subset \mathrm{Aff}_h(M)$. Then there exists a unique K-V structure (∇', h') on M/Γ such that

$$p : (M, \nabla, h) \to (M/\Gamma, \nabla', h'),$$

is a K-V map.

Conversely, let (∇, h) be a K-V structure on M/Γ. Then the structure (∇, h) can be uniquely lifted to a Γ-invariant K-V structure $(\widetilde{\nabla}, \widetilde{h})$ on M.

In summary, we have established the following result.

Proposition 3.1.7. *There is a natural one-to-one correspondence between Γ-invariant K-V structures on M and K-V structures on M/Γ.*

The above Proposition 3.1.7 can be illustrated as follows.

Example 3.1.8 (Construction by suspension)**.** *Let (N, ∇, h) be a K-V manifold and $\gamma \in \mathrm{Aff}_h(N)$. We define an action of $\mathbb{Z}$ on $\mathbb{R} \times N$ by setting $n.(t, x) := (t+n, \gamma^n(x))$. Obviously $\mathbb{Z}$ acts freely and properly on $\mathbb{R} \times N$. Hence the quotient space $M := \mathbb{R} \times_{\mathbb{Z}} N$ is a smooth manifold. Now we endow $\mathbb{R} \times N$ with the K-V structure given by the product structure $(\mathbb{R}, \nabla^\circ, 0) \times (N, \nabla, h)$ (see [1, Proposition 2.9]), where ∇° is the canonical flat torsionless connection on $\mathbb{R}$. Hence M can be endowed with a K-V structure.*

We say that a K-V manifold is complete when the connection is complete. From Theorem 3.1.7 and [57, Corollary 1.9.6 pp. 45] we get the following corollary.

Corollary 3.1.9. *Complete K-V manifolds are just the quotient manifold $\mathbb{R}^n/\Gamma$, where Γ is a subgroup of affine diffeomorphisms of $\mathbb{R}^n$, which preserve a K-V structure (∇, h) on $\mathbb{R}^n$ acting freely and properly discontinuously on $\mathbb{R}^n$.*

Example 3.1.10. *Denote by ∇ the canonical connection on $\mathbb{R}^2$ and h the K-V bivector field on $\mathbb{R}^2$ given by*

$$h_\# = \begin{pmatrix} \cos(2\pi x) & -\cos(2\pi x) \\ -\cos(2\pi x) & \cos(2\pi x) \end{pmatrix}.$$

The K-V structure (∇, h) is $\mathbb{Z}^2$-invariant with respect to the linear action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. Hence we get a K-V structure (∇', h') on $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. The diffeomorphism $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$, $[(x, y)] \mapsto [(x, y + \sqrt{2})]$ preserve (∇', h'). Hence it follows from the construction by suspension that the compact manifold $M := \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2$ can be endowed with a K-V structure.

3.2 Invariant K-V structures on homogeneous affine manifolds

A G-invariant K-V structure on a homogeneous manifold $M := G/H$ is a K-V structure (∇, h) on M, such that the left action of the Lie group G on G/H preserves both ∇ and h. This is also equivalent to the fact that the group $\lambda(G) := \{\lambda_g /\ g \in G\}$ is a subgroup of the Lie group $\mathrm{Aff}_h(G/H)$.

Now we give an algebraic characterisation of K-V bivector fields on homogeneous affine manifolds. Let $(M := G/H, \nabla)$ be a G-homogeneous affine manifold. Denote by

$$L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h}),$$

the Lie algebra representation associated to ∇, and $L^* : \mathfrak{g} \to \mathrm{End}((\mathfrak{g}/\mathfrak{h})^*)$ its contragredient representation.

Let h be a G-invariant symmetric bivector field on M, $r \in \otimes^2 \mathfrak{g}/\mathfrak{h}$ its associated H-invariant symmetric bivector and $\widetilde{r} \in \otimes^2 \mathfrak{g}$ any symmetric bivector satisfying $q \circ \widetilde{r}_\# \circ q^* = r_\#$, where $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection.

We introduce a product $\bullet_{\widetilde{r}}$ on $(\mathfrak{g}/\mathfrak{h})^*$, a bracket $[,]_{\widetilde{r}}$ on $(\mathfrak{g}/\mathfrak{h})^*$, and a trilinear map $[[r, r]] : \otimes^3 (\mathfrak{g}/\mathfrak{h})^* \to \mathbb{R}$ by the following formulas

$$\epsilon^1 \bullet_{\widetilde{r}} \epsilon^2 := L^*_{\widetilde{r}_\#(q^*\epsilon^1)}(\epsilon^2), \quad [\epsilon^1, \epsilon^2]_{\widetilde{r}} := \epsilon^1 \bullet_{\widetilde{r}} \epsilon^2 - \epsilon^2 \bullet_{\widetilde{r}} \epsilon^1,$$

and

$$[[r, r]](\epsilon^1, \epsilon^2, \epsilon^3) := \prec \epsilon^3, r_\# \left([\epsilon^1, \epsilon^2]_{\widetilde{r}}\right) - q\left([\widetilde{r}_\#(q^*\epsilon^1), \widetilde{r}_\#(q^*\epsilon^2)]\right) \succ .$$

Lemma 3.2.1.

1. *The trilinear map $[[r, r]]$ only depends on r.*
2. *The $(3, 0)$-tensor $[h, h]$ on M given by*

$$[h, h](\alpha, \beta, \gamma) := \prec \gamma, (\nabla_{\alpha^\#}\beta)^\# - \left(\nabla_{\beta^\#}\alpha\right)^\# - [\alpha^\#, \beta^\#] \succ$$

is G-invariant and its associated trilinear map is $[[r, r]]$.

Proof. 1. Let $\widetilde{r}_1, \widetilde{r}_2 \in \otimes^2 \mathfrak{g}$ be two symmetric bivectors such that

$$q \circ \widetilde{r}_{i,\#} \circ q^* = r_\#.$$

Hence

$$\begin{aligned} r\left(\epsilon^3, [\epsilon^1, \epsilon^2]_{\widetilde{r}_1} - [\epsilon^1, \epsilon^2]_{\widetilde{r}_2}\right) = &\prec \epsilon^1, L_{\widetilde{r}_{1,\#}(q^*\epsilon^2) - \widetilde{r}_{2,\#}(q^*\epsilon^2)}(r_\#(\epsilon^3)) \succ \\ &- \prec \epsilon^2, L_{\widetilde{r}_{1,\#}(q^*\epsilon^1) - \widetilde{r}_{2,\#}(q^*\epsilon^1)}(r_\#(\epsilon^3)) \succ . \end{aligned}$$

Since $\widetilde{r}_{1,\#}(q^*\epsilon^i) - \widetilde{r}_{2,\#}(q^*\epsilon^i) \in \mathfrak{h}$ and from equation (A.4.1) it follows that

$$\begin{aligned} r\left(\epsilon^3, [\epsilon^1, \epsilon^2]_{\widetilde{r}_1} - [\epsilon^1, \epsilon^2]_{\widetilde{r}_2}\right) = &\prec \epsilon^1, \overline{ad}_{\widetilde{r}_{1,\#}(q^*\epsilon^2) - \widetilde{r}_{2,\#}(q^*\epsilon^2)}(r_\#(\epsilon^3)) \succ \\ &- \prec \epsilon^2, \overline{ad}_{\widetilde{r}_{1,\#}(q^*\epsilon^1) - \widetilde{r}_{2,\#}(q^*\epsilon^1)}(r_\#(\epsilon^3)) \succ \\ = &- r(\epsilon^3, \overline{ad}^*_{\widetilde{r}_{1,\#}(q^*\epsilon^2) - \widetilde{r}_{2,\#}(q^*\epsilon^2)}(\epsilon^1)) \\ &+ r(\epsilon^3, \overline{ad}^*_{\widetilde{r}_{1,\#}(q^*\epsilon^1) - \widetilde{r}_{2,\#}(q^*\epsilon^1)}(\epsilon^2)). \end{aligned}$$

Now, from equation (A.3.1) we obtain

$$\begin{aligned} r\left(\epsilon^3, [\epsilon^1, \epsilon^2]_{\widetilde{r}_1} - [\epsilon^1, \epsilon^2]_{\widetilde{r}_2}\right) = &r(\epsilon^1, \overline{ad}^*_{\widetilde{r}_{1,\#}(q^*\epsilon^2) - \widetilde{r}_{2,\#}(q^*\epsilon^2)}(\epsilon^3)) \\ &- r(\epsilon^2, \overline{ad}^*_{\widetilde{r}_{1,\#}(q^*\epsilon^1) - \widetilde{r}_{2,\#}(q^*\epsilon^1)}(\epsilon^3)). \\ = &- \prec \epsilon^3, q([\widetilde{r}_{1,\#}(q^*\epsilon^2) - \widetilde{r}_{2,\#}(q^*\epsilon^2), \widetilde{r}_{1,\#}(q^*\epsilon^1)]) \succ \\ &+ \prec \epsilon^3, q([\widetilde{r}_{1,\#}(q^*\epsilon^1) - \widetilde{r}_{2,\#}(q^*\epsilon^1), \widetilde{r}_{2,\#}(q^*\epsilon^2)]) \succ \\ = &\prec \epsilon^3, q([\widetilde{r}_{1,\#}(q^*\epsilon^1), \widetilde{r}_{1,\#}(q^*\epsilon^2)]) \succ \\ &- \prec \epsilon^3, q([\widetilde{r}_{2,\#}(q^*\epsilon^1), \widetilde{r}_{2,\#}(q^*\epsilon^2)]) \succ . \end{aligned}$$

This shows that $[[r,r]]$ only depends on r.

