Cadi Ayyad University
Faculty of Sciences and Technologies

Bachelor Thesis

The Gauss-Bonnet Theorem

Author:
Rachad Bentbib

Supervisor:
Pr. Abdelhak Abouqateb

Jury members:
Pr. Rachid Belfadli
Pr. Mohamed Boucetta
Pr. Aziz Ikemakhen

Submitted in fulfillment of the requirements
for the degree of Licence Sciences et Techniques MIASI
(Mathématiques et Informatique Appliqués aux Sciences de l'Ingénieur)

in the

Department of Mathematics

20th June 2017

Acknowledgements

All praise is due to Allah, Lord of the worlds. Peace and blessings be upon his Prophet Muhammad, blessings of Allah be upon him and his family and peace.

I would like to thank my parents for everything they did for me, for teaching me morals, for supporting me for my studies, for being on my side during happy moments and during difficult moments.

I express my sincere thanks to my advisor Professor Abdelhak Abouqateb. I was honored to be his student and have him supervise me for the bachelor thesis. He dedicated a lot of time to teach me a lot of things. His relevant remarks were key for the quality of this report.

I sincerely thank the members of the Jury, Professors Mohamed Boucetta, Aziz Ikemakhen and Rachid Belfadli. I'm honored by their presence in the Jury.

I also express my sincere gratitude to all the Professors who helped me and encouraged me during my three years in the Faculty of Sciences and Technologies Gueliz Marrakesh.

Abstract

The Gauss-Bonnet theorem is an important result in differential geometry. It gives a link between the geometry of some regular surfaces, namely the Gaussian curvature, the geodesic curvature of the boundary of the surface and the angles at the vertices of that boundary, and a topologically invariant number named the Euler-Poincaré characteristic of the surface. We will approach the subject in the following way:

1. In Chapter 1, we will present a proof of a theorem called Hopf's Umlaufsatz. It applies to simple closed regular curves in the plane to form a particular case of the Gauss-Bonnet theorem for planar surfaces. We will achieve this after studying notions such as liftings, the rotation index of a closed curve and path homotopies. We will then apply the theorem to find some well known results about the sum of interior angles of a polygon in Euclidean geometry.

2. In Chapter 2, we will study the theory of classical differential geometry on regular surfaces to set up the ground for presenting a proof of a local version of the theorem: as regular surfaces being locally homeomorphic to the plane, we can use Hopf's Umlfausatz locally. We will apply the theorem to generalize the results of Chapter 1 about polygons to regular surfaces. We will also give some insight about the theorem in Riemannian geometry and apply the results to hyperbolic geometry.

3. In Chapter 3, we will globalize the local theorem of Chapter 2 using what is called triangulation. We will then present some of the many applications of this theorem, from studying possible ways to design a soccer ball to proving the fundamental theorem of algebra.

The document was typeset in LaTeX. Figures and animations were designed using Asymptote [12], Convertio [16], Geogebra [25], RecordMyDesktop [32], Adobe Photoshop [38], TikZ [39] and Wolfram Mathematica [44]. The following website contains a PDF version of this document as well as the animations accompanying it:

`http://www.tinyurl.com/BATHGB`

Please don't hesitate to contact me for any remark or error.

My e-mail: bentbibrachad@gmail.com
My advisor's e-mail: a.abouqateb@uca.ac.ma

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To my parents Naila and Abdeslem Hafid,
my brother Yassine and Karima.

Chapter 1

Hopf's Umlaufsatz[1]

In this chapter, we present the proof of Hopf's Umlaufsatz, a particular case of the Gauss-Bonnet Theorem that will be, however, used to prove the latter.

1.1 Motivation

Consider a triangle ABC as shown below:

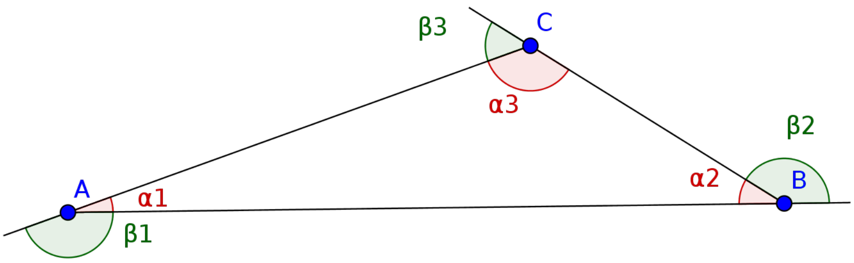


FIGURE 1.1: A triangle with its interior and exterior angles

In red, the angles α_1, α_2 and α_3 are the interior angles of the triangle and in green, the angles β_1, β_2 and β_3 are the exterior angles of the triangle, defined by

$$\begin{aligned}\beta_1 &= \pi - \alpha_1\\ \beta_2 &= \pi - \alpha_2\\ \beta_3 &= \pi - \alpha_3.\end{aligned}$$

Recall that the sum $\sum_{i=1}^{3} \alpha_i$ of the interior angles of a triangle is π. Summing the three equations above, we get

$$\begin{aligned}\sum_{i=1}^{3} \beta_i &= 3\pi - \sum_{i=1}^{3} \alpha_i\\ &= 3\pi - \pi\\ &= 2\pi.\end{aligned}$$

[1]From the German words *Umlauf* (circulation) and *Satz* (theorem). Mentionned in other books as *The Turning Tangents Theorem* or *The Rotation Angle Theorem*.

A more general result is that the sum of the oriented interior angles of any simple n-sided polygon[2] is $\pm(n-2)\pi$. There is a proof that relies on the fact that any such polygon can be partitioned into triangles which vertices are the polygone's vertices and which edges do not intersect each other unless they're common between two triangles (such a partitionning is called a triangulation of a polygon), and that the number of these triangles is always $n-2$ [17, p. 47].

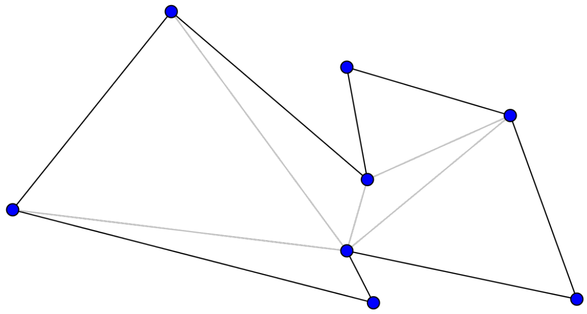


FIGURE 1.2: Triangulation of a simple polygon

Thus the sum of the oriented interior angles of the polygon is the sum of the oriented interior angles of all the $n-2$ triangles, which is $\pm(n-2)\pi$. As in above, let α_i and β_i denote, respectively, the interior and exterior angles of a simple n-gon. Then

$$\begin{aligned}\sum_{i=1}^{n} \beta_i &= \sum_{i=1}^{n} \pm(\pi - \alpha_i) \\ &= \pm n\pi \mp \sum_{i=1}^{n} \alpha_i \\ &= \pm n\pi \mp (n-2)\pi \\ &= \pm 2\pi.\end{aligned} \tag{1.1}$$

This proves the theorem below:

Theorem 1.1. *The sum of the oriented exterior angles of any simple polygon is* $\pm 2\pi$*.*

See `2.1.gif` for an animation [34].

The exterior angle at a vertex of a simple polygon are in fact the angle difference between the left and right tangent lines to the curve at that point. Imagine a vehicle V moving on the polygon in a counter-clockwise way. Keep track of the angle θ between the tangent lines at his different positions and the tangent line he had at his starting point V_0. Whenever the vehicle changes direction at a vertex, the exterior angle at this point adds up to θ. When the point reaches back V_0, θ is 2π.

[2] A polygon which edges intersect only at vertices.

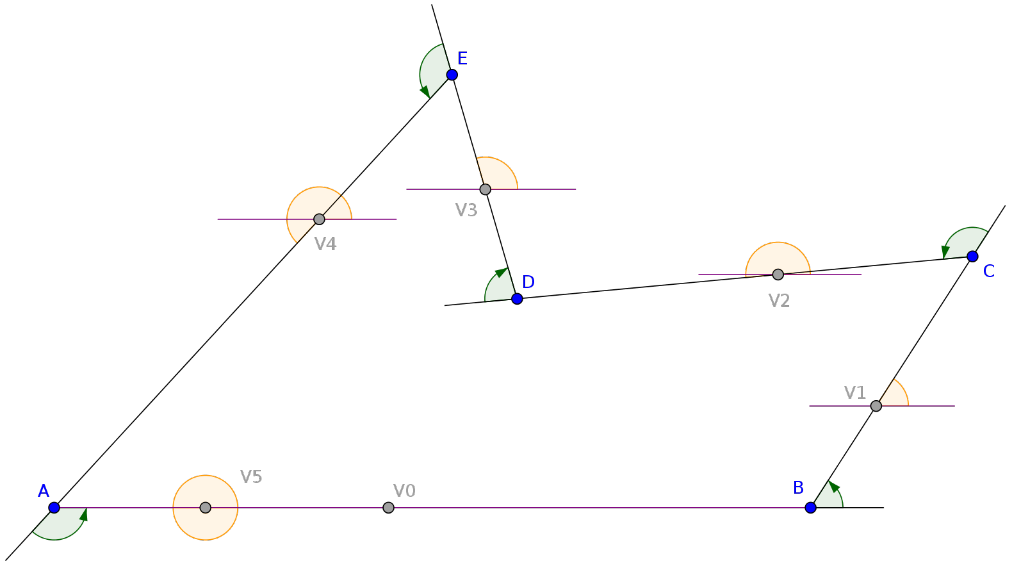


FIGURE 1.3: Another way to perceive the result

What happens to this theorem when we bend the edges of a simple polygon? Before answering this question, we recall the following:

Definition 1.2.

- A ***plane curve*** is a continuous map $\gamma : I \to \mathbb{R}^2$ where I is an interval of $\mathbb{R}$.

- A ***regular curve*** of class C^k is a plane curve of class C^k such that $\forall t \in I, \gamma'(t) \neq 0$. If the curve is injective, we say that it is ***simple***. If $I = [a, b]$ is a closed bounded interval and $\forall n \in \{0, \dots, k\}, \gamma^{(n)}(a) = \gamma^{(n)}(b)$, the curve is said to be ***closed***. If $I = [a, b]$ and the curve is closed and injective on $[a, b)$, it is said to be a ***simple closed curve***. If the class of the regular curve isn't stated, it shall be considered of class C^∞.

- A ***polygonal curve*** of class C^k is a closed piecewise regular curve that has nonzero derivative at the left and the right of the points $t_1, \dots, t_n$ of I where it is not differentiable. The points $\gamma(t_1), \dots, \gamma(t_n)$ are called ***vertices*** of γ, and for any $i \in \{1, \dots, n-1\}$, $\gamma\,|_{[t_i, t_{i+1}]}$ is called an ***edge*** of γ.

- A regular curve γ of class C^k is said to be an ***arc length parametrization*** if
$\forall t \in I, \|\gamma'(t)\| = 1$.

- For an arc length parametrization γ, let $T = \gamma'$. $T(t)$ is the ***tangent vector*** to γ at $\gamma(t)$. The composition of T with the rotation of angle $\dfrac{\pi}{2}$ is noted N. $N(t)$ is called the ***normal vector*** to γ at $\gamma(t)$. The map $k = \langle T', N \rangle$ is called the ***curvature*** of γ.

Definition 1.3. Let $\gamma : I = [a, b] \to \mathbb{R}^2$ and $\tilde{\gamma} : \tilde{I} = [c, d] \to \mathbb{R}^2$ be parametrized curves of class C^k. We say that $\tilde{\gamma}$ is a ***reparametrization*** of γ if there exists a C^k diffeomorphism $\varphi : [c, d] \to [a, b]$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{I} \arrow[r, "\varphi"] \arrow[dr, "\tilde{\gamma}"'] & I \arrow[d, "\gamma"] \\ & \mathbb{R}^2 \end{tikzcd}$$

If furthermore φ is strictly increasing, then $\tilde{\gamma}$ is said to be an ***orientation preserving reparametrization***.

Definition 1.4.

- The ***argument*** of a unit vector $v = (v_x, v_y)$ is

$$\arg v = \begin{cases} 2\arctan\left(\dfrac{v_y}{1+v_x}\right), & v_x \neq -1; \\ \pi, & v_x = -1. \end{cases}$$

  For a nonunit vector, we set $\arg v = \arg \dfrac{v}{\|v\|}$.

- The ***angle*** or ***directed angle*** from a vector $u = (u_x, u_y)$ to a vector $v = (v_x, v_y)$ is

$$\angle(u, v) = \arg\left(\frac{v_x + iv_y}{u_x + iu_y}\right).$$

- Let $\gamma : I \to \mathbb{R}^2$ be a polygonal curve of class C^k with vertices $\gamma(t_1), \dots, \gamma(t_n)$. Let $i \in \{1, \dots, n\}$. The ***interior angle*** of γ at $\gamma(t_i)$ is $\angle(T(t_i^-), T(t_i^+))$ if it is different than $\pm\pi$. If its value is $\pm\pi$, $\gamma(t_i)$ is called a ***cusp*** of γ. The exterior angle is $\pm\pi$, the sign being the sign of $\lim\limits_{\varepsilon\to 0^+} \dfrac{\angle(T(t_i-\varepsilon), T(t_i+\varepsilon))}{|\angle(T(t_i-\varepsilon), T(t_i+\varepsilon))|}$. If β_i is the exterior angle of γ at $\gamma(t_i)$, the ***interior angle*** of γ at $\gamma(t_i)$ is $\alpha_i = \pi - \beta_i$.

Let us answer now the previous question. For illustration, let's bend the edge BC of the triangle in figure 1.1 to make it an arc of a circle.

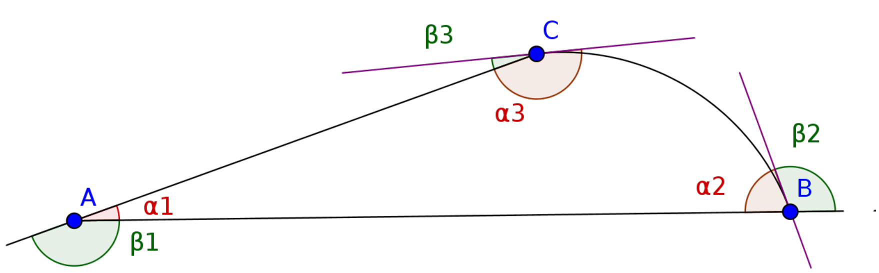


FIGURE 1.4: Triangle in 1.1 with a curved edge

In the figure above, the purple lines are the tangents of the circle arc at B and C. When we compare this figure with figure 1.1, we see that the interior angles α_2 and α_3 became larger, and thus the exterior angles β_2 and β_3 became smaller. Hence the sum of the exterior angles in this case is smaller than 2π, i.e,

$$\sum_{i=1}^{3} \beta_i < 2\pi.$$

Therefore there is a positive number x such that

$$x + \sum_{i=1}^{3} \beta_i = 2\pi.$$

In order to find out what does x represent, consider as previously a vehicle on this curve. The angle θ doesn't only vary at the vertices of this curve, but also on the bended edge. The animation `2.2.gif` suggests that x is the difference between the angle $\theta(t_2^-)$ of the tangent vector at C from the left and the angle $\theta(t_1^+)$ of the tangent vector at B. Since θ is differentiable on $[t_1, t_2]$, we infer that:

$$\begin{aligned} 2\pi &= \theta(t_2^-) - \theta(t_1^+) + \sum_{i=1}^{3} \beta_i \\ &= \int_{t_1}^{t_2} \theta'(t)\,\mathrm{d}t + \sum_{i=1}^{3} \beta_i. \end{aligned}$$

Notice that since θ is constant on the edges AB and AC, θ' vanishes on $[t_0, t_1]$ and $[t_2, t_3]$. Thus the equation above becomes

$$\int_a^b \theta'(t)\,\mathrm{d}t + \sum_{i=1}^{3} \beta_i = 2\pi. \tag{1.2}$$

Another thing that the animation `2.2.gif` suggests is that θ' is related to how the segment bends, namely its curvature.

Proposition 1.5. Let $\gamma : I \to \mathbb{R}^2$ be an arc length curve and θ be a smooth function such that $\forall t \in I, T(t) = (\cos\theta(t), \sin\theta(t))$. Then $k = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$.

Then equation (1.2) becomes:

$$\int_a^b k(t)\,\mathrm{d}t + \sum_{i=1}^{3} \beta_i = 2\pi.$$

The German mathematician Heinz Hopf generalized the equation above to any polygonal curve ([24],1935). This theorem is called Hopf's Umlaufsatz.

FIGURE 1.5: Heinz Hopf (1894 - 1971) [1]

Theorem 1.6 (Hopf's Umlaufsatz)**.** *Let $\gamma : [a, b] \to \mathbb{R}^2$ be an arc length polygonal curve, k its curvature, n the number of its vertices and θ_i the exterior angle of γ at its ith vertex, where $i \in \{1, \cdots, n\}$. Then $I(\gamma) = \pm 1$ and*

$$\int_a^b k(t)\,\mathrm{d}t + \sum_{i=1}^{n} \theta_i = 2\pi I(\gamma).$$

(see Definition 1.22 for the meaning of $I(\gamma)$)
If the curve is of class C^1 but not C^2, we still have the following:

Theorem 1.7 (Hopf's Umlaufsatz)**.** *Let $\gamma : [a, b] \to \mathbb{R}^2$ be a polygonal curve of class C^1. Then $I(\gamma) = \pm 1$.*

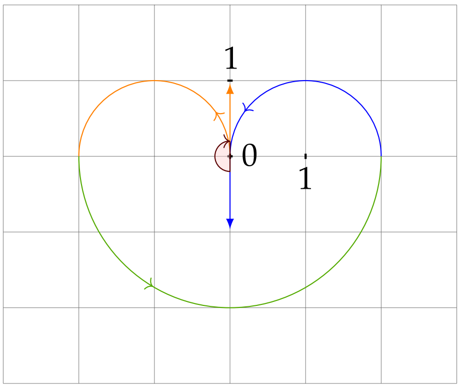
We will give Hopf's proof of this theorem in section 1.3.
Here are examples that illustrate the theorem.

Example 1.8. Consider the parametrized curve

$$\gamma : [0, 4\pi] \to \mathbb{R}^2$$
$$t \mapsto \begin{cases} (1,0) + (\cos t, \sin t)\,, & t \in [0, \pi]; \\ (-1,0) + (\cos(t-\pi), \sin(t-\pi))\,, & t \in [\pi, 2\pi]; \\ 2\left(\cos\dfrac{t}{2}, \sin\dfrac{t}{2}\right), & t \in [2\pi, 4\pi]. \end{cases}$$

γ is parametrized by arc length. Its curvature is given by:

$$k : [0, 4\pi] \to \mathbb{R}$$
$$t \mapsto \begin{cases} 1, & t \in [0, \pi] \cup [\pi, 2\pi]; \\ \dfrac{1}{2}, & t \in [2\pi, 4\pi]. \end{cases}$$

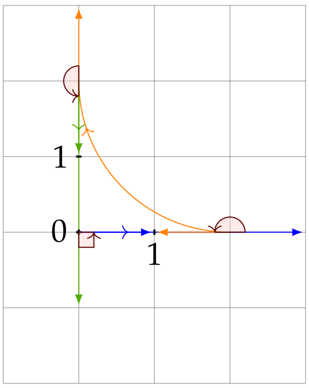


The curve has one vertex: the point $(0,0)$. It is in fact a cusp, and the exterior angle at it is $-\pi$, as represented in red in the figure above. We have

$$\begin{aligned}\int_0^{4\pi} k(t)\,\mathrm{d}t - \pi &= \int_0^{2\pi} \mathrm{d}t + \int_{2\pi}^{4\pi} \frac{1}{2}\,\mathrm{d}t - \pi \\ &= 2\pi - \pi + \pi \\ &= 2\pi\end{aligned}$$

Example 1.9. Consider the parametrized curve

$$\begin{aligned}\gamma : [0, 4+\pi] &\to \mathbb{R}^2 \\ t &\mapsto \begin{cases} (t,0), & t \in [0,2]; \\ (2,2) + 2\left(\cos\left(-\dfrac{t-2+\pi}{2}\right), \sin\left(-\dfrac{t-2+\pi}{2}\right)\right), & t \in [2, 2+\pi]; \\ (0, -t+4+\pi), & t \in [2+\pi, 4+\pi]. \end{cases}\end{aligned}$$



γ is parametrized by arc length. Its curvature is given by:

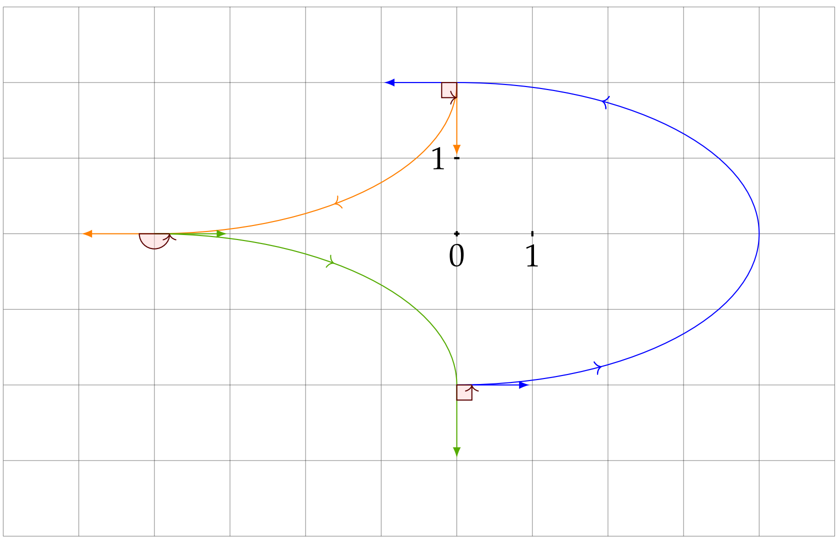
$$\begin{aligned} k : [0, 4+\pi] &\to \mathbb{R} \\ t &\mapsto \begin{cases} 0, & t \in [0,2] \cup [2+\pi, 4+\pi]; \\ -\dfrac{1}{2}, & t \in [2, 2+\pi]. \end{cases} \end{aligned}$$

The curve has three vertices: two cups at which the exterior angle is π, and the point $(0,0)$ at which the exterior angle is $\dfrac{\pi}{2}$. We have:

$$\begin{aligned} \int_0^{4+\pi} k(t)\,\mathrm{d}t + \pi + \pi + \frac{\pi}{2} &= \int_2^{2+\pi} -\frac{1}{2}\,\mathrm{d}t + \frac{5\pi}{2} \\ &= -\frac{\pi}{2} + \frac{5\pi}{2} \\ &= 2\pi. \end{aligned}$$

Example 1.10. Consider the parametrized curve

$$\begin{aligned} \gamma : \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] &\to \mathbb{R}^2 \\ t &\mapsto \begin{cases} (4\cos t, 2\sin t), & t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; \\ (-4, 2) + \left(4\cos\left(-t+\frac{\pi}{2}\right), 2\sin\left(-t+\frac{\pi}{2}\right)\right), & t \in \left[\frac{\pi}{2}, \pi\right]; \\ (-4,-2) + \left(4\cos\left(-t+\frac{3\pi}{2}\right), 2\sin\left(-t+\frac{3\pi}{2}\right)\right), & t \in \left[\pi, \frac{3\pi}{2}\right]. \end{cases} \end{aligned}$$



We get:

$$
k : \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}
$$

$$
t \mapsto \begin{cases} \dfrac{1}{\sqrt{1+3\sin^2 t}^3}, & t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]; \\ -\dfrac{1}{\sqrt{1+3\sin^2\left(-t+\dfrac{\pi}{2}\right)}^3}, & t \in \left[\dfrac{\pi}{2}, \pi\right]; \\ -\dfrac{1}{\sqrt{1+3\sin^2\left(-t+\dfrac{3\pi}{2}\right)}^3}, & t \in \left[\pi, \dfrac{3\pi}{2}\right]. \end{cases}
$$

Since $k\,|_{\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]}$ is even, we have

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k(t)\,\mathrm{d}t = 2\int_{0}^{\frac{\pi}{2}} k(t)\,\mathrm{d}t. \tag{1.3}
$$

Using the change of variable $u = t - \dfrac{\pi}{2}$, we get

$$
\int_{\frac{\pi}{2}}^{\pi} k(t)\,\mathrm{d}t = -\int_{0}^{\frac{\pi}{2}} k(u)\,\mathrm{d}u. \tag{1.4}
$$

Using the change of variable $u = -t + \dfrac{3\pi}{2}$, we get

$$
\int_{\pi}^{\frac{3\pi}{2}} k(t)\,\mathrm{d}t = -\int_{0}^{\frac{\pi}{2}} k(u)\,\mathrm{d}u. \tag{1.5}
$$

From (1.3), (1.4) and (1.5), we conclude that

$$
\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} k(t)\,\mathrm{d}t = 0.
$$

There are three vertices, at which the exterior angles are $\dfrac{\pi}{2}$, π and $\dfrac{\pi}{2}$. Hence:

$$
\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} k(t)\,\mathrm{d}t + 2\cdot\frac{\pi}{2} + \pi = 2\pi.
$$

1.2 Lifting, Degree and Homotopy

In this section, we introduce some mathematical notions that will be used to prove Hopf's Umlaufsatz.

