On Quasi-Alternating Links

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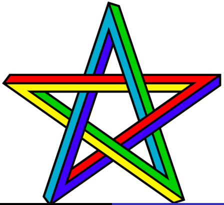

Plan

Definition

A link L with n components is a submanifold of $\mathbb{R}^3$ or S^3 which is homeomorphic to the disjoint union of n circles $S^1 \sqcup ... \sqcup S^1$. If $n = 1$, we call L a knot and denote it by K.

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Notation: We denote by $L = K_1 \sqcup ... \sqcup K_n$ a link with n components.

$$S^1 \sqcup ... \sqcup S^1 \hookrightarrow \mathbb{R}^3 \text{ or } S^3$$

Examples -:

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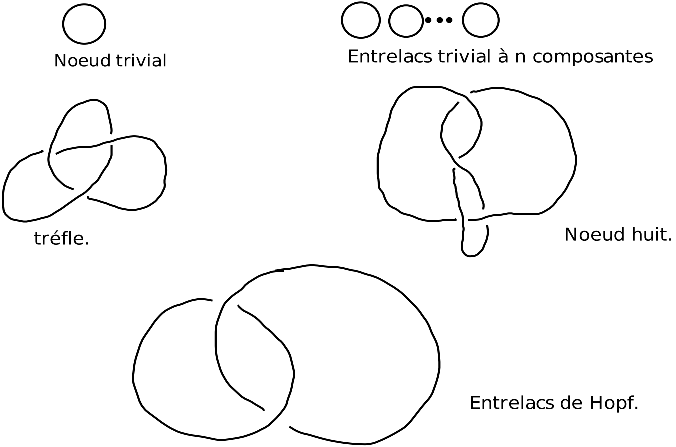

Figure : Exemples d'entrelacs.

Isotopy equivalence

For convenience, we restrict ourselves to smooth knots.

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Definition

Two knots K_1 and K_2 are **ambient isotopic** if there exists a family of diffeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0, 1]$ such that $h_0 = id_{\mathbb{R}^3}$ and $h_1(K_1) = K_2$. If they are so, we say that K_1 and K_2 are equivalent.

Proposition

Two knots K_1 and K_2 are equivalent if and only if there exists a preserving orientation diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = h(K_2)$.

Example -:

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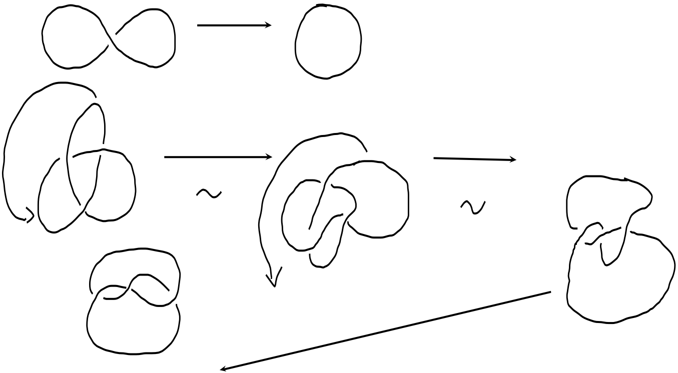

Figure : Nœuds isotopes.

Plan

Definition

Let K be a knot. Let $\mathcal{P}$ be a plane in the space. Let p be a perpendicular projection on $\mathcal{P}$. We say that $p(K)$ is a **regular projection** of K on $\mathcal{P}$ if it satisfies

1. the tangent lines to the knot at all points are projected onto lines on the plane. (i.e. the projections of the tangents never degenerate into points);
2. No more than two distinct points of the knot are projected on one and the same point of the plane;
3. The set of **crossing points** (those on which two points project) is finite and at each crossing point the projections of the two tangents do not coincide.

Definition

A diagram D of a knot K is its image by a regular projection.

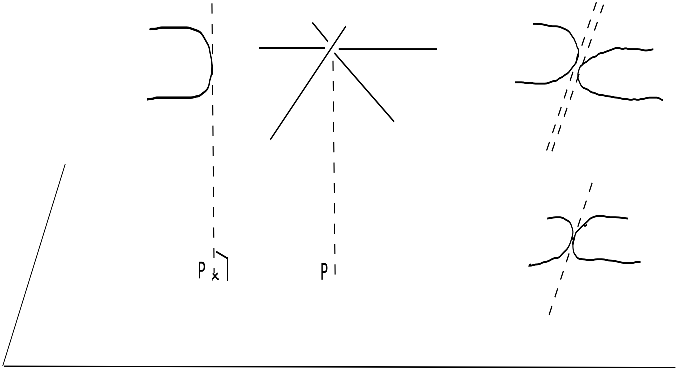

Figure : Forbidden projections.