2. From the G-invariance of h we get that for any $g \in G$, $(g.\alpha)^{\#} = g^{-1}.\alpha^{\#}$. The G-invariance of ∇ implies that for any $g \in G$,

$$\prec g.\gamma, \nabla_{(g.\alpha)^{\#}}(g.\beta)^{\#} \succ = \prec \gamma, \nabla_{\alpha^{\#}}\beta^{\#} \succ, \prec g.\gamma, (\nabla_{(g.\alpha)^{\#}}(g.\beta))^{\#} \succ = \prec \gamma, (\nabla_{\alpha^{\#}}\beta)^{\#} \succ .$$

Hence, for any $g \in G$,

$$[h,h](g.\alpha, g.\beta, g.\gamma) = [h,h](\alpha,\beta,\gamma).$$

Which means that $[h,h]$ is G-invariant. Now, we will compute the value of $[h,h]$ at $\bar{e}$. From (A.4.3), we get

$$(\nabla_{\alpha^{\#}}\beta)^{\#}_{\bar{e}} = \Phi_e\left(r_{\#}\left(d_eF^{\beta}(\widetilde{r}_{\#}(q^*\epsilon^1)) + \epsilon^1 \bullet_{\widetilde{r}} \epsilon^2\right)\right),$$

where $\epsilon^1 = \Phi_e^{-1}(\alpha_{\bar{e}})$, $\epsilon^2 = \Phi_e^{-1}(\beta_{\bar{e}})$. According to (A.4.2), we have

$$\begin{aligned}(\nabla_{\alpha^{\#}}\beta)^{\#}_{\bar{e}} - \left(\nabla_{\alpha^{\#}}\beta^{\#}\right)_{\bar{e}} =& \Phi_e(d_eF^{\beta^{\#}}(\widetilde{r}_{\#}(q^*\epsilon^1)) - r_{\#}(d_eF^{\beta}(\widetilde{r}_{\#}(q^*\epsilon^1)))\\ &+ L_{\widetilde{r}_{\#}(q^*\epsilon^1)}(r_{\#}(\epsilon^2)) - r_{\#}(\epsilon^1 \bullet_{\widetilde{r}} \epsilon^2)).\end{aligned}$$

On the other hand, for any $g \in G$,

$$F^{\alpha^{\#}}(g) = g^{-1}.\alpha^{\#}(\bar{g}) = (g^{-1}.\alpha(\bar{g}))^{\#} = r_{\#}(F^{\alpha}(g)).$$

Hence we get

$$d_eF^{\alpha^{\#}}(\widetilde{r}_{\#}(q^*\epsilon^1)) = r_{\#}(d_eF^{\alpha}(\widetilde{r}_{\#}(q^*\epsilon^1))).$$

This implies that

$$(\nabla_{\alpha^{\#}}\beta)^{\#}_{\bar{e}} - \left(\nabla_{\alpha^{\#}}\beta^{\#}\right)_{\bar{e}} = \Phi_e(L_{\widetilde{r}_{\#}(q^*\epsilon^1)}(r_{\#}(\epsilon^2)) - r_{\#}(\epsilon^1 \bullet_{\widetilde{r}} \epsilon^2)).$$

This shows that

$$[h,h]_{\bar{e}}(\alpha,\beta,\gamma) = [[r,r]](\epsilon^1,\epsilon^2,\epsilon^3),$$

where $\epsilon^3 = \Phi_e^{-1}(\gamma_{\bar{e}})$.

∎

Definition 3.2.2. *Let $(G/H,\nabla)$ be a G-homogeneous affine manifold. An H-invariant symmetric bivector $r \in \otimes^2(\mathfrak{g}/\mathfrak{h})$ satisfying $[[r,r]] = 0$ is called an H-invariant S-matrix on $\mathfrak{g}/\mathfrak{h}$.*

From the above Definition 3.2.2 we can see that K-V structures on an affine Lie group (G,∇) corresponds to symmetric bivectors $r \in \otimes^2\mathfrak{g}$, satisfying $[[r,r]] = 0$ (see [1]). In what follows we give a more general situation which is our main result in this section.

Theorem 3.2.3. *Let $(G/H,\nabla)$ be a G-homogeneous affine manifold, h a symmetric G-invariant bivector field and $r \in \otimes^2\mathfrak{g}/\mathfrak{h}$ its associated H-invariant symmetric bivector. Then h is a K-V bivector field if and only if r is an H-invariant S-matrix.*

Proof. Lemma 3.2.1 implies that $[h,h]$ is G-invariant and $[h,h]_{\bar{e}} = [[r,r]]$. Hence h is a K-V bivector field if and only if, $[[r,r]] = 0$. ∎

Now, we are going to give a situation where we can apply this Theorem 3.2.3. Let G be a connected Lie group endowed with a bi-invariant flat torsionless connection D and H a connected Lie subgroup of G. Suppose that the Lie algebra $\mathfrak{h} := \mathrm{Lie}(H)$ is a left ideal of $(\mathfrak{g}, \bullet)$, where $\bullet$ is the canonical associative product associated to D and put $l_u(v) := u \bullet v$. The linear map

$$L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h}), u \mapsto L_u(v + \mathfrak{h}) = q(u \bullet v),$$

is well defined and it is a Lie algebra representation, which obviously satisfies the conditions of Nomizu's Theorem. Therefore, there is a unique G-invariant flat torsionless connection ∇ on G/H such that

$$\nabla_{u^*} v^* := (v \bullet u)^*.$$

Moreover the canonical projection $p : (G, D) \to (G/H, \nabla)$ is an affine map.

Proposition 3.2.4. *Under the above hypothesis, let s be an $\mathrm{Ad}(H)$-invariant S-matrix on $(\mathfrak{g}, \bullet)$, and consider the symmetric bivector, $r_\# := q \circ s_\# \circ q^*$. Then r is an H-invariant S-matrix on $\mathfrak{g}/\mathfrak{h}$, and the projection $p : G \to G/H$ is a K-V map.*

Proof. For any $\epsilon^1, \epsilon^2 \in (\mathfrak{g}/\mathfrak{h})^*$ and $u \in \mathfrak{g}$,

$$\prec q^* L^*_{s_\#(q^*\epsilon^1)} \epsilon^2, u \succ = - \prec \epsilon^2, s_\#(q^*\epsilon^1) \bullet u + \mathfrak{h} \succ = \prec l^*_{s_\#(q^*\epsilon^1)} q^*\epsilon^2, u \succ .$$

Hence

$$q^* \left([\epsilon^1, \epsilon^2]_s \right) = l^*_{s_\#(q^*\epsilon^1)} q^*\epsilon^2 - l^*_{s_\#(q^*\epsilon^2)} q^*\epsilon^1.$$

Since s is an S-matrix on $(\mathfrak{g}, \bullet)$ we get

$$s_\# \left(l^*_{s_\#(q^*\epsilon^1)} q^*\epsilon^2 - l^*_{s_\#(q^*\epsilon^2)} q^*\epsilon^1 \right) = [s_\#(q^*\epsilon^1), s_\#(q^*\epsilon^2)].$$

Hence

$$r_\#([\epsilon^1, \epsilon^2]_s) = q \circ s_\# \circ q^* \left([\epsilon^1, \epsilon^2]_s \right) = q([s_\#(q^*\epsilon^1), s_\#(q^*\epsilon^2)]).$$

∎

The following example is an illustration of the above result.

Example 3.2.5. *Let $G = GL_3^+(\mathbb{R})$ and H be the connected closed Lie subgroup of G given by*

$$H = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & a \end{pmatrix} \middle| \; a \in \mathbb{R}^*_+, b, c \in \mathbb{R} \right\}.$$

Denote by $\mathfrak{g} = \mathcal{M}_3(\mathbb{R})$ and $\mathfrak{h} = Vect(E_{13}, E_{23}, E_{33})$ the Lie algebras of G and H respectively, where (E_{ij}) is the canonical basis of $\mathcal{M}_3(\mathbb{R})$. Obviously, $\mathfrak{h}$ is a left ideal of $(\mathfrak{g}, \bullet)$, where $\bullet$ is the canonical associative product of $\mathcal{M}_3(\mathbb{R})$. Using a software program, we can show *that the only* $\mathrm{ad}(\mathfrak{h})$*-invariant* S*-matrix on* $(\mathfrak{g}, \bullet)$ *are*

$$s^1_\# = \begin{pmatrix} \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \alpha \end{pmatrix} s^2_\# = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

Hence we get two H*-invariant* S*-matrix on* $\mathfrak{g}/\mathfrak{h}$*,* r^1*, and* r^2 *respectively associated to* s^1 *and* s^2

$$r^1_\# = \begin{pmatrix} \alpha & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} r^2_\# = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence we get two non-equivalent G*-invariant K-V bivector fields on the* 6*-dimensional affine manifold* G/H*,* h^1 *and* h^2 *respectively associated to* r^1 *and* r^2*.*

Let $(G/H, \nabla)$ be a G-homogeneous affine manifold, g a G-invariant pseudo-Riemannian metric on G/H and denote by φ its associated H-invariant pseudo-Euclidean scalar product on $\mathfrak{g}/\mathfrak{h}$. According to the Theorem 3.2.3, we get the following corollary which is an improvement of [49, Lemma 9.2].

Corollary 3.2.6. *g is a G-invariant pseudo-Hessian metric on $(G/H, \nabla)$ if and only if, for all $u, v, w \in \mathfrak{g}$,*

$$\varphi([u, v] + \mathfrak{h}, w + \mathfrak{h}) = \varphi(u + \mathfrak{h}, L_v(w + \mathfrak{h})) - \varphi(v + \mathfrak{h}, L_u(w + \mathfrak{h})). \tag{3.2.1}$$

Proof. Denote by r the inverse of φ. From Theorem 3.2.3 it follows that g is a G-invariant pseudo-Hessian metric if and only if, for any $\epsilon^1, \epsilon^2, \epsilon^3 \in (\mathfrak{g}/\mathfrak{h})^*$,

$$\prec \epsilon^3, r_\# \left([\epsilon^1, \epsilon^2]_{\widetilde{r}}\right) - q\left([\widetilde{r}_\#(q^*\epsilon^1), \widetilde{r}_\#(q^*\epsilon^2)]\right) \succ = 0,$$

where $\widetilde{r} \in \otimes^2\mathfrak{g}$ is any symmetric bivector satisfying $q \circ \widetilde{r}_\# \circ q^* = r_\#$.