Recall that any regular curve has an arc length reparametrization. In this case, we consider that γ' has the unit circle S^1 as codomain. Since we need to work with angles, it is important for us that for any $t \in [a,b]$, there exists $\theta \in \mathbb{R}$ such that $\gamma'(t) = (\cos\theta, \sin\theta)$. According to the axiom of choice, there exists a function $\theta : [a,b] \to \mathbb{R}$ such that

$$\forall t \in [a,b], \gamma'(t) = (\cos\theta(t), \sin\theta(t)) \,. \tag{1.6}$$

Axiom of Choice. *For any relation R, there exists a function f such that $f \subset R$ and dom f = dom R.*

Letting $p : \mathbb{R} \to S^1$ defined by $p(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$, (1.6) simply becomes $\gamma' = p \circ \theta$. The function θ is not unique by the periodicity of $\cos$ and $\sin$. However, γ' and p being continuous, one might wonder if it is possible to find a continuous function θ that satisfies (1.6). One might be tempted to consider a continuous right inverse q of p and then take $\theta = q \circ \gamma'$ so that $p \circ \theta = (p \circ q) \circ \gamma' = \gamma'$. Unfortunately, this is not the case:

Proposition 1.11. There is no continuous function $q : S^1 \to \mathbb{R}$ such that $p \circ q = id_{S^1}$.

Proof. Suppose by way of contradiction that such a function q exists. As q has a left inverse, it is injective. Let $I = q(S^1)$. Since q is continuous and S^1 is connected, so is I. Thus I is an interval in $\mathbb{R}$. Furthermore, if we change the codomain of q to I, q becomes a continuous bijection between S^1 and I, which inverse, $p\,|_I$, is continuous . Hence q is a homeomorphism between I and S^1. S^1 being compact, so is I. Hence $I = [a,b]$ for some $a, b \in \mathbb{R}$.

Now let $c = \dfrac{a+b}{2}$ and $d = p(c)$. Consider $\tilde{q} = q\,|_{S^1 \setminus \{d\}}$ and with $[a,b] \setminus \{c\}$ as a codomain. Then $\tilde{q}$ is a homeomorphism between $S^1 \setminus d$, a connected space, and $[a,c) \cup (c,b]$, which is not connected. This is a contradiction as connectedness is a topological property. $\square$

The proposition above is however true when restrecting the domain. Using this, it turns out that a continuous function θ that satisfies (1.6) does exist as we shall prove in Theorem 1.17. This leads to the following definitions:

Definition 1.12. Let $p : X \to Y$ be a continuous surjective function. An open set U of X is said to be ***evenly covered*** by p if $p^{-1}(U)$ can be written as the union of a collection $\mathcal{S}$ of disjoint open sets such that for any $V \in \mathcal{S}$, $p\,|_V$ is a homeomorphism between V and U. Such a collection $\mathcal{S}$ is called a partition of $p^{-1}(U)$ into ***slices***.

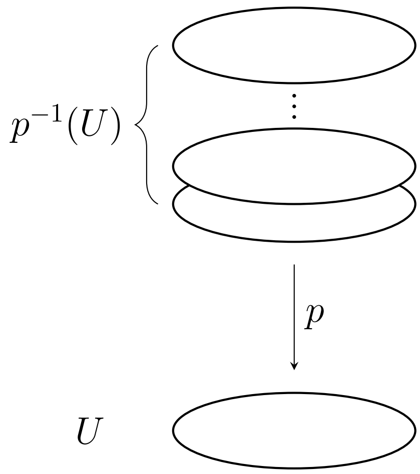


FIGURE 1.6: [45]

Definition 1.13. Let $p : X \to Y$ be a continuous surjective function. If every element $y \in Y$ has a neighborhood evenly covered by p, then p is called a ***covering map***, and X is said to be a ***covering space*** of Y.

In the definition above, the fact that every element in Y has a neighborhood evenly covered by p can be simply reformulated as follows: Y has an open covering consisting of evenly covered sets by p.

Proposition 1.14. The function

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ t &\mapsto (\cos t, \sin t) \end{aligned}$$

is a covering map.

Proof. Consider the following open sets of S^1:

$$\begin{aligned} U_1 &= p\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right) \\ U_2 &= p\left(\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[\right) \\ U_3 &= p\left(]0, \pi[\right) \\ U_4 &= p\left(]\pi, 2\pi[\right). \end{aligned}$$

It is easy to see that $\mathcal{C} = \{U_1, U_2, U_3, U_4\}$ is an open covering of S^1. Let's prove that its elements are evenly covered by p.

For all $n \in \mathbb{Z}$, let:

$$
\begin{aligned}
A_n &= \left]-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right[\\
B_n &= \left]\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right[\\
C_n &= \left]2\pi n, \pi + 2\pi n\right[\\
D_n &= \left]\pi + 2\pi n, 2\pi + 2\pi n\right[.
\end{aligned}
$$

We have:

$$
\begin{aligned}
p^{-1}(U_1) &= \bigcup_{n\in\mathbb{Z}} A_n \\
p^{-1}(U_2) &= \bigcup_{n\in\mathbb{Z}} B_n \\
p^{-1}(U_3) &= \bigcup_{n\in\mathbb{Z}} C_n \\
p^{-1}(U_4) &= \bigcup_{n\in\mathbb{Z}} D_n.
\end{aligned}
$$

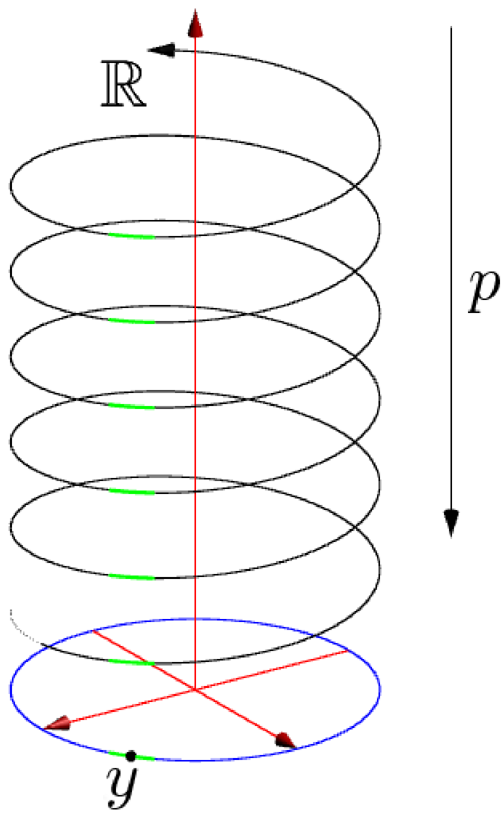


FIGURE 1.7: A covering map of S^1 [7]

We will show that U_1 is evenly covered by p. The proof is similar for the other sets. We have for any $n \in \mathbb{N}$, A_n is an open set, and for any $m \in \mathbb{N}$ such that $m \neq n$, $A_n \cap A_m = \emptyset$. Indeed, A_0 is an interval of length $\pi < 2\pi$ and for all $n \in \mathbb{N}$, A_n is the translation of A_0 by $2\pi n$. Since p is 2π-periodic, it is clear that if $p\,|_{A_0}$ is a homeomorphism between A_0 and U_1, then so will be $p\,|_{A_n}$ for any $n \in \mathbb{N}^*$.

Let

$$\begin{aligned} f : \overline{A_0} &\to \overline{U_1} \\ t &\mapsto p(t). \end{aligned}$$

f is clearly a bijective continuous function. Since $\overline{A_0}$ is compact and $\overline{U_1}$ is a Hausdorff space[3], f is a homeomorphism. Since $f(A_0) = U_1$, it clearly follows that $p\,|_{A_0}$ is a homeomorphism between A_0 and U_1. □

We're now ready for the proof of theorem 1.17. We start by proving the following lemma:

Lemma 1.15 (The Lebesgue number lemma). *Let (X, d) be a compact metric space and let $\mathcal{C}$ be an open covering of X. Then there exists a real number $\delta > 0$ such that every subset of X of diameter[4] less than δ is included in some element of $\mathcal{C}$.*

*The number δ is called a **Lebesgue number** for the covering $\mathcal{C}$.*

Proof. Suppose by way of contradiction that such a number doesn't exist. Then for any $n \in \mathbb{N}^*$, $\dfrac{1}{n}$ is not a Lebesgue number, i.e, there exists a subset S_n of X such that

$$\operatorname{diam} S_n < \frac{1}{n} \text{ and } \forall O \in \mathcal{C}, S_n \not\subset O. \tag{1.7}$$

By the axiom of choice, there is a sequence $(x_n)_{n\in\mathbb{N}^*}$ such that

$$\forall n \in \mathbb{N}^*, x_n \in S_n.$$

Since (X, d) is compact, there is a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ of $(x_n)_{n\in\mathbb{N}^*}$ that converges. Call x its limit. Since $\mathcal{C}$ is an open covering of X, there exists $O \in \mathcal{C}$ that is a neighborhood[5] of x. Thus there exists $\delta > 0$ such that $B(x, \delta) \subset O$. Since $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ converges to x, we have

$$\exists N \in \mathbb{N}^*, \forall n \geq N, x_{\varphi(n)} \in B\left(x, \frac{\delta}{2}\right).$$

[3] A topological space (X, τ) such that any two distinct elements x and y of X have disjoint neighborhoods is called a Hausdorff space.

[4] In a metric space (X, d), the diameter of a subset S of X is: $\operatorname{diam} S = \sup_{x,y\in S} d(x, y)$

[5] A neighborhood of a point is an open set containing it. We recall this for clarification, as some authors don't require the set to be open.

Let $m \geq N$ such that $\varphi(m) > \dfrac{2}{\delta}$. We claim that $S_{\varphi(m)} \subset B(x, \delta)$. Indeed, let $s \in S_{\varphi(m)}$. Since $\operatorname{diam} S_{\phi(m)} < \dfrac{1}{\varphi(m)}$, we have $d(s, x_{\varphi(m)}) < \dfrac{1}{\varphi(m)}$. Thus

$$\begin{aligned} d(s,x) &\leq d(s, x_{\varphi(m)}) + d(x_{\varphi(m)}, x) \\ &\leq \frac{1}{\varphi(m)} + \frac{\delta}{2} \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} \\ &\leq \delta. \end{aligned}$$

Hence $S_{\varphi(m)} \subset B(x, \delta) \subset O$, contradicting (1.7). We conclude that such a number exists. $\square$

Definition 1.16. Let X, Y and Z be topological spaces and $f : X \to Y$ and $p : Z \to Y$ be a continuous functions. We call a ***lifting*** of f accross p a continuous function $\tilde{f} : X \to Z$ such that $f = p \circ \tilde{f}$.

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Theorem 1.17. *Let Y and Z be topological spaces, $p : Z \to Y$ be a covering map, $y_0 \in Y$ and $z_0 \in Z$ such that $p(z_0) = y_0$. Let $I = [a, b]$ be a compact interval of $\mathbb{R}$. Then any continuous function $f : I \to Y$ with $f(a) = y_0$ has a unique lifting $\tilde{f}$ accross p such that $\tilde{f}(a) = z_0$.*

Proof.

- **Existence.**
  Let $\mathcal{C}$ be a covering of Y with sets evenly covered by p (which exists since p is a covering map). Let $\mathcal{A} = \{f^{-1}(U) \mid U \in \mathcal{C}\}$. Since f is continuous, $\mathcal{A}$ is an open covering of I. Since I is a compact metric space, $\mathcal{A}$ has a Lebesgue number δ. Let $n = \left\lfloor \dfrac{b-a}{\delta} \right\rfloor + 1$ and for all $k \in \{0, \ldots, n-1\}$, let $I_k = \left[a + k\,\dfrac{b-a}{n}, a + (k+1)\dfrac{b-a}{n}\right]$. These intervals have diameter less than δ. Thus

  $$\forall k \in \{0, \ldots, n-1\}, \exists U \in \mathcal{C}, I_k \subset f^{-1}(U).$$

  Since for any $U \in \mathcal{C}$, $f(f^{-1}(U)) \subset U$, the statement above becomes

  $$\forall k \in \{0, \ldots, n-1\}, \exists U \in \mathcal{C}, f(I_k) \subset U.$$

  For simplicity, let $s_0, \ldots, s_n$ be such that

  $$\forall k \in \{0, \ldots, n-1\}, I_k = [s_k, s_{k+1}].$$

The lifting $\tilde{f}$ will be constructed on these intervals recursively.

Firstly, we let $\tilde{f}(a) = z_0$. Now let $k \in \{0, n-1\}$ and suppose that $\tilde{f}$ is already defined on $[s_0, s_k]$. We have $\exists U \in \mathcal{C}, f(I_k) \subset U$. Since U is evenly covered by p, there exists a collection $\mathcal{S}$ of disjoint open sets in Z which union is $p^{-1}(U)$ and such that for any $V \in \mathcal{S}$, $p\,|_V$ is a homeomorphism between V and U. Since $f(s_k) = p(\tilde{f}(s_k)) \in U$, we have $\tilde{f}(s_k) \in p^{-1}(U) = \cup\mathcal{S}$. Thus there is a unique $V_k \in \mathcal{S}$ that contains $\tilde{f}(s_k)$. Then we define $\tilde{f}$ on I_k by:

$$\forall s \in I_k, \tilde{f}(s) = (p\,|_{V_k})^{-1}\,(f(s)).$$

Since f is continuous and $f\,|_{V_k}$ is a homeomorphism, $\tilde{f}\,|_{I_k}$ is continuous.

It clearly follows from above that $\tilde{f}$ is continuous on I and that $f = p \circ \tilde{f}$.

- **Uniqueness.**
  Suppose that g and h are two liftings of f accross p such that $g(a) = h(a) = z_0$. Let $A = \{t \in [a,b] \mid g(t) = h(t)\}$. As $A \subset I$ and $A \neq \emptyset$ because $a \in A$, and since I is connected, it's enaugh to show that A is clopen[6] to conclude that $A = I$, and this shows that $g = h$.

  - A is open: Let $t \in A$ and $x = g(t) = h(t)$. Since p is a covering map, there exists an open set U of Y that contains $p(x)$ such that $p^{-1}(U)$ is the union of a collection $\mathcal{S}$ of disjoint open sets such that p restricted on every element of $\mathcal{S}$ is a homeomorphism between it and U. As $x \in p^{-1}(U)$, we have $\exists V \in \mathcal{S}, x \in V$. Since g and h are continous and $g(t) = h(t)$, there exists an open interval J such that $g(J), h(J) \subset V$. Since $f = p \circ g = p \circ h$ and p is a homeomorphism between V and U, we get $g\,|_J = h\,|_J$. Hence $J \subset A$. Therefore A is open.
  - A is closed: g and h are continuous, $\{0\}$ is a closed set of $\mathbb{R}$ and $A = (g-h)^{-1}(\{0\})$. Therefore A is closed.

$\square$

Corollary 1.18. Let $f : [a,b] \to S^1$ be a continuous map and let p be the map from proposition 1.14. Then there are functions $\tilde{f} : [a,b] \to \mathbb{R}$ such that $f = p \circ \tilde{f}$. Furthermore, for any two such functions g and h, there is a unique $k \in \mathbb{Z}$ such that

$$\forall x \in [a,b], g(x) - h(x) = 2\pi k.$$

Proof. The existence of such functions clearly follows from the previous theorem. Now let $g, h : [a,b] \to \mathbb{R}$ such that $f = p \circ g = p \circ h$. We have $\forall x \in [a,b], \dfrac{g(x) - h(x)}{2\pi} \in \mathbb{Z}$ because $p(g(x)) = p(h(x))$. Since the function $x \mapsto \dfrac{g(x) - h(x)}{2\pi}$ is continuous and its domain, $[a,b]$, is connected, so is its

[6] A clopen set is a set that is both open and closed.

image. But since its image is a subset of $\mathbb{Z}$ which is totally disconnected, the image is a singleton $\{k\}$. Hence

$$\forall x \in [a, b], g(x) - h(x) = 2\pi k.$$

□

Definition 1.19. Let $f : I \to S^1$ be a continuous function. A continuous map $\theta : I \to \mathbb{R}$ that satisfies $f = p \circ \theta$ is called an ***angle function*** of f.

For functions that are of class C^k with $k \geq 2$, we can construct liftings of class C^k:

Proposition 1.20. Let $f : I \to S^1$ be a function of class C^k with $k \geq 2$, f_1 and f_2 its components, $t_0 \in I$ and $\theta_0 \in \mathbb{R}$ such that $f(t_0) = p(\theta_0)$. Then

$$\begin{aligned} \theta : I &\to \mathbb{R} \\ t &\mapsto \theta_0 + \int_{t_0}^{t} (f_1 f_2' - f_1' f_2)(t)\,\mathrm{d}t \end{aligned}$$

is a lifting of f of class C^k.

In the previous section, we motived Hopf's Umlaufsatz using the angle variation of the tangent lines of a simple closed curve $\gamma : [a, b] \to \mathbb{R}^2$, which are directed by γ'. If we restrict our attention for the moment to arc length curves γ, there are no vertices in their trace. Using Hopf's Umlaufsatz, we can say that the difference in angle between $\gamma'(a)$ and $\gamma'(b)$ is $\pm 2\pi$, i.e, the tangent line performed a complete turn when moving along the curve. It can be more if the curve is not simple, as shown in figure below.

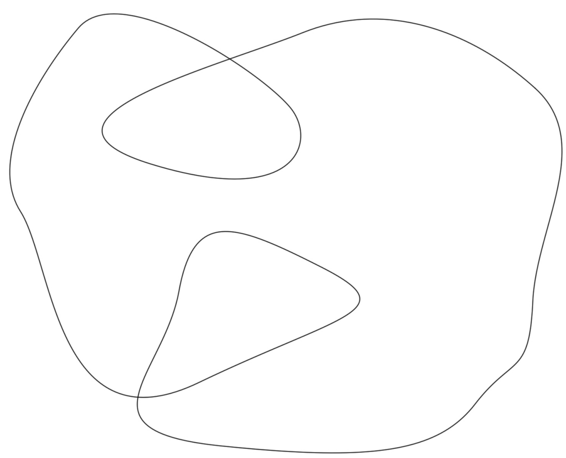


FIGURE 1.8: A non simple closed curve

This motivates the following definitions:

Definition 1.21. Let θ be an angle function of a continuous map $f : [a, b] \to S^1$. If $f(a) = f(b)$, f is called a ***closed path*** on S^1. In this case, the ***degree*** of f is the integer $\deg f = \dfrac{\theta(b) - \theta(a)}{2\pi}$.

The fact that the degree of closed path doesn't depend on the chosen angle function follows by Corollary 1.18.

Definition 1.22. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed arc length curve. An angle function of T is called a ***tangent angle function*** of γ. The degree of T is called the ***rotation index*** of γ and is noted $I(\gamma)$. If $I(\gamma)$ is positive, we say that γ is ***positively oriented***.

For arc length polygonal curves, the tangent vector is piecewise continuous. We need to apply Definition 1.22 to such curves in such a way that the rotation index still matches our intuition. For this purpose, it is enough to define the angle function of the tangent vector in such a case.

Definition 1.23. Let $\gamma : I = [a,b] \to \mathbb{R}^2$ be an arc length polygonal curve, $t_1, \ldots, t_n$ the points of I where γ is not differentiable and $\theta_1, \ldots, \theta_n$ be the exterior angles at, respectively, $\gamma(t_1), \ldots, \gamma(t_n)$. We can assume without loss of generality that $t_1 \neq a$ and $t_n \neq b$. We define an ***angle function*** θ of T recursively: firstly, we let $\theta\,|_{[a,t_1)}$ be any angle function of $T\,|_{[a,t_1)}$. Set $t_0 = a$ and $t_{n+1} = b$. Then for any $i \in \{0, \ldots, n-1\}$, we let $\theta\,|_{[t_{i+1},t_{i+2})}$ be the angle function of $\theta\,|_{[t_{i+1},t_{i+2})}$ that satisfies $\theta(t_{i+1}) = \theta_{i+1} + \theta(t_{i+1}^-)$. Finally, we let $\theta(b) = \theta(b^-)$.

We now introduce the notion of homotopy. It will be crucial for the proof of Hopf's Umlaufsatz.

Definition 1.24. Let X and Y be topological spaces and f and g two continuous functions from X into Y. A ***homotopy*** between f and g is a continuous map $h : X \times [0,1] \to Y$ such that $\forall x \in X, h(x,0) = f(x) \wedge h(x,1) = g(x)$. When such a map exists, f and g are said to be ***homotopic***.

Definition 1.25. Two paths[7] $f, g : I = [a,b] \to X$ are said to be ***path homotopic*** if $x_0 = f(a) = g(a)$ and $x_1 = f(b) = g(b)$ and there exists a homotopy $h : I \times [0,1] \to X$ such that $\forall t \in [0,1], h(a,t) = x_0$ and $h(b,t) = x_1$. In this case, h is called a ***path homotopy*** between f and g.

Homotopies and path homotopies can be lifted as well.

Theorem 1.26. *Let $p : Z \to Y$ be a covering map, $y_0 \in Y$ and $z_0 \in Z$ such that $p(z_0) = y_0$. Let $I = [a,b]$, $J = [0,1]$ be compact intervals of $\mathbb{R}$. Then any continuous function $h : I \times J \to Y$ with $h(a,0) = y_0$ has a unique lifting $\tilde{h}$ accross p such that $\tilde{h}(a,0) = z_0$. Furthermore, if h is a path homotopy, then so is $\tilde{h}$.*

Proof.

- **Existence.** We shall construct $\tilde{h}$ recursively. Firstly, we let $\tilde{h}(a,0) = z_0$. Then we use Theorem 1.17 to define $\tilde{h}$ on $I \times \{0\}$ and $\{a\} \times J$ as, respectively, the lifting of $x \mapsto h(x,0)$ and $t \mapsto h(a,t)$ accross p that satisfie $\tilde{h}(a,0) = z_0$. For what remains of $I \times J$, we firstly use the Lebesgue number lemma to construct, similarly to the proof of Theorem 1.17, intervals

[7] A path on a topological space X is a continuous function mapping a compact interval of $\mathbb{R}$ to X.

$I_i = [x_i, x_{i+1}]$, $i \in \{0, \dots, m\}$ and $J_j = [t_j, t_{j+1}]$, $j \in \{0, \dots, n\}$ such that every $I_i \times J_j$ is mapped by h into an open set of Y that is evenly covered by p. Then we construct $\tilde{h}$ recursively as follows: let $i_0 \in \{0, \dots, m\}$ and $j_0 \in \{0, \dots, n\}$ and assume that $\tilde{h}$ is defned on all $I_i \times J_j$ such that $j = j_0 \wedge i < i_0$ or $j < j_0$. Let U be an open set of Y that is evenly covered by p and contains $h(I_{i_0} \times J_{j_0})$. Let $\mathcal{S}$ be a partition of $p^{-1}(U)$ into slices. Note that $\tilde{h}$ is already defined in $R = \{x_{i_0}\} \times J_{j_0} \cup I_{i_0} \times \{j_{j_0}\}$, which is connected. Its image by $\tilde{h}$ is thus connected. Hence $\exists! V \in \mathcal{S}, \tilde{h}(R) \subset V$. Then let $\tilde{h}\,|_{I_{i_0} \times J_{j_0}} = (p\,|_V)^{-1} \circ h$. Clearly $\tilde{h}$ is a lifting of h accross p.

- **Uniqueness.** Similar to the proof uniqueness proof in Theorem 1.17.

It is also clear that if h is a path homotopy, then so is $\tilde{h}$. □

An important link between degrees and path homotopies is the following:

Theorem 1.27. *Let $f_0, f_1 : [a, b] \to S^1$ be two closed paths on S^1. f_0 and f_1 are path homotopic if and only if their degrees are equal.*

For the proof, we will need some facts.

