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Pseudo-Euclidean Alternative Algebras and Malcev-Poisson-Jordan Algebras

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Abstract

The aim of this thesis is to study certain nonassociative algebras which carrying an invariant scalar product, such algebras are called pseudo-euclidean or quadratic algebras.

In the first part of this thesis, we transfer the notion of double extension, introduced by Medina and Revoy for quadratic Lie algebras (see [35]), and extended by Benayadi and Baklouti for pseudo-euclidean Jordan algebras (see [7],[5]), to the case of pseudo-euclidean alternative algebras. We show that every pseudo-euclidean alternative algebra, which is irreducible and neither simple nor nilpotent, is a suitable double extension. Moreover, we introduce the notion of generalized double extension of pseudo-euclidean alternative algebras by the one dimensional alternative algebra with zero product. This leads to an inductive classification of nilpotent pseudo-euclidean alternative algebras. A short review of the basics on alternative algebras and their connections to some other algebraic structures is also provided.

In the second part of this thesis, we introduce the notion of a Malcev-Poisson-Jordan algebra (MPJ-algebra for short), that is defined to be a vector space endowed with a Malcev bracket and a Jordan structure which are satisfying the Leibniz rule. We describe such algebras in terms of a single bilinear operation, this class strictly contains alternative algebras. For a given Malcev algebra $(P, [\ ,\ ])$, it is interesting to classify the Jordan structure $\circ$ on the underlying vector space of P such that $(P, [\ ,\ ], \circ)$ is a MPJ-algebra ($\circ$ is called a MPJ structure on Malcev algebra $(P, [\ ,\ ])$. We explicitly give all MPJ structures on some interesting classes of Malcev algebras. Further, we introduce the concept of pseudo-euclidean MPJ-algebras (PEMPJ-algebras for short) and we show how one can construct new interesting quadratic Lie algebras and pseudo-euclidean Malcev (non Lie) algebras from PEMPJ-algebras. Finally, we give inductive descriptions of nilpotent PEMPJ-algebras.

Contents

1

Introduction **5**

1 Preliminaries and results on nonassociative algebras **11**
1.1 Basic concepts on nonassociative algebras . . . 11
1.2 Power-associative algebras . . . 14
1.3 Jordan algebras . . . 16
1.4 Lie algebras . . . 18
1.5 Malcev algebras and related Lie triple systems . . . 20

2 Invariant scalar products on alternative algebras **26**
2.1 Alternative algebras . . . 26
2.2 Pseudo-euclidean alternative algebras . . . 28
2.3 Peirce decomposition of pseudo-euclidean alternative algebras . . . 31
2.4 Pseudo-euclidean Alternative algebras and related $\mathbb{Z}_2$-graded quadratic Lie algebra . . . 32

3 Inductive descriptions of pseudo-euclidean alternative algebras **35**
3.1 Double extension of pseudo-euclidean alternative algebras . . . 35
3.1.1 Semi-direct product of alternative algebras . . . 35
3.1.2 Central extension of alternative algebras and interpretation of the second cohomology group . . . 36
3.2 Generalized double extension of pseudo-euclidean alternative algebras by the one dimensional algebra with zero product . . . 50

4 Malcev-Poisson-Jordan algebras (MPJ-algebras) **54**
4.1 Definitions and preliminary results . . . 54
4.2 MPJ-admissible algebras such that the associated Malcev algebra is reductive 57
4.3 PEMPJ-algebras and related quadratic Lie algebras . . . 60
4.4 Construction of pseudo-euclidean Malcev algebras from PEMPJ-admissible algebras . . . 65
4.5 Inductive description of nilpotent PEMPJ-admissible algebras . . . 71
4.6 Double extension of PEMPJ-admissible algebras . . . 74

5 Lie-Yamaguti algebras from MPJ-algebras **76**
5.1 Definitions and preliminary results . . . 76
5.2 LY-algebras with a unique quadratic structure . . . 81

Conclusion and perspectives **85**

Version abrégée en français **87**

1 Algèbres alternatives pseudo-euclidiennes et algèbres de Lie quadratiques $\mathbb{Z}_2$-graduées **89**
1.0.1 Premières définitions 89
1.0.2 Algèbres alternatives 90
1.0.3 Algèbres alternatives pseudo-euclidiennes 91