If we take $u + \mathfrak{h} = r_\#(\epsilon^1)$, $v + \mathfrak{h} = r_\#(\epsilon^2)$ and $w + \mathfrak{h} = r_\#(\epsilon^3)$ in the last equation, we obtain

$$\prec \epsilon^1, L_v(w + \mathfrak{h}) \succ - \prec \epsilon^2, L_u(w + \mathfrak{h}) \succ = \prec \epsilon^3, [u, v] + \mathfrak{h} \succ.$$

Hence

$$\varphi(u + \mathfrak{h}, L_v(w + \mathfrak{h})) - \varphi(v + \mathfrak{h}, L_u(w + \mathfrak{h})) = \varphi(w + \mathfrak{h}, [u, v] + \mathfrak{h}).$$

∎

The following result is known for the Hessian manifolds [49, Theorem 9.2]. The next theorem tells us that this result remains true if we suppose just h is non-degenerate.

Theorem 3.2.7. *Let G/H be a G-homogeneous space of a semisimple Lie group G. Then G/H does not admit any non trivial G-invariant pseudo-Hessian structure.*

Proof. Suppose that G/H admits a non trivial G-invariant pseudo-Hessian structure (∇, g). Denote by d_L the coboundary operator for the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in $(\mathfrak{g}/\mathfrak{h}, L)$. Regarding $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ as a 1-dimensional $(\mathfrak{g}/\mathfrak{h}, L)$-cochain we have,

$$(d_L q)(u, v) = L_u(q(v)) - L_v(q(u)) - q([u, v]) = 0,$$

that is q is a $(\mathfrak{g}/\mathfrak{h}, L)$-cocycle. Since $H^1(\mathfrak{g}, (\mathfrak{g}/\mathfrak{h}, L)) = 0$ because of the semisimplicity of $\mathfrak{g}$, there exists $w_0 \in \mathfrak{g}$ such that

$$q = d_L(w_0 + \mathfrak{h}),$$

which means that, for any $u \in \mathfrak{g}$,

$$u + \mathfrak{h} = L_u(w_0 + \mathfrak{h}).$$

Now, let $z \in \mathfrak{g}$. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, there exists $u, v \in \mathfrak{g}$ such that $z = [u, v]$. Then, according to the formula (3.2.1), we get that for any $w \in \mathfrak{h}$

$$\varphi(z + \mathfrak{h}, w + \mathfrak{h}) = \varphi(u + \mathfrak{h}, L_v(w + \mathfrak{h})) - \varphi(v + \mathfrak{h}, L_u(w + \mathfrak{h})).$$

If we take $w = w_0$ in this last equation, we obtain

$$\begin{aligned}\varphi(z + \mathfrak{h}, w_0 + \mathfrak{h}) &= \varphi(u + \mathfrak{h}, L_v(w_0 + \mathfrak{h})) - \varphi(v + \mathfrak{h}, L_u(w_0 + \mathfrak{h})) \\ &= \varphi(u + \mathfrak{h}, v + \mathfrak{h}) - \varphi(v + \mathfrak{h}, u + \mathfrak{h}) \\ &= 0.\end{aligned}$$

Since φ is non degenerate, this implies that $w_0 \in \mathfrak{h}$, and then $\mathfrak{h} = \mathfrak{g}$, which is a contradiction. ∎

In what follows we give a characterization of K-V structures on reductive homogeneous affine manifolds. Let $(G/H, \nabla)$ be a reductive G-homogeneous affine manifold, with the decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}$. Denote by $\mathfrak{b}$ the associated products to ∇ and $\mathfrak{b}_u : \mathfrak{m} \to \mathfrak{m}$, the map given by

$$\prec \mathfrak{b}_u^*(\xi), v \succ = - \prec \xi, \mathfrak{b}_u(v) \succ .$$

Hence the Lie algebra representation $L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ associated to $\mathfrak{b}$ is given by

$$L_u(v + \mathfrak{h}) = [u_\mathfrak{h}, v_\mathfrak{m}] + \mathfrak{b}(u_\mathfrak{m}, v_\mathfrak{m}) + \mathfrak{h}.$$

Theorem 3.2.8. *Let $(G/H, \nabla)$ be a reductive G-homogeneous affine manifold, with the decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}$. Then there is a one to one correspondence between G-invariant K-V bivector field h on $(G/H, \nabla)$ and H-invariant symmetric bivector $s \in \otimes^2\mathfrak{m}$ satisfying,*

$$s_\# \left(\mathfrak{b}^*_{s_\#(\epsilon^1)}(\epsilon^2) - \mathfrak{b}^*_{s_\#(\epsilon^2)}(\epsilon^1) \right) = [s_\#(\epsilon^1), s_\#(\epsilon^2)]_\mathfrak{m}.$$

Proof. Let h be a G-invariant symmetric bivector field on $(G/H, \nabla)$, and $s \in \otimes^2\mathfrak{m}$ its associated H-invariant symmetric bivector. Denote by $r \in \otimes^2\mathfrak{g}/\mathfrak{h}$ the H-invariant symmetric bivector given by,

$$r_{\#} := q \circ \iota \circ s_{\#} \circ \iota^{*} \circ q^{*},$$

where $\iota : \mathfrak{m} \hookrightarrow \mathfrak{g}$ is the inclusion map. From Theorem 3.2.3 it follows that h is a K-V bivector field if and only if, $[[r, r]] = 0$.

Now, let's compute the brackets $[[r, r]]$. For that, we take the symmetric bivector $\widetilde{r} \in \otimes^2\mathfrak{g}$ given by

$$\widetilde{r}_{\#} = \iota \circ s_{\#} \circ \iota^{*}.$$

One can easily see that $q \circ \widetilde{r}_{\#} \circ q^{*} = r_{\#}$. Hence, for any $\epsilon^1, \epsilon^2 \in (\mathfrak{g}/\mathfrak{h})^{*}$ and $u \in \mathfrak{m}$ we have,

$$\begin{aligned} \prec L^{*}_{\widetilde{r}_{\#}(q^{*}\epsilon^1)}(\epsilon^2), u + \mathfrak{h} \succ &= - \prec \epsilon^2, L_{\widetilde{r}_{\#}(q^{*}\epsilon^1)}(u + \mathfrak{h}) \succ \\ &= - \prec \epsilon^2, \mathfrak{b}_{s_{\#}((q\circ\iota)^{*}\epsilon^1)}(u) + \mathfrak{h} \succ \\ &= \prec \mathfrak{b}^{*}_{s_{\#}((q\circ\iota)^{*}\epsilon^1)}((q \circ \iota)^{*}\epsilon^2), u \succ . \end{aligned}$$

If we take $\xi^1 = (q \circ \iota)^{*}\epsilon^1$, $\xi^2 = (q \circ \iota)^{*}\epsilon^2$ and $\xi^3 = (q \circ \iota)^{*}\epsilon^3$, we obtain

$$(q \circ \iota)^{*} \left(L^{*}_{\widetilde{r}_{\#}(q^{*}\epsilon^1)}(\epsilon^2) \right) = \mathfrak{b}^{*}_{s_{\#}(\xi^1)}(\xi^2).$$

Hence

$$[[r, r]](\epsilon^1, \epsilon^2, \epsilon^3) = \prec \xi^3, s_{\#} \left(\mathfrak{b}^{*}_{s_{\#}(\xi^1)}(\xi^2) - \mathfrak{b}^{*}_{s_{\#}(\xi^2)}(\xi^1) \right) - [s_{\#}(\xi^1), s_{\#}(\xi^2)]_{\mathfrak{m}} \succ .$$

Hence, $[[r, r]] = 0$ if and only if

$$s_{\#} \left(\mathfrak{b}^{*}_{s_{\#}(\xi^1)}(\xi^2) - \mathfrak{b}^{*}_{s_{\#}(\xi^2)}(\xi^1) \right) = [s_{\#}(\xi^1), s_{\#}(\xi^2)]_{\mathfrak{m}}.$$

■

In what follows, we bring some corollaries, which are a direct consequence of Theorem 3.2.8.

Corollary 3.2.9. *Let $\Gamma \subset G$ be a discrete subgroup of a connected Lie group G. Then there is a one-to-one correspondence between the G-invariant K-V structures on G/Γ and the pairs $(\bullet, s)$, where $\bullet$ is an Γ-invariant left symmetric compatible product on $\mathfrak{g}$ and $s \in \otimes^2\mathfrak{g}$ is an Γ-invariant S-matrix on $(\mathfrak{g}, \bullet)$.*

Example 3.2.10. *Denote by*

$$\mathcal{H}_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$$

the 3-dimensional Heisenberg Lie group and by Γ the lattice in $\mathcal{H}_3$ given by

$$\Gamma = \left\{ \gamma := \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \middle| \; m, n, p \in \mathbb{Z} \right\}.$$

The Lie algebra of $\mathcal{H}_3$ is given by

$$\mathfrak{h}_3 = \{e_1, e_2, e_3 | [e_1, e_3] = e_2\},$$

where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can easily check that the product given by $e_1 \bullet e_3 = e_2$ is an Γ-invariant associative product on $\mathfrak{h}_3$. And a direct computation shows that the only Γ-invariant S-matrix on $(\mathfrak{h}_3, \bullet)$ are

$$s_{\#}(e_1^*) = s_{\#}(e_3^*) = 0, \; s_{\#}(e_2^*) = \lambda e_2.$$

Hence, each s induce an $\mathcal{H}_3$-invariant K-V bivector field on the compact homogeneous affine manifold $\mathcal{H}_3/\Gamma$.