Lemma 1.28. *Let $I = [a, b]$ be an interval of length at least two, i.e, $b - a \geq 2$. Then the interval contains two successive integers. In particular, it contains odd and even integers.*

Proof.

$$b \geq a + 2 \geq \lfloor a \rfloor + 2 \geq a + 1 \geq \lfloor a \rfloor + 1 \geq a.$$

□

Proposition 1.29. Let $f_1, f_2 : [a, b] \to S^1$ be continuous functions such that $f_1(a) = f_1(b)$ and $f_2(a) = f_2(b)$. If $\deg f_1 \neq \deg f_2$, then $\exists t \in [a, b],\ f_1(t) = -f_2(t)$.

Proof. Assume that $deg f_1 \neq \deg f_2$, which is equivalent to $|\deg f_2 - \deg f_1| \geq 1$ because the degrees are integers.

Let $\theta_1, \theta_2 : [a, b] \to \mathbb{R}$ be angle functions for f_1 and f_2. Let $\delta = \theta_2 - \theta_1$ We have

$$|\delta(b) - \delta(a)| = |(\theta_2(b) - \theta_2(a)) - (\theta_1(b) - \theta_1(a))| = 2\pi |\deg f_2 - \deg f_1| \geq 2\pi.$$

Thus $\left| \dfrac{\delta(b)}{\pi} - \dfrac{\delta(a)}{\pi} \right| \geq 2$, which means that the interval with endpoints $\dfrac{\delta(a)}{\pi}$ and $\dfrac{\delta(b)}{\pi}$ has length at least 2. According to Lemma 1.28, there exists an odd integer k between $\dfrac{\delta(a)}{\pi}$ and $\dfrac{\delta(b)}{\pi}$. Hence, $k\pi$ is an element of the interval with endpoints $\delta(a)$ and $\delta(b)$, which is the image of δ. Since δ is continuous on $[a, b]$, the intermediate value theorem implies that $\exists t_0 \in [a, b],\ \delta(t_0) = k\pi$.

Thus $\theta_2(t_0) = k\pi + \theta_1(t_0)$. Therefore

$$\begin{aligned} f_2(t_0) &= (\cos\theta_2(t_0), \sin\theta_2(t_0)) \\ &= (\cos(k\pi + \theta_1(t_0)), \sin(k\pi + \theta_1(t_0))) \\ &= -(\cos\theta_1(t_0), \sin\theta_1(t_0)) && (k \text{ is odd}) \\ &= -f_1(t_0). \end{aligned}$$

□

Definition 1.30. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be non-empty topological spaces.

A function $f : X \to Y$ is said to be ***locally constant*** if every point of X has a neighborhood on which the function is constant.

Proposition 1.31. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be non-empty topological spaces and f a locally constant function from X to Y. Then:

1. The preimages of the singletons in $\mathcal{P}(Y)$ under f are open.

2. f is continuous.

3. f is constant on every connected subspace of $\boldsymbol{X}$.

Proof.

1. Let $y \in Y$ and $x \in F = f^{-1}(\{y\})$. Since f is locally constant, there exists a neighborhood N of x on which f is constant. Thus $x \in N \subset F$, showing that F is open.

2. Let U be an open set of $\boldsymbol{Y}$. Clearly $f^{-1}(U) = \bigcup_{u \in U} f^{-1}(\{u\})$. Using 1 in this proposition, we conclude that $f^{-1}(U)$ is open as a union of open sets. Hence f is continuous.

3. Let $\boldsymbol{A}$ be a connected subspace of $\boldsymbol{X}$. Let $a_0 \in X$ and $y_0 = f(a_0)$. Let $U = f^{-1}(\{a_0\}) \cap A$ and $V = A \backslash U$. U and V are disjoint and their union is X. Furthermore, as $f^{-1}(\{a_0\})$ is open in $\boldsymbol{X}$ according to 1, U is open in $\boldsymbol{A}$. Since $U \neq \emptyset$ because $a_0 \in U$, if we show that V is open in $\boldsymbol{A}$, then as this space is connected, it will follow that $V = A \backslash U = \emptyset$, and we get $U = A$, showing that f is constant on A. So, let $x \in V \subset A$. There exists a neighborhood W of x in τ_X on which f is constant. Thus $\forall w \in W,\ f(w) = f(x) \neq y_0$. Thus $x \in W \cap A \subset V$, which shows that V is open in $\boldsymbol{A}$;

□

Now we prove the theorem.

Proof of Theorem 1.27.

$\Rightarrow$
We present two proofs:

1. Let $h : [a,b] \times [0,1] \to S^1$ be a path homotopy between f_0 and f_1. Let $\tilde{h}$ be a lifting of h. Note that $\theta_0 : x \mapsto \tilde{h}(x,0)$ and $\theta_1 : x \mapsto \tilde{h}(x,1)$ are, respectively, angle functions of f_0 and f_1. Let $\theta \in \mathbb{R}$ such that $f_0(a) = f_1(a) = p(\theta)$. We have $\tilde{h}(\{a\} \times [0,1]) \subset \{\theta + 2k\pi \mid k \in \mathbb{Z}\}$. Since $\tilde{h}$ is continuous and $\{a\} \times [0,1]$ is connected, $t \mapsto \tilde{h}(a,t)$ is constant. In particular:

$$\theta_0(a) = \tilde{h}(a,0) = \tilde{h}(a,1) = \theta_1(a).$$

Similarly, $\theta_0(b) = \theta_1(b)$. Hence $\deg f_0 = \deg f_1$.

2. Let $h : [a,b] \times [0,1] \to S^1$ be a path homotopy between f_0 and f_1. Let

$$\begin{aligned} \forall s \in [0,1], f_s : [a,b] &\to S^1 \\ t &\mapsto h(t,s) \end{aligned}$$

and

$$\begin{aligned} D : [0,1] &\to \mathbb{Z} \\ s &\mapsto \deg f_s. \end{aligned}$$

Our goal is to show that $D(0) = D(1)$. We will achieve this through Proposition 1.29 by proving that D is locally constant, and then using Proposition 1.31.

Let $s \in (0,1)$ and $\varepsilon = 1$. h is continuous on $[a,b] \times [0,1]$ which is a compact of $\mathbb{R}^2$. Thus h is uniformly continuous on $[a,b] \times [0,1]$. $\mathbb{R}^2$ being a finite-dimensional normed vector space, its norms are equivalent. We can thus work with the ∞-norm. By uniform continuity:

$$\exists \alpha > 0, \forall x,y \in [a,b] \times [0,1], \|x-y\|_\infty < \alpha \implies \|h(x)-h(y)\| < \varepsilon = 1 \tag{1.8}$$

and we can take α small enaugh so that $(s-\alpha, s+\alpha) \subset (0,1)$, because $(0,1)$ is an open set. Notice that for any $u \in (s-\alpha, s+\alpha)$ and $t \in [0,1]$,

$$\|(t,u) - (t,s)\|_\infty = \|(0, u-s)\|_\infty = |u-s| < \alpha.$$

Using this and (1.8), we get

$$\forall u \in (s-\alpha, s+\alpha), \forall t \in [0,1], \|h(t,u) - h(t,s)\| < 1,$$

or equivalently,

$$\forall u \in (s-\alpha, s+\alpha), \forall t \in [0,1], \|f_u(t) - f_s(t)\| < 1. \tag{1.9}$$

Let $u \in (s_\alpha, s+\alpha)$ and assume that

$$\exists t \in [0,1], f_u(t) = -f_s(t).$$

Then according to (1.9), $2\|f_s(t)\| < 1$. Therefore $\|f_s(t)\| < \dfrac{1}{2} < 1$, which contradicts the assumption ran $f_s \subset S^1$. Therefore

$$\forall t \in [0,1], f_u(t) \neq -f_s(t).$$

According to Proposition 1.29, $D(u) = \deg f_u = \deg f_s = D(s)$. Since u was arbitrary in the neighborhood $(s-\alpha, s+\alpha)$ of s, and since s was arbitrary in $[0,1]$, we conclude that D is locally constant on $(0,1)$. We can do the same for the cases $s = 0$ and $s = 1$, taking neighborhoods of the form, respectively, $[0,\alpha)$ and $(1-\alpha, 1]$ (these sets are open in the subspace topology on $[0,1]$). Hence, we conclude that D is locally constant on $[0,1]$, which is a connected space. Hence D is contant on $[0,1]$ according to Proposition 1.31. We conclude that $D(0) = D(1)$ as desired.

$\Leftarrow$
Suppose that $\deg f_0 = \deg f_1$. Let θ_0 be an angle function of f_0, and let θ_1 be the angle function of f_1 that satisfies $\theta_1(a) = \theta_0(a)$. Since $\deg f_0 = \deg f_1$, we have $\theta_1(b) = \theta_0(b)$. Then let

$$\begin{aligned} \psi : [a,b] \times [0,1] &\to \mathbb{R} \\ (x,t) &\mapsto (1-t)\theta_0(x) + t\theta_1(x) \end{aligned}$$

and $h = p \circ \psi$. h is a path homotopy between f_0 and f_1.

$\square$

1.3 Hopf's Umlaufsatz

In this section, we will prove Hopf's Umlaufsatz. We start by proving the following lemma:

Lemma 1.32. *Let $\gamma : [a,b] \to \mathbb{R}^2$ be a polygonal curve, and p_2 be the second canonical projection of $\mathbb{R}^2$, i.e,*

$$\begin{aligned} p_2 \colon \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto y. \end{aligned}$$

Then $\exists t_0 \in [a,b]$, $p_2(\gamma(t_0)) = \min\{p_2(\gamma(t)) \mid t \in [a,b]\}$.

Proof. This is because $p_2 \circ \gamma$ is continuous on $[a,b]$ which is a compact of $\mathbb{R}$. $\square$

Theorem 1.7 (Hopf's Umlaufsatz). *Let $\gamma : [a,b] \to \mathbb{R}^2$ be a polygonal curve of class C^1. Then $I(\gamma) = \pm 1$.*

Theorem 1.6 (Hopf's Umlaufsatz). *Let $\gamma : [a,b] \to \mathbb{R}^2$ be an arc length polygonal curve, k its curvature, n the number of its vertices and θ_i the exterior angle of γ at its ith vertex, where $i \in \{1, \cdots, n\}$. Then $I(\gamma) = \pm 1$ and*

$$\int_a^b k(t)\,\mathrm{d}t + \sum_{i=1}^{n} \theta_i = 2\pi I(\gamma).$$

Proof. We begin by proving the theorem for a simple closed curve. Let C be the trace of γ and $p = \gamma(t_0) \in C$, where $t_0 \in [a, b]$, the point which ordinate is the lowest in C, and which existence was established in Lemma 1.32. We can assume without loss of generality that $\gamma(a) = p$. In fact, if that's not the case, let $\tilde{I} = [\tilde{a}, \tilde{b}] = [t_0, t_0 + b - a]$, $I = [a, b]$, $\phi : t \mapsto t + a - t_0$ and $\tilde{\gamma}$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{I} \arrow[r, "\phi"] \arrow[dr, "\tilde{\gamma}"'] & I \arrow[d, "\gamma"] \\ & \mathbb{R}^2 \end{tikzcd}$$

$\tilde{\gamma}$ is an arc length reparametrization of γ by the diffeomorphism ϕ satisfying $\tilde{\gamma}(\tilde{a}) = p$, having the same rotation index as γ. Indeed, let θ and $\tilde{\theta}$ be the tangent angle functions of, respectively, γ and $\tilde{\gamma}$. Since $\tilde{\gamma}' = \gamma' \circ \phi$, it is clear that $\tilde{\theta} = \theta \circ \phi$. Thus

$$\begin{aligned} I(\tilde{\gamma}) &= \frac{\tilde{\theta}(\tilde{b}) - \tilde{\theta}(\tilde{a})}{2\pi} \\ &= \frac{\theta(\phi(\tilde{b})) - \theta(\phi(\tilde{a}))}{2\pi} \\ &= \frac{\theta(b) - \theta(a)}{2\pi} \\ &= I(\gamma) \end{aligned}$$

as desired. So, let's make this assumption.

Let $T = \{(t_1, t_2) \in \mathbb{R}^2 \mid a \leq t_1 \leq t_2 \leq b\}$ and

$$\begin{aligned} \psi \colon T &\to S^1 \\ (t_1, t_2) &\mapsto \begin{cases} \gamma'(t_1), & t_1 = t_2; \\ -\gamma'(a), & (t_1, t_2) = (a, b); \\ \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|}, & \text{otherwise.} \end{cases} \end{aligned}$$

For any $(t_1, t_2) \in T \backslash (\{(a, b)\} \cup \Delta)$, where $\Delta = \{(u, v) \in T \mid u = v\}$, $\psi(t_1, t_2)$ is the unit vector which initial point is $\gamma(t_1)$ and points towards $\gamma(t_2)$. When t_2 gets closer to t_1, we can see by geometric intuition that $\psi(t_1, t_2)$ gets closer to $\gamma'(t_1)$. The case where $(t_1, t_2) \to (a, b)$ can also be viewed inututively (see [31] for nice animations). This convinces us that ψ is continuous on T. Let's prove it.

It is clear that ψ is continous on $T \backslash (\Delta \cup \{(a, b)\})$. The function is also continuous on Δ. Indeed if $t \in [a, b]$ then according to the mean value theorem we have:

$$\forall (t_1, t_2) \in T, t_1 < t_2 \implies \exists c_1, c_2 \in (t_1, t_2), \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1} = (\cos\theta(c_1), \sin\theta(c_2)).$$

Thus

$$\lim_{\substack{(t_1,t_2)\to(t,t)\\ t_1<t_2}} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1} = (\cos\theta(t), \sin\theta(t)) \\ = \gamma'(t).$$

Therefore

$$\begin{aligned} \lim_{\substack{(t_1,t_2)\to(t,t)\\ t_1<t_2}} \psi(t_1,t_2) &= \lim_{\substack{(t_1,t_2)\to(t,t)\\ t_1<t_2}} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}\left|\frac{t_2-t_1}{\gamma(t_2)-\gamma(t_1)}\right| \\ &= \frac{1}{\|g'(t)\|}\gamma'(t) \\ &= \gamma'(t). \end{aligned} \tag{1.10}$$

To conclude, let $\varepsilon > 0$. By continuity of γ' at t, we have

$$\exists\delta_1 > 0, \forall x \in (t-\delta_1, t+\delta_1), \|\psi(x,x) - \gamma'(t)\| < \varepsilon.$$

On the other hand, by (1.10), we have

$$\exists\delta_2 > 0, \forall (t_1,t_2) \in T\backslash\Delta, \|(t_1,t_2)-(t,t)\|_\infty < \delta_2 \implies \|\psi(t_1,t_2) - \gamma'(t)\| < \varepsilon.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, we conclude that

$$\forall (t_1,t_2) \in T, \|(t_1,t_2)-(t,t)\|_\infty < \delta \implies \|\psi(t_1,t_2) - \gamma'(t)\| < \varepsilon$$

which shows that ψ is continuous at (t,t). For continuity at (a,b), it is slightly more complicated as if we use the mean value theorem just as we did earlier, the numbers c_1 and c_2 belong to the interval (t_1,t_2), and when we take the limit $(t_1,t_2) \to (a,b)$, we can't conclude anything about their behavior. To solve this issue, we extend γ periodically to a function $\tilde\gamma$ defined as follows: Let $L = b - a$. We know that

$$\forall t \in \mathbb{R}, \exists! r \in [a,b), \exists! k \in \mathbb{Z}, t = kL + r.$$

This defines a function $r : \mathbb{R} \to [a,b)$. Now let $\tilde\gamma = \gamma \circ r$. Since γ is a closed curve, it is easy to see that $\tilde\gamma$ is smooth and $\tilde\gamma' = \gamma' \circ r$. We have by the mean value theorem applied to $\tilde\gamma$:

$$\forall (t_1,t_2) \in T, \exists c_1, c_2 \in (t_2-L, t_1), \frac{\gamma(t_2)-\gamma(t_1)}{t_2-L-t_1} = \frac{\tilde\gamma(t_2-L)-\tilde\gamma(t_1)}{t_2-L-t_1} \\ = (\cos\theta(r(c_1)), \sin\theta(r(c_2))).$$

Now $c_1, c_2 \to a$ as $(t_1,t_2) \to (a,b)$. Therefore

$$\lim_{(t_1,t_2)\to(a,b)} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-L-t_1} = (\cos\theta(a), \sin\theta(a)) \\ = \gamma'(a).$$

Hence

$$\begin{aligned}\lim_{(t_1,t_2)\to(a,b)} \frac{\gamma(t_2)-\gamma(t_1)}{|\gamma(t_2)-\gamma(t_1)|} &= \lim_{(t_1,t_2)\to(t,t)} -\frac{\gamma(t_2)-\gamma(t_1)}{t_2-L-t_1}\left|\frac{t_2-L-t_1}{\gamma(t_2)-\gamma(t_1)}\right| \\ &= -\frac{1}{|\gamma'(a)|}\gamma'(a) \\ &= -\gamma'(a).\end{aligned}$$

We conclude that ψ is continuous on T.

Why bother with this function? Notice that for

$$\begin{aligned}\alpha_0 : [0,1] &\to \Delta \\ t &\mapsto (1-t)(a,a)+t(b,b)\end{aligned}$$

which trace is exaclty Δ, $\psi\circ\alpha_0=\gamma'$. On the other hand, for

$$\begin{aligned}\alpha_1 : [0,1] &\to T \\ t &\mapsto \begin{cases}(1-2t)(a,a)+2t(a,b), & t\in\left[0,\frac{1}{2}\right]; \\ 2(1-t)(a,b)+2\left(t-\frac{1}{2}\right)(b,b), & t\in\left[\frac{1}{2},1\right],\end{cases}\end{aligned}$$

the degree of the function $f=\psi\circ\alpha_1$ is easy to calculate. See [31] for nice animations. If we show that γ' and f are pathhomotopic, we would be able to find the rotation index of γ using Theorem 1.27.

Let

$$\begin{aligned}\beta : [0,1] &\to T \\ t &\mapsto \frac{1-t}{2}(a+b,a+b)+t(a,b)\end{aligned}$$

and

$$\begin{aligned}h : [0,1]\times[0,1] &\to T \\ (t,s) &\mapsto \begin{cases}(1-2t)(a,a)+2t\beta(s), & t\in\left[0,\frac{1}{2}\right]; \\ 2(1-t)\beta(s)+(2t-1)(b,b), & t\in\left[\frac{1}{2},1\right].\end{cases}\end{aligned}$$

It is clear that h is a path homotopy between α_0 and α_1. To visualize β and h, let $A=(a,a)$, $C=(a,b)$ and $B=(b,b)$. The triangle ABC is the boundary of T. It has a right angle at C and is isoscele, as $CA=CB=b-a$. The segment $[AB]$ is the trace of α_0 while the reunion of the segments $[AC]$ and $[CB]$ is the trace of α_1. Let H be the perpendicular projection of C on $[AB]$. Then $[CH]$ is the trace of β. At each instant t, $\beta(t)$ is a point on $[CH]$. β goes up from H to C. Then at each moment $s\in[0,1]$, the curve $t\mapsto h(t,s)$ is the reunion of the segments $[A\beta(s)]$ and $[\beta(s)B]$. See the figure below for an illustration, and `Homotopy_in_Hopf's_Umlaufsatz.gif` for an animation.

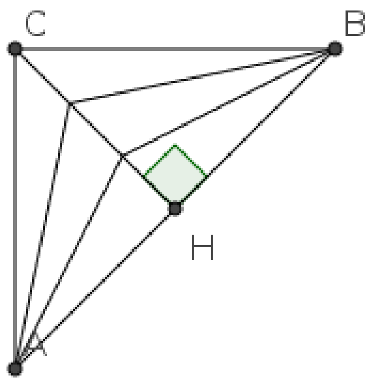


FIGURE 1.9: Homotopy between α_0 and α_1

Since h and ψ are continuous and ran $h \subset$ dom ψ, so is $\psi \circ h$. Hence $\psi \circ h$ is a path homotopy between $\psi \circ \alpha_0 = \gamma'$ and $\psi \circ \alpha_1 = f$. Therefore

$$\begin{aligned} I(\gamma) &= \deg \gamma' \\ &= \deg f. \end{aligned}$$

Let's calculate $\deg f$. We have $p_2(\gamma(a)) = \min p_2 \circ \gamma([a,b]) = \min p_2 \circ \tilde{\gamma}(\mathbb{R})$. Thus $(p_2 \circ \tilde{\gamma})'(a) = 0$. We have:

$$\begin{aligned} (p_2 \circ \tilde{\gamma})'(a) &= \mathrm{d}p_2(\tilde{g}(a)) \cdot \tilde{\gamma}'(a) \\ &= p_2(\tilde{\gamma}'(a)) \qquad (p_2 \text{ is a linear map}) \\ &= p_2(\gamma'(a)). \end{aligned}$$

Therefore $p_2(\gamma'(a)) = 0$. Hence $f(0) = \gamma'(a) = \pm e_1$ where $\mathcal{B}_c = (e_1, e_2)$ is the canonical basis of $\mathbb{R}^2$. We can aassume without loss of generality that $f(0) = e_1$. Now let θ be the angle function of f that satisfies $\theta(0) = 0$. All the points of γ are in the higher half plane $\{(x,y) \in \mathbb{R} \mid y \geq 0\}$. Thus $\forall t \in \left[0, \frac{1}{2}\right], \theta(t) \in [0, \pi]$. Furthermore, as $f\left(\frac{1}{2}\right) = -f(0)$, we have $\theta\left(\frac{1}{2}\right) = \pi$. Notice that f is antisymmetric at $\frac{1}{2}$, i.e, $\forall t \in [0,1], f\left(\frac{1}{2} - t\right) = -f(t)$. Thus we have $\forall t \in \left[\frac{1}{2}, 1\right], \theta(t) \in [\pi, 2\pi]$ and $f(1) = -f\left(\frac{1}{2}\right)$, thus $\theta(1) = 2\pi$. Hence $\deg f = 1$. Similarly, we find that if $f(0) = -e_1$, $\deg f = -1$. Therefore $I(\gamma) = \pm 1$. Lastly, let $\tilde{\theta}$ be a tangent angle function for γ. We have:

$$\begin{aligned} \int_a^b k(t)\,\mathrm{d}t &= \tilde{\theta}(b) - \tilde{\theta}(a) \\ &= 2\pi I(\gamma). \end{aligned}$$

Now, assume that γ is a polygonal curve without cusps[8] and let θ be a tangent angle function for γ. Let $t_1, \ldots, t_n$ be the numbers in $[a,b]$ for which

[8]The theorem still holds even when there are cusps

$\gamma(t_i)$ is a vertex, and $\theta_1, \ldots, \theta_n$ be the external angles of γ at, respectively, $\gamma(t_1), \ldots, \gamma(t_n)$. We can assume without loss of generality that $\gamma(a)$ is not a vertex. Let $i \in \{1, \ldots, n\}$ and $\varepsilon = \dfrac{\pi - |\theta_i|}{4}$. We have $\lim_{t \to t_i^-} \theta(t) = \theta_(t_i) - \theta_i$ and $\lim_{t \to t_i^+} \theta(t) = \theta(t_i)$. Thus there exists $\delta > 0$ such that

$$\forall t \in (t_i - \delta, t_i), |\theta(t) - (\theta(t_i) - \theta_i)| < \varepsilon$$

and

$$\forall t \in (t_i, t_i + \delta), |\theta(t) - \theta(t_i)| < \varepsilon.$$

Since γ is continuous, $S = \gamma([a, b] \backslash (t_i - \delta, t_i + \delta)$ is compact. It also doesn't contain $\gamma(t_i)$. Thus $\exists r > 0, C \cap S = \emptyset$ where

$$C = \bar{B}(\gamma(t_i), r) = \{p \in \mathbb{R}^2 \mid \|p - \gamma(t_i)\| \leq r\}.$$

Let $t_{i1} = \min\{t \in (t_i - \delta, t_i + \delta) \mid \|\gamma(t) - \gamma(t_i)\| = r\}$. It is easy to see that such a number does exist and that it describes the first moment in $(t_i - \delta, t_i + \delta)$ where γ enters C. Similarly, let $t_{i2} = \max\{t \in (t_i - \delta, t_i + \delta \mid \|\gamma(t) - \gamma(t_i)\| = r\}$ be the last moment in $(t_i - \delta, t_i + \delta)$ where γ exits C. We have $|\theta(t_{i2}) - \theta(t_i)| < \varepsilon$ and $|\theta(t_i) - \theta_i - \theta(t_{i1})| < \varepsilon$. Thus

$$\begin{aligned}
|\theta(t_{i2}) - \theta(t_{i1})| &\leq |\theta(t_{i2}) - \theta(t_{i1}) - \theta_i| + |\theta_i| \\
&\leq |\theta(t_{i2}) - \theta(t_i)| + |\theta(t_i) - \theta_i - \theta(t_{i1})| + |\theta_i| \\
&< 2\varepsilon + |\theta_i| \\
&< \frac{|\theta_i| + \pi}{2} \\
&< \pi
\end{aligned}$$