2 Double extension des algèbres alternatives **95**
2.1 Produit semi-direct d'algèbres alternatives 95
2.2 Extensions centrales des algèbres alternatives 96
2.3 Description inductive des algèbres alternatives pseudo-euclidiennes 98
2.4 Double extensions généralisées des algèbres alternatives pseudo-euclidiennes par l'algèbre de dimension 1 à produit nul. 99

3 Algèbres de Malcev-Poisson-Jordan (Algèbres-MPJ) **102**
3.1 Définitions et premières résultats 102
3.2 Algèbres-MPJ admissibles avec l'algèbre de Malcev associée est réductive . 103
3.3 Relations entre les algèbres-PEMPJ et les algèbres de Lie quadratiques . . 105
3.4 Construction des algèbres de Malcev pseudo-euclidiennes à partir des algèbres-PEMPJ admissibles 107
3.5 Description inductive des algèbres-PEMPJ admissibles nilpotentes 109

4 Algèbres de Lie Yamaguti à partir des algèbres-MPJ **111**
4.1 Définitions et premières resultats 111
4.2 Algèbres-LY avec un unique produit scalaire invariant 113

Introduction

An algebra is a vector space A over a field $\mathbb{F}$ together with a bilinear multiplication $(x, y) \mapsto xy$ from $A \times A$ to A. In the general literature, an algebra is often referred to a nonassociative algebra in order to emphasize that the condition:

$$(xy)z = x(yz), \text{for all } \ x, y \in A, \tag{1}$$

is not being assumed. Use of this term doesn't mean that (1) fails to hold. If (1) is actually not satisfied in an algebra, we say that the algebra is not associative, rather than nonassociative. We shall use the name associative algebra for algebra in which the associative law (1) holds. Nonassociative algebras are well studied by several authors [57, 46, 14, 21, 33].

In this thesis, all algebras, vector spaces are finite-dimensional and defined over a commutative field $\mathbb{F}$ of characteristic 0. Nonassociative algebras considered in this thesis are Jordan algebras, Lie algebras, Malcev algebras, alternative algebras, and noncommutative Jordan algebras. A Jordan algebra A is an algebra in which products are commutative:

$$xy = yx, \text{ for all } \ x, y \in A,$$

and satisfy the Jordan identity,

$$x(yx^2) = (xy)x^2, \text{ for all } \ x, y \in A.$$

A natural generalization to noncommutative algebras is the class of algebras A satisfying the Jordan identity. A noncommutative Jordan algebra A is a flexible algebra (*i.e.* $(xy)x = x(yx)$) satisfying the Jordan identity. Clearly any Jordan algebra can be considered as a noncommutative Jordan algebra. An alternative algebra A is an algebra defined by the identities,

$$x^2y = x(xy) \text{ and } yx^2 = (yx)x, \text{ for all } \ x, y \in A,$$

known respectively as the left and right alternative law. Clearly any alternative algebra is a noncommutative Jordan algebra. A Lie algebra A is an algebra in which the multiplication is anticommutative, that is,

$$xy = -yx, \text{ for all } \ x, y \in A,$$

and the jacobi identity,

$$(xy)z + (yz)x + (zx)y = 0, \text{ for all } \ x, y, z \in A,$$

is satisfied. Any Lie algebra is obviously a noncommutative Jordan algebra. Malcev algebras are a class of algebras which generalizes the class of Lie algebras. An algebra A

is said to be a Malcev algebra if it satisfies the anticommutative law $xy = -yx$ and the Malcev identity

$$J(x, y, xz) = J(x, y, z)x,$$

for all $x, y, z \in A$. Where J corresponds to Jacobi's identity, that is,

$$J(x, y, z) = (xy)z + (yz)x + (zx)y.$$

Further, we will introduce a new class on nonassociative algebras (which are interesting as we will see later) called Malcev-Poisson-Jordan algebras [1] (MPJ-algebras for short). A MPJ-algebra A is a $\mathbb{F}$-vector space equipped with two bilinear multiplications $\circ$ and $[\ ,\ ]$ such that:

i) $(A, \circ)$ is a Jordan algebra,

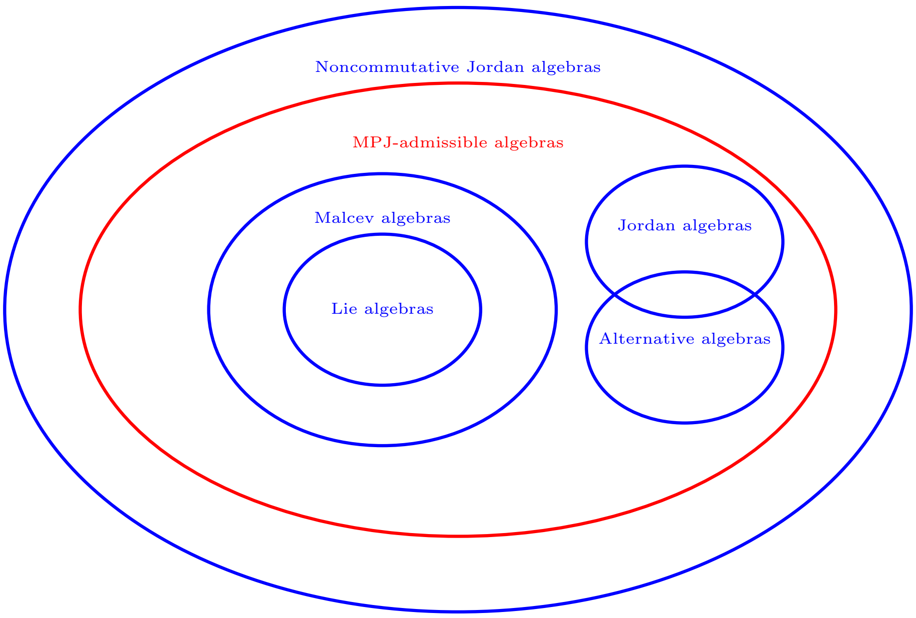
ii) $(A, [\ ,\ ])$ is a Malcev algebra,

iii) These two operations are required to satisfy Leibniz condition :

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z],\ \forall x, y, z \in A.$$

We denote a MPJ-algebra by $(A, [\ ,\ ], \circ)$. In the particular case when $(A, \circ)$ is associative commutative, $(A, [\ ,\ ], \circ)$ is called a Malcev-Poisson algebra. In [48], I. P. Shestakov was the first who generalized the notion of a Poisson algebra to that of a Malcev–Poisson algebra and studied this class of algebras and some related problems. It is clear that the class of MPJ-algebras also contains the class of Poisson algebras (ie. the case when $(P, [\ ,\ ])$ is a Lie algebra, $(P, \circ)$ is an associative commutative algebra and the Leibniz rule is satisfied).

We describe such algebras in terms of a single bilinear operation, this class called MPJ-admissible algebras, it strictly contains alternative algebras and it is strictly included in the class of noncommutative Jordan algebras. The links between all these algebras can be exprimed by the Scheme as follows:



The theory of alternative algebras has attracted a lot of interest from mathematicians after having discovered its close connection with the theory of projective plans. It was found that alternative algebras are connected to associative ones. The essence of this cloness is exhibited by the theorem of Artin, which asserts that in every alternative algebra the subalgebra generated by any two elements, is associative (see [46]). In particular, associative algebras are alternative. However, there are plenty of nonassociative alternative algebras such as the octonions.

Alternative algebras are related to Malcev algebras (also called Moufang-Lie algebras) as associative algebras are related to Lie algebras. Indeed, every alternative algebra A is Malcev-admissible, i.e., the commutator algebra A^- is a Malcev algebra.

A similar connection exists between alternative algebras and Jordan algebras. They appeared in the work of the German physicist Jordan, that dealt with the axiomatization of the foundations of quantum mechanics. We can find this relationship by defining a new multiplication in an alternative algebra A such as

$$x \circ y = xy + yx.$$

The obtained algebra A^+ is a Jordan algebra. Moreover, alternative algebras are as much important in theory as in applications. These can be found in projective geometry, buildings, and algebraic groups, (see for example [50]).

A very important class of alternative algebras are the pseudo-euclidean alternative algebras [2], carrying an invariant symmetric and nondegenerate bilinear form, are also called orthogonal alternative algebras ([13]). In this thesis, we prove that in the case of pseudo-euclidean alternative algebras there are equivalence between left and right alternative algebras. Such bilinear forms on nonassociative algebras have a great value in the study of their structures. For example, the Killing form plays a key role in the theory of semi-simple Lie algebras, Albert forms in the case of semi-simple Jordan algebras and trace form in the case of semi-simple alternative algebras.

In particular, we give a concept of double extension (central extension followed by semi-direct product) in the case of pseudo-euclidean alternative algebras, this concept was introduced by A. Medina and Ph. Revoy for quadratic Lie algebras (see [35]), and extended by S. Benayadi and A. Baklouti for pseudo-euclidean Jordan algebras (see [7],[5]).