Plan

1. Links, Diagrams and the Reidemeister Theorem
   - Links
   - Diagrams
   - Reidemeister theorem
2. Some Invariants
3. Quasi-alternating links
   - Definition and examples
   - Detection of some non quasi-alternating links
4. Questions

Definition

Two diagrams D_1 and D_2 of a knot K are said equivalent if we can obtain D_1 from D_2 by a finite sequence of

1. ambient plane isotopies and
2. Ω_1-moves, Ω_2-moves and Ω_3-moves.

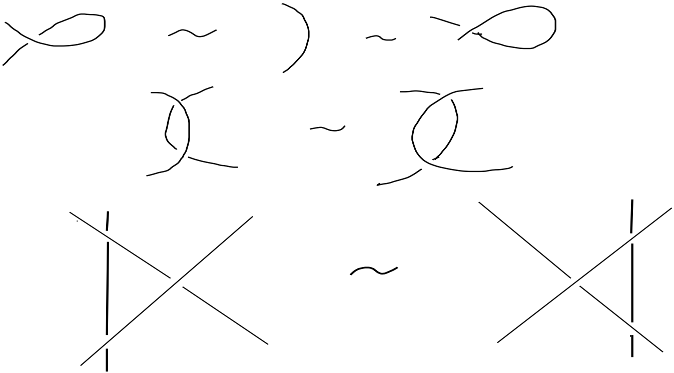

Figure : Reidemeister moves.

Theorem

Two knot diagrams correspond to isotopic knots if and only if one can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies.

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Corollary

There is a one-to-one correspondence between the equivalence classes of knots and the equivalence classes of diagrams.

Operation on Links: The connected sum

Some families of links:

1. Alternating knots.

Figure : Alternating links.

2. Torus knot $T_{(p,q)}$ where p and q are coprime integers.

The crossing and the unknotting numbers

1. The crossing number $c(L)$.
   - In fact for each positive integer $c \leq 3$ there exists an (irreductible) knot such that $c(K) = c$.
   - $c(T(p,q)) = (p-1)q$.
2. The unknotting number $u(L)$.
   - $u(T(p,q)) = \frac{1}{2}(p-1)(q-1)$.
3. The 3-coloring number.

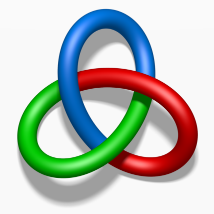

Figure : Nœud Tréfle colorié.

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We denote $F^{\pm}$ the surface

$$F^{\pm} = F \times \{\pm 1\}$$

If α is a curve in F, its copy in $F^{\pm}$ is denoted by $\alpha^{\pm}$.

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If $\{a_i\}$ is a basis of $H_1(F)$, we consider the integers $lk(a_i, a_j^+)$ and then the square matrix $M = (lk(a_i, a_j^+))$ called the Seifert matrix of L:

$$\begin{array}{rccc} \Theta : & H_1(F) \times H_1(F) & \longrightarrow & \mathbb{Z} \\ & (a_i, a_j) & \longmapsto & lk(a_i, a_j^+) \end{array}$$

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Definition

The determinant of a link L, written $\det L$ is the positive integer $|\det(M + M^T)|$.

$$\det L := |\det(M + M^T)|$$

Theorem

Let M be a Seifert matrix of a link L constructed from a Seifert matrix F spanning L. Then $\det L$ *is a link invariant.*

Example -:

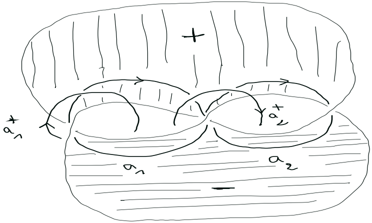

Figure : Seifert Matrix of the Trefoil

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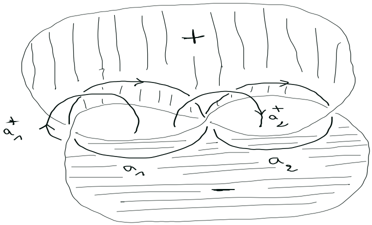

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$$M = \begin{pmatrix} lk(a_1, a_1^+) & lk(a_1, a_2^+) \\ lk(a_2, a_1^+) & lk(a_2, a_2^+) \end{pmatrix} = \begin{pmatrix} +1 & -1 \\ 0 & +1 \end{pmatrix}$$