Corollary 3.2.11. *Let $(G/H, \nabla)$ be a reductive G-homogeneous affine manifold with decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}$. Then there exists a one to one correspondence between G-invariant pseudo-Hessian metric g on $(G/H, \nabla)$ and H-invariant pseudo-Euclidean scalar product $\langle , \rangle$ on $\mathfrak{m}$ satisfying,*

$$\langle [u, v]_{\mathfrak{m}}, w \rangle = \langle u, \mathfrak{b}_v(w) \rangle - \langle v, \mathfrak{b}_u(w) \rangle.$$

Corollary 3.2.12. *Let G/H be a symmetric G-homogeneous space with canonical decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Then there exists a one to one correspondence between G-invariant K-V structure (∇, h) on G/H and the pair $(\bullet, s)$ where $\bullet$ is an H-invariant commutative product on $\mathfrak{m}$ satisfying,*

$$[\mathfrak{b}_u, \mathfrak{b}_v] = \mathrm{ad}_{[u,v]},$$

and s is an H-invariant symmetric bivector on $\mathfrak{m}$ satisfying,

$$\prec \epsilon^1, \mathfrak{b}(s_{\#}(\epsilon^2), s_{\#}(\epsilon^3)) \succ = \prec \epsilon^2, \mathfrak{b}(s_{\#}(\epsilon^1), s_{\#}(\epsilon^3)) \succ .$$

Corollary 3.2.13. *Let G/H be a symmetric G-homogeneous space with canonical decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Then there exists a one-to-one correspondence between G-invariant pseudo-Hessian structure (∇, g) on G/H and the pair $(\bullet; \langle \cdot, \cdot \rangle)$ where $\bullet$ is an H-invariant commutative product on $\mathfrak{m}$ satisfying,*

$$[\mathfrak{b}_u, \mathfrak{b}_v] = \mathrm{ad}_{[u,v]},$$

and $\langle \cdot, \cdot \rangle$ is an H-invariant pseudo-Euclidean scalar product on $\mathfrak{m}$ satisfying,

$$\langle u \bullet v, w \rangle = \langle v, u \bullet w \rangle.$$

Example 3.2.14 ($\mathrm{GL}_n^+(\mathbb{R})$-Homogeneous K-V structure on $\mathcal{S}_n^{++}(\mathbb{R})$).
Denote by $\mathcal{S}_n^{++}(\mathbb{R})$ the space of real symmetric positive definite which is an open subset of $\mathcal{S}_n(\mathbb{R})$, the vector space of real symmetric $n \times n$-matrices. We know that the connected Lie group $G := \mathrm{GL}_n^+(\mathbb{R})$ acts transitively on $\mathcal{S}_n^{++}(\mathbb{R})$, $g \cdot x := gxg^t$. The isotropy subgroup in I_n is $H := \mathrm{SO}_n(\mathbb{R})$. Hence we have the following identification

$$G/H \xrightarrow{\sim} \mathcal{S}_n^{++}(\mathbb{R}), \; \bar{g} \mapsto gg^t.$$

The Lie algebra of H is $\mathfrak{h} = \mathfrak{so}_n(\mathbb{R})$ and with $\mathfrak{m} := \mathcal{S}_n(\mathbb{R})$ we have a canonical decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \; \mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}, \; [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Define the following product

$$\bullet : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}, u \bullet v := uv + vu.$$

And the canonical scalar product

$$\langle\cdot,\cdot\rangle : \mathfrak{m}\times\mathfrak{m}\to\mathbb{R}, \langle u,v\rangle := \mathrm{tr}(u^t v).$$

Obviously the pair $(\bullet, \langle\cdot,\cdot\rangle)$ *satisfies the conditions of the Corollary 3.2.13. Hence we get a* G*-invariant Hessian structure* (∇, g) *on* G/H*. By pushing forward* (∇, g) *under the identification* $G/H \xrightarrow{\sim} \mathcal{S}_n^{++}(\mathbb{R})$ *we get a* G*-invariant Hessian structure* (∇, g) *on* $\mathcal{S}_n^{++}(\mathbb{R})$*. And one can see that* ∇ *is the restriction of the canonical flat torsion less connection of* $S_n(\mathbb{R})$ *to the open subset* $S_n^{++}(\mathbb{R})$*, and* g *is the canonical metric of* $S_n^{++}(\mathbb{R})$*.*

Now we are going to apply Corollary 3.2.12 to the manifold G seen as a symmetric $(G\times G)$-homogeneous space, where the action is given by $(a,b).x := axb^{-1}$. The isotropy group of e is $H := \mathrm{diag}(G) = \{(a,a)/\ a\in G\}$, which is isomorphic to G. The Lie algebra $\mathrm{Lie}(G\times G) = \mathfrak{g}\oplus\mathfrak{g}$ can be decomposed as, $\mathfrak{h}\oplus\mathfrak{m}$, where $\mathfrak{h} := \{(u,u)/\ u\in\mathfrak{g}\}$ and $\mathfrak{m} := \{(u,-u)/\ u\in\mathfrak{g}\}$. The adjoint representation

$$\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{m}),\ \mathrm{Ad}_{(a,a)}(u,-u) = (\mathrm{Ad}_a(u), -\mathrm{Ad}_a(u)),$$

which is equivalent to the usual adjoint representation of the Lie group $G\to\mathrm{GL}(\mathfrak{g})$ when H is identified with G and $\mathfrak{m}\cong\mathfrak{g}$ as a vector space. In addition, we have $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$. So, from the Proposition A.4.3 we get

Corollary 3.2.15. *Let* G *be a connected Lie group endowed with a bi-invariant flat torsionless connection* ∇*. Then there exists a one to one correspondence between the following sets*

1. *Bi-invariant K-V structures on* G*.*
2. $\mathrm{Ad}(G)$*-invariant symmetric bivectors* s *on* $\mathfrak{g}$ *satisfying,*

$$\prec \epsilon^1, \mathfrak{b}(s_\#(\epsilon^2), s_\#(\epsilon^3)) \succ = \prec \epsilon^2, \mathfrak{b}(s_\#(\epsilon^1), s_\#(\epsilon^3)) \succ .$$

3. $\mathrm{Ad}(G)$*-invariant* S*-matrices* s *on* $(\mathfrak{g}, \mathfrak{b}')$*.*

3.3 $\mathrm{SL}_2(\mathbb{R})$-invariant K-V structures on $\mathrm{SL}_2(\mathbb{R})$-homogeneous surfaces

Denote by $\mathrm{SL}_2(\mathbb{R})$ the Lie group of 2×2-matrices with determinant equal to 1. Its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is given by

$$\mathfrak{sl}_2(\mathbb{R}) = (e_1, e_2, e_3 |\, [e_1,e_2] = 2e_2; [e_1,e_3] = -2e_3; [e_2,e_3] = e_1),$$

where

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let's consider the two following closed subgroup of $\mathrm{SL}_2(\mathbb{R})$

$$N := \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} | \, t\in\mathbb{R}\right\}, H := \left\{\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} | \, t\in\mathbb{R}_+^*\right\}.$$

Denote by $\mathfrak{n}$ the Lie algebra of N and $\mathfrak{h}$ the Lie algebra of H.

Lemma 3.3.1.

1. *There exists a unique non trivial* $\mathrm{SL}_2(\mathbb{R})$*-invariant flat torsionless connection on the* $\mathrm{SL}_2(\mathbb{R})$*-homogeneous space* $\mathrm{SL}_2(\mathbb{R})/N$.

2. *There is no non trivial* $\mathrm{SL}_2(\mathbb{R})$*-invariant flat torsionless connection on the* $\mathrm{SL}_2(\mathbb{R})$*-homogeneous space* $\mathrm{SL}_2(\mathbb{R})/H$.

Proof. 1. A direct computation shows that the only Lie algebra representation $L : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{End}(\mathfrak{sl}_2(\mathbb{R})/\mathfrak{n})$ satisfying the conditions of Nomizu's Theorem A.4.1 is given by

$$\begin{cases} L_{e_1}(e_1+\mathfrak{h}) = e_1+\mathfrak{h} \\ L_{e_1}(e_3+\mathfrak{h}) = -e_3+\mathfrak{h}, \end{cases} \quad \begin{cases} L_{e_2}(e_1+\mathfrak{h}) = 0, \\ L_{e_2}(e_3+\mathfrak{h}) = e_1+\mathfrak{h}, \end{cases} \quad \text{and} \quad \begin{cases} L_{e_3}(e_1+\mathfrak{h}) = e_3+\mathfrak{h} \\ L_{e_3}(e_3+\mathfrak{h}) = 0. \end{cases}$$

Hence the result follows from Nomuzi's Theorem A.4.1.

2. Since $\mathrm{SL}_2(\mathbb{R})$ is a semisimple real Lie group and $\mathrm{SL}_2(\mathbb{R})/H$ is reductive $\mathrm{SL}_2(\mathbb{R})$-homogeneous space. The results follow from [24].

■

Remark 3.3.2. *The unique non trivial* $\mathrm{SL}_2(\mathbb{R})$*-invariant flat trosionless connection on* $\mathrm{SL}_2(\mathbb{R})/N$ *is the pullback of the canonical flat torsionless connection on* $\mathbb{R}^2\backslash\{0\}$ *under the diffeomorphism* $\mathrm{SL}_2(\mathbb{R})/N \xrightarrow{\sim} \mathbb{R}^2\backslash\{0\}$, $\bar{g} \mapsto g(1,0)^t$.

Let Σ be a surface on which $\mathrm{SL}_2(\mathbb{R})$ acts effectively and transitively and $x \in \Sigma$ be a fixed point. According to [14] there is twos situation,

1. If Σ is non-compact. Then
   (a) If the isotropy subgroup at x is isomorphic to H. Then Σ is $\mathrm{SL}_2(\mathbb{R})$-equivariantly diffeomorphic to $\mathrm{SL}_2(\mathbb{R})/H$.
   (b) If the isotropy subgroup at x is isomorphic to N. Then Σ is $\mathrm{SL}_2(\mathbb{R})$-equivariantly diffeomorphic to $\mathrm{SL}_2(\mathbb{R})/N$.
2. If Σ is compact. Then Σ is $\mathrm{SL}_2(\mathbb{R})$-equivariantly diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

Hence, according to Lemma 3.3.1 and Theorem 3.2.3, we get the following theorem.

Theorem 3.3.3. *Under the above assumptions, we have.*

1. *If the isotropy subgroup at* x *is isomorphic to* N*. Then there is a unique non trivial* $\mathrm{SL}_2(\mathbb{R})$*-invariant K-V structure on* Σ.

2. *If the isotropy subgroup at* x *is isomorphic to* H*. Then there is no non trivial* $\mathrm{SL}_2(\mathbb{R})$*-invariant K-V structure on* Σ.