We consider a new curve $\tilde{\gamma}_i$ which is the result of replacing $\gamma\,|_{[t_{i1}, t_{i2}]}$ in γ with an arc length regular curve such that the tangent angle function $\tilde{\theta}_i$ of $\tilde{\gamma}_i$ is either increasing or decreasing in $[t_{i1}, t_{i2}]$. Hence $\tilde{\theta}_i(t_{i2}) - \tilde{\theta}_i(t_{i1}) = \theta(t_{i2}) - \theta(t_{i1})$. Doing this for all $i \in \{1, \ldots, n\}$, we end up with a simple closed curve $\tilde{\gamma}$ that is equal to γ outside of the intervals $[t_{i1}, t_{i2}]$ for $i \in \{1, \ldots, n\}$ such that $I(\tilde{\gamma}) = I(\gamma)$. Let $\tilde{\theta}$ denote its tangent angle function that verifies $\tilde{\theta}(a) = \theta(a)$, and $\tilde{k}$ its curvature. According to what we showed earlier, we have

$$\begin{aligned}
2\pi I(\tilde{\gamma}) &= \int_a^b \tilde{k}(t)\,\mathrm{d}t \\
&= \int_{[a, t_{11}] \cup [t_{n2}, b]} \tilde{k}(t)\,\mathrm{d}t + \int_{t_{11}}^{t_{n2}} \tilde{k}(t)\,\mathrm{d}t \\
&= \int_{[a, t_{11}] \cup [t_{n2}, b]} k(t)\,\mathrm{d}t + \sum_{i=1}^{n} \int_{t_{i1}}^{t_{i2}} \tilde{k}(t)\,\mathrm{d}t + \sum_{i=1}^{n-1} \int_{t_{i2}}^{t_{(i+1)1}} \tilde{k}(t)\,\mathrm{d}t \\
&= \int_{[a, t_{11}] \cup [t_{n2}, b]} k(t)\,\mathrm{d}t + \sum_{i=1}^{n} \int_{t_{i1}}^{t_{i2}} \tilde{k}(t)\,\mathrm{d}t + \sum_{i=1}^{n-1} \int_{t_{i2}}^{t_{(i+1)1}} k(t)\,\mathrm{d}t
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
\sum_{i=1}^{n}\int_{t_{i1}}^{t_{i2}}\tilde{k}(t)\,\mathrm{d}t &= \sum_{i=1}^{n}\int_{t_{i1}}^{t_{i2}}\tilde{\theta}'(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{n}\tilde{\theta}(t_{i2})-\tilde{\theta}(t_{i1}) \\
&= \sum_{i=1}^{n}\theta(t_{i2})-\theta(t_{i1}) \\
&= \sum_{i=1}^{n}(\theta(t_{i2})-\theta(t_i))+(\theta(t_i)-\theta_i-\theta(t_{i1}))+\theta_i \\
&= \sum_{i=1}^{n}\int_{t_i}^{t_{i2}}\theta'(t)\,\mathrm{d}t+\int_{t_{i1}}^{t_i}\theta'(t)\,\mathrm{d}t+\sum_{i=1}^{n}\theta_i \\
&= \sum_{i=1}^{n}\int_{t_{i1}}^{t_{i2}}k(t)\,\mathrm{d}t+\sum_{i=1}^{n}\theta_i.
\end{aligned} \tag{1.12}$$

Combining (1.11) and (1.12), we conclude that

$$\begin{aligned}
2\pi I(\gamma) &= 2\pi I(\tilde{\gamma}) \\
&= \int_{[a,t_{11}]\cup[t_{n2},b]}k(t)\,\mathrm{d}t+\sum_{i=1}^{n}\int_{t_{i1}}^{t_{i2}}k(t)\,\mathrm{d}t+\sum_{i=1}^{n-1}\int_{t_{i2}}^{t_{(i+1)1}}k(t)\,\mathrm{d}t+\sum_{i=1}^{n}\theta_i \\
&= \int_a^b k(t)\,\mathrm{d}t+\sum_{i=1}^{n}\theta_i.
\end{aligned}$$

□

We give now an a proof of the result below using Hopf's Umlaufsatz instead of triangulations as mentioned in the beginning of this chapter.

Corollary 1.33. The sum of the interior angles of any simple n-sided polygon is $\pm(n-2)\pi$.

Proof. Let $\gamma:[a,b]\to\mathbb{R}^2$ be an arc length polygonal curve which trace is the boundary of the polygon. Then the exterior angles $\beta_1,\ldots,\beta_n$ of γ are those of the polygon. Since the edges have no curvature, we have $\int_a^b k(t)\,\mathrm{d}t=0$. According to Hopf's Umlaufsatz

$$\begin{aligned}
\sum_{i=1}^{n}\beta_i &= 2\pi I(\gamma) \\
&= \pm 2\pi.
\end{aligned}$$

Then we use (1.1) to find the desired result. □

Chapter 2

The Local Gauss-Bonnet Theorem

In this chapter, we present a proof of a local version of the Gauss-Bonnet theorem. The proof will involve Hopf's Umlaufsatz, the main theorem of the previous chapter. Historically, however, the local Gauss-Bonnet theorem was proved much earlier. The German mathematician Johann Carl Friedrich Gauss proved a special case of it [1] ([22],1827), while the French mathematician Pierre Ossian Bonnet proved the local Gauss-Bonnet theorem ([11],1848).

(A) Carl Friedrich Gauss (1777 - 1855) [26]

(B) Pierre Ossian Bonnet (1819 - 1892) [2]

FIGURE 2.1: Gauss and Bonnet

2.1 Regular Surfaces

In this section, we recall some elementary concepts on regular surfaces.

Definition 2.1. A ***regular curve***[2] is a smooth map $\gamma : I \to \mathbb{R}^3$, where I is an interval, such that $\forall t \in I, \gamma'(t) \neq 0$.

[1] See Corollary 2.69.
[2] Whether the curve is plane or in the space ($\mathbb{R}^3$) is usually clear from context.

Definition 2.2. A ***regular surface*** is a subset S of $\mathbb{R}^3$ such that for each point p on S there is a neighborhood $V \subset \mathbb{R}^3$ of p, an open set $U \subset \mathbb{R}^2$ and a map $\varphi : U \to V \cap S$ such that:

1. φ is a homeomorphism;
2. φ is of class C^∞ on U;
3. for any $q \in U$, $\mathrm{d}\varphi_q$ is injective.

Such a map φ is called a ***local parametrization*** of S at p. The neighborhood $V \cap S$ is called a ***coordinate neighborhood***. A collection of local parametrization which range covers S is called an ***atlas*** for S.

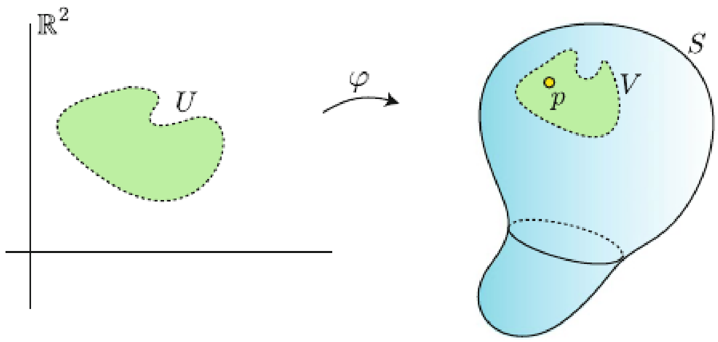


FIGURE 2.2: [40, p. 126]

Example 2.3. An ***affine plane*** $\mathcal{P}$ in $\mathbb{R}^3$ given by $p + \text{span}(u, v)$ where u and v are two linearly independent vectors of $\mathbb{R}^3$ and $p \in \mathbb{R}^3$ is a regular surface. In fact, it has a global parametrization

$$\begin{aligned} \varphi : \mathbb{R}^2 &\to \mathcal{P} \\ (x, y) &\mapsto p + xu + yv \end{aligned}$$

(see figure 2.3) that satisfies the three conditions in the previous definition.

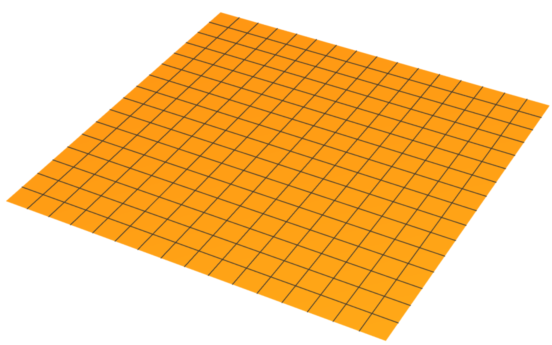


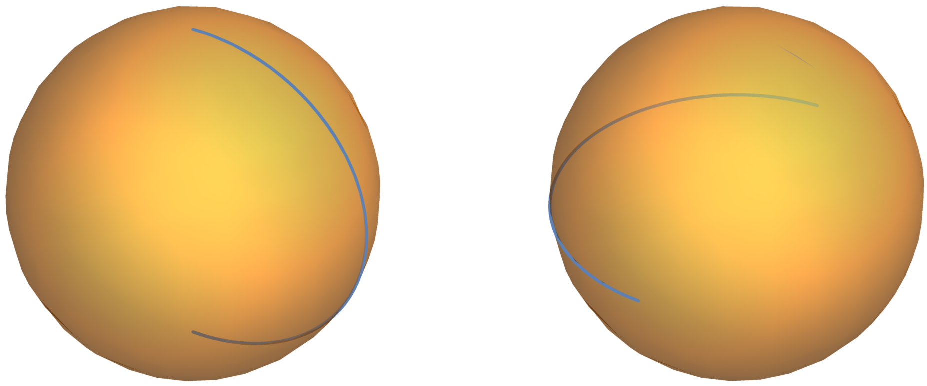
FIGURE 2.3: A plane in $\mathbb{R}^3$

Example 2.4. The sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is a regular surface. In fact, it has an atlas consisting of two parametrizations:

$$\begin{aligned} \varphi_1 : (0, \pi) \times (0, 2\pi) &\to \operatorname{Im} \varphi_1 \\ (\theta, \phi) &\mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \end{aligned}$$

and $\varphi_2 = g \circ \phi_1$ (see figure 2.4 with $g = (r_2 \circ r_1) |_{\operatorname{Im} \varphi_1}$ where r_1 and r_2 are the rotation operators which corresponding matrices are

$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$



(A) $\operatorname{Im} \varphi_1$ (B) $\operatorname{Im} \varphi_2$

FIGURE 2.4: Coordinate neighborhoods of S^2

We recall now some theorems that can be used to prove that a subset of $\mathbb{R}^3$ is a regular surface.

Proposition 2.5. Let $f : U \to \mathbb{R}$ be a smooth function on the open set U of $\mathbb{R}^2$. Then the graph of f, that is the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$, is a regular surface.

Example 2.6. Let

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x^2 + y^2. \end{aligned}$$

The graph S of f (figure 2.5), called an elliptic paraboloid, is a regular surface.

Proposition 2.7. Let $F : U \to \mathbb{R}$ be a smooth function on the open set U of $\mathbb{R}^3$, and let $a \in \mathbb{R}$. Suppose that $\forall M \in F^{-1}(\{a\}), \vec{\nabla} F(M) \neq \vec{0}$. Then $F^{-1}(\{a\})$ is a regular surface.

Example 2.8. Let

$$\begin{aligned} F : \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto x^4 + y^3 + z^2 - 1. \end{aligned}$$

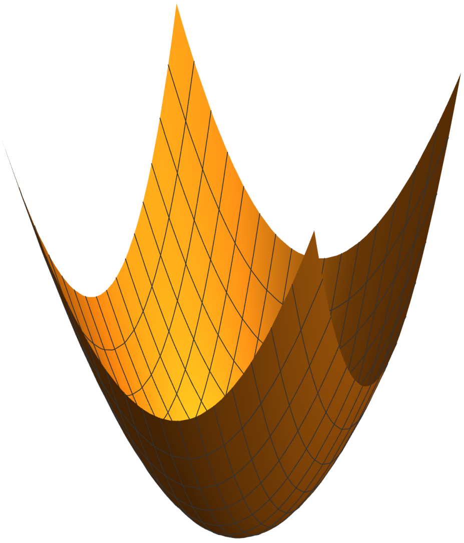


FIGURE 2.5: Elliptic Paraboloid

We have $\forall(x,y,z) \in \mathbb{R}^3, \vec{\nabla}F(x,y,z) = (4x^3, 3y^2, 2z)$. Thus

$$\forall(x,y,z) \in \mathbb{R}^3, \vec{\nabla}F(x,y,z) = \vec{0} \Leftrightarrow (x,y,z) = 0,$$

and as $F(0,0,0) = -1$, we conclude that $\forall M \in F^{-1}(\{0\}), \vec{\nabla}F(M) \neq \vec{0}$. Hence $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^4 + y^3 + z^2 = 1\}$(figure 2.6) is a regular surface.

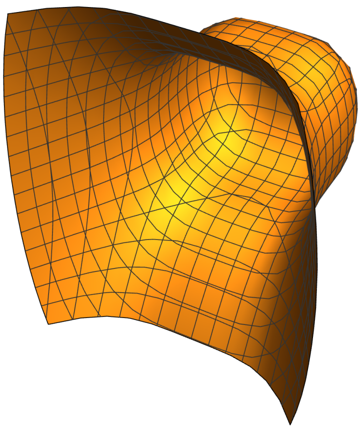


FIGURE 2.6: The surface $x^4 + y^3 + z^2 = 1$

Proposition 2.9. Let

$$\begin{aligned} \gamma : I = (a,b) &\to \mathbb{R}^3 \\ t &\mapsto (x(t), 0, z(t)) \end{aligned}$$

be a regular curve in the xz-plane of classe C^1 such that

1. γ never interesects the z-axis, which means that either $x > 0$ or $x < 0$;
2. γ is injective, which means that the curve never interesects itself.

Let $S = \{(x(v)\cos u, x(v)\sin u, z(v)) \mid u \in (0, 2\pi), v \in I\}$. Then S is a regular surface that has the global parametrization:

$$\begin{aligned} \varphi : (0, 2\pi) \times I &\to S \\ (u, v) &\mapsto (x(v)\cos u, x(v)\sin u, z(v)). \end{aligned}$$

S is called a ***surface of revolution***. The curve γ is called the ***generating curve*** of S.

Example 2.10. Let

$$\begin{aligned} \gamma : (1, +\infty) &\to \mathbb{R}^3 \\ t &\mapsto \left(\frac{1}{t}, 0, t\right). \end{aligned}$$

This curve is defined on an unbounded interval. But since $x \mapsto \dfrac{1}{x}$ is a diffeomorphism between $(1, +\infty)$ and $(0, 1)$, we conclude that γ has a reparametrization with a bounded open interval as a domain. The surface of revolution that we obtain is called Torricelli[3]'s trumpet (figure 2.7) and is famous for being a surface with infinite area and finite volume π.

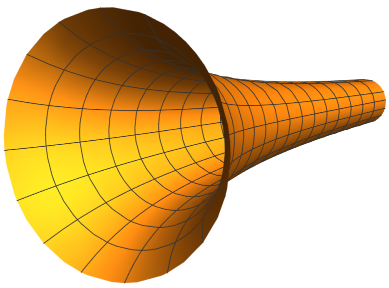


FIGURE 2.7: Torricelli's Trumpet

Remarks 2.11. Let $\gamma : [a, b] \to \mathbb{R}^3$ be a regular curve in the xz-plane of classe C^1 such that $\gamma|_{(a,b)}$ satisfies the conditions of the previous theorem. Let $S = \{(x(v)\cos u, x(v)\sin u, z(v)) \mid u \in (0, 2\pi), v \in [a, b]\}$. Then, in each of these cases, S is a regular surface:

1. γ is a simple closed curve, and γ still never intersects the z-axis;
2. $\gamma(a)$ and $\gamma(b)$ are in the z-axis, and $\gamma'(a)$ and $\gamma'(b)$ are parallel to the x-axis. Furthermore, if we had only $\gamma(a)$ in the z-axis but not $\gamma(b)$, then the result still holds but this time for

$$S = \{(x(v)\cos u, x(v)\sin u, z(v)) \mid u \in (0, 2\pi), v \in [a, b)\}.$$

[3]Named after the italian physicist and mathematician Evangelista Torricelli (1608-1647).

Example 2.12. Let

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{R}^3 \\ t &\mapsto (2 + \sin t, 0, \cos t) \end{aligned}$$

The surface of revolution that we get is a torus (figure 2.8).

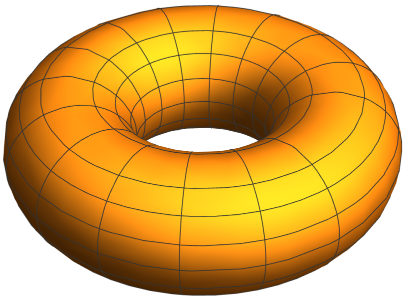


FIGURE 2.8: Torus

Example 2.13. Let

$$\begin{aligned} \gamma : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] &\to \mathbb{R}^3 \\ t &\mapsto (\cos t, 0, 2\sin t). \end{aligned}$$

The surface of revolution that we get is an ellipsoid (figure 2.9).

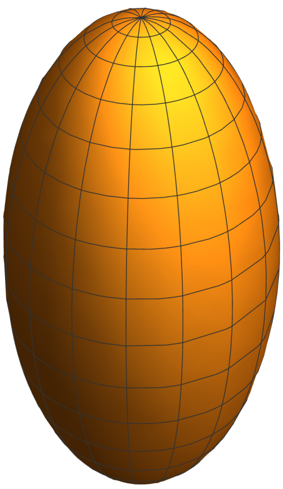


FIGURE 2.9: Ellispoid

We give as a final example a surface that is not regular.

Example 2.14. Let

$$\begin{aligned} \gamma : [0, 1) &\to \mathbb{R}^3 \\ t &\mapsto (t, 0, t). \end{aligned}$$

The surface of revolution that we get is a cone (figure 2.10). This surface is not regular as there is no local parametrization at the apex that is differentiable.

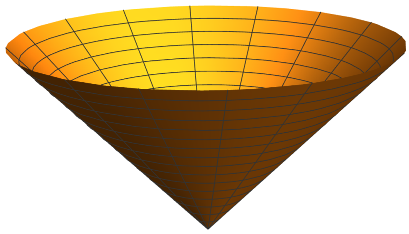


FIGURE 2.10: Cone

2.2 Curvature

In the previous chapter, we involved the tangent and normal vectors to a parametrized curve in order to define curvature. The same applies to surfaces as we shall see in this section, but in a much less easy way. In fact, there will be multiple curvatures to define.

If tangents to curves in $\mathbb{R}^2$ are lines, tangents to surfaces in $\mathbb{R}^3$ are planes spanned by vectors that we call tangent vectors. This gives the following intuitive definition:

Definition 2.15. A vector $v \in \mathbb{R}^3$ is a ***tangent vector*** to S at p if there exists a smooth parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. The set of all tangent vectors of S at p is called the ***tangent plane*** to S at p and is denoted $T_p(S)$.

An equivalent way to define the tangent plane is given by the following theorem:

Theorem 2.16. *Let $\varphi : U \subset \mathbb{R}^2 \to V \cap S$ be a local parametrization at p. Then $T_p(S) = \mathrm{d}\varphi_p\left(\mathbb{R}^2\right)$.*

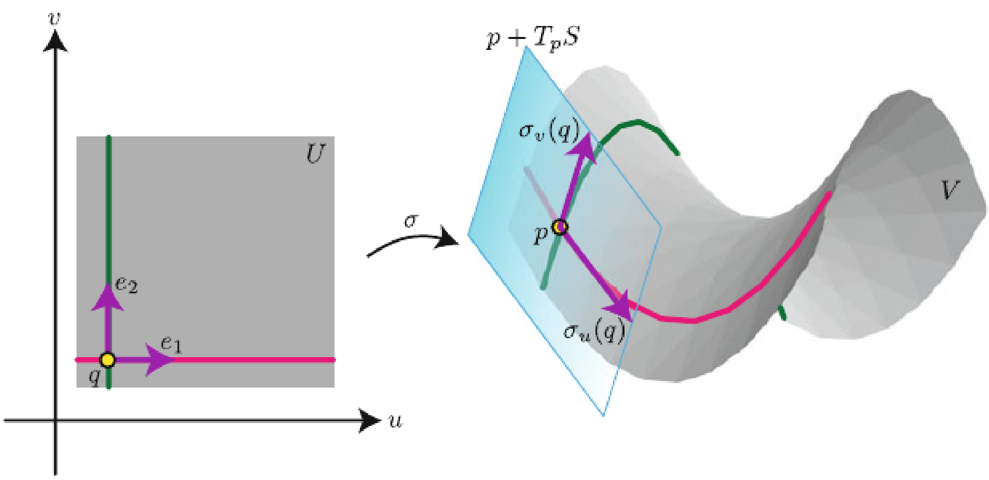


FIGURE 2.11: Tangent plane [40, p. 142]

In particular, the tangent planes are vector spaces of dimension 2. Notice that when we work on $\mathbb{R}^2$ and $\mathbb{R}^3$, we don't think of them as mere vector spaces, but rather as endowed with a norm induced from an inner product. We want to think of tangent planes the same way:

Definition 2.17. Let $p \in S$ and $\langle\, ,\, \rangle_p$ denote the restriction of the usual inner product of $\mathbb{R}^3$ on $T_p(S)$. The ***first fundamental form*** of S at p is the quadratic form

$$\begin{aligned} \mathrm{I}_p : T_p(S) &\to \mathbb{R} \\ v &\mapsto \langle v, v\rangle_p = \|v\|^2. \end{aligned}$$

Let φ be a local parametrization of S at p, $q = \varphi^{-1}(p)$ and let $\varphi_u(q) = \dfrac{\partial\varphi}{\partial u}(q)$ and
$\varphi_v(q) = \dfrac{\partial\varphi}{\partial v}(q)$. Then the matrix of I_p in the basis $\{\varphi_u(q), \varphi_v(q)\}$ is given by

$$\begin{pmatrix} E(q) & F(q) \\ F(q) & G(q) \end{pmatrix}$$

where $E = \langle\varphi_u, \varphi_u\rangle_p$, $F = \langle\varphi_u, \varphi_v\rangle_p$ and $G = \langle\varphi_v, \varphi_v\rangle_p$ are called the ***metric coefficients*** of S. Note that E, F and G are smooth functions. Also note that the matrix of I_p is invertible as it is symmetric and I_p is positive.

Remark 2.18. Note that if ϕ is the angle between φ_u and φ_v then

$$\cos\phi = \frac{\langle\varphi_u, \varphi_v\rangle}{\|\varphi_u\|\|\varphi_v\|} = \frac{F}{\sqrt{EG}}.$$

Therefore, $\forall q \in U$, $\langle\varphi_u(q), \varphi_v(q)\rangle = 0 \Leftrightarrow F(q) = 0$. In this case, we say that φ is an ***orthogonal parametrization***.

Notation 2.19. As shown in the previous definition, if

$$\begin{aligned} \varphi : U &\to V \cap S \\ (u, v) &\mapsto \varphi(u, v) \end{aligned}$$

is a local parametrization and f is any function defined on U, it is convenient to denote $f_u = \dfrac{\partial f}{\partial u}$ and $f_v = \dfrac{\partial f}{\partial v}$. If $\gamma : I \to U$ is a regular curve on U, we might as well denote $f_u = \dfrac{\partial f}{\partial u} \circ \gamma$ and $f_v = \dfrac{\partial f}{\partial v} \circ \gamma$. We might even use f instead of $f \circ \gamma$. Although these notations are ambiguous, it will be clear from the context what they mean.