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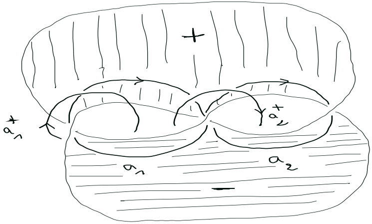

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$$\det K = 3$$

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For any link L we have

$$\det(-L) = \det L = \det(L^*).$$

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If $L_1 \sqcup L_2$ is a split link then

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Theorem

If a link can be factorised as $L_1 \# L_2$ then

$$\det(L_1 \# L_2) = \det L_1 \det L_2.$$

Plan

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Definition

A closed three-manifold Y is called an L-space if $H_1(Y;\mathbb{Q}) = 0$ and $\widehat{HF}(Y)$ is a free abelian group whose rank coincides with the number of elements in $H_1(Y;\mathbb{Z})$ denoted by $|H_1(Y;\mathbb{Z})|$

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The most important property of quasi-alternating links is the fact that their double branched covers are L-spaces, and then are rational homology 3-spheres with the simplest Heegaard Floer homology.

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The most important property of quasi-alternating links is the fact that their double branched covers are L-spaces, and then are rational homology 3-spheres with the simplest Heegaard Floer homology.
The converse statement is not true, but quasi-alternating links are recognised recently as an important class of knots and links.

Definition

The set $\mathcal{Q}$ of quasi-alternating links is the smallest set of links which satisfies the following properties:

1. the unknot belongs to $\mathcal{Q}$.
2. If L is a link with a diagram D containing a crossing c s.t.
   - both smoothings of the diagram D at the crossing c, L_0 and L_∞ as in the figure 8 belong to $\mathcal{Q}$ and
   - $\det L = \det(L_0) + \det(L_\infty)$;

   then L is in $\mathcal{Q}$ and we say that L is quasi-alternating at the crossing c with quasi-alternating diagram D.

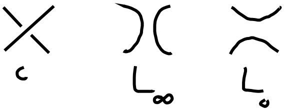

Figure : 8. Smoothing of the crossing c

Proposition

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$\square$

Example -The 9_{47} knot has a diagram which is quasi-alternating at a crossing c.

The first non alternating knot in the standard knot table is the 8_{19} which is the Torus Knot $T(4,3)$.

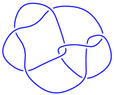

Figure : The 8_{19} knot

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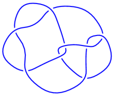
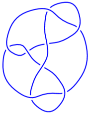


Figure : The 8_{19} knot

The first non-alternating quasi-alternating knot in the standard knot table is the 8_{20}.



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Theorem

1. *The Pretzel link $P(p_1, ..., p_n, -q)$ is quasi-alternating for $n \geq 1$, $p_i \geq 1$ $\forall i$ and $q > \min\{p_1, ..., p_n\}$.*
2. *The Pretzel link $P(p_1, ..., p_n, -q_1, ..., -q_m)$ is not quasi-alternating if $m, n \geq 1$, and all $p_i, q_j \geq 2$.*

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Remark -: in fact the knot 8_{20} is the Pretzel knot $P(2, 3, -3)$.

Plan

The Kauffman polynomial

Let L be a link.

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We denote by L_+, L_-, L_∞ and L_0 the links which are identical to L except in a small disc as shown below.

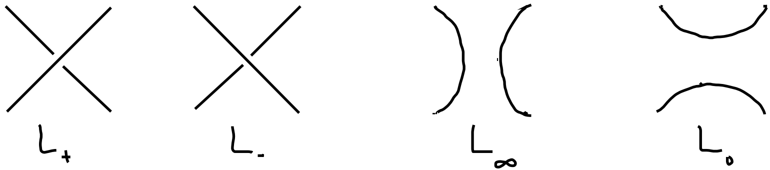


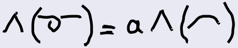
Figure : The skein quadruple

Theorem

There exists a function

$$\Lambda : \{\text{Unoriented links diagrams } S^2\} \longrightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

that is defined uniquely by the following

1. $\Lambda(U) = 1$, *where U is the zero-crossing diagram of the unknot;*
2. $\Lambda(D)$ *is unchanged by Reidemeister moves of Types II and III on the diagram D;*
3. 
4. *If* D_+, D_-, D_0 *and* D_∞ *are four diagrams exactely the same except near a point where they are as shown in figure, then*

$$\Lambda(D_+) + \Lambda(D_-) = z(\Lambda(D_0) + \Lambda(D_\infty))$$

Definition

The Kauffman polynomial is the function

$$F : \{\text{Unoriented links in } \mathrm{S}^3\} \longrightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

defined by $F_L = a^{-w(D)}\Lambda(D)$ where D is a diagram with writhe $w(D)$ of the oriented link L.