We recall that the Hopf torus does not admit any non trivial Hessian metric [49, Corollary 7.4 pp. 129]. Now, we consider the $\mathbb{Z}$-action on $\mathbb{C}^*$ given by

$$\psi : \mathbb{Z} \times \mathbb{C}^* \to \mathbb{C}^*, \ (a,z) \mapsto a.z := e^{2\pi a}z.$$

The orbit space $\mathbb{C}^*/\mathbb{Z}$ can be identified with the two torus $\mathbb{S}^1 \times \mathbb{S}^1$ via the following diffeomorphism

$$f : \mathbb{C}^*/\mathbb{Z} \xrightarrow{\cong} \mathbb{S}^1 \times \mathbb{S}^1, \ [z] \mapsto (\frac{z}{|z|}, e^{i\ln(|z|)}).$$

This allows us to transfer the action ψ to an $\mathrm{SL}_2(\mathbb{R})$-action on $\mathbb{S}^1 \times \mathbb{S}^1$, such that f becomes an equivariant diffeomorphism. Moreover the unique non trivial K-V structure on $\mathbb{C}^* := \mathbb{R}^2 \backslash \{0\}$ given by Theorem 3.3.3 is $\mathbb{Z}$-invariant. Therefore, it induces a non trivial K-V structure on $\mathbb{S}^1 \times \mathbb{S}^1$ denoted by (∇, h).

Theorem 3.3.4. *(∇, h) is the unique non trivial $\mathrm{SL}_2(\mathbb{R})$-invariant K-V structure on $\mathbb{S}^1 \times \mathbb{S}^1$.*

Proof. Let $(\overline{\nabla}, \bar{h})$ be a $\mathrm{SL}_2(\mathbb{R})$-invariant K-V structure on $\mathbb{S}^1 \times \mathbb{S}^1$. The projection

$$p : \mathbb{C}^* \to \mathbb{S}^1 \times \mathbb{S}^1, z \mapsto \big(\frac{z}{|z|}, e^{i \ln(|z|)}\big),$$

is a $\mathbb{Z}$-cover and it is $\mathrm{SL}_2(\mathbb{R})$-equivariant. Then its lift a K-V structure $(\widetilde{\nabla}, \widetilde{h})$ on $\mathbb{C}^* := \mathbb{R}^2 \backslash \{0\}$. From Proposition 3.1.7 it follows that $(\widetilde{\nabla}, \widetilde{h})$ is $\mathrm{SL}_2(\mathbb{R})$-invariant. Finally Theorem 3.3.3 permits us to conclude. ∎

3.4 Description of the regular affine foliation

We recall that any K-V manifold (M, ∇, h) is foliated by an affine foliation, moreover, each leaf carries a pseudo-Hessian structure [1, Theorem 3.3]. The G-invariance of the K-V bivector field implies that the affine foliation is regular.

Let $M{=}G/H$ be a G-homogeneous manifold endowed with a G-invariant K-V structure (∇, h). Denote by $r \in \otimes^2 \mathfrak{g}/\mathfrak{h}$ the H-invariant symmetric bivector associated to h, $\widetilde{r} \in \otimes^2 \mathfrak{g}$ any symmetric bivector such that $q \circ \widetilde{r}_\# \circ q^* = r_\#$, this means the commutativity of the following diagram

$$\begin{array}{ccc} T^*_{\bar{e}}M & \xrightarrow{h_{\#,\circ}} & T_{\bar{e}}M \\ \big\downarrow \Phi_e^* & & \Phi_e \big\uparrow \\ (\mathfrak{g}/\mathfrak{h})^* & \xrightarrow{r_\#} & \mathfrak{g}/\mathfrak{h} \\ \big\downarrow q^* & & q \big\uparrow \\ \mathfrak{g}^* & \xrightarrow{\widetilde{r}_\#} & \mathfrak{g} \end{array} .$$

And $\mathfrak{a} := q^{-1}(Im(r_\#))$ be the vector subspace of $\mathfrak{g}$.

Proposition 3.4.1. *$\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. Let $u, v \in \mathfrak{a}$, and $\epsilon^1, \epsilon^2 \in (\mathfrak{g}/\mathfrak{h})^*$ such that $\widetilde{r}_\#(q^*\epsilon^1) + k_1 = u$, $\widetilde{r}_\#(q^*\epsilon^2) + k_2 = v$, where $k_1, k_2 \in \mathfrak{h}$. Then we have

$$\begin{aligned} [u, v] + \mathfrak{h} &= [\widetilde{r}_\#(q^*\epsilon^1) + k_1, \widetilde{r}_\#(q^*\epsilon^2) + k_2] + \mathfrak{h} \\ &= q\left([\widetilde{r}_\#(q^*\epsilon^1), \widetilde{r}_\#(q^*\epsilon^2)]\right) + q \circ \mathrm{ad}_{k_1}(\widetilde{r}_\#(q^*\epsilon^2)) - q \circ \mathrm{ad}_{k_2}(\widetilde{r}_\#(q^*\epsilon^1)) \\ &= q\left([\widetilde{r}_\#(q^*\epsilon^1), \widetilde{r}_\#(q^*\epsilon^2)]\right) + \overline{\mathrm{ad}}_{k_1}(r_\#(\epsilon^2)) - \overline{\mathrm{ad}}_{k_2}(r_\#(\epsilon^1)). \end{aligned}$$

Since $[[r, r]] = 0$ and r is H-invariant, we get

$$[u, v] + \mathfrak{h} = r_\#\left([\epsilon^1, \epsilon^2]_{\widetilde{r}}\right) - r_\#(\overline{\mathrm{ad}}^*_{k_1}\epsilon^2) + r_\#(\overline{\mathrm{ad}}^*_{k_2}\epsilon^1).$$

Hence $[u, v] \in \mathfrak{a}$. ∎

Denote by A the connected immersed Lie subgroup of G which integrates the Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$.

Lemma 3.4.2. *For any $b \in H$ we have $c_b(A) = A$, where $c_b(a) = bab^{-1}$.*

Proof. Since A is connected, then it suffices to show that $\mathrm{Ad}_b(\mathfrak{a}) = \mathfrak{a}$ for any $b \in H$. Since Ad_b is an isomorphism, it suffices to show that $\mathrm{Ad}_b(\mathfrak{a}) \subset \mathfrak{a}$. Let $u \in \mathfrak{a}$, then there exists $\epsilon \in (\mathfrak{g}/\mathfrak{h})^*$ such that $q(u) = u + \mathfrak{h} = r_\#(\epsilon)$. Hence

$$\begin{aligned} q(\mathrm{Ad}_b(u)) &= \overline{\mathrm{Ad}}_b(q(u)) \\ &= \overline{\mathrm{Ad}}_b(r_\#(\epsilon)) \\ &= -r_\#(\overline{\mathrm{Ad}}^*_{b^{-1}}(\epsilon)). \end{aligned}$$

Hence $\mathrm{Ad}_b(u) \in \mathfrak{a}$. ∎

From the G-invariance of the K-V bivector field h it follows that $E{:=}\mathrm{Im}h_\# \subset TM$ is a homogeneous G-vector subbundle. Hence we get the following isomorphism of homogeneous G-vector bundle

$$G \times_H E_{\bar{e}} \xrightarrow{\cong} E, (g, u) \mapsto T_{\bar{e}}(\lambda_g)(u).$$

Now, from Lemma 3.4.2 it follows that for any $b \in H$, we have

$$\mathrm{Ad}_b(\mathfrak{a}) = \mathfrak{a},\ \mathrm{Ad}_b(\mathfrak{h}) = \mathfrak{h}.$$

Then we get a linear representation

$$\overline{\mathrm{Ad}} : H \to \mathrm{End}(V),$$

where $V := \mathfrak{a}/\mathfrak{h}$. Hence

$$G \times_H V \to M,$$

is a homogeneous G-vector bundle.

Theorem 3.4.3. *The regular affine foliation E is given by the homogenous G-vector bundle isomorphism*

$$G \times_H V \xrightarrow{\cong} E,\ (g, u + \mathfrak{h}) \mapsto T_{\bar{e}}(\lambda_g) \circ T_e p(u).$$

Proof. Denote by $\psi : V \to E_{\bar{e}},\ u + \mathfrak{h} \mapsto T_e p(u)$. Obviously, ψ is a linear isomorphism of vector spaces. In particular, we get the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\overline{\mathrm{Ad}}_a} & V \\ \psi \downarrow & & \downarrow \psi \\ E_{\bar{e}} & \xrightarrow{T_{\bar{e}}(\lambda_a)} & E_{\bar{e}} \end{array}.$$

Hence the two linear representations $\overline{\lambda} : H \to \mathrm{End}(E_{\bar{e}})$ and $\overline{\mathrm{Ad}} : H \to \mathrm{End}(V)$ are equivalent. This means that the bundle map

$$G \times_H V \longrightarrow E,\ (g, u + \mathfrak{h}) \mapsto T_{\bar{e}}(\lambda_g) \circ T_e p(u),$$

is an isomorphism of homogeneous G-vector bundles over M. ∎

In the following, we describe the leaf spaces.

Proposition 3.4.4.

1. *The pseudo-Hessian leaf* $\mathcal{F}^{\bar{g}}$ *passing through* $\bar{g} \in M$ *is given by*

$$\mathcal{F}^{\bar{g}} = \{gaH,\ a \in A\} = c_g(A).gH$$

2. *For any* $\bar{g} \in M$ *the pseudo-Hessian leaf* $\mathcal{F}^{\bar{g}}$ *is a* $c_g(A)$*-homogeneous pseudo-Hessian manifold, which is isomorphic to the homogeneous space* A/H.

3. *The leaf spaces* $M/\mathcal{F}$ *can be identified with* G/A *through the map* $gA \mapsto \mathcal{F}^{\bar{g}}$.