In the previous chapter, we had to rely on the canonical orientation of $\mathbb{R}^2$ to define a continuous function that associates each point on the curve to the normal vector to the curve at that point. The situation is similar for surfaces: at each point $p \in S$ and for each local parametrization $\varphi : U \to V \cap S$ at

$p = \varphi(q)$, we have $T_pS = \text{span}\{\varphi_u(q), \varphi_v(q)\}$, thus

$$T_pS^\perp = \text{span}\left\{\frac{\varphi_u(q) \wedge \varphi_v(q)}{\|\varphi_u(q) \wedge \varphi_v(q)\|}\right\}.$$

Hence, we can choose $N = \dfrac{\varphi_u \wedge \varphi_v}{\|\varphi_u \wedge \varphi_v\|}$ or $N = -\dfrac{\varphi_u \wedge \varphi_v}{\|\varphi_u \wedge \varphi_v\|}$ on $V \cap S$. But this choice relies on φ, which is a local parametrization. As each point has a coordinate neighborhood, can we always "stick" together these choices in a way that gives us a global continous function $N : p \mapsto N(p) \in T_pS^\perp$?

For the moment, let's make the following definition:

Definition 2.20. A regular surfaces S is called ***orientable*** if there exists a continuous function $N : S \to \mathbb{R}^3$ such that $\forall p \in S, N(p) \in T_pS^\perp$. The choice of such a function is called an ***orientation*** of S and we say in this case that S is ***oriented***. If such a function doesn't exist, we say that S is ***nonorientable***. In the case where $\varphi : U \to S \cap V$ is a local parametrization and $N|_{S\cap V} /\!/ \varphi_u \wedge \varphi_v$, we say that φ is ***compatible with the orientation*** of S.

The answer to the previous question with this new terminology is: not always; nonorientable surfaces do exist.

Example 2.21. Let

$$\begin{aligned}\psi : D = (-0.5, 0.5) \times \mathbb{R} &\to \varphi(D)\\ (t, \theta) &\mapsto (1.5\cos\theta, 1.5\sin\theta, 0) + t\left(\cos\frac{\theta}{2}, 0, \sin\frac{\theta}{2}\right).\end{aligned}$$

$S = \varphi(D)$ is called a Mobius strip. It has an atlas consisting of two local parametrizations: $\psi|_{U_1}$ and $\psi|_{U_2}$ where $U_1 = (-0.5, 0.5) \times (0, 2\pi)$ and $U_2 = (-0.5, 0.5) \times (\pi, 3\pi)$. Let's show that it is nonorientable.

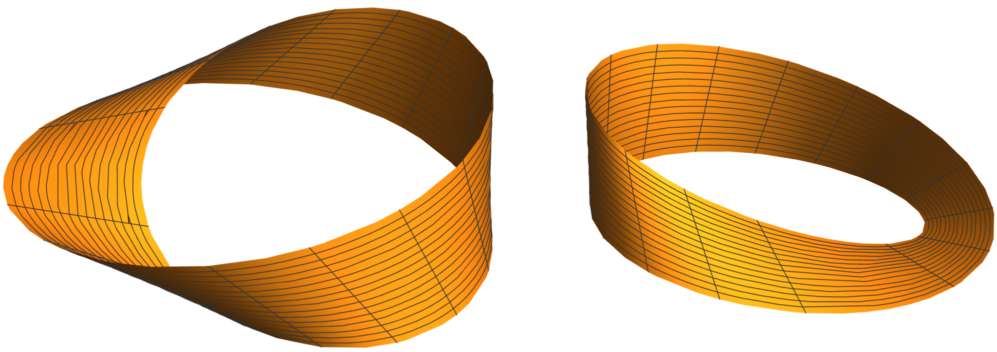


FIGURE 2.12: Mobius strip

Suppose that it has an orientation N. We can assume without loss of generality that $N\,|_{U_1} = \dfrac{\psi_t \wedge \psi_\theta}{\|\psi_t \wedge \psi_\theta\|}$. Let $p = \psi(0,0) \in U_1$, and consider

$$\begin{aligned}(x_n)_{n\in\mathbb{N}^*} &= \left(\left(0, \frac{\pi}{n}\right)\right)_{n\in\mathbb{N}^*} \\ (y_n)_{n\in\mathbb{N}^*} &= \left(\left(0, 2\pi - \frac{\pi}{n}\right)\right)_{n\in\mathbb{N}^*}.\end{aligned}$$

We have:

$$\begin{cases} \dfrac{\partial\psi}{\partial t}(t,\theta) = \left(\cos\dfrac{\theta}{2}, 0, \sin\dfrac{\theta}{2}\right) \\ \dfrac{\partial\psi}{\partial\theta}(t,\theta) = \left(-1.5\sin\theta - \dfrac{t}{2}\sin\dfrac{\theta}{2}, 1.5\cos\theta, \dfrac{t}{2}\cos\dfrac{\theta}{2}\right). \end{cases}$$

After calculations, we get

$$(\psi_t \wedge \psi_\theta)(0,\theta) = -1.5\left(\cos\theta\sin\frac{\theta}{2}, \sin\theta\sin\frac{\theta}{2}, -\cos\theta\cos\frac{\theta}{2}\right)$$

and we see that $\lim_{n\to+\infty} N(\psi(x_n)) = e_3$ while $\lim_{n\to+\infty} N(\psi(y_n)) = -e_3$, where $e_3 = (0,0,1)$. This contradicts the continuity of N as both $\psi(x_n)$ and $\psi(y_n)$ converge to p. See [18] for a nice animation of this proof.

For orientable surfaces, one has the following:

Proposition 2.22. Every connected orientable regular surface has exactly two orientations.

Definition 2.23. Let S be a regular surface with orientation N. If

$$\forall p \in S, \|N(p)\| = 1,$$

we have $N(S) \subset S^2$. By restrecting the codomain of N to S^2, the map $N : S \to S^2$ is called the ***Gauss map*** of S.

A corollary of the previous proposition is:

Corollary 2.24. Every connected orientable regular surface has exactly two Gauss maps.

Example 2.25. The Gauss maps of S^2 are id_{S^2} and $-id_{S^2}$.

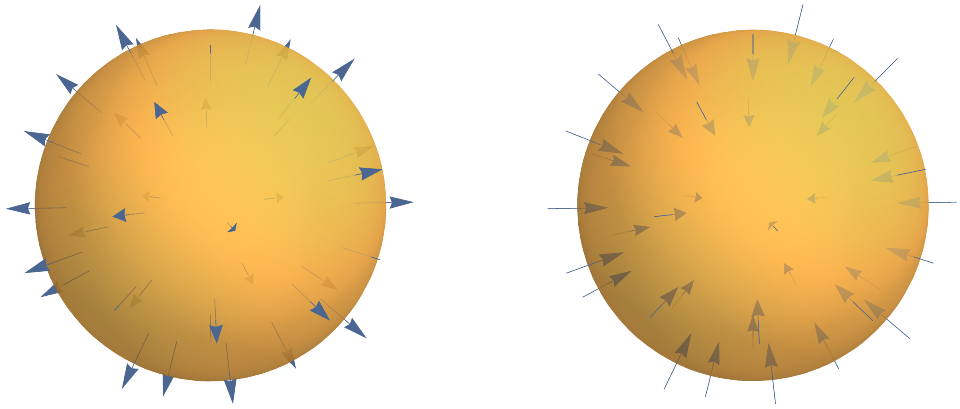


FIGURE 2.13: The Gauss maps of S^2

We recall now the notion of differentiability of maps defined on open sets of regular surfaces.

Definition 2.26. Let $f : V \cap S \to \mathbb{R}^3$ be a function, where S is a regular surface and S is an open set of $\mathbb{R}^3$. f is said to be ***differentiable*** at $p \in V \cap S$ if there exists a local parametrization $\varphi : U \to V \cap S$ at p such that $f \circ \varphi$ is differentiable at $\varphi^{-1}(p)$. When f is differentiable at every point of $V \cap S$, we say that f is ***differentiable*** on $V \cap S$.

In this case, the ***differential*** of f at p is

$$\begin{aligned} \mathrm{d}f_p : T_pS &\to \mathbb{R}^3 \\ v &\mapsto \frac{\mathrm{d}f \circ \alpha}{\mathrm{d}t}(0) \end{aligned}$$

where $\alpha : (-\varepsilon, \varepsilon) \to S$ is a curve such that $\alpha(0) = p$ and $\alpha'(0) = v$. This map is proven to be well defined and linear.

Remark 2.27. If $f : S_1 \to S_2$ is a differentiable function at $p \in S_1$ between two regular surfaces then $\operatorname{Im} \mathrm{d}f_p \subset T_{f(p)}S_2$. In such cases, we set as a convention to restrict the codomain of $\mathrm{d}f_p$ to become a map from T_pS_1 to $T_{f(p)}S_2$.

A remarkable property of a Gauss map is:

Proposition 2.28. The differential of a Gauss map N at a point $p \in S$ is a self-adjoint operator.

Proof. We have $\mathrm{d}N_p : T_pS \to T_{N(p)}S^2$. Since $\mathrm{d}N_p$ is a linear map and

$$T_{N(p)}S^2 = \{N(p)\}^\perp = T_pS$$

because $N(p) \in T_pS^\perp$, $\mathrm{d}N_p$ is an operator.
To show that $\mathrm{d}N_p$ is self-adjoint, let $\varphi : U \to V \cap S$ be a local parametrization of S at p, $q = \varphi^{-1}(p)$, $v_1 = \varphi_u(q)$ and $v_2 = \varphi_v(q)$. Note that (v_1, v_2) is a basis of T_pS. Hence, it is enough to show that $\langle \mathrm{d}N_p(v_1), v_2 \rangle = \langle v_1, \mathrm{d}N_p(v_2) \rangle$. We have:

$$\begin{cases} \langle N \circ \varphi, \varphi_u \rangle = 0 \\ \langle N \circ \varphi, \varphi_v \rangle = 0 \end{cases}$$

Taking the partial derivative with respect to v in the first equation and the partial derivative with respect to u in the second one, we get:

$$\begin{cases} \langle N \circ \varphi, \varphi_{vu} \rangle = -\langle (N \circ \varphi)_v, \varphi_u \rangle \\ \langle N \circ \varphi, \varphi_{uv} \rangle = -\langle (N \circ \varphi)_u, \varphi_v \rangle \end{cases} \tag{2.1}$$

We conclude by using the fact that $\varphi_{vu} = \varphi_{uv}$. □

Remark 2.29. The proof above also yields to (2.1).

We're now ready to tackle the curvature of regular surfaces. Firstly, we recall the following:

Definition 2.30. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve of class C^2 parametrized by arc length. The function

$$\begin{aligned} k_\gamma : I &\to \mathbb{R} \\ t &\mapsto \|\gamma''(t)\| \end{aligned}$$

is the ***curvature*** of γ. The map

$$\begin{aligned} n : I &\to \mathbb{R}^3 \\ t &\mapsto \frac{1}{k_\gamma(t)}\gamma''(t) \end{aligned}$$

assigns to each $t \in I$ the ***normal vector*** to γ at $\gamma(t)$.

We now define the first curvature that relates to surfaces:

Definition 2.31. Let $\gamma : I \to S$ be a regular curve parametrized by arc length in an oriented surface S, k_γ the curvature of γ, θ the angle between n and $N \circ \gamma$ so that $\cos\theta = \langle n, N \circ \gamma \rangle$, where n and N are, respectivly, the normal vector to γ and the Gauss map of S. The function $k_n = k\cos\theta$ is called the ***normal curvature*** of γ.

Suppose now that $\gamma : I = (-\varepsilon, \varepsilon) \to S$ with $p = \gamma(0) \in S$ and $v = \gamma'(0) \in T_pS$. We have $\langle N \circ \gamma, \gamma' \rangle = 0$, thus $\langle N \circ \gamma, \gamma'' \rangle = -\langle (N \circ \gamma)', \gamma' \rangle$. Hence:

$$\begin{aligned} k_n(p) &= \langle k(0)n(0), N(p) \rangle \\ &= \langle \gamma''(0), N(\gamma(0)) \rangle \\ &= -\langle \gamma'(0), \mathrm{d}N_p(\gamma'(0)) \rangle \\ &= -\langle v, \mathrm{d}N_p(v) \rangle . \end{aligned}$$

Therefore we can define the normal curvature independently of any curve on S, and simply say that $k_n(p)$ is the ***normal curvature*** of S at p along the direction v (where $v = \gamma'(0)$). This is known as Meusnier's theorem. The notation is quite inconvinient as $k_n(p)$ depends as well on the direction. This motivates the following definition:

Definition 2.32. Let S be an oriented regular surface and $p \in S$. The bilinear symmetric form

$$\begin{aligned}\mathrm{II_p} : \mathrm{T_pS} \times \mathrm{T_pS} &\to \mathbb{R}\\ (u, v) &\mapsto -\langle \mathrm{d}N_p(u), v\rangle\end{aligned}$$

is called the ***second fundamental form*** of S at p. The associated quadratic form maps each $v \in T_pS$ to the normal curvature at p along the direction v.

Since $\mathrm{d}N_p$ is a self-adjoint operator, it is diagonalizable in an orthonormal basis (v_1, v_2) of T_pS. Let $k_1, k_2 \in \mathbb{R}$ such that $\mathrm{d}N_p(v_1) = -k_1 v_1$, $\mathrm{d}N_p(v_2) = -k_2 v_2$ and $k_1 \geq k_2$ (we introduced the minus sign so that it disapears when manipulating the second fundamental form). Hence

$$\mathrm{II_p(v_1, v_1) = k_1 \wedge II_p(v_2, v_2) = k_2}.$$

Now consider any direction $v = \cos\theta v_1 + \sin\theta v_2$. A simple calculation shows that the curvature of S at p along the direction v is

$$\mathrm{II_p(v, v) = k_1} \cos^2\theta + \mathrm{k_2} \sin^2\theta.$$

Hence, we see that the curvature of S at p along any direction is between k_1 and k_2, which justifies the following definition:

Definition 2.33. Using the notations in the paragraph above, k_1 and k_2 are called respectively the ***maximum normal curvature*** and the ***minimum normal curvature*** of S at p. They are called the ***principal curvatures*** at p. The corresponding directions v_1 and v_2 are called ***principal directions*** at p.

Definition 2.34. The function

$$\begin{aligned}K : S &\to \mathbb{R}\\ p &\mapsto \det \mathrm{d}N_p\end{aligned}$$

is called the ***Gaussian curvature*** of S. The function

$$\begin{aligned}H : S &\to \mathbb{R}\\ p &\mapsto -\frac{\operatorname{tr} \mathrm{d}N_p}{2}\end{aligned}$$

is called the ***mean curvature*** of S.

The following proposition gives a relation between the fundamental forms and the differential of the Gauss map.

Proposition 2.35. Recall that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is the matrix of the first fundamental form with respect to the basis (φ_u, φ_v). Let

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

be the matrix of the second fundamental form with respect to the new basis, and

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

be the matrix of $\mathrm{d}N_p$, with $p \in S$. Then:

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Corollary 2.36. The Gaussian curvature is given by: $K = \dfrac{eg - f^2}{EG - F^2}$.

We admit the following result [37, p. 112]:

Corollary 2.37 (Brioschi formula)**.** The Gaussian curvature is given by:

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}. \tag{2.2}$$

In particular, if $F = 0$:

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right). \tag{2.3}$$

Brioschi's formula shows that the Gaussian curvature depends only on the metric coefficients of the surface. Recall the following:

Definition 2.38. Let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth curve. The ***length*** of γ is

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t$$

Definition 2.39. Let S_1 and S_2 be two regular surfaces and $f : S_1 \to S_2$ a diffeomorphism. f is said to be an ***isometry*** between S_1 and S_2 if f preserves the lengths of the curves, that is for every smooth curve $\gamma : [a, b] \to S_1$, $l(f \circ \gamma) = l(\gamma)$. We say that S_1 and S_2 are ***isometric***.

Proposition 2.40. Let S_1 and S_2 be two regular surfaces and $f : S_1 \to S_2$ a diffeomorphism. f is an isometry if and only if for every local parametrization φ of S_1, φ and $f \circ \varphi$ have the same metric coefficients.

Since the Gaussian curvature depends only on the metric coefficients of the surface which are invariant by isometry, this proves Gauss's Theorema Egregium[4].

Theorem 2.41 (Gauss's Theorema Egregium)**.** *Let S_1 and S_2 be two regular surfaces and $f : S_1 \to S_2$ an isometry. Let K_1 and K_2 be the Gaussian curvature of, respectively, S_1 and S_2. Then $K_1 = K_2 \circ f$.*

[4] Latin for "Remarkable Theorem"

It is often said that the curvature is the amount by which a geometric object such as a surface deviates from being flat, or a plane curve from being straight as in the case of a line. But what about curves on surfaces? How to define a straight curve on a curved surface? At a point $p = \gamma(t)$ of a regular curve γ on a regular surface S, we defined the curvature $k(t)$ of γ and the normal curvature $k_n(t_0)$ of S towards $\gamma'(t)$. We have

$$\gamma''(t) = k_n(t)N(p) + xv \tag{2.4}$$

where $v \in T_pS$ is a unit vector. x is simply $\langle \gamma''(t), v \rangle$. But if we take $-v$ instead, the sign changes (unless $x = 0$). For the moment, let's call its absolute value the ***unsigned geodesic curvature***.

We have the following relations:

$$\begin{cases} k_\gamma & = \|\gamma''\| \\ k_n & = \langle \gamma'', N \circ \gamma \rangle \,. \end{cases} \tag{2.5}$$

Seeking for a nice relation between the unsigned geodesic curvature and γ'' and aspiring for defining the (algebraic) geodesic curvature, we must introduce some new notions.

Definition 2.42. Let S be a regular surface. A ***vector field*** on S is a smooth function $W : S \to \mathbb{R}^3$ that assigns to each $p \in U$ a tangent vector $W(p) \in T_pS$ at p.
If $\gamma : I \to S$ is a regular curve, we shall call $w = W \circ \gamma$ its ***restriction*** to γ.
Let $p \in S$, $v \in T_pS$ and $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. The orthogonal projection of $\dfrac{\mathrm{d}w}{\mathrm{d}t}(0)$ onto T_pS is called the ***covariant derivative*** W at p relative to the vector v, and will be denoted by $\mathrm{D}_v W(p)$.

Note that if $w : \operatorname{dom}\alpha \to \mathbb{R}^3$ such that $\forall t \in \operatorname{dom}\alpha, w(t) \in T_{\alpha(t)}S$ isn't the result of the restriction of a vector field to α, we can still define the covariant derivative.

Definition 2.43. Let $\gamma : I \to S$ be a regular curve and $w : I \to \mathbb{R}^3$ be a smooth map such that $\forall t \in I, w(t) \in T_{\gamma(t)}S$. w is called a ***vector field*** along γ. The orthogonal projection of $\dfrac{\mathrm{d}w}{\mathrm{d}t}(t)$ onto $T_{\gamma(t)}S$ is called the ***covariant derivative*** of w at t and is denoted by $\dfrac{\mathrm{D}w}{\mathrm{d}t}(t)$.

According to this definition, every restriction w of a vector field W on S to a regular curve $\gamma : I \to S$ is a vector field along γ.

Let us now express the geodesic curvature using covariant derivatives. Consider again the relation $\gamma''(t) = k_n(t)N(p) + xv$ where $v \in T_pS$ is a unit vector. Assume furthermore that γ is parametrized by arc length. This means that $\langle \gamma'', \gamma' \rangle = 0$. But since $\langle N \circ \gamma, \gamma' \rangle = 0$, we have $\langle \gamma'(t), v \rangle = 0$, and as $v \perp N(\gamma(t))$, we conclude that $v /\!/ (\gamma'(t) \wedge N(\gamma(t)))$. As $\|\gamma'(t)\| = \|N(\gamma(t))\| = 1$ and $\gamma'(t) \perp N(\gamma(t))$, we have therefore $v = \pm(\gamma'(t) \wedge N(\gamma(t)))$. This observation can be easly generalized to lead to the following:

Definition 2.44. Let w be a ***unit vector field*** over γ, i.e, $\forall t \in I$, $\|w(t)\| = 1$. For any $t \in I$, there is a unique $\lambda(t) \in \mathbb{R}$ such that

$$\frac{\mathrm{D}w}{\mathrm{d}t}(t) = \lambda(t)\left(N(\gamma(t)) \wedge w(t)\right).$$

This defines a function $\lambda : I \to \mathbb{R}$ that is called the ***algebraic value of the covariant derivative*** of w at t and is denoted $\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right]$. Keep in mind that its sign depends on the orientation of the surface.

We can now define the geodesic curvature:

Definition 2.45. The ***geodesic curvature*** of an arc length regular curve γ on a regular oriented surface S with a Gauss map N at a point $\gamma(t)$ is the algebraic value of the covariant derivative of γ' at t.

(2.4) becomes

$$\gamma'' = k_n N \circ \gamma + k_g((N \circ \gamma) \wedge \gamma'). \tag{2.6}$$

This allows us to extend (2.5) to:

$$\begin{cases} k_\gamma &= \|\gamma''\| \\ k_n &= \langle \gamma'', N \circ \gamma \rangle \\ k_g &= \left[\frac{\mathrm{D}\gamma'}{\mathrm{d}t}\right] \\ k_\gamma^2 &= k_n^2 + k_g^2. \end{cases}$$

Definition 2.46. A regular curve which geodesic curvature is 0 is called a ***geodesic curve***.

More generally, we have the following definition:

Definition 2.47. A vector field along a regular curve is said to be a ***parallel*** if its covariant derivative is 0.

In the remaining of this section, we present some necessary facts that will allow us to find a formula for the algebraic value of the covariant derivative when we have an orthogonal parametrization.

Lemma 2.48. *Let v and w be two unit vector fields along a regular curve $\gamma : I \to S$ and $\theta : I \to \mathbb{R}$ be a lifting of*

$$\begin{aligned} f : I &\to S^1 \\ t &\mapsto \left(\langle w(t), v(t)\rangle, \langle w(t), \bar{v}(t)\rangle\right) \end{aligned}$$

*accross p, where $\bar{v}$ is such that $(v(t), \bar{v}(t))$ is a directed orthonormal basis of $T_{\gamma(t)}S$ and p is the map defined in Proposition 1.14 (we say that θ is an **angle function from** v **to** w). Then*

$$\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] - \left[\frac{\mathrm{D}v}{\mathrm{d}t}\right] = \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Proof. Clearly, we have:

$$w = \cos\theta\, v + \sin\theta\, \bar{v}. \tag{2.7}$$

Let

$$\begin{aligned} \bar{w} &= N \wedge w = \cos\theta N \wedge v + \sin\theta N \wedge \bar{v} && (2.8) \\ &= \cos\theta\, \bar{v} - \sin\theta\, v && (2.9) \end{aligned}$$

as $\bar{v} = N \wedge v$. By differentiating (2.7), we get

$$w' = -\theta' \sin\theta\, v + \cos\theta\, v' + \theta' \cos\theta\, \bar{v} + \sin\theta\, \bar{v}'.$$

By taking the inner product of w' with $\bar{w}$ and using the relation above and (2.8), we get:

$$\langle w', \bar{w} \rangle = \theta' + \langle v', \bar{v} \rangle \cos^2\theta - \langle \bar{v}', v \rangle \sin^2\theta.$$

As $\langle v, \bar{v} \rangle = 0$, we get by differentiating: $\langle v, \bar{v}' \rangle = -\langle v', \bar{v} \rangle$. Hence the relation above becomes

$$\langle w', \bar{w} \rangle = \theta' + \langle v', \bar{v} \rangle .$$

As $\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] = \langle w', \bar{w} \rangle$ and $\left[\frac{\mathrm{D}v}{\mathrm{d}t}\right] = \langle v', \bar{v} \rangle$, we conclude that

$$\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] - \left[\frac{\mathrm{D}v}{\mathrm{d}t}\right] = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

□

Remarks 2.49. If we apply this lemma to $w = \gamma'$ and v a parallel vector field along γ, we get

$$k_g = \left[\frac{\mathrm{D}\gamma'}{\mathrm{d}t}\right] = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Hence the geodesic curvature is the rate of change of the angle between the tangent to the curve and any parallel vector field. Of course, for this to make sense, we must ensure that there does exist at least one parallel vector field. There are in fact infinitely many as shown in Proposition 2 of [15, p. 242].

We will now give the promised formula of the algebraic value of the covariant derivative.