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Remark -: The polynomial

$$Q_L(z) = F_L(1, z)$$

called the Brandt-Lickorish-Millet Polynomial is a link invariant. It happens $\deg Q_L < \deg_z F_L$.

Theorem

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Examples -:

1. $\det(8_{19}) = 3$ and $\deg Q_{8_{19}} = 6$, so the knot 8_{19} is not quasi-alternating.
2. $\det(8_{20}) = 9$ and $\deg Q_{8_{20}} = 6$, so the knot 8_{20} satsfies the needed condition to be quasi-alternating.

Lemma

Let L be a link, then

$$\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1,$$

where L_0, L_∞ are the smoothings of the link L at any crossing c.

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If $n = 1$, then D_- is a diagram of the unlink with k components. We have that $Q_{L_-}(z) = (2z^{-1} - 1)^{k-1}$, and then
$Q_{L_+}(z) = z(Q_{L_0}(z) + Q_{L_\infty}(z)) - (2z^{-1} - 1)^{k-1}$.

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$Q_{L_+}(z) = z(Q_{L_0}(z) + Q_{L_\infty}(z)) - (2z^{-1} - 1)^{k-1}$.
Assume the result true for all link diagrams with a crossing switches less than n, in particular for the link L_-. The result follows from the identity $Q_L(z) = z(Q_{L_0}(z) + Q_{L_\infty}(z)) - Q_{L_-}(z)$

Proof.

Induction on $\det L$.

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The result is obvious if $\det L = 1$.
Now assume that the result is true for each quasi-alternating link L such that $\det L \leq m$. If $\det L = m + 1$, then L_0 and L_∞ satisfy $\deg L_0 < \det L_0$ and $\deg L_\infty < \det L_\infty$. We conclude by using the last lemma as follows:

$$\begin{aligned}
\deg Q_L & \\
&\leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1 \\
&< \max\{\det L_0, \det L_\infty\} + 1 \\
&< \det L_0 + \det L_\infty \\
&= \det L.
\end{aligned}$$

□

Theorem

Let L be a non-split alternating link. Then either,

1. *L is a $(2, n)$-torus link for $n \neq 0$, and $\deg_z F_L = \det L - 1$;*
2. *L is the figure-eight knot or the connected sum of two Hopf links, and $\deg_z F_L = \det L - 2$; or*
3. $\deg_z F_L \leq \det L - 3$.

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3. $\deg_z F_L \leq \det L - 3$.

Theorem

Let L be a non-alternating, quasi-alternating link. Then either,

1. $\deg_z F_L \leq \det L - 3$;*or*
2. *L has exactly three components, each of which is unknotted. Moreover, L is obtained from the Hopf link by a banding on one component.*

Let L be an oriented link. The **breadth** of the Jones polynomial $V_L(t)$ is the difference between the maximal degree of t and the minimal degree of t that occur in $V_L(t)$. We denote it by $B(V_L)$. Inspired by their computations of the breadth and the determinants of a large number of links, K. Qazaqzeh and N. Chbili conjectured the following:

Conjecture

If L is a quasi-alternating link, then $B(V_L) \leq \det(L)$.

Since we know that the breadth of the Jones polynomial is always less than or equal to $c(L)$, the crossing number of the link L (see theorem 5.9. in [6]), the last conjecture is weaker than the one in [7] which states that

Conjecture

For any quasi-alternating link L, we have $c(L) \leq \det L$.

The latter is true for any non-split alternating link L as proved in [7].
However, Conjecture 1 has the advantage that it involves the breadth of the Jones polynomial which is, in general, easier to compute than the crossing number. Conjecture 2 is true for all quasi-alternating links that have been checked to satisfy the conjecture $c(L) \leq \det(L)$. Chbili and Qazaqzeh proved both conjectures for quasi-alternating closed 3-braids.
The importance of the conjecture 2 does also come from that it

In [3], Chbili and Qazaqzeh asked also the two following questions:

Question -: Can we determine all Kanenobu knots that are quasi-alternating? They conjectured that $K(0,0)$, $K(1,0)$, $K(1,-1)$ are the only Kanenobu knots that are quasi-alternating.

Question -: Can we characterize all quasi-alternating knots with crossing number less than or equal to 11?

In the same direction, Teragaito conjecture

Conjecture

If L is a non-alternating, quasi-alternating link, then $\det L \geq 8$.

One can also ask the following

Question -: Can one measure the default of a quasi-alternating knot to be alternating ? Or what is the difference between quasi-alternating and almost alternating knots?

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THANKS FOR YOUR ATTENTION