Using this technique of double extension, we obtain information about structures of these algebras, this enables us to give an inductive descriptions.

This study is limited to the irreducible ones. The main result is that any pseudo-euclidean alternative algebra is the orthogonal direct sum of irreducible ideals.
One of our motivations to classify such algebras is that we can construct a $\mathbb{Z}_2$-graded quadratic Lie algebra from a pseudo-euclidean alternative algebra. In fact, it is well known that there is a correspondence between simply-connected pseudo-Riemannian symmetric spaces and symmetric triples (see e.g. [15],[22]). In addition, one can construct at least one triple symmetric from a $\mathbb{Z}_2$-graded quadratic Lie algebra (Proposition III.2.6 p.31 of [7]). The technique of double extension is also a tool to construct a new quadratic Lie algebra from a quadratic Lie algebra (S, ψ) and a Lie algebra $\mathcal{W}$ (not necessarily quadratic) which acts on S by skew-symmetric derivations with respect to ψ. Let us remark that the non-trivial new quadratic Lie algebra will be obtained if $\mathcal{W}$ acts by non-inner skew-symmetric derivation on (S, ψ). In general, it is difficult to find a Lie algebra $\mathcal{W}$ of dimension upper or equal to 2 in order to make such double extension.

For a nonassociative algebra $(P, .)$, denote by P^- (resp. P^+) the algebra with multiplication $[x, y] = x.y - y.x$ (resp. $x \circ y = \frac{1}{2}(x.y + y.x)$) defined on the vector space P. $(P, .)$ is called MPJ-admissible algebra if $(P, [\ ,\ ], \circ)$ is a MPJ-algebra. So $(P, .)$ is a MPJ-admissible algebra if and only if $(P, .)$ is a Malcev-admissible algebra (ie. $(P, [\ ,\ ])$ is a Malcev algebra), $(P, .)$ is a Jordan-admissible algebra (ie. $(P, \circ)$ a Jordan algebra) and $(P, .)$ is a flexible algebra (ie. for all $x \in P$, $[x, .]$ is a derivation of $(P, \circ)$). Conversely, if $(P, [\ ,\ ], \circ)$ is a Malcev-Poisson-Jordan algebra, then the vector space P endowed with the product defined by:

$$x.y := \frac{1}{2}[x, y] + x \circ y, \ \ \forall\, x, y \in P,$$

is a MPJ-admissible algebra. Algebraic studies of Poisson-admissible algebras $(P, .)$, (ie. P^- is a Lie algebra, P^+ is an associative commutative algebra and $(P, .)$ is a flexible algebra), can be found in [8] and [19]. Other interesting geometrical results were obtained, in [8], on a Lie group such that its Lie algebra is P^- where $(P, .)$ is a Poisson admissible algebra. The well-known example of MPJ-admissible algebras are the alternative algebras. However, in [19], there are examples of algebras of dimension 3 which are Poisson-admissible algebras (thus MPJ-admissible algebras) but not alternative algebras. This proves that the class of MPJ-admissible algebras strictly contains the class of alternative algebras. Next, we study the structure of MPJ-algebra $(P, [\ ,\ ], \circ)$ endowed with a symmetric nondegenerate bilinear form ψ such that ψ is invariant (ie. $\psi([x, y], z) = \psi(x, [y, z])$ and $\psi(x \circ y, z) = \psi(x, y \circ z)$, $\forall x, y, z \in P$). The motivation for studying these algebras comes from the fact that the nonassociative algebras endowed by invariant symmetric nondegenerate bilinear forms appear in several areas of Mathematics and Physics [13]. We explicitly give all MPJ structures on some interesting classes of Malcev algebras. Further, we show how one can construct new interesting quadratic Lie algebras and pseudo-euclidean Malcev (non Lie) algebras from PEMPJ-algebras. We introduce the concept of double extension of PEMPJ-admissible algebras by the one-dimensional algebra with null product by using some results of [4],[5], in order to give an inductive description of nilpotent PEMPJ-admissible algebras. Next, we introduce the concept of double extension of PEMPJ-admissible algebras, we prove in Theorem 4.42 that if a PEMPJ-admissible algebra is decomposed as direct sum of the orthogonal of an ideal totally isotropic and a subalgebra then it is a suitable double extension.