Proposition 2.50. Let $\varphi : U \to S \cap V$ be a local parametrization of S,

$$\begin{aligned} \gamma : I &\to U \\ t &\mapsto (x(t), y(t)) \end{aligned}$$

be a regular curve and w be a vector field along γ. Assume furthermore that S is oriented and that φ is an orthogonal parametrization compatible compatible with the orientation of S. Let θ be an angle function from φ_u restricted to γ to w. Then:

$$\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] = \frac{1}{2\sqrt{EG}}\left(G_u \frac{\mathrm{d}y}{\mathrm{d}t} - E_v \frac{\mathrm{d}x}{\mathrm{d}t}\right) + \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Proof. Let $e_1 = \dfrac{\varphi_u}{\sqrt{E}}$ and $e_2 = \dfrac{\varphi_v}{\sqrt{G}}$. Since the parametrization is orthogonal and compatible with the orientation of S, we have $e_1(t) \wedge e_2(t) = N(u(t), v(t))$. According to Lemma 2.2, we have

$$\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] = \left[\frac{\mathrm{D}e_1}{\mathrm{d}t}\right] + \frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{2.10}$$

All we need is to find a suitable expression of $\left[\dfrac{\mathrm{D}e_1}{\mathrm{d}t}\right]$ to conclude the proof. We have:

$$\begin{aligned}\left[\frac{\mathrm{D}e_1}{\mathrm{d}t}\right] &= \left\langle \frac{\mathrm{d}e_1}{\mathrm{d}t}, N(u(t), v(t)) \wedge e_1 \right\rangle \\ &= \left\langle \frac{\mathrm{d}e_1}{\mathrm{d}t}, e_2 \right\rangle \\ &= \frac{\mathrm{d}x}{\mathrm{d}t} \left\langle (e_1)_u, e_2 \right\rangle + \frac{\mathrm{d}y}{\mathrm{d}t} \left\langle (e_1)_v, e_2 \right\rangle . \end{aligned} \tag{2.11}$$

Let's calculate these inner products. The parametrization being orthogonal, we have $\langle \varphi_u, \varphi_v \rangle = F = 0$. Differentiating with respect to u, we get $\langle \varphi_{uu}, \varphi_v \rangle = -\langle \varphi_u, \varphi_{vu} \rangle = -\langle \varphi_u, \varphi_{uv} \rangle$. Differentiating $\langle \varphi_u, \varphi_u \rangle = E$ by v, we get $\langle \varphi_u, \varphi_{uv} \rangle = \frac{1}{2} E_v$. Therefore $\langle \varphi_{uu}, \varphi_v \rangle = -\frac{1}{2} E_v$. Hence

$$\langle (e_1)_u, e_2 \rangle = -\frac{1}{2\sqrt{EG}} E_v.$$

A similar argument shows that

$$\langle (e_1)_v, e_2 \rangle = \frac{1}{2\sqrt{EG}} G_u.$$

Replacing in (2.10) and (2.11), we get the desired result. □

In the proposition above, we used an orthogonal parametrization on an arbitrary regular surface. We shall prove in the remaining of this section that such a parametrization exists.

Definition 2.51. Let $U \subset \mathbb{R}^2$. A ***vector field*** w on U is a smooth map that assigns to each point $p \in U$ a vector $w(p) \in \mathbb{R}^2$.

A ***trajectory*** of w is a differentiable parametrized curve $\alpha : I \to U$ such that $\alpha' = w \circ \alpha$.

An important theorem is the local existence of trajectories:

Theorem 2.52. *Let w be a vector field on an open set $U \subset \mathbb{R}^2$ and $p \in U$. There exists locally a unique trajectory $\alpha : I \to U$ of w such that $\alpha(0) = p$, where I is an open interval containing 0.*

There is even more:

Theorem 2.53. *Let w be a vector field on an open set $U \subset \mathbb{R}^2$ and $p \in U$. There exists an open interval I containing 0, a neighborhood $V \subset U$ of p and a differentiable map $\alpha : V \times I \to U$ such that for every $q \in V$, the curve*

$$\begin{aligned} \alpha_q : I &\to U \\ t &\mapsto \alpha(q,t) \end{aligned}$$

*is a trajectory of w such that $\alpha(q,0) = q$. Such a map is called the **local flow of** w **at** p.*

The proofs of these theorems rely entirely on the theory of ordinary differential equations [15, p. 176-177]. They allow us to prove the following:

Proposition 2.54. Let w be a vector field on an open set $U \subset \mathbb{R}^2$ and let $p \in U$. Suppose that $w(p) \neq 0$. Then there exist a neighborhood $V \subset U$ of p and a differentiable function $f : W \to \mathbb{R}$ such that f is constant along each trajectory of w and $\forall q \in V, \mathrm{d}f_q \neq 0$. Such a function f is called a ***first integral*** of w at p.

Proof. We can assume without loss of generality that $p = (0,0)$ and $w(p)$ is in the direction of the x-axis. Let $\alpha : V \times I \to U$ be a local flow at p, $W = \{y \in \mathbb{R} \mid (0,y) \in V\}$ and

$$\begin{aligned} \tilde{\alpha} : W \times I &\to U \\ (y,t) &\mapsto \alpha(0,y,t). \end{aligned}$$

We have $\dfrac{\partial \tilde{\alpha}}{\partial t}(0,0) = w(\alpha(p,0)) = w(p)$, which belongs to the x-axis. On the other hand, as $\forall y \in W, \tilde{\alpha}(y,0) = (0,y)$, we have $\dfrac{\partial \tilde{\alpha}}{\partial t}(0,0) = (0,1)$, which belongs to the y-axis. Hence $\mathrm{d}\tilde{\alpha}_{(0,0)}$ is nonsingular. According to the implicit function theorem, there exists a neighborhood $\tilde{W} \subset U$ of p where $\tilde{\alpha}^{-1}$ is a diffeomorphism such that $\forall q \in \tilde{W}, (\mathrm{d}\tilde{\alpha}^{-1})_q$ is invertible. Let

$$\begin{aligned} P : W \times I &\to W \\ (y,t) &\mapsto y \end{aligned}$$

and let $f = P \circ \tilde{\alpha}^{-1}$. f is constant along each trajectory of w and

$$\forall q \in \tilde{W}, \mathrm{d}f_q = P \circ (\mathrm{d}\tilde{\alpha}^{-1})_q \neq 0.$$

□

Remark 2.55. The results we just proved for vector fields hold as well for vector fields over a coordinate neighborhood of a regular surface because such a set is diffeomorphic to an open set of $\mathbb{R}^2$.

We now prove a theorem that will lead us to the desired result.

Theorem 2.56. *Let w_1 and w_2 be two smooth vector fields in an open set $V \cap S$ of s and $p \in V \cap S$. Suppose that $(w_1(p), w_2(p))$ is linearly independent. Then there is a* *local parametrization $\varphi : U \to W \cap S \subset V \cap S$ of S at p such that for every $q \in U$, $\varphi_u(q) \,/\!/\, w_1(\varphi(q))$ and $\varphi_v(q) \,/\!/\, w_2(\varphi(q))$.*

Proof. Let $\tilde{W} \cap S$ be a neighborhood of p where the first integrals f_1 and f_2 of, respectively, w_1 and w_2 are defined. Let

$$\begin{aligned}\psi : \tilde{W} \cap S &\to \mathbb{R}^2\\ q &\mapsto (f_1(q), f_2(q)),\end{aligned}$$

$e_1 = (1,0)$ and $e_2 = (0,1)$. Since f_1 and f_2 are constant on the trajectories of, respectively, w_1 and w_2, we have $(\mathrm{d}f_1)_p(w_1(p)) = (\mathrm{d}f_2)_p(w_2(p)) = 0$. On the other hand, as $(w_1(p), w_2(p))$ is a basis for T_pS, we have $(\mathrm{d}f_2)_p(w_1(p)) = a \neq 0$. Hence $\mathrm{d}\psi_p(w_1(p)) = ae_2 \neq 0$. Similarly, $(\mathrm{d}f_1)_p(w_2(p)) = b \neq 0$ and thus $\mathrm{d}\psi_p(w_2(p)) = be_1 \neq 0$. Hence $\mathrm{d}\psi_p$ is nonsingular. According to the inverse function theorem, there exists a neighborhood $U \subset \mathbb{R}^2$ of $\psi(p)$ and a neighborhood $V \cap S$ of p such that $\varphi = \left(\psi\,|_{V\cap S}\right)^{-1} : U \to V \cap S$ is a diffeomorphism and for every $q \in U$, $\mathrm{d}\varphi_q$ is nonsingular and thus injective, and $w_1(q)$ and $w_2(q)$ are linearly independent, with $(\mathrm{d}f_1)_q(w_2(q') = b(q) \neq 0$ and $(\mathrm{d}f_2)_q(w_1(q') = a(q)$, where $q' = \varphi(q)$. Let $q \in U$ and $q' = \varphi(q)$. We have $\mathrm{d}\varphi_q = (\mathrm{d}\psi_{q'})^{-1}$. Thus $\varphi_u(q) = (\mathrm{d}\psi_{q'})^{-1}(e_1)$ and $\varphi_v(q) = (\mathrm{d}\psi_{q'})^{-1}(e_2)$. Since $\forall v \in T_pS, \mathrm{d}\psi_{q'}(v) = ((\mathrm{d}f_1)_{q'}(v), (\mathrm{d}f_2)_{q'}(v))$, we see that $\mathrm{d}\psi_{q'}\left(\frac{1}{b}w_2(q')\right) = e_1$ and that $\mathrm{d}\psi_{q'}\left(\frac{1}{a}w_1(q')\right) = e_2$. Hence

$$\varphi_u(q) = \frac{1}{b(q)}w_2(q') \wedge \varphi_v(q') = \frac{1}{a(q)}w_1(q').$$

This shows that the partial derivatives of φ are of class C^∞ and that

$$\forall q \in U, \varphi_u(q) \,/\!/\, w_1(q) \wedge \varphi_v(q) \,/\!/\, w_2(q).$$

□

Corollary 2.57. Let S be a regular surface and $p \in S$. Then there exists an orthogonal local parametrization $\varphi : U \to V \cap S$ of S at p.

Proof. Let $\tilde{\varphi} : \tilde{U} \to \tilde{V} \cap S$ be a local parametrization of S at p. If it is orthogonal, we're done. Otherwise, let $w_1 = \tilde{\varphi}_{\tilde{u}} \circ \tilde{\varphi}^{-1}$ and $w_2 = -\frac{\tilde{F}}{\tilde{E}}\tilde{\varphi}_{\tilde{u}} + \tilde{\varphi}_{\tilde{v}}$, where $\tilde{E}$ and $\tilde{F}$ are the coefficients of the matrix of the first fundamental form relative to $\tilde{\varphi}$. According to the previous theorem, there exists a local parametrization $\varphi : U \to V \cap S \subset \tilde{V} \cap S$ at p such that for any $q \in U$, $\varphi_u(q) \,/\!/\, w_1(q)$ and $\varphi_v(q) \,/\!/\, w_2(q)$. As $\forall q \in U, w_1(q) \perp w_2(q)$, we conclude that φ is an orthogonal parametrization. □

2.3 The Local Gauss-Bonnet Theorem

In this section, we present a proof of the local Gauss-Bonnet Theorem.

Definition 2.58. Let S be an oriented regular surface. A ***region*** $R \subset S$ is a set that can be written as the union of a connected open set and its boundary. The region is called a ***simple region*** if R is homeomorphic to a disk and the boundary δR of R is the trace of a polygonal curve $\gamma : I \to \delta R$.

We can extend Definitions 1.22 and 1.23 to curves in the space if we define the exterior angle.

Definition 2.59. Let S be a regular surface, $\gamma : I \to S$ be a polygonal curve. Let $t \in I$ such that $p = \gamma(t)$ is a vertex of γ. Let φ be a local orthogonal parametrization of S at p, $v_1 = \dfrac{\varphi_u(p)}{\|\varphi_u(p)\|}$ and $v_2 = \dfrac{\varphi_v(p)}{\|\varphi_v(p)\|}$. We identify $\mathcal{B}_p = (v_1, v_2)$ with the canonical basis $\mathcal{B}_c = (e_1, e_2)$ of $\mathbb{R}^2$ by the isomorphism $\psi_p : T_pS \to \mathbb{R}$ defined by $\psi_p(v_1) = e_1$ and $\psi_p(v_2) = e_2$. The ***interior angle*** of γ at $\gamma(t)$ is the ***directed angle*** $\angle(\gamma'(t^-), \gamma'(t^+)) = \angle(\psi(\gamma'(t^-)), \psi(\gamma'(t^+)))$.

Definition 2.60. Let S be a regular surface and $\gamma : I \to V \cap S$ be a closed curve where $V \cap S$ is a coordinate neighborhood of a local orthogonal parametrization of S. γ is said to be ***positively oriented*** if $I(\gamma) > 0$.

Definition 2.61. Let S be a regular surface, $\varphi : U \to S \cap V$ be a local parametrization of φ, $f : S \cap V \to \mathbb{R}$ be a continuous function. The ***integral*** of f over $D \subset S \cap V$ is

$$\iint_D f \, \mathrm{d}\sigma = \iint_{\varphi^{-1}(D)} f(\varphi(u, v)) \, \|\varphi_u \wedge \varphi_v\| \, (u, v) \, \mathrm{d}u \, \mathrm{d}v. \tag{2.12}$$

Remark 2.62. Let θ be the oriented angle between φ_u and φ_v. We have:

$$\begin{aligned}
\|\varphi_u \wedge \varphi_v\|^2 &= \|\varphi_u\|^2 \, \|\varphi_v\|^2 \sin^2\theta \\
&= \|\varphi_u\|^2 \, \|\varphi_v\|^2 - (\|\varphi_u\| \, \|\varphi_v\| \cos\theta)^2 \\
&= EG - F^2.
\end{aligned}$$

Hence (2.12) becomes:

$$\iint_D f \, \mathrm{d}\sigma = \iint_{\varphi^{-1}(D)} f(\varphi(u, v)) \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v.$$

We recall the following theorem:

Theorem 2.63 (Green's Theorem). *Let $P, Q : R \to \mathbb{R}$ be differentiable functions defined on a simple region R,*

$$\begin{aligned}
\gamma : I &\to \delta R \\
t &\mapsto (x(t), y(t))
\end{aligned}$$

be a simple closed piecewise regular curve which trace is δR, that is positively oriented. Then

$$\int_I \left(P(\gamma(t)) \frac{\mathrm{d}x}{\mathrm{d}t} + Q(\gamma(t)) \frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t = \iint_{\mathring{R}} \left(Q_u - P_v\right)(u,v)\, \mathrm{d}u \, \mathrm{d}v$$

where $\mathring{R}$ is the interior of R.

Theorem 2.64 (The Local Gauss-Bonnet Theorem)**.** *Let $\varphi : U \to V \cap S$ be an orthogonal parametrization of a surface S oriented on $V \cap S$ by the orientation compatible with φ, with $U \subset \mathbb{R}^2$ being homeomorphic to an open disk. Let $R \subset S \cap V$ be a simple region of S and let $\gamma : I \to S \cap V$ be a positively oriented arc length parametrized regular curve which trace is ∂R, and let $\gamma(t_1), \cdots, \gamma(t_n)$ and $\theta_1, \cdots, \theta_n$ be, respectively, the vertices and external angles of γ. Then*

$$\int_I k_g(t)\, \mathrm{d}t + \iint_R K \, \mathrm{d}\sigma + \sum_{i=1}^{n} \theta_i = 2\pi.$$

For the proof, we need the following lemma:

Lemma 2.65. *Let S be a regular surface, $\varphi : U \to V \cap S$ be a local parametrization of S and $\gamma : I \to V \cap S$ be a polygonal curve. Let $\alpha = \varphi^{-1} \circ \gamma$. Then $I(\gamma) = I(\alpha)$.*

Proof. Let $t_1, \ldots, t_n$ be the points of I which images by γ are vertices. Assume without loss of generality that $t_1 = \min I$ and let $t_{n+1} = \max I$ and $\forall i \in \{1, \ldots, n\}, I_i = [t_i, t_{i+1})$. Also, let, for every $i \in \{1, \ldots, n\}$, θ_i be the exterior angle of α at $\alpha(t_i)$. For any point $t \in I$ and $\lambda \in J = [0,1]$, let $\langle \cdot, \cdot \rangle_t^\lambda$ be the inner product on $\mathbb{R}^2$ defined by:

$$\forall x, y \in \mathbb{R}^2, \langle x, y \rangle_t^\lambda = (1-\lambda) \langle x, y \rangle + \lambda \left\langle \mathrm{d}\varphi_{\alpha(t)}(x), \mathrm{d}\varphi_{\alpha(t)}(y) \right\rangle_{\gamma(t)},$$

$\|\cdot\|_t^\lambda$ be its associated norm and let $(v_1^\lambda(t), v_2^\lambda(t))$ be the result of applying the Gram-Schmidt process to (e_1, e_2), the canonical basis of $\mathbb{R}^2$. Clearly, $v_1^\lambda(t)$ and $v_2^\lambda(t)$ depend continuously on t and λ. Also, let, for $i \in \{1, 2\}$,

$$a_i^\lambda(t) = \frac{\left\langle \alpha'(t), v_i^\lambda(t) \right\rangle_t^\lambda}{\|\alpha'(t)\|_s^\lambda}$$

when $t \notin \{t_1, \ldots, t_n\}$ and

$$\begin{aligned} \beta : I \backslash \{t_1, \ldots, t_n\} \times J &\to S^1 \\ (t, \lambda) &\mapsto (a_1^\lambda(t), a_2^\lambda(t)). \end{aligned}$$

We construct a "lifting" $\phi : I \times J \to \mathbb{R}$ of β recursivly as follows: firstly, we let $\phi|_{I_1 \times J}$ be a lifting of $\beta|_{I_1 \times J}$ (see Theorem 1.26). Then for every $i \in \{1, \ldots, n-1\}$, let $\phi|_{I_{i+1} \times J}$ be the lifting of $\beta|_{I_i \times J}$ satisfying

$$\phi(t_{i+1}, 0) = \lim_{t \to t_{i+1}^-} \phi(t, 0) + \theta_{i+1},$$

and finally let $\phi(t_{n+1},0) = \lim_{t\to t_{n+1}^-}\phi(t,0)+\theta_1$. Note that $t\mapsto\phi(t,0)$ is an angle function of α and $t\mapsto\phi(t,1)$ is an angle function of γ. In fact, the function

$$\begin{aligned} f : J &\to \mathbb{R} \\ \lambda &\mapsto \frac{\phi(t_{n+1},\lambda)-\phi(t_1,\lambda)}{2\pi} \end{aligned}$$

is continuous on the interval J and integer valued, with $f(0) = I(\alpha)$ and $f(1) = I(\gamma)$. Hence, f is constant and $I(\alpha) = I(\gamma)$. $\square$

Proof of the local Gauss-Bonnet theorem. Suppose without loss of generality that $t_1 = \min I$. Let $t_{n+1} = \sup I$. Since γ is closed, $\gamma(t_{n+1}) = \gamma(t_1)$. Let for every $i\in\{1,\cdots,n\}, I_i = [t_i,t_{i+1}]$. We have $I = \bigcup_{i=1}^n I_i$. Let $\alpha = \varphi^{-1}\circ\gamma$ and $x,y : I\to\mathbb{R}$ such that $\forall t\in I, \alpha(t) = (x(t),y(t))$. According to Proposition 2.50, we have for every $i\in\{1,\cdots,n\}$ and $t\in I_i$,

$$k_g = \frac{1}{2\sqrt{EG}}\left(G_u\frac{\mathrm{d}y}{\mathrm{d}t}-E_v\frac{\mathrm{d}x}{\mathrm{d}t}\right)+\frac{\mathrm{d}\varphi_i}{\mathrm{d}t}$$

where φ_i is the angle function from φ_u to γ' on I_i. Note that these angle functions can be chosen in such a way that

$$\forall i\in\{1,\ldots,n-1\}, \varphi_{i+1}(t_{i+1}) = \varphi_i(t_{i+1})+\theta_{i+1}.$$

By integrating:

$$\int_I k_g\,\mathrm{d}t = \int_I\left(\frac{G_u}{2\sqrt{EG}}\frac{\mathrm{d}y}{\mathrm{d}t}-\frac{E_v}{2\sqrt{EG}}\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t+\sum_{i=1}^n\int_{I_i}\frac{\mathrm{d}\varphi_i}{\mathrm{d}t}\,\mathrm{d}t. \tag{2.13}$$

According to Green's formula and (2.3):

$$\begin{aligned} \int_I\left(\frac{G_u}{2\sqrt{EG}}\frac{\mathrm{d}y}{\mathrm{d}t}-\frac{E_v}{2\sqrt{EG}}\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t &= \iint_{\varphi^{-1}(\mathring{R})}\left(\left(\frac{G_u}{2\sqrt{EG}}\right)_u+\left(\frac{E_v}{2\sqrt{EG}}\right)_v\right)\mathrm{d}u\,\mathrm{d}v \\ &= -\iint_{\varphi^{-1}(\mathring{R})}K\sqrt{EG}\,\mathrm{d}u\,\mathrm{d}v \\ &= -\iint_R K\,\mathrm{d}\sigma. \qquad \text{(because } F=0\text{)} \end{aligned} \tag{2.14}$$

On the other hand, we have:

$$
\begin{aligned}
\sum_{i=1}^{n} \int_{I_i} \frac{\mathrm{d}\varphi_i}{\mathrm{d}t}\,\mathrm{d}t &= \sum_{i=1}^{n} \varphi_i(t_{i+1}) - \varphi_i(t_i) \\
&= \sum_{i=1}^{n} \varphi_i(t_{i+1}) - \sum_{i=1}^{n} \varphi_{i+1}(t_{i+1}) \\
&= \sum_{i=1}^{n-1} \varphi_i(t_{i+1}) - \sum_{i=2}^{n} \varphi_{i+1}(t_{i+1}) + \varphi_n(t_{n+1}) - \varphi_1(t_1) \\
&= \varphi_n(t_{n+1}) - \varphi_1(t_1) - \sum_{i=1}^{n-1} \varphi_{i+1}(t_{i+1}) - \varphi_i(t_{i+1}) \\
&= ((\theta_1 + \varphi_n(t_{n+1})) - \varphi_1(t_1)) - \theta_1 - \sum_{i=1}^{n-1} \theta_{i+1} \\
&= 2\pi I(\gamma) - \sum_{i=1}^{n} \theta_i \\
&= 2\pi I(\alpha) - \sum_{i=1}^{n} \theta_i \quad \text{(according to Lemma 2.65)} \\
&= 2\pi - \sum_{i=1}^{n} \theta_i .
\end{aligned}
\tag{2.15}
$$

Replacing (2.14) and (2.15) in (2.13), we get

$$
\int_I k_g(t)\,\mathrm{d}t + \iint_R K\,\mathrm{d}\sigma + \sum_{i=1}^{n} \theta_i = 2\pi
$$

as desired. □

Remark 2.66. Hopf's Umlaufsatz is a special case of this theorem. Indeed, the surface S in this case is a plane, which Gaussian curvature is 0. According to the Jordan-Schoenflies theorem [13, Theorem 4.1 on p. 864], the trace of a simple closed curve is the boundary of a simple region. Finally, the geodesic curvature is the same as the curvature we defined in the previous chapter.