4. *If we assume that* (G, H) *is a reductive pair with decomposition:*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}.$$

Then (A, H) *is a reductive pair with decomposition:*

$$\mathfrak{a} = \mathfrak{h} \oplus (\mathfrak{a} \cap \mathfrak{m}),\ \mathrm{Ad}(H)(\mathfrak{a} \cap \mathfrak{m}) = \mathfrak{a} \cap \mathfrak{m}.$$

5. *If we assume that* (G, H) *is a symmetric pair with canonical decomposition:*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m},\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

Then (A, H) *is a symmetric pair with canonical decomposition:*

$$\mathfrak{a} = \mathfrak{h} \oplus (\mathfrak{a} \cap \mathfrak{m}),\ \mathrm{Ad}(H)(\mathfrak{a} \cap \mathfrak{m}) = (\mathfrak{a} \cap \mathfrak{m}),\ [\mathfrak{a} \cap \mathfrak{m}, (\mathfrak{a} \cap \mathfrak{m})] \subset \mathfrak{h}.$$

Proof. 1. Let $b \in H$. From Lemma 3.4.2 it follows that

$$c_{gb}(A).gbH = c_g(c_b(A)).gH = c_g(A).gH.$$

Hence $c_g(A).gH$ is well defined. Now, let's compute $T_{\bar{e}}\mathcal{F}^{\bar{e}}$.

$$\begin{aligned} T_{\bar{e}}\mathcal{F}^{\bar{e}} &= T_e p(\mathfrak{a}) \\ &= \Phi_e \circ q(\mathfrak{a}) \\ &= \Phi_e \circ r_\#((\mathfrak{g}/\mathfrak{h})^*) \\ &= \Phi_e \circ r_\# \circ \Phi_e^*(T_{\bar{e}}^* M) \\ &= h_{\#,\bar{e}}(T_{\bar{e}}^* M). \end{aligned}$$

Hence, for any $a \in A$,

$$\begin{aligned} T_{\bar{a}}\mathcal{F}^{\bar{e}} &= T_{\bar{e}}(\lambda_a)(T_{\bar{e}}\mathcal{F}^{\bar{e}}) \\ &= T_{\bar{e}}(\lambda_a) \circ h_{\#,\bar{e}}(T_{\bar{e}}^* M) \\ &= T_{\bar{e}}(\lambda_a) \circ h_{\#,\bar{e}} \circ T_{\bar{e}}^*(\lambda_a)(T_{\bar{a}}^* M) \\ &= h_{\#,\bar{a}}(T_{\bar{a}}^* M). \end{aligned}$$

This shows that the leaf passing through $\bar{e}$ is given by

$$\mathcal{F}^{\bar{e}} = \{aH,\ a \in A\} = A\,.$$

Hence the leaf passing through any $\bar{g} \in M$ is given by

$$\mathcal{F}^{\bar{g}} = \lambda_g(\mathcal{F}^{\bar{e}}) = \{gaH,\, a \in A\} = c_g(A).H\,.$$

2. Let $g \in G$. Since $c_g(A)$ is a subgroup of G it follows that $c_g(A)$ act on $\mathcal{F}^{\bar{g}}$ by K-V transformation, which also act transitively on $\mathcal{F}^{\bar{g}}$. Hence $\mathcal{F}^{\bar{g}}$ is a $c_g(A)$-homogeneous pseudo-Hessian manifold.

3. Let $g' = gba$, where $b \in H$ and $a \in A$. From Lemma 3.4.2 we get that

$$\mathcal{F}^{\bar{g'}} = \{gbaa'H,\, a \in A\} = \{gbaH,\, a \in A\} = \{gaH,\, a \in A\} = \mathcal{F}^{\bar{g}}.$$

This means the map $gAH \mapsto \mathcal{F}^{\bar{g}}$ is well defined. The other assumptions on this map are obvious.

∎

As a corollary of Proposition 3.4.4 we get.

Corollary 3.4.5. *The following assertions are equivalent:*

1. *The leaf passing through $\bar{e}$ is closed in M.*
2. *A is closed in G.*
3. *The leaf space $M/\mathcal{F}$ is a Hausdorff space.*

Now, we give a characterization of the A-invariant pseudo-Hessian structure $(\nabla^{\bar{e}}, g^{\bar{e}})$ on A/H. For that, we consider the following notation

- $L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ the Nomizu representation associated to ∇.
- $L^{\bar{e}} : \mathfrak{a} \to \mathrm{End}(\mathfrak{a}/\mathfrak{h})$ the Nomizu representation associated to $\nabla^{\bar{e}}$.
- $\varphi^{\bar{e}}$ the pseudo-Euclidean scalar product on $\mathfrak{a}/\mathfrak{h}$ associated to the pseudo-Hessian metric $g^{\bar{e}}$.
- $\iota : \mathfrak{a}/\mathfrak{h} \hookrightarrow \mathfrak{g}/\mathfrak{h}, u + \mathfrak{h} \mapsto u + \mathfrak{h}$ and $\tilde{\iota} : T_{\bar{e}}\mathcal{F}^{\bar{e}} \hookrightarrow T_{\bar{e}}M$ the canonical injections. Hence we have the following commutative diagram:

$$\begin{array}{ccc} T_{\bar{e}}\mathcal{F}^{\bar{e}} & \xhookrightarrow{\tilde{\iota}} & T_{\bar{e}}M \\ {\scriptstyle \Phi_e^{\bar{e}}}\big\uparrow & & {\scriptstyle \Phi_e}\big\uparrow \\ \mathfrak{a}/\mathfrak{h} & \xhookrightarrow{\iota} & \mathfrak{g}/\mathfrak{h} \end{array}$$

Proposition 3.4.6.

1. *$L^{\bar{e}}$ is the unique map satisfying, for any $u \in \mathfrak{a}$,*

$$\iota \circ L^{\bar{e}}_u = L_u \circ \iota.$$

2. *For any $u, v \in \mathfrak{a}$,*

$$\varphi^{\bar{e}}(u + \mathfrak{h}, v + \mathfrak{h}) := r(\epsilon^1, \epsilon^2),$$

where $\epsilon^1, \epsilon^2 \in (\mathfrak{g}/\mathfrak{h})^*$ such that $r_\#(\epsilon^1) = u + \mathfrak{h}, r_\#(\epsilon^2) = v + \mathfrak{h}$.

Proof. 1. Let $u, v \in \mathfrak{a}$,

$$\begin{aligned} \tilde{\iota} \circ \Phi_e^{\bar{e}}(L_u^{\bar{e}}(v + \mathfrak{a} \cap \mathfrak{h})) &= \tilde{\iota}((\nabla_{v^*}^{\bar{e}} u^*)_{\bar{e}}) \\ &= (\nabla_{v^*} u^*)_{\bar{e}} \\ &= \Phi_e(L_u^{\bar{e}}(v + \mathfrak{h})). \end{aligned}$$

Then

$$\Phi_e \circ \iota(L_u^{\bar{e}}(v + \mathfrak{h})) = \Phi_e(L_u^{\bar{e}}(v + \mathfrak{h})).$$

Hence we get

$$\iota \circ L_u^{\bar{e}} = L_u \circ \iota.$$

The uniqueness of $L^{\bar{e}}$ follows from the injectivity of ι.

2. For any $\alpha, \beta \in \Omega^1(M)$, we have

$$g^{\bar{e}}(h_{\#}(\alpha), h_{\#}(\beta)) = h(\alpha, \beta).$$

Let $u, v \in \mathfrak{a}$ and $\epsilon^1, \epsilon^2 \in (\mathfrak{g}/\mathfrak{h})^*$ such that $r_{\#}(\epsilon^1) = u + \mathfrak{h}, r_{\#}(\epsilon^2) = v + \mathfrak{h}$. Hence

$$\begin{aligned} \varphi^{\bar{e}}(u + \mathfrak{h}, v + \mathfrak{h}) &= g^{\bar{e}}(\Phi_e^{\bar{e}}(u + \mathfrak{h}), \Phi_e^{\bar{e}}(v + \mathfrak{h})) \\ &= h((\Phi_e^*)^{-1}(\epsilon^1), (\Phi_e^*)^{-1}(\epsilon^2)) \\ &= r(\epsilon^1, \epsilon^2). \end{aligned}$$

∎

Chapter 4

Perspectives

1. According to [section 2.5, chapter 2] it would be very interesting to see whether coisotropic reduction can be done in this situation, and which information can be gained in the reduced space once singularities can be controlled. Or is there an analog of Meyer-Marsden-Weinstein reduction using Lie group actions in this general setting?

2. In mathematics, very often while working with a certain class $\mathcal{A}$ of objects, we want to distinguish the pairs $A, \widetilde{A} \in \mathcal{A}$ which are compatible in a certain sense. Examples include:

   (a) Poisson structures.

   (b) Nijenhuis operators.

   (c) Poisson-Nijenhuis structures.

   For each of these situations it makes sense to introduce the following problems:

   (a) Define compatible K-V structures. (Such definition should entail the compatibility of Poisson structures).

   (b) Determine compatibility between K-V and Nijenhuis tensors. (This makes sense to introduce the notions of K-V-Nijenhuis structures).

3. The integrability of Lie algebroids is a longstanding problem of differential geometry that has been solved in [18]. It is known that Lie's third theorem fails to hold for Lie algebroids. More precisely, for any Lie groupoids we can associate a Lie algebroids for the converse there is some computable obstructions which given in [18] and in [19] for the special case's of Poisson manifolds, moreover, in the case's of Poisson manifolds if a Poisson manifold is integrable than its Lie groupoid is a symplectic Lie groupoid. Thinking at the similarity between K-V manifolds and Poisson manifolds one can ask:

   (a) Is there any others obstructions to the integrability of K-V manifolds?

   (b) If a K-V manifold is integrable what is the geometric structures on its Lie groupoid which integrate the K-V strcuture?

Appendix A

Homogeneous spaces

In this appendix, we collect some technical results needed in chapter 3. Some of these results are well-known and we basically just cite them here, some other are modifications of well-known results or even new, but technical in nature. For more details about the geometry of homogeneous spaces, one can consult [31, 15].

Appendix A.1 Tensor fields on homogeneous manifold

From now on, G is a connected Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ is a closed subgroup with Lie algebra $\mathfrak{h}$ and $M := G/H$.

Denote by $p : G \to M$, $p(g) = \bar{g} = gH$ the canonical projection and $\bar{e} = H$. The homogeneous action of G on M is given by

$$\lambda : G \times M \to M, \quad (g, \overline{g'}) \mapsto g \cdot \overline{g'} = \lambda_g(\overline{g'}) = \overline{gg'}. \tag{A.1.1}$$

The *isotropy representation* is given by

$$\mathrm{Ad}^M : H \to \mathrm{GL}(T_{\bar{e}}M), \quad a \mapsto T_{\bar{e}}(\lambda_a).$$

Hence we have a bundle isomorphism

$$G \times_H T_{\bar{e}}M \xrightarrow{\cong} TM, \quad (g, X_{\bar{e}}) \mapsto g \cdot X_{\bar{e}} = T_{\bar{e}}(\lambda_g)(X_{\bar{e}}),$$

where $G \times_H T_{\bar{e}}M$ is the orbit space of $G \times T_{\circ}M$ under the action of H given by $(a, v).h = (ah, h^{-1}v)$. This fact will be explained below in a more general setting of G-vector bundles over M.