This thesis is composed of five chapters that are organized in the following way:

In the first chapter, we give the basic definitions of nonassociative algebras studied in this thesis and we review the different interesting results related to these algebras. In the first section of this chapter we recall general definitions and examples related to a nonassociative algebra, however in the second section we study a class of nonassociative algebras satisfying a weak associativity which called power-associative algebras. The flexible law in an algebra plays an important role in its structure, in particular a flexible algebra is a third power-associative. We recall an interesting result given in [40], that is an algebra is power-associative if and only if it is third and fourth power-associative. In the third section, we recall basic facts on Jordan algebras and give some equalities related to this class and we prove that any Jordan algebra is a power-associative algebra. The fourth section is devoted to give basic notions of Lie algebras and to recall an interesting construction called Tits-Kantor-Koecher construction (TKK construction) [27, 28]. This construcion is an interesting process to construct Lie algebras with large dimensions from Jordan algebras with small dimensions. The last section OF this chapter is an investigation of a class of nonassociative algebras which generalizes the class of Lie algebras. We recall some interesting identities which will be used in Chapter 4. Moreover, we review the construction of Loos [32] which allows us to construct Lie triple systems from Malcev algebras.

In the second chapter, we study alternative algebras endowed with an invariant scalar products, we recall in the first section some definitions and properties of alternative algebras, it follows that the concept of nilpotent algebra, solvable algebra and nilalgebra coincides for alternative algebras. Then, we define a bilinear form (called the trace form) on an alternative algebra which characterizes the semisimplicity case. In the second section, we prove that the left (resp. right) alternative algebras endowed with an invariant scalar product are olso right (resp. left) alternative algebras. Then, we discuss the links between alternative algebras and some other algebraic structures such as Jordan algebras and Malcev algebras. In the third section we review some properties of the Peirce decomposition relative to a single idempotent e and we give others in case of a pseudo-euclidean alternative algebra. The last section is reserved to discuss the relationship between alternative algebras and Lie triple systems. we prove that a pseudo-euclidean Malcev algebra gives rise to a Lie triple system with invariant scalar product and $\mathbb{Z}_2$-graded quadratic Lie algebra.

In the third chapter, we introduce the notion of double extension of alternative algebras and we use this notion to describe inductively the pseudo-euclidean alternative algebras. In the first section, we define two types of extensions the so-called semi-direct product and central extension of alternative algebras. Then, we prove that the second cohomology group $H(A, \mathbb{F})$ can be interpreted as the set of classes of one dimensionel central extensions of the alternative algebra A. Moreover, we prove in Corollary 3.10 that every finite dimensional pseudo-euclidean irreducible alternative algebra, which is neither simple nor nilpotent, is a suitable double extension. In the second section, we introduce the notion of generalized double extension of pseudo-euclidean alternative algebras by the one dimensional alternative algebra with zero product. The main results of this section are, in one hand, many finite-dimensional pseudo-euclidean alternative algebras including the irreducible nilpotent pseudo-euclidean alternative algebras, which are not the one dimensional alternative algebras with zero product, are in fact isometric to certain generalized double extensions. On the other hand, we claim that the results concerning descriptions of pseudo-euclidean Jordan algebras by double extension and generalized extensions can be extended to those of alternative algebras.

In the fourth chapter, we study MPJ-algebras and highlight the links between these algebras and some other algebraic structures (we often consider these algebras in terms of a single bilinear operation (ie. MPJ-admissible algebras)). In the first section, we prove that a MPJ-admissible algebra is a noncommutative Jordan algebra. We also construct a generalized triple system from a MPJ-admissible algebra with an additional condition. To do this, we use results obtained in [16] on the connection between a class of noncommutative Jordan algebras and a class of some generalized Jordan triple systems. An interesting problem which is treated in the second section, that is to classify the Jordan structures $\circ$ on the underlying vector space of an arbitrary Malcev algebra $(P, [\ ,\ ])$. These Jordan structures $\circ$ are called a MPJ structures on Malcev algebra $(P, [\ ,\ ])$. By using the classification of flexible Malcev-admissible structures on the reductive Malcev algebras obtained by H.C. Myung in [38], we explicitly give all MPJ structures on reductive Malcev algebras over an algebraically closed field with characteristic zero. Recall that $\mathbb{O}^-$ is a reductive Malcev algebra where $\mathbb{O}$ is the octonions algebra. In the third section, we prove how from a PEMPJ-algebra one can construct non-trivial examples of quadratic Lie algebras by using the notion of double extension for quadratic Lie algebras