Example 2.67. Consider the torus S described in Example 2.12. Let

$$R = \left\{((2+\sin t)\cos\theta, (2+\sin t)\sin\theta, \cos t) \mid (t,\theta) \in \left[0, \frac{\pi}{3}\right] \times \left[0, \frac{\pi}{4}\right]\right\}.$$

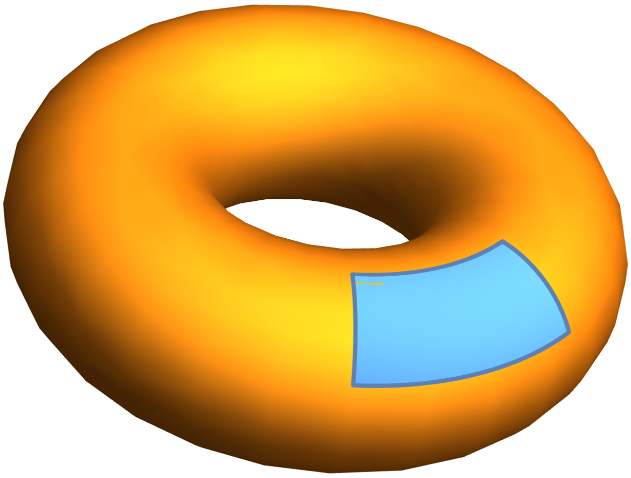


FIGURE 2.14: Simple region on a torus

We illustrate the formula in this example. We work with the following local parametrization:

$$\begin{aligned} \varphi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2 &\to S \\ (t,\theta) &\mapsto ((2+\sin t)\cos\theta, (2+\sin t)\sin\theta, \cos t). \end{aligned}$$

This parametrization is orthogonal. Thus we can use formula (2.3) to calculate the Gaussian curvature. We have:

$$\begin{aligned} E &= 1; \\ E_t &= E_\theta = 0; \\ G &= (2+\sin t)^2; \\ G_t &= 2\cos t(2+\sin t); \\ G_\theta &= 0; \\ \sqrt{EG} &= 2+\sin t; \\ K &= \frac{\sin t}{2+\sin t}; \\ \iint_R K\,\mathrm{d}\sigma &= \frac{\pi}{4}\int_0^{\frac{\pi}{3}} \sin t\,\mathrm{d}t \\ &= \frac{\pi}{8} \end{aligned}$$

Now let's calculate the total geodesic curvature. Note that by rotational symmetry, the two "vertical" curves of δR in Figure 2.14 have, in absolute value, the same geodesic curvature. As, when integrating, they will be parcoured in opposing directions, the total geodesic curvature over them will vanish (actually, these curves are geodesics). Hence, it's enaugh to calculate the geodesic curvature of the two "horizontal" curves parametrized by arc length as follows:

$$\begin{aligned}\gamma_1 : \left[0, \frac{\pi}{2}\right] &\to S\\ \theta &\mapsto \left(2\cos\left(\frac{\theta}{2}\right), 2\sin\left(\frac{\theta}{2}\right), 1\right);\end{aligned}$$

$$\begin{aligned}\gamma_2 : \left[0, \left(2+\frac{\sqrt{3}}{2}\right)\frac{\pi}{4}\right] &\to S\\ \theta &\mapsto \left(\left(2+\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\theta}{2+\frac{\sqrt{3}}{2}}\right), \left(2+\frac{\sqrt{3}}{2}\right)\sin\left(\frac{\theta}{2+\frac{\sqrt{3}}{2}}\right), \frac{1}{2}\right).\end{aligned}$$

From (2.6), we have the following formula:

$$k_g = \langle \gamma'', (N \circ \gamma) \wedge \gamma' \rangle .$$

We get:

$$\begin{aligned}k_{g_1} &= \frac{1}{2}\cos 0 = \frac{1}{2};\\ k_{g_2} &= \frac{1}{2+\frac{\sqrt{3}}{2}}\cos\frac{\pi}{3} = \frac{1}{2\left(2+\frac{\sqrt{3}}{2}\right)};\\ \int_{\delta R} k_g(\theta)\,\mathrm{d}\theta &= -\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{2} + \int_0^{(2+\frac{\sqrt{3}}{2})\frac{\pi}{4}} \frac{\mathrm{d}\theta}{2\left(2+\frac{\sqrt{3}}{2}\right)}\\ &= -\frac{\pi}{8}.\end{aligned}$$

Hence

$$\int_I k_g(t)\,\mathrm{d}t + \iint_R K\,\mathrm{d}\sigma = 0.$$

Finally, it is easy to see that the sum of the exterior angles $\theta_1, \ldots, \theta_4$ is 2π. Therefore:

$$\int_I k_g(t)\,\mathrm{d}t + \iint_R K\,\mathrm{d}\sigma + \sum_{i=1}^{4}\theta_i = 2\pi.$$

Definition 2.68. Let S be a regular surface. A ***geodesic triangle*** on S is a triangle which edges are geodesics.

Corollary 2.69. Let S be a regular surface and T be an geodesic triangle contained in a coordinate neighborhood of an orthogonal parametrization of S with interior angles α, β and γ. Then

$$\iint_T K\,\mathrm{d}\sigma = \alpha + \beta + \gamma - \pi.$$

This number is called the ***angle defect of*** T as it expresses the difference between the sum of the interior angles of T and π, the sum of the interior angles of any triangle in the plane.

Example 2.70. Consider a geodesic triangle[5] T on a sphere S with radius R with interior angles α, β and γ. Girard's theorem states that

$$\mathcal{A}(T) = (\alpha + \beta + \gamma - \pi)R^2$$

where $\mathcal{A}(T)$ is the area of T.

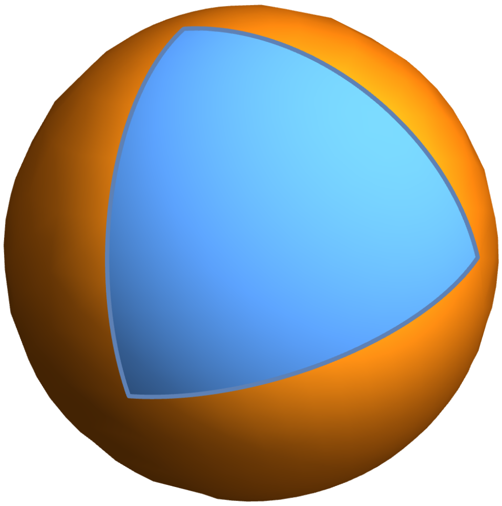


FIGURE 2.15: A geodesic triangle on a sphere

This result can be proven independently from the local Gauss-Bonnet theorem (see [42]), but can be easly found using the previous corollary. Indeed, the Gaussian curvature of the sphere is constant and equals $\dfrac{1}{R^2}$.

The previous corollary can be easly generalized to polygones.

Definition 2.71. Let S be a regular surface. A ***geodesic polygon*** on S is a polygon which edges are geodesics.

Corollary 2.72. Let S be a regular surface and P be an n-sided simple geodesic polygon contained in a coordinate neighborhood of an orthogonal parametrization of S with interior angles $\alpha_1, \ldots, \alpha_n$. Then

$$\iint_P K \, \mathrm{d}\sigma = \sum_{i=1}^{n} \alpha_i - (n-2)\pi.$$

This number is called the ***angle defect of*** P.

As noted in Corollary 2.37, the Gaussian curvature is entirely determined by the metric coefficients of a regular surface, which in turn are determined

[5] A triangle which edges are geodesics of the surface on which it is drawn.

using the usual inner product of $\mathbb{R}^3$ for the tangent space at each point of the surface. One can consider abstract regular surfaces by considering different inner products.

Definition 2.73. Let S be a regular surface. A ***Riemannian metric*** on S is a set g of inner products $g_p : T_pS \times T_pS \to \mathbb{R}$, $p \in S$ which varies smoothly with p in the following sense: for every vector fields X and Y on S, the map $p \to g_p(X(p), Y(p))$ is smooth. A regular surface endowed with a Riemannian metric is an example of a ***smooth Riemannian manifold***. We call such a structure a ***metric surface***[6].

Definition 2.74. Let (S, g) be a metric surface. The ***first fundamental form*** of S at $p \in S$ is the quadratic form

$$\begin{aligned} \mathrm{I}_p : T_p(S) &\to \mathbb{R} \\ v &\mapsto g_p(v, v) \end{aligned}$$

Let φ be a local parametrization of S at p, $q = \varphi^{-1}(p)$ and let $\varphi_u(q) = \dfrac{\partial \varphi}{\partial u}(q)$ and
$\varphi_v(q) = \dfrac{\partial \varphi}{\partial v}(q)$. Then the matrix of I_p in the basis $\{\varphi_u(q), \varphi_v(q)\}$ is given by

$$\begin{pmatrix} g_{11}(q) & g_{12}(q) \\ g_{21}(q) & g_{22}(q) \end{pmatrix}$$

where $g_{11} = E = g_p(\varphi_u, \varphi_u)$, $g_{12} = g_{21} = F = g_p(\varphi_u, \varphi_v)$ and $g_{22} = G = g_p(\varphi_v, \varphi_v)$ are called the ***metric coefficients*** of S. Note that the g_{ij} are smooth functions. Also note that the matrix of I_p is invertible as it is symmetric and I_p is positive.

Definition 2.75. Let (S, g) be a metric surface. The ***Gaussian curvature*** of S is the function $K : S \to \mathbb{R}$ defined by (2.2).

The geodesic curvature can also be defined for metric surfaces, but it requires materials beyond the scope of this document. The local Gauss-Bonnet theorem does apply for smooth Riemannian manifolds [27, Theorem 9.3 on p. 164]. Hence the two previous corollaries hold for metric surfaces. Geodesics on a regular surface are the curves that minimize lengths locally as shown in [30, Theorem 4.3.1 on p. 59]. The length of a curve on a regular surface involves only its first fundamental form. Hence, we can extend this notion to smooth Riemannian manifolds. It turns out that, for smooth Riemannian manifolds as well, geodesics are the curve that minimize lengths locally [27, Theorems 6.6 and 6.12 on p. 100 and p. 107].

Example 2.76. Let $\mathbb{H} = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$. $\mathbb{H}$ can be homeomorphically identified with $S = \{(u, v, 0) \in \mathbb{R}^2 \times \{0\} \mid v > 0\}$, which is a regular surface

[6]We wanted to use the name "Riemannian surfaces" instead, but it turns out that this name is standardly used to denote a different structure.

for which the homeomorphism

$$\begin{aligned}\varphi : \mathbb{H} &\to S\\ (u, v) &\mapsto (u, v, 0)\end{aligned}$$

between $\mathbb{H}$ and S is a global parametrization. Hence, we work on $\mathbb{H}$ as a regular surface instead of S. Notice that all the tangent planes of S are parallel and equal $\mathbb{R}^2 \times \{0\}$. Hence, we consider $\mathbb{R}^2$ the tangent plane of $\mathbb{H}$ at every point, and we endow $\mathbb{H}$ with the Riemannian metric at every point $p = (u, v)$ defined by

$$\begin{aligned}g_p : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R}\\ (v_1, v_2) &\mapsto \frac{\langle v_1, v_2\rangle}{v^2}\end{aligned}$$

where $\langle , \rangle$ is the usual inner product on $\mathbb{R}^2$. The metric surface $(\mathbb{H}, g)$ is called the ***hyperbolic plane*** because it is a model of hyperbolic geometry. It is also called the ***Poincaré half-plane model***. The metric coefficients are

$$\begin{aligned}g_{11}(p) = g_{22}(p) &= \frac{1}{v^2}\\ g_{12}(p) = g_{21}(p) &= 0\end{aligned}$$

Since the parametrization is orthogonal, we can use (2.3) to calculate the Gaussian curvature. Since $v > 0$, we have

$$\frac{1}{2\sqrt{EG}} = \frac{v^2}{2}.$$

We also have $G_u = 0$ and

$$\begin{aligned}E_v &= -\frac{2}{v^3}\\ \frac{E_v}{\sqrt{EG}} &= \frac{2}{v}\\ \left(\frac{E_v}{\sqrt{EG}}\right)_v &= -\frac{2}{v^2}.\end{aligned}$$

Therefore $K(p) = -1$. Hence, the hyperbolic plane has constant curvature -1. According to Corrolary 2.69, for a triangle T in $\mathbb{H}$ with interior angles α, β and γ, we have

$$\mathcal{A}(T) = -(\alpha + \beta + \gamma - \pi).$$

More generally, according to Corrolary 2.72, for a simple n-sided polygon P in $\mathbb{H}$ with interior angles $\alpha_1, \ldots, \alpha_n$, we have

$$\mathcal{A}(P) = -(\sum_{i=1}^{n} \alpha_i - \pi).$$

Geodesics in $\mathbb{H}$ are parts of vertical lines or arcs of semicircls centered at the u-axis [35, p. 92]. Here are examples of geodesic polygones in $\mathbb{H}$.

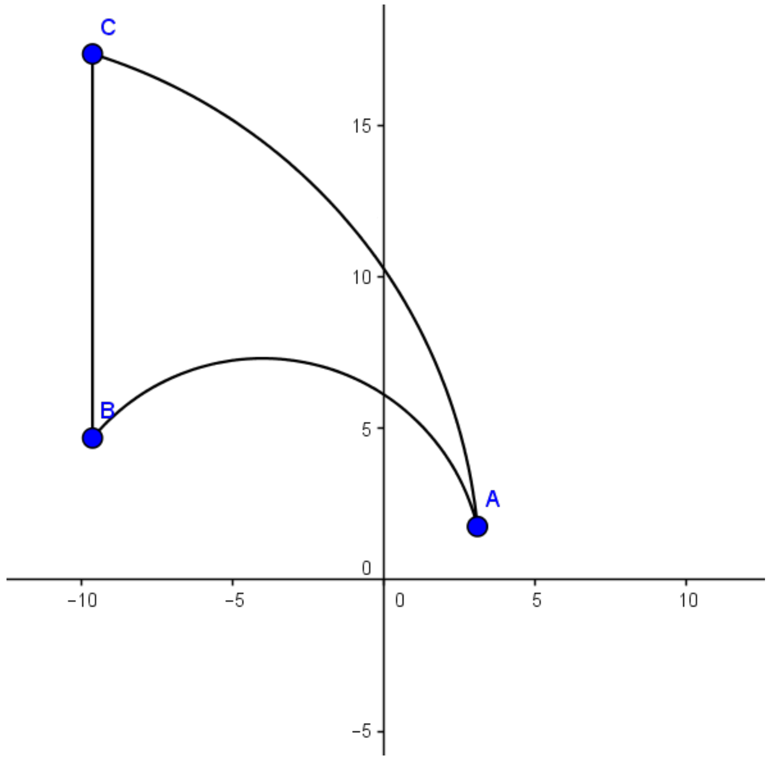


FIGURE 2.16: A geodesic triangle in $\mathbb{H}$

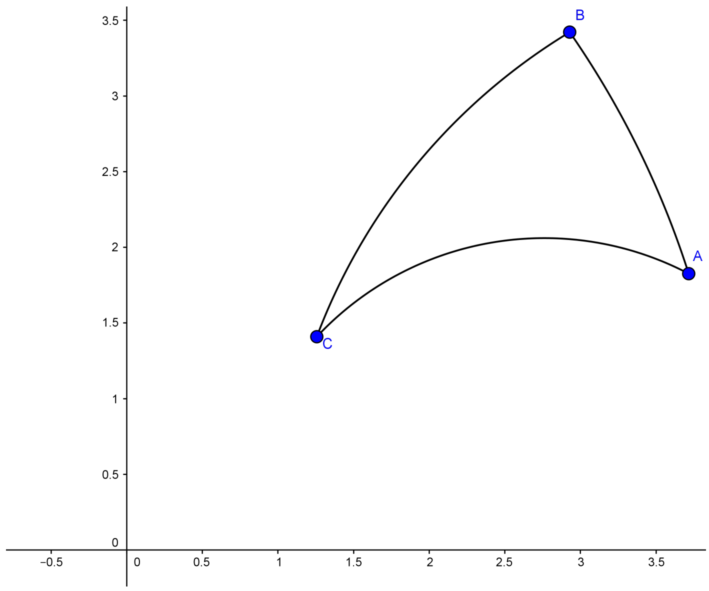


FIGURE 2.17: Another geodesic triangle in $\mathbb{H}$

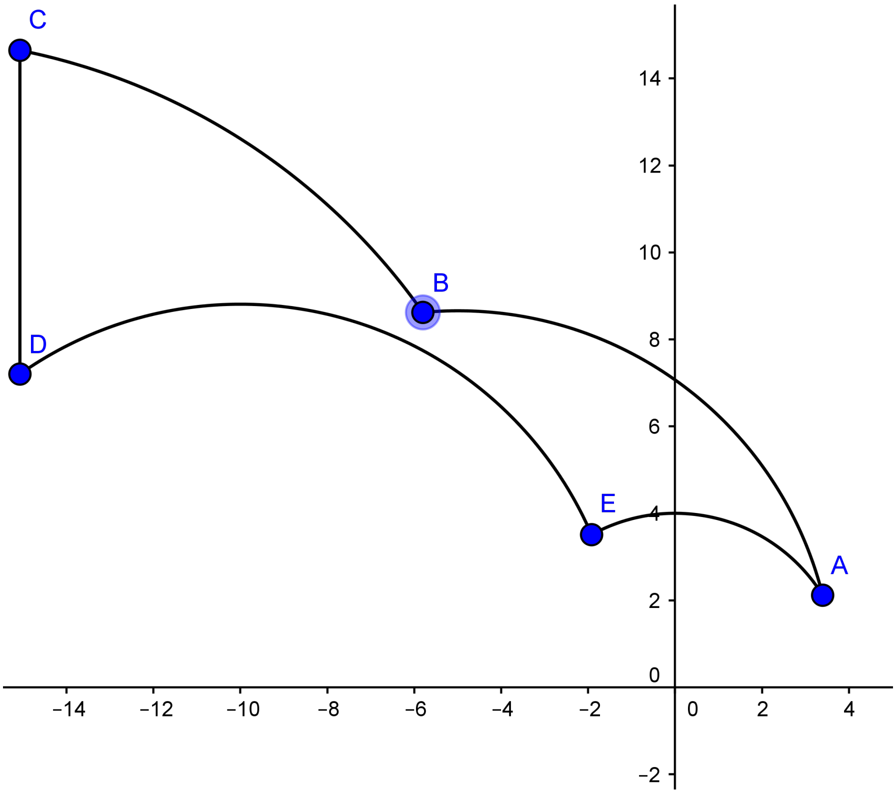


FIGURE 2.18: A geodesic pentagon in $\mathbb{H}$

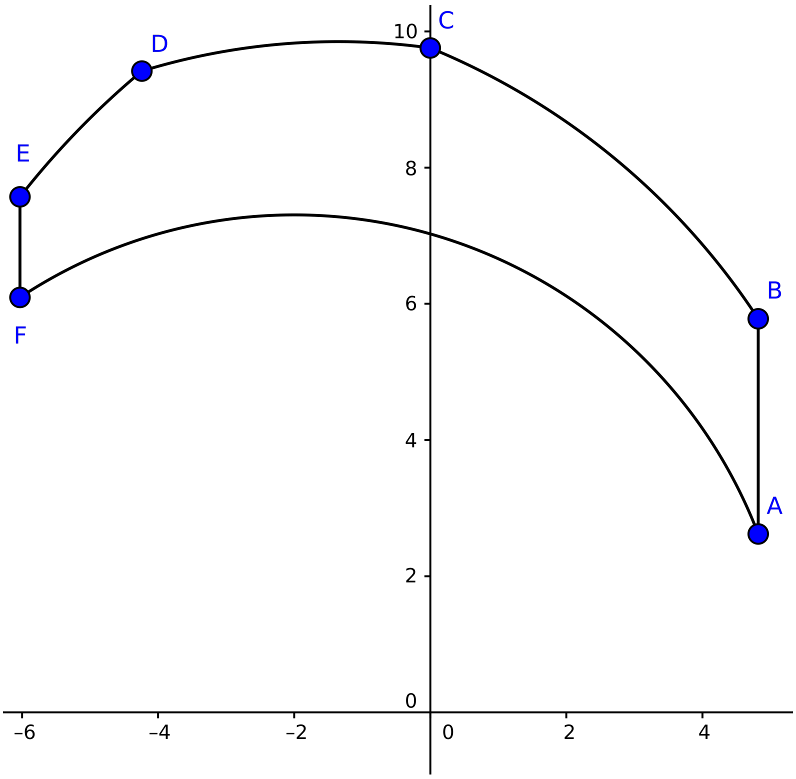


FIGURE 2.19: A geodesic hexagon in $\mathbb{H}$

Chapter 3

The Global Gauss-Bonnet Theorem

In this chapter, we prove the global Gauss-Bonnet theorem. It is due to the German mathematician Walther Franz Anton von Dyck [19, p. 1888].

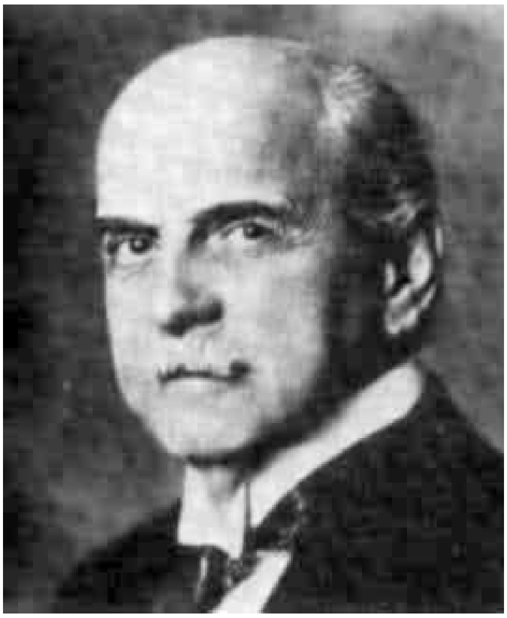


FIGURE 3.1: Walther von Dyck (1856 - 1934) [3]

3.1 Triangulations

In the beginning of Chapter 1, we said that the sum of the interior angles of any simple polygon can be calculated by triangulizing it, hence working on "smaller parts of it" that are triangles which sum of interior angles is known. We can use this technic to globalize the Gauss-Bonnet theorem.

Let S be a regular surface.

Definition 3.1. A region R of S is said to be ***regular*** if R is compact and δR is a finite union of traces of polygonal curves that do not intersect. A compact connected regular surface is considered as a regular region without boundary.

Definition 3.2. A ***triangle*** is a simple region which boundary has exactly three vertices.

A ***triangulation*** of a regular region R of S is a finite set $\mathbf{T}$ of triangles such that:

1. $\bigcup_{T \in \mathbf{T}} T = R$;

2. for every $T_1, T_2 \in \mathbf{T}$, if $T_1 \cap T_2 \neq \emptyset$ then $T_1 \cap T_2$ is either a common edge or a common vertex of the two triangles;

3. for every $T \in \mathbf{T}$, if $T \cap \delta R \neq \emptyset$ then $T \cap \delta R$ consists of vertices and edges of T.

4. every vertex of δR is the vertex of at least one triangle of $\mathbf{T}$.

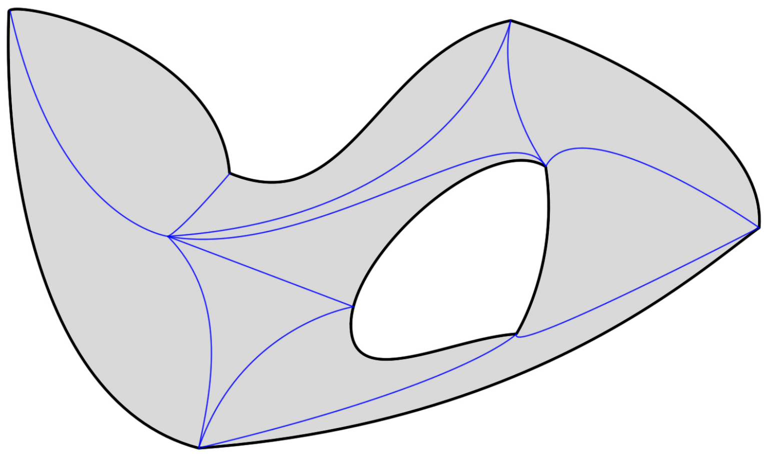


FIGURE 3.2: Triangulation of a regular region

Some vocabulary concerning triangulations:

Definition 3.3. Let $\mathbf{T}$ be a triangulation of a regular region R of S. Each triangle is also called a ***face***, and $F = \mathrm{card}\mathbf{T}$ represents the number of faces in $\mathbf{T}$.
E and V are respectively the number of edges and vertices in $\mathbf{T}$. A vertex that belongs to δR is called an ***exterior vertex***, while a vertex that doesn't belong to δR (and hence belongs to the interior of R) is called an ***interior vertex***. We define similarly an ***exterior edge*** and an ***interior edge***.
We denote by:

1. V_e : the number of exterior vertices;

2. V_i : the number of interior vertices;

3. E_e : the number of exterior edges;

4. E_i : the number of interior edges.

The following proposition gives relations between the numbers given in the previous definition:

Proposition 3.4. Let $\mathbf{T}$ be a triangulation of a regular region R of S. Then:

1. $V = V_e + V_i$;
2. $E = E_e + E_i$;
3. $V_e = E_e$;
4. $3F = 2E_i + E_e$.

Proof. 1. Trivial.

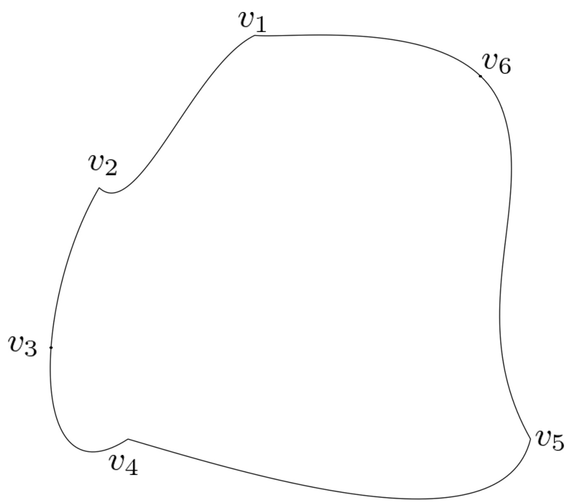
2. Trivial.