The tangent linear map $T_e p : \mathfrak{g} \to T_{\bar{e}}M$ is surjective and then induces a linear isomorphism

$$\Phi_e : \mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T_{\bar{e}}M, \quad \Phi_e(u + \mathfrak{h}) = T_e p(u). \tag{A.1.2}$$

For any $a \in H$, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h} & \xrightarrow{\overline{\mathrm{Ad}}_a} & \mathfrak{g}/\mathfrak{h} \\ \Phi_e \downarrow & & \downarrow \Phi_e \\ T_{\bar{e}}M & \xrightarrow{T_{\bar{e}}(\lambda_a)} & T_{\bar{e}}M \end{array},$$

where $\overline{\mathrm{Ad}}_a(v + \mathfrak{h}) = \mathrm{Ad}_a(v) + \mathfrak{h}$. This means that the isotropy representation Ad^M is equivalent to the representation $\overline{\mathrm{Ad}} : H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Hence, we get the following bundle isomorphism

$$\Phi : G \times_H \mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} TM, \quad (g, u + \mathfrak{h}) \mapsto T_{\bar{e}}\lambda_g \circ T_e p(u).$$

Which means that the tangent bundle $TM \to M$ is identified with the vector bundle associated to the H-principal bundle $G \to M$ and the representation $\overline{\mathrm{Ad}} : H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. In particular, for any $g \in G$ we have a canonical isomorphism

$$\Phi_g : \mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T_{\bar{g}}M, \quad \Phi_g(u + \mathfrak{h}) = T_{\bar{e}}\lambda_g \circ T_e p(u). \tag{A.1.3}$$

The tangent bundle $TM \to M$, the cotangent bundle $T^*M \to M$, and more generally for each $p, q \in \mathbb{N}$ with $p + q > 0$ the (p, q)-*tensor bundle*:

$$T^{p,q}M := \otimes^p TM \otimes \otimes^q T^*M \to M$$

are examples of homogeneous G-vector bundles (Definition A.1.1). The action of G on $T^{(p,q)}M$ is given by

$$\tilde{\lambda} : G \times T^{(p,q)}M \to T^{(p,q)}M, \tag{A.1.4}$$

where

$$\tilde{\lambda}_g(u_1 \otimes \cdots \otimes u_p \otimes l_1 \otimes \cdots \otimes l_q) := g.u_1 \otimes \cdots \otimes g.u_p \otimes g.l_1 \otimes \cdots \otimes g.l_q.$$

The smooth sections of the (p, q)-tensor bundle are called (p, q)-*tensor fields*, and are denoted by $\mathcal{T}^{p,q}M := \Gamma(T^{p,q}M)$. In particular $\Omega^1(M) := \mathcal{T}^{0,1}M$ is the space of differential 1-forms and $\Gamma(TM) := \mathcal{T}^{1,0}M$ is the space of vector fields. There is a canonical linear isomorphism between $\mathcal{T}^{p,q}M$ and the space of $C^\infty(M)$-multilinear maps

$$\underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{p \text{ factors}} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{q \text{ factors}} \longrightarrow C^\infty(M) \tag{A.1.5}$$

Definition A.1.1 (Homogeneous vector bundles)**.**
A homogeneous G-vector bundle over M is a vector bundle $\pi : E \to M$, together with an action of G on E, such that

1. *π is a G-equivariant map, i.e. $\pi(g \cdot u) = g \cdot \pi(u)$*
2. *If $g \in G$ and $x \in M$ then $g : \pi^{-1}(x) \to \pi^{-1}(gx)$ is linear map.*

The main examples are the associated vector bundles $\pi : G \times_H V \to M$, where $H \to \mathrm{GL}(V)$ is a finite dimensional linear representation, $\pi[a, v] = aH$ and the action of G is given by $g \cdot [a, v] = [ga, v]$. Conversely, every G-vector bundle is of this type. Indeed, if we start with a homogeneous G-vector bundle $\pi : E \to M$ and we put $V = \pi^{-1}(eH)$, then by definition $\pi(g \cdot u) = g \cdot \pi(u)$ and in particular $\pi(h \cdot v) = eH$ for any $h \in H, v \in V$. So the left action of G on E restricts to a linear representation $H \to \mathrm{GL}(V)$ and the map $G \times V \to E$, $(g, u) \mapsto g \cdot u$, factors to an isomorphism of G-vector bundles

$$G \times_H V \xrightarrow{\cong} E.$$

In summary, we have a correspondence between the category of G-vector bundles over M and the category of linear representations $H \to GL(V)$. In particular, the tensor bundle $T^{(p,q)}M \to M$ is identified with the G-vector bundle

$$G \times_H (\mathfrak{g}/\mathfrak{h})^{p,q} \to M,$$

where $(\mathfrak{g}/\mathfrak{h})^{p,q} := \otimes^p(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^q(\mathfrak{g}/\mathfrak{h})^*$. Hence, there is a natural linear isomorphism between the space of tensor fields $\mathcal{T}^{p,q}M$ and the space of smooth maps $F : G \to (\mathfrak{g}/\mathfrak{h})^{p,q}$ which are H-equivariant.

For any $u \in \mathfrak{g}$ we assign the fundamental vector field $u^* \in \Gamma(TM)$,

$$(u^*)_{\overline{g}} = \frac{d}{dt}_{|t=0} \exp(-tu)gH.$$

The associated H-equivariant map $F : G \to \mathfrak{g}/\mathfrak{h}$ is given by

$$\begin{aligned} F(g) &= g^{-1} \cdot u^*_{\overline{g}} \\ &= \Phi_e^{-1}(g^{-1}\frac{d}{dt}_{|t=0}(\exp(-tu))gH) \\ &= \Phi_e^{-1}(\frac{d}{dt}_{|t=0}(\exp(-tAd_{g^{-1}}u))H) \\ &= -Ad_{g^{-1}}(u) + \mathfrak{h}. \end{aligned}$$

Appendix A.2 Invariant tensor fields

The canonical actions (A.1.1) and (A.1.4) induce an action of G on the space of tensor fields $\mathcal{T}^{p,q}M$,

$$g.\tau := \tilde{\lambda}_g \circ \tau \circ \lambda_{g^{-1}}.$$

Here τ is considered as a section of the vector bundle $T^{p,q}M \to M$, but when it is interpreted as a $C^\infty(M)$-multilinear mappings (cf. A.1.5) the action becomes

$$g.\tau(\omega_1, \cdots, \omega_p, X^1, \cdots, X^q) := \tau(g.\omega_1, \cdots, g.\omega_p, g.X^1, \cdots, g.X^q) \circ \lambda_{g^{-1}}.$$

A tensor field $\tau \in \mathcal{T}^{p,q}M$ will be called G-invariant if $g.\tau = \tau$ for any $g \in G$. We consider the *Lie derivative* of any tensor field τ with respect to the fundamental vector field u^*,

$$(\mathcal{L}_{u^*}\tau)_x := \frac{d}{dt}_{|t=0}[\exp(-tu).\tau]_x.$$

Hence τ is G-invariant if and only if, $\mathcal{L}_{u^*}\tau = 0$ for any $u \in \mathfrak{g}$.

The following classical result, reduces the question of G-invariant tensor fields on M to the problem of H-invariant tensors on the finite-dimensional vector space $\mathfrak{g}/\mathfrak{h}$. More precisely, as explained in [15, Theorem 1.4.4 p.55], we have

Theorem A.2.1. *There is a natural isomorphism between the space $(\mathcal{T}^{p,q}(M))^G$ of G-invariant tensor fields of type (p,q) on M and the vector space $((\mathfrak{g}/\mathfrak{h})^{p,q})^H$ of H-invariant tensors of type (p,q) on $\mathfrak{g}/\mathfrak{h}$.*

Appendix A.3 Invariant Symmetric bivector fields

Let h be a symmetric bivector field on G/H. Then the following conditions are equivalent

1. h is G-invariant, i.e., for any $g \in G$ and $\alpha, \beta \in \Omega^1(G/H)$,

$$h(\lambda_g^*\alpha, \lambda_g^*\beta) = h(\alpha, \beta) \circ \lambda_g.$$

2. For any $g \in G$,
$$h_{\#,\overline{g}} = T_{\overline{e}}(\lambda_g) \circ h_{\#,\overline{e}} \circ (T_{\overline{e}}(\lambda_g))^*.$$

3. If G is connected. For any $u \in \mathfrak{g}$,
$$\mathcal{L}_{u^*} h = 0.$$

On the other hand, the H-invariance of a symmetric bivector $r \in \otimes^2(\mathfrak{g}/\mathfrak{h})$ means that one of the following equivalent conditions is satisfied

1. For any $\varepsilon^i \in (\mathfrak{g}/\mathfrak{h})^*$, $a \in H$,
$$r(\overline{\mathrm{Ad}}_a^*(\varepsilon^1), \overline{\mathrm{Ad}}_a^*(\varepsilon^2)) = r(\varepsilon^1, \varepsilon^2).$$

2. If H is connected, for any $u \in \mathfrak{h}$,
$$r(\overline{ad}_u^*(\varepsilon^1), \varepsilon^2) = -r(\varepsilon^1, \overline{ad}_u^*(\varepsilon^2)). \tag{A.3.1}$$

If r is an H-invariant symmetric bivector on $\mathfrak{g}/\mathfrak{h}$, then its associated G-invariant symmetric bivector field h is defined by
$$h(\alpha, \beta)(\overline{g}) = r(\Phi_g^* \alpha_{\overline{g}}, \Phi_g^* \beta_{\overline{g}}).$$

Appendix A.4 Invariant flat torsionless connections

A connection ∇ on $TM \to M$ is said to be G-invariant if for any $X, Y \in \Gamma(TM)$ and $g \in G$,
$$g \cdot (\nabla_X Y) = \nabla_{g \cdot X} g \cdot Y.$$
This means that for any $g \in G$, the transformation $\lambda_g : M \to M$ preserves ∇. In particular, if ∇ is a G-invariant flat torsionless connection on M, then the canonical action of the Lie group $\mathrm{Aff}(M, \nabla)$ on M is transitive and (M, ∇) is a homogeneous affine manifold [49, 59, 43]

The following theorem is due to Nomizu [41] in the case of reductive pair (G, H) and the connection is not necessarily flat, but in the general context, we refer the reader to [15, 28].