[35]. The aim of the fourth chapter is to give interesting constructions of pseudo-euclidean Malcev algebra related to PEMPJ-algebras by using the concept of double extension in the case of pseudo-euclidean Malcev algebras [4]. In order to give an inductive description of nilpotent PEMPJ-admissible algebras, we introduce in the fifth section the concept of double extension of PEMPJ-admissible algebras by the one-dimensional algebra with null product by using some results of [4],[5] . In the last section, we introduce the concept of double extension of PEMPJ-admissible algebras, we prove in Theorem 4.42 that if a PEMPJ-admissible algebra is decomposed as direct sum of the orthogonal of an ideal totally isotropic and a subalgebra then it is a suitable double extension.

In the last chapter, we would like to extend the concept of MPJ-algebras to Lie–Yamaguti algebras. The notion of a Lie-Yamaguti algebra is a natural abstraction made by K. Yamaguti [52] of Nomizu's considerations. Yamaguti called these systems general Lie triple systems, while Kikkawa [23] termed them Lie triple algebras. The term Lie–Yamaguti algebra, adopted here, appeared for the first time in [26]. They have been studied by several authors [24], [25], [43], [44], [45], although there is not a general structure theory. In particular, a classification of the simple Lie-Yamaguti algebras seems to be a very difficult task. In the first section, We recall some definitions and results concerning Lie-Yamaguti algebras, we prove that we can construct a Lie-Yamaguti algebra from a MPJ-algebra. Moreover we give a definition of a pseudo-euclidean Lie–Yamaguti algebra and prove that we can construct a pseudo-euclidean Lie-Yamaguti algebra from a PEMPJ-algebra. In the second section, we study LY-algebras admitting a unique, up to a constant, invariant scalar product. We prove that any LY-algebra $\mathcal{T}$ over $\mathbb{F}$ (with an additional condition) admitting a unique, up to a constant, invariant scalar product is necessarily a simple LY-algebra. If the field $\mathbb{F}$ is algebraically closed and $\mathcal{T}$ is irreducible such that $\mathcal{T}$ and $L(\mathcal{T},\mathcal{T})$ are not isomorphic as $L(\mathcal{T},\mathcal{T})$-modules, then $\mathcal{T}$ admits a unique, up to a constant, invariant scalar product.

We conclude this thesis by a conclusion and perspectives.

Chapter 1

Preliminaries and results on nonassociative algebras

In this chapter we will recall some basic definitions and results on nonassociative algebras. All algebras, vector spaces are finite-dimensional and defined over a commutative field $\mathbb{F}$ of characteristic 0.

1.1 Basic concepts on nonassociative algebras

Definition 1.1

1. An algebra is a vector space A together with a bilinear multiplication $(x, y) \mapsto xy$ from $A \times A$ to A.
2. An algebra A is commutative (resp. anticommutative) if for all $x, y \in A$, $xy = yx$ (resp. $xy = -yx$).
3. The associator in an algebra A is the trilinear function $(x, y, z) = (xy)z - x(yz)$, for $x, y, z \in A$.
4. The algebra A is associative (resp. nonassociative) if the associator vanishes identically $(x, y, z) = 0$, for all $x, y, z \in A$ (resp. the associator is not necessarily vanishes).
5. An algebra A is flexible (weaker than associativity) if for all $x, y \in A$, the identity $(x, y, x) = 0$ holds.

Examples 1.2

1. All commutative (resp. anticommutative) algebras are flexible.
2. There are other algebras which are flexible as Lie algebras, Jordan algebras, Malcev algebras, and alternative algebras that we will see later.

3. The set of $n \times n$ matrices with coefficients in $\mathbb{F}$, endowed with the sum and matrix product and scalar multiplication is an associative algebra.

4. The polynomials with coefficients in $\mathbb{F}$ form an associative, commutative algebra.

5. Let $\mathfrak{sl}_2 := span\{X, H, Y\}$ be the vector space of dimension 3 spanned by X, H, and Y such that

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The vector space $\mathfrak{sl}_2$ endowed with the following product:

$$XY = -YX = H, \quad HX = -XH = 2X, \quad HY = -YH = -2Y, \quad XX = YY = HH = 0,$$

is a nonasociative algebra.