3. The boundary of R can be written as

$$\delta R = \bigcup_{j=1}^{n} C_j$$

where each $(C_j)_{1\leq j\leq n}$ is a collection of disjoint traces of polygonal curves. Hence, it is clearly enaugh to establish that formula for an arbitrary C_j.

Let $j \in \{1, \ldots, n\}$ and let m be the number of vertices $v_1, \ldots, v_m$ of $\mathbf{T}$ in C_j.



The edges of $\mathbf{T}$ on C_j are the ones which endpoints are

$$\{v_1, v_2\}, \ldots, \{v_{m-1}, v_m\}, \{v_m, v_1\}.$$

Hence the number of edges of $\mathbf{T}$ on $C_j$is m, which is the number of vertices of $\mathbf{T}$ on C_j. Therefore $V_e = E_e$.

4. Let $T_1, \ldots, T_n$ be the elements of $\mathbf{T}$ and let $j \in \{1, \ldots, n\}$. Let E_{i_j} and E_{e_j} be, respectively, the interior and exterior edges of T_j. We have $3 = E_{i_j} + E_{e_j}$. Thus $3F = \sum_{j=1}^{n} E_{i_j} + \sum_{j=1}^{n} E_{e_j}$. Since each exterior edge belongs to one and only one triangle, we have $\sum_{j=1}^{n} E_{e_j} = E_e$.

On the other hand, every interior edge belongs to two triangles, hence $\sum_{j=1}^{n} E_{i_j} = 2E_i$. Therefore $3F = 2E_i + E_e$.

□

We will state a number of results about regular regions and triangulations without proofs. These can be found in [9, Section 6.2] and [5, Section 6.2].

Proposition 3.5. Every regular region of a regular surface has a triangulation.

For our purposes however, we need to make sure that we can use the local Gauss-Bonnet theorem on every triangle and that the geodesic curvature of the interior edges is not taken into consideration. Fortunately, the following proposition will assure this.

Proposition 3.6. Suppose that S is an oriented surface with a family $(\varphi_j)_{j\in J}$ of local parametrizations compatible with the orientation of S that cover a regular region $R \subset S$. Then there exists a triangulation $\mathbf{T}$ of R such that every triangle of $\mathbf{T}$ is contained in a coordinate neighborhood of $(\varphi_j)_{j\in J}$ and its boundary is positively oriented. Furthermore, triangles that share a common edge determine opposite orientations on it.

Definition 3.7. Let R be a regular region of S and $\mathbf{T}$ be a triangulation of R. The number $\chi(\mathbf{T}) = F - E + V$ is called the ***Euler-Poincaré characteristic*** of $\mathbf{T}$.

An important fact about the Euler-Poincaré characteristic is the following:

Theorem 3.8. *Let R be a regular region of S, $\mathbf{T}_1$ and $\mathbf{T}_2$ be two triangulations of R. Then $\chi(\mathbf{T}_1) = \chi(\mathbf{T}_2)$.*

This means that the Euler-Poincaré characteristic doesn't depend of the triangulation but rather on the region itself. Hence we make the following definition:

Definition 3.9. Let R be a regular region of S. The ***Euler-Poincaré characteristic*** of R is $\chi(R) = \chi(\mathbf{T})$ where $\mathbf{T}$ is any triangulation of R (which existence is established by Proposition 3.5).

There is even more:

Theorem 3.10. *Let R and R' be two regular regions of, respectively, two regular surfaces S and S'. If R and R' are homeomorphic then $\chi(R) = \chi(R')$.*

Hence, the Euler-Poincaré characteristic is a topological invariant.

Example 3.11. The Euler-Poincaré characteristic of a simple regular region is 1. This is because such a region is homeomorphic to a disk which can be triangulated as follows:

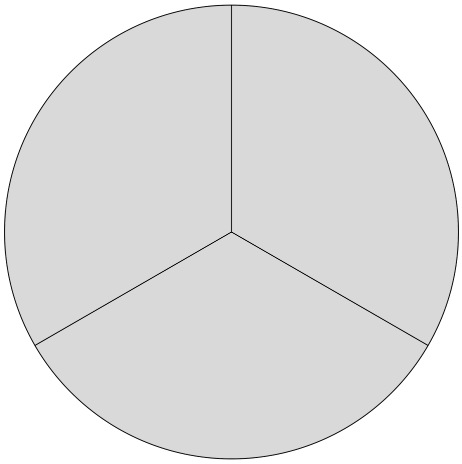


FIGURE 3.3: A triangulation of a disk

We have $F = 3$, $E = 6$ and $V = 4$. Thus $F - E + V = 1$.

Example 3.12. The unit sphere S^2 is a compact region of itself with empty boundary. Thus it is a regular region of itself. It can be triangulized as follows:

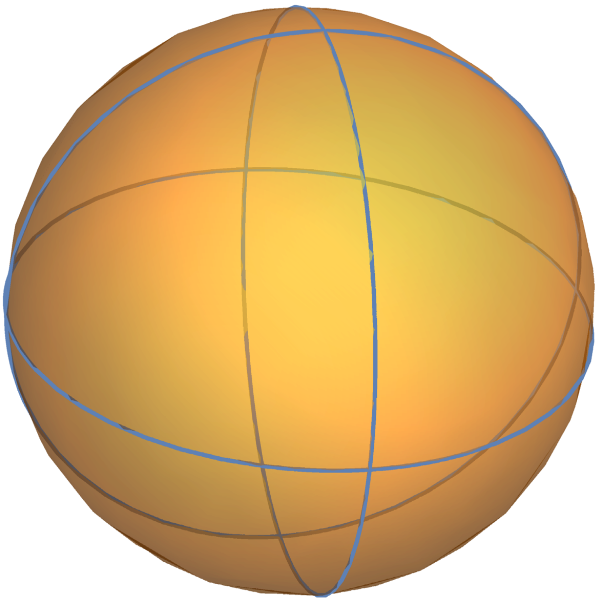


FIGURE 3.4: A triangulation of a sphere

We have $F = 8$, $E = 12$ and $V = 6$. Thus $\chi(S^2) = 2$.

To conclude this section, we note that triangulations allow us to define integrals of functions over larger domains.

Definition 3.13. Let $(\varphi_j)_{j\in J}$ be a family of local parametrizations of S that cover a regular region $R \subset S$. Let $D \subset R$ and $f : R \to \mathbb{R}$ be a continuous function. The ***integral*** of f over D is

$$\iint_D f \, \mathrm{d}\sigma = \sum_{i=1}^{n} \iint_{D\cap T_i} f \, \mathrm{d}\sigma$$

where $\{T_1, \dots, T_n\}$ is any triangulation of R such that every triangle is contained in a coordinate neighborhood of $(\varphi_j)_{j\in J}$.

3.2 The Global Gauss-Bonnet Theorem

We can now state and prove the global Gauss-Bonnet theorem:

Notation 3.14. Let $R \subset S$ be a regular region of an oriented surface S whose boundary consists of the positively oriented polygonal curves $\gamma_1, \dots, \gamma_n$ defined on, respectively, the compact intervals $I_1, \dots, I_n$, and which geodesic curvatures are, respectively, $k_{g_1}, \dots, k_{g_n}$. Let

$$\int_{\delta R} k_g(t)\,\mathrm{d}t = \sum_{i=1}^{n} \int_{I_i} k_{g_i}(t)\,\mathrm{d}t.$$

Theorem 3.15 (The Global Gauss-Bonnet Theorem)**.** *Let S be a regular oriented surface, $R \subset S$ be a regular region. Let $\theta_1, \dots, \theta_m$ be the external angles of these curves. Then:*

$$\int_{\delta R} k_g(t)\,\mathrm{d}t + \iint_R K\,\mathrm{d}\sigma + \sum_{i=1}^{m} \theta_i = 2\pi\chi(R). \tag{3.1}$$

Proof. Let $(\varphi_j)_{j\in J}$ be a family of orthogonal parametrizations of S that cover S and that are compatible with the orientation of S. Let $\mathbf{T} = \{T_1, \dots, T_p\}$ be a triangulation of R as given by Proposition for the family $(\varphi_j)_{j\in J}$. Let α_{ij} and β_{ij} denote, respectively, the i-th internal and external angle of the j-th triangle. According to the local Gauss-Bonnet theorem:

$$\forall j \in \{1, \dots, p\},\ \int_{\delta T_j} k_g(t)\,\mathrm{d}t + \iint_{T_j} K\,\mathrm{d}\sigma + \sum_{i=1}^{3} \beta_{ij} = 2\pi.$$

Thus:

$$\sum_{j=1}^{p} \int_{\delta T_j} k_g(t)\,\mathrm{d}t + \sum_{j=1}^{p} \iint_{T_j} K\,\mathrm{d}\sigma + \sum_{j=1}^{p}\sum_{i=1}^{3} \beta_{ij} = 2\pi p. \tag{3.2}$$

Since every interior edge is shared by two triangles that determine opposite directions on it, we have

$$\sum_{j=1}^{p} \int_{\delta T_j} k_g(t)\,\mathrm{d}t = \int_{\delta R} k_g(t)\,\mathrm{d}t. \tag{3.3}$$

Since triangles intersect on edges or vertices and $\bigcup_{j=1}^{p} T_j = R$, we have

$$\sum_{j=1}^{p} \iint_{T_j} K\,\mathrm{d}\sigma = \iint_R K\,\mathrm{d}\sigma.$$

We also note that $p = F$. Hence (3.2) becomes

$$\int_{\delta R} k_g(t)\,\mathrm{d}t + \iint_R K\,\mathrm{d}\sigma + \sum_{j=1}^{p}\sum_{i=1}^{3}\beta_{ij} = 2\pi F. \tag{3.4}$$

For the remaining double sum, we will work with internal angles instead. We have

$$\begin{aligned}\sum_{j=1}^{p}\sum_{i=1}^{3}\beta_{ij} &= \sum_{j=1}^{p}\sum_{i=1}^{3}\pi - \alpha_{ij}\\ &= 3\pi F - \sum_{j=1}^{p}\sum_{i=1}^{3}\alpha_{ij}\end{aligned}$$

Note that for each vertex of $\mathbf{T}$, the sum of internal angles around is

- 2π if the vertex is internal;
- π if the vertex is external but is not a vertex of a curve in $\{\gamma_i \mid i \in \{1,\ldots,n\}\}$ (the number of such vertices will be denoted by V_{et});
- $\pi - \theta_i$ if the vertex is the vertex of a curve in $\{\gamma_i \mid i \in \{1,\ldots,n\}\}$ (the number of such vertices will be denoted by V_{ec}).

Therefore

$$\begin{aligned}\sum_{j=1}^{p}\sum_{i=1}^{3}\alpha_{ij} &= 2\pi V_i + \pi V_{et} + \sum_{i=1}^{m}\pi - \theta_i\\ &= 2\pi V_i + \pi V_{et} + \pi V_{ec} - \sum_{i=1}^{m}\theta_i\\ &= 2\pi V_i + \pi V_e - \sum_{i=1}^{m}\theta_i. \qquad (V_e = V_{et} + V_{ec})\end{aligned}$$

Hence:

$$\sum_{j=1}^{p}\sum_{i=1}^{3}\beta_{ij} = \sum_{i=1}^{m}\theta_i + \pi(3F - 2V_i - V_e). \tag{3.5}$$

We have

$$\begin{aligned}3F - 2V_i - V_e &= 2E_i + E_e - 2V_i - V_e\\ &= 2E_i + 2E_e - 2V_i - V_e - E_e\\ &= 2E - 2V_i - 2V_e \qquad (V_e = E_e)\\ &= 2E - 2V.\end{aligned}$$

Thus (3.5) becomes

$$\sum_{j=1}^{p}\sum_{i=1}^{3}\beta_{ij} = \sum_{i=1}^{m}\theta_i + 2\pi(E - V).$$

Replacing in (3.4), we get

$$\begin{aligned}\int_{\delta R} k_g(t)\,\mathrm{d}t + \iint_R K\,\mathrm{d}\sigma + \sum_{i=1}^{p} \theta_i &= 2\pi(F - E + V)\\ &= 2\pi\chi(R)\end{aligned}$$

as desired. □

Example 3.16. Consider the torus S described in Example 2.12. Let

$$R = \left\{((2+\sin t)\cos\theta, (2+\sin t)\sin\theta, \cos t) \mid (t,\theta) \in [0, 2\pi] \times \left[0, \frac{\pi}{4}\right]\right\}.$$

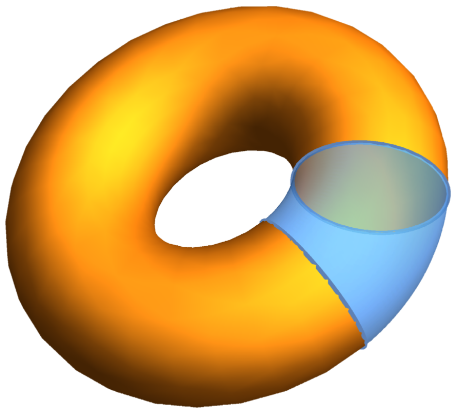


FIGURE 3.5: Regular region on a torus

Let's calculate $\chi(R)$. The boundary of this region has no exterior angles and is the reunion of the trace of two geodesic curves on the torus. Hence

$$\chi(R) = \frac{1}{2\pi}\iint_R K\,\mathrm{d}\sigma.$$

In Example 2.67, we saw that $K = \dfrac{\sin t}{2+\sin t}$. Hence

$$\begin{aligned}\iint_R K\,\mathrm{d}\sigma &= \frac{\pi}{4}\int_0^{2\pi} \sin t\,\mathrm{d}t\\ &= 0.\end{aligned}$$

Therefore $\chi(R) = 0$.

Remark 3.17. Using Example 3.11, we see that the local Gauss-Bonnet theorem is a particular case of the global.

Remark 3.18. The global Gauss-Bonnet theorem involves the Euler-Poincaré characteristic of a regular region, which is ensured to be well defined by Theorem 3.8. This theorem can be proven independently from the global Gauss-Bonnet theorem [5, Theorem 6.2.10], but it can also be retrieved using the global Gauss-Bonnet theorem for regions of oriented surfaces. Indeed, let R be a regular region of a regular oriented surface S, and let $\mathbf{T}_1$ and $\mathbf{T}_2$ be two triangulations of R. The proof of the global theorem holds for an arbitrary triangulation of R, and $\chi(R)$ in (3.1) is actually the Euler-Poincaré characteristic of that triangulation. So we get (3.1) with $\chi(\mathbf{T}_1)$ when using $\mathbf{T}_1$, and with $\chi(\mathbf{T}_2)$ when using $\mathbf{T}_2$. Since the LHS of (3.1) doesn't depend on the triangulation of R, we conclude that $\chi(\mathbf{T}_1) = \chi(\mathbf{T}_2)$.

Example 3.19. Soccer is undoubtly one of the most popular sports in the world. Here's a classic soccer ball:

FIGURE 3.6: A soccer ball [4]

What if we wanted to make a soccer ball with only one kind of pentagon, with the furthur assumption that each vertex is common to exactly three faces? Suppose that such a ball exists. Let F, E and V be the number of, respectively, pentagons, edges (of pentagons) and vertices (of pentagons). We triangulate every pentagon in the following way:

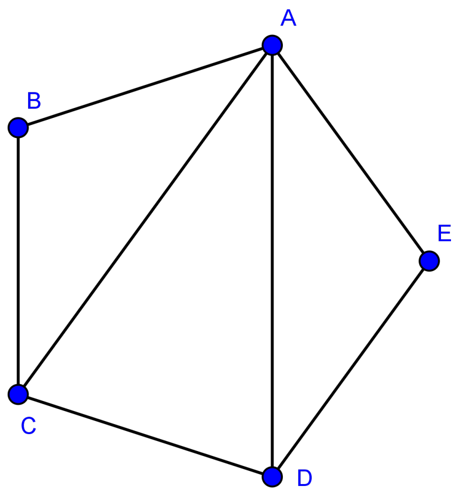


FIGURE 3.7: A triangulation of a pentagon

We get a triangulation $\mathbf{T}$ of the ball. With a triangulation similar to the one in Figure 3.4, the Euler-Poincaré characteristic of the ball B is 2. Now let F_T, E_T and V_T be the number of, respectively, triangles, edegs and vertices in $\mathbf{T}$. According to Remark 3.18, we have $2 = \chi(B) = F_T - E_T + V_T$. Every pentagon corresponds to by three triangles, thus $F_T = 3F$. Every edge in $\mathbf{T}$ is either the edge of a pentagon or an edge introduced when triangulating a pentagon. For each pentagon, two additional edges were used to triangulate it. Hence $E_T = E + 2F$. Finally, as no vertex was introduced when triangulating, we have $V_T = V$. Therefore $\chi(B) = F - E + V$. Every edge (of a pentagon) is common to two pentagons and every pentagon contains 5 edges. Thus $5F = 2E$. Every vertex is common to three pentagons, and every pentagon contains 5 vertices. Thus $5F = 3V$. Hence $\chi(B) = \dfrac{1}{6}F = 2$. Therefore $F = 12$, which means that if such a ball exists, the only way possible number of pentagons on it is 12. Actually, such a ball does exist:

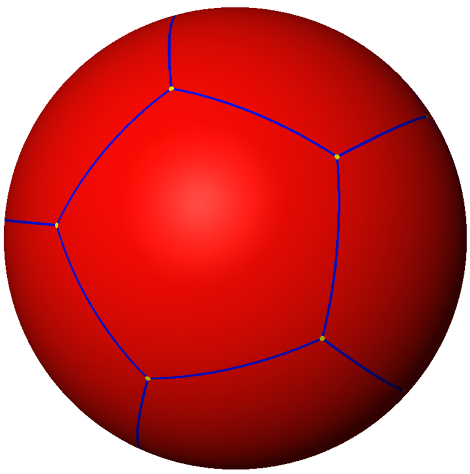


FIGURE 3.8: A ball with pentagons [10]

Now what if we wanted to make a soccer ball with only hexagons? Suppose that such a ball B exists. With similar notations as in the case of pentagons above, and with similar reasoning, we get the following formulaes:

$$\begin{aligned} \chi(B) &= F - E + V \\ 6F &= 2E \\ 6F &= 3V \end{aligned}$$

Hence $\chi(B) = 0 \neq 2$, which is a contradiction. Hence, it is impossible to construct such a ball.

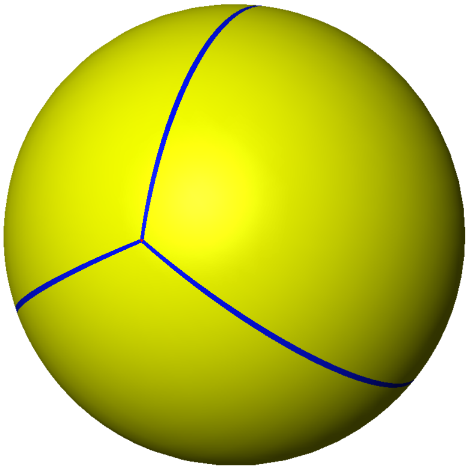
More generally, for a ball constructed with n-gons where $n \geq 3$, we have:

$$\begin{aligned} \chi(B) &= F - E + V \\ nF &= 2E \\ nF &= 3V \end{aligned}$$

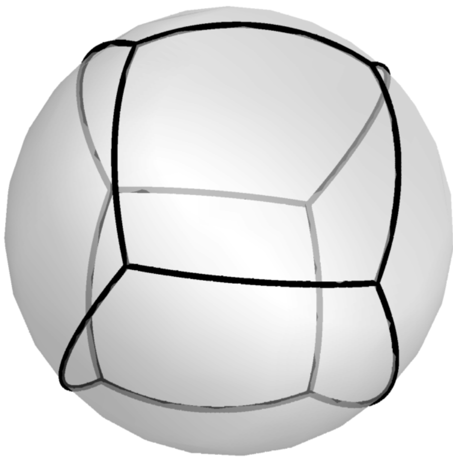
Hence $\chi(B) = \dfrac{6-n}{6} F = 2$, thus

$$F = \frac{12}{6-n}. \tag{3.6}$$

We see that we must have $3 \leq n \leq 5$ and $6 - n \mid 6$. Hence n must be either 3, 4 or 5. Figure 3.8 shows that it is possible for $n = 5$. For $n \in \{3, 4\}$, the following figure shows that it is again possible:



(A) A ball with triangles [41]



(B) A ball with squares

The corollary below is what some authors call the global Gauss-Bonnet theorem.

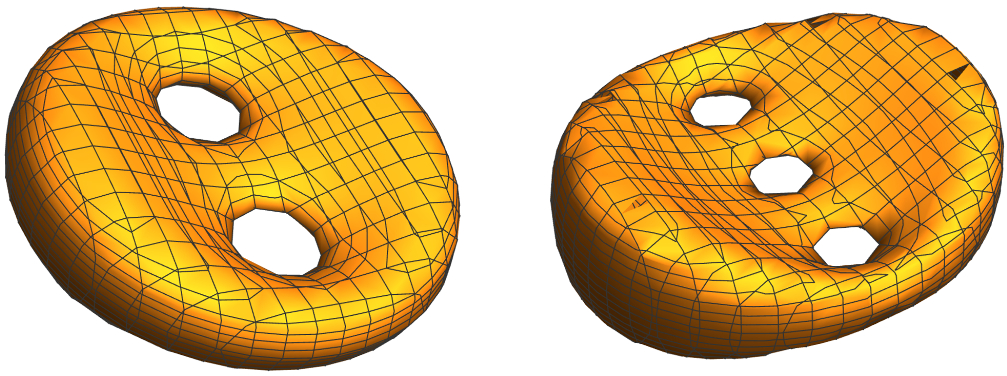
Corollary 3.20. Let S be an orientable compact connected regular surface. Then

$$\iint_S K \, \mathrm{d}\sigma = 2\pi\chi(S).$$

Proof. Let $\mathbf{T}$ be a triangulation of S. All the edges of $\mathbf{T}$ are interior edges. Hence, in the proof of the Gauss-Bonnet theorem for this surface and this triangulation, the sum in (3.3) will vanish. As there are no exterior angles, we get the desired result. □

For the next corollary, we need to introduce the ***genus*** of a compact connected orientable regular surface. Intuitively, it represents the number of handles or "holes" in the surface (see [5, Definition 6.2.13 on p. 314]).

Example 3.21. Using the formula in [6, Exercice 1.4.3], we represent the 2-torus and the 3-torus, which have a genus of, respectively, 2 and 3.



(A) A 2-torus (B) A 3-torus

FIGURE 3.10: Surfaces with genus 2 and 3

Theorem 3.22 (Classification of compact connected regular surfaces)**.** *For every compact connected orientable regular surface S there exists $g \in \mathbb{N}$ (the genus of S) such that*

$$\chi(S) = 2 - 2g.$$

Such surfaces with the same genus are homeomorphic.

This theorem gives the following equivalent definition for compact connected orientable regular surfaces:

Definition 3.23. Let S be a compact connected orientable regular surface. The ***genus*** of S is $g = \dfrac{2 - \chi(S)}{2}$.

Corollary 3.24. Every compact connected oriented regular surface of which Gaussian curvature is nonnegative and positive at at least one point is homeomorphic to the sphere.

Proof. According to Corollary 3.20, the Euler-Poincaré characteristic of such a surface is positive. According to Theorem 3.22, all such surfaces with positive Euler-Poincaré characteristic are homeomorphic to the sphere. □

Other interesting applications can be found in [15, p. 277–282] and [5, p. 322–333].

For the last application, we mention that the global Gauss-Bonnet does hold for smooth Riemannian manifolds [27, Theorem 9.7 on p. 167]. Hence Corollary 3.20 holds for metric surfaces.

Example 3.25. Let g be any Riemannian metric on S^2. We have

$$\iint_{S^2} K \, \mathrm{d}\sigma = 4\pi.$$

This is because S^2 is an orientable compact connected regular surface with Euler-Poincaré characteristic 2 as shown in Example 3.12.

Definition 3.26. A field F is said to be ***algebrically closed*** if every nonconstant polynomial in $F[x]$ has at least one root in F.

Theorem 3.27 (The fundamental theorem of algebra)**.** *The field* $(\mathbb{C}, +, \times)$ *is algebrically closed.*

Proof. We present the outline of the proof in [8]:

1. Suppose $p(X) = \sum_{i=0}^{n} a_i X^i$ has no root, $n \geq 1$.
2. Construct from $p(X)$ a Riemannian metric on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, that is identified to S^2, such that $K = 0$.
3. $\iint_{S^2} K \, \mathrm{d}\sigma = 0 \neq 4\pi$, contradiction.

$\square$

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