Theorem A.4.1 (Nomizu). *There is a one-to-one correspondence between G-invariant flat torsionless connections on M and Lie algebra representations*
$$L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h}),$$
satisfying

1. *For any $u, v \in \mathfrak{g}$,*
$$L_u(v + \mathfrak{h}) - L_v(u + \mathfrak{h}) = [u, v] + \mathfrak{h}. \tag{A.4.1}$$

2. *For any $u \in \mathfrak{g}, a \in H$,*
$$L_{\mathrm{Ad}_a(u)} = \overline{\mathrm{Ad}}_a \circ L_u \circ \overline{\mathrm{Ad}}_{a^{-1}}.$$

Let's clarify this correspondence. Denote by ∇ a G-invariant flat torsionless connection on M, the corresponding representation L is given by:
$$L_u(v + \mathfrak{h}) = -\Phi_e^{-1}\left(A_{u^*}(\overline{e})(\Phi_e(v + \mathfrak{h}))\right),$$

where $u^* \in \Gamma(TM)$ is the fundamental vector field induced by $\exp(-tu)$, A_{u^*} the Koszul operator given by $A_{u^*} := \mathcal{L}_{u^*} - \nabla_{u^*}$ and Φ_e is the canonical isomorphism A.1.2.

Conversely, the connection ∇ corresponding to L is given by

$$(\nabla_X Y)_{\overline{g}} = \Phi_g \left(\tilde{X}_g . F^Y + L_{g^{-1}.\tilde{X}_g}(F^Y(g)) \right), \tag{A.4.2}$$

where $F^Y : G \to \mathfrak{g}/\mathfrak{h}$ is the H-equivariant function associated to the vector field Y, the expression of Φ_g is given by (A.1.3) and $\tilde{X}_g \in T_g G$ is any vector satisfying $p_*(\tilde{X}_g) = X_{\overline{g}}$. In particular, we have

$$(\nabla_{u^*} v^*)_{\overline{g}} = \Phi_g \left(L_{\mathrm{Ad}_{g^{-1}}(v)}(\mathrm{Ad}_{g^{-1}}(u) + \mathfrak{h}) \right).$$

The dual connection associated to L is given by

$$(\nabla_X \alpha)_{\overline{g}} = (\Phi_g^{-1})^* \left(\tilde{X}_g . F^\alpha + L^*_{g^{-1}.\tilde{X}_g}(F^\alpha(g)) \right), \tag{A.4.3}$$

where $F^\alpha : G \to (g/\mathfrak{h})^*$ is the H-equivariant function associated to the one form α, $\tilde{X}_g \in T_g G$ is any vector satisfying $p_*(\tilde{X}_g) = X_{\overline{g}}$ and L^* be the contragredient representation of L.

In the case of reductive homogeneous space we have the following consequence of Theorem 3.1.7 (see [28]).

Corollary A.4.2. *If (G, H) is a reductive pair with the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $Ad(H)(\mathfrak{m}) = \mathfrak{m}$. Then there is a one-to-one correspondence between G-invariant flat torsionless connections on M and bilinear products*

$$\mathfrak{b} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}, (u, v) \mapsto \mathfrak{b}_u(v) = \mathfrak{b}(u, v)$$

satisfying

1. $\mathfrak{b}(u, v) - \mathfrak{b}(v, u) = [u, v]_{\mathfrak{m}}$, [1]
2. $[\mathfrak{b}_u, \mathfrak{b}_v] = \mathfrak{b}_{[u,v]_{\mathfrak{m}}} + \mathrm{ad}_{[u,v]_{\mathfrak{h}}}$,
3. $\mathrm{Ad}_a(\mathfrak{b}(u, v)) = \mathfrak{b}(\mathrm{Ad}_a(u), \mathrm{Ad}_a(v))$, *for any* $a \in H$.

The connection ∇ corresponding to $\mathfrak{b}$ is determined by

$$(\nabla_{u^*} v^*)_{\overline{g}} = \Phi_g \left([\mathrm{Ad}_{g^{-1}}(v)_{\mathfrak{h}}, \mathrm{Ad}_{g^{-1}}(u)_{\mathfrak{m}}] + \mathfrak{b}(\mathrm{Ad}_{g^{-1}}(v)_{\mathfrak{m}}, \mathrm{Ad}_{g^{-1}}(u)_{\mathfrak{m}}) + \mathfrak{h} \right).$$

This last correspondence allows us to give the following characterization of bi-invariant flat torsionless connections (if they exist) on Lie groups. Denote by u^l the left invariant vector field associated to $u \in \mathfrak{g}$.

Proposition A.4.3. *Let G be a connected Lie group endowed with a connection ∇. Then the following assertions are equivalent*

1. *∇ is a bi-invariant flat torsionless connection on G.*
2. *The product $\mathfrak{b} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $\mathfrak{b}(u, v) := 2\left(\nabla_{u^l} v^l\right)_e - [u, v]$, is an $\mathrm{Ad}(G)$-invariant commutative product on $\mathfrak{g}$ satisfying*

$$\mathfrak{b}(u, \mathfrak{b}(v, w)) - \mathfrak{b}(v, \mathfrak{b}(u, w)) = [[u, v], w]. \tag{A.4.4}$$

[1] We denote by $w_{\mathfrak{h}}$ (resp. $w_{\mathfrak{m}}$) the projection of w on $\mathfrak{h}$ (resp. on $\mathfrak{m}$).

3. *The product* $\mathfrak{b}' : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *given by* $\mathfrak{b}'(u,v) := \left(\nabla_{u^l} v^l\right)_e$*, is an associative product on* $\mathfrak{g}$ *satisfying*
$$\mathfrak{b}'(u,v) - \mathfrak{b}'(v,u) = [u,v].$$

Proof. The equivalence between 1 and 2 is a direct consequence of Corollary 3.7.

Now assume that 2 is satisfied. We pose

$$\mathfrak{b}'(u,v) := \frac{1}{2}\left(\mathfrak{b}(u,v) + [u,v]\right).$$

Obviously,
$$\mathfrak{b}'(u,v) - \mathfrak{b}'(v,u) = [u,v].$$

Let's show that the product $\mathfrak{b}'$ is associative. The $\mathrm{Ad}(G)$-invariance of the product $\mathfrak{b}$ implies that for all $u,v,w \in \mathfrak{g}$, we have

$$\mathfrak{b}([u,v],w) + \mathfrak{b}(v,[u,w]) = [u,\mathfrak{b}(v,w)].$$

Hence

$$\begin{aligned}
\mathfrak{b}'(\mathfrak{b}'(u,v),w) &= \frac{1}{2}\mathfrak{b}'(\mathfrak{b}(u,v) + [u,v], w) \\
&= \frac{1}{4}\left(\mathfrak{b}(\mathfrak{b}(u,v),w) + [\mathfrak{b}(u,v),w] + \mathfrak{b}([u,v],w) + [[u,v],w]\right) \\
&= \frac{1}{4}\left(\mathfrak{b}(w,\mathfrak{b}(u,v)) - \mathfrak{b}([w,u],v) - \mathfrak{b}(u,[w,v]) + \mathfrak{b}([u,v],w) + [[u,v],w]\right).
\end{aligned}$$

A similar computation gives

$$\mathfrak{b}'(u,\mathfrak{b}'(v,w)) = \frac{1}{4}\left(\mathfrak{b}(u,\mathfrak{b}(w,v)) + \mathfrak{b}([u,v],w) + \mathfrak{b}(v,[u,w]) + \mathfrak{b}(u,[v,w]) - [[v,w],u]\right).$$

Hence

$$\mathfrak{b}'(\mathfrak{b}'(u,v),w) - \mathfrak{b}'(u,\mathfrak{b}'(v,w)) = \frac{1}{4}\left([[w,u],v] + [[u,v],w] + [[v,w],u]\right) = 0.$$

Conversely, assume that 3 is satisfied. We pose

$$\mathfrak{b}(u,v) := 2\mathfrak{b}'(u,v) - [u,v].$$

Obviously $\mathfrak{b}(u,v) - \mathfrak{b}(v,u) = 0$. Let's show that the product $\mathfrak{b}$ is $\mathrm{Ad}(G)$-invariant. From the associativity of $\mathfrak{b}'$ and since $\mathfrak{b}'(u,v) - \mathfrak{b}'(v,u) = [u,v]$, it follows that

$$\mathfrak{b}'([u,v],w) + \mathfrak{b}'(v,[u,w]) = [u,\mathfrak{b}'(v,w)].$$

Since G is connected, then for any $g \in G$,

$$\mathfrak{b}'(\mathrm{Ad}_g(u),\mathrm{Ad}_g(v)) = \mathrm{Ad}_g(\mathfrak{b}'(u,v)).$$

Now let's show that $\mathfrak{b}$ satisfies (A.4.4).

$$\begin{aligned}
\mathfrak{b}(u,\mathfrak{b}(v,w)) &= 2\mathfrak{b}(u,\mathfrak{b}'(v,w) - [v,w]) \\
&= 4\mathfrak{b}'(u,\mathfrak{b}'(v,w)) - 2[u,\mathfrak{b}'(v,w)] - 2\mathfrak{b}'(u,[v,w]) + [u,[v,w]] \\
&= 4\mathfrak{b}'(\mathfrak{b}'(u,v),w) - 2\mathfrak{b}'([u,v],w) - 2\mathfrak{b}'(v,[u,w]) - 2\mathfrak{b}'(u,[v,w]) + [u,[v,w]].
\end{aligned}$$

In a similar way, we get

$$\mathfrak{b}(v, \mathfrak{b}(u, w)) = 4\mathfrak{b}'(\mathfrak{b}'(v, u), w) - 2\mathfrak{b}'([v, u], w) - 2\mathfrak{b}'(u, [v, w]) - 2\mathfrak{b}'(v, [u, w]) + [v, [u, w]].$$

Finally, we have

$$\mathfrak{b}(u, \mathfrak{b}(v, w)) - \mathfrak{b}(v, \mathfrak{b}(u, w)) = [u, [v, w]] + [v, [w, u]] = [[u, v], w],$$

which completes the proof. ■

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