Remark 1.3

1. If A is any associative algebra, then the algebra A^- (resp.A^+) obtained by defining a new multiplication $[x, y] = xy - yx$ (resp. $x.y = xy + yx$), in the same vector space as A, is a nonassociative algebra.

2. Let a be any element of an algebra A. The right (resp. the left) multiplication R_a (resp. L_a) of A which is determined by a is defined by $R_a : x \to xa$ (resp. $L_a : x \to ax$), for all $x \in A$. Then the algebra generated by the left and right multiplications of A is associative. This associative algebra is called the multiplication algebra of A.

Definition 1.4

Let A be an algebra and I be a vector subspace of A.

1. I is called a subalgebra of A if it is stable for the multiplication, that is $II \subseteq I$.

2. I is called a left ideal (resp. right ideal) if $xy \in I$ for all $x \in A$ and $y \in I$ (resp. for all $x \in I$ and $y \in A$). A left and right ideal is called a two-sided ideal.

In all this thesis we will mean by ideal a two-sided ideal.

Definition 1.5

Given two algebras A and B, then a morphism of algebras $f : A \longrightarrow B$ is simply a linear map such that $f(xy) = f(x)f(y), \quad \forall x, y \in A$.
An isomorphism is a homomorphism which is a linear isomorphism of vector spaces.

Proposition 1.6

The morphism $A \longrightarrow A/I$ from an algebra A to the quotient algebra A/I is a morphism of algebras.

Definition 1.7

we say that an algebra A is unital if there exists an element $e \in A$ for which $ex = xe = x$, for all $x \in A$.

Remark 1.8

If an algebra A does not contain an identity element e there is a standard construction for obtaining an algebra $(A_1, .)$ from A containing e. To do this, Let $A_1 = \mathbb{F} \times A$ as a vector space endowed with the following product: for $(\alpha, a), (\beta, b) \in \mathbb{F} \times A$, put

$$(\alpha, a).(\beta, b) = (\alpha\beta, \beta a + \alpha b + ab).$$

Then A_1 is an algebra with identity element $e = (1, 0)$. We say that we have adjoined an identity element to A to obtain A_1. Moreover, $\{0\} \times A$ is an ideal of A_1 which is isomorphic as an algebra to A itself. Moreover, A_1 is commutative (resp. associative) iff A is.

Definition 1.9

Let A be an algebra and $f : A \longrightarrow A$ be a linear map.

1. f is called a derivation of A if $f(xy) = xf(y) + f(x)y$.
2. f is called an involution of A if $f(xy) = f(y)f(x)$ and $f(f(x)) = f^2(x) = x$,

for all $x, y \in A$.

Examples 1.10

1. Let $\mathfrak{sl}_2$ be the nonassociative algebra defined above, the linear map $f_X : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2$ defined by $f_X(Z) = XZ$, for all $Z \in \mathfrak{sl}_2$ and $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, is a derivation of $\mathfrak{sl}_2$.

2. Let A be a nonassociative algebra, the vector space $A \times A$ endowed with the following product:

$$(x, y)(z, t) = (xz, ty), \quad \forall x, y, z, t \in A$$

is a nonassociative algebra and the linear map $\sigma : A \times A \longrightarrow A \times A$ given by

$$\sigma(x, y) = (y, x), \text{ for all } x, y \in A.$$

is an involution of $A \times A$

Remark 1.11

The set of all derivations of an algebra A, noted by $Der(A)$, is a vector subspace of $End(A)$ (The space of endomorphisms of A).

Proposition 1.12

Let f_1 and f_2 be two derivations of an algebra A, then the commutator $[f_1, f_2] = f_1 f_2 - f_2 f_1$ is again a derivation of A.

Definition 1.13

Let A be an algebra. Set $A^1 = A^{(1)} = A$, and then by induction define

$$A^{n+1} = \sum_{i+j=n+1} A^i A^j \text{ and } A^{(n+1)} = A^{(n)}A^{(n)}, \text{ for } n \geq 1.$$

The algebra A is nilpotent if $A^n = \{0\}$ for some n and solvable if $A^{(s)} = \{0\}$ for some s. The smallest natural number n (respectively s) with this property is the nilpotency index (respectively solvability index) of A.

Remark 1.14

Every nilpotent algebra is solvable, the converse is not generally true as proves the following example:

Example 1.15

Let A be an algebra with basis x, y, and multiplication given by $x^2 = y^2 = 0$ and $xy = yx = y$. Then, we have $A^{(2)} = \{0\}$ but $A^n \neq \{0\}$ for any $n \geq 1$ Thus A is solvable but not nilpotent.

Proposition 1.16

If I and J are solvable ideals of an algebra A, then $I + J$ is a solvable ideal of A. Hence A has a unique maximal solvable ideal R. We call R the radical of A.

Definition 1.17

An algebra A is called simple if $AA \neq 0$ and A has no ideals apart from 0 and A.
An algebra A is called semisimple if it is the direct sum of simple algebras, this is equivalent to say that the radical of A is 0.

1.2 Power-associative algebras

In this section we study a class of nonassociative algebras satisfying a weak associativity, that is the subalgebra generated by any element is associative.

Definition 1.18

Let A be an algebra. For $x \in A$, define $x^1 = x$ and $x^{m+1} = x^m x^1$ for positive integers m.
An element $x \in A$ is said to be nth power-associative if $x^p x^q = x^{p+q}$ for all positive integers p, q such that $p + q = n$.
x is called power-associative if it is nth power-associative for all positive integer n.
A is called power-associative If every element of A is power-associative.
In case every element of A is n-th power-associative, A itself is called nth power-associative.

Remark 1.19

1. Let A be a flexible algebra, we have for all $x \in A$, $(x, x, x) = x^2x - xx^2 = 0$, then all flexible algebras are third power-associative.

2. Two important power-associativities are third power-associativity and fourth power-associativity, as we will see in the Lemma 1.21.

$$x^2x = xx^2 \tag{1.1}$$

$$x^2x^2 = (x^2x)x = x(x^2x) = x(xx^2) = (xx^2)x \tag{1.2}$$

for all $x \in A$.

Notation:
For an algebra A, one can denote by A^- (resp. A^+) the anticommutative algebra (resp. the commutative algebra) with multiplication $[x, y] = xy - yx$ (resp. $x \circ y = \frac{1}{2}(xy + yx)$) defined on the vector space A.

Lemma 1.20

Assume that A is a third power-associative algebra. Then

i) A satisfies the identity $x^3x = xx^3$ and hence fourth power-associativity (1.2) if and only if $x^2x^2 = x^3x$,

ii) A satisfies the relation $(x^2x)x - x^2x^2 = ((x \circ x) \circ x) \circ x - (x \circ x) \circ (x \circ x)$ for all $x \in A$. Hence, A is fourth power-associative if and only if A^+ is fourth power-associative .

Proof: i) The third power-associativity (1.1) can be linearized to the identity

$$[x \circ y, z] + [y \circ z, x] + [z \circ x, y] = 0, \quad \text{for all } x, y, z \in A.$$

Letting $y = x$ and $z = x^2$, then by using (1.1) we obtain $[x^3, x] = 0$.

ii) Using $[x^3, x] = 0$, we have

$$\begin{aligned}(x^2x)x - x^2x^2 &= \frac{1}{2}((x^2x)x - x(x^2x)) - x^2 \circ x^2 \\ &= (x^2x) \circ x - x^2 \circ x^2 \\ &= \frac{1}{2}(x^2x + xx^2) \circ x - x^2 \circ x^2 \\ &= (x^2 \circ x) \circ x - x^2 \circ x^2 \\ &= ((x \circ x) \circ x) \circ x - (x \circ x) \circ (x \circ x).\end{aligned}$$

Thus, A is fourth power-associative if and only if A^+ is. ■

The following result is useful.

Lemma 1.21

An algebra A is power-associative if and only if it is third and fourth power-associative, that is, A satisfies (1.1) and (1.2).

Proof: Since A is a fourth power-associative, then by the Lemma 1.20, A^+ is a power-associative. Now, for each $x \in A$, let x^p denote the p-th power of x in A^+ for each positive integer p. Hence, $x^{p+q} = x^p \circ x^q = \frac{1}{2}(x^px^q + x^px^q))$ for all positive integers p and q, and we need to show that $x^{p+q} = x^px^q$. It is clearly sufficient to show that $[x^p, x^q] = 0$, for all positive integers p and q. To do this we proceed by induction on $n = p + q$. for more details (see. [40], Proposition 9.7). ■

Remark 1.22

Not every algebra is power-associative as shown by the following examples.

Examples 1.23

1. Let A be an algebra which have a basis $a,\ b,\ c,\ d,\ e,\ f$ with multiplication given by
$$ab = c, \quad cd = e, \quad bd = \alpha f, \quad af = e,$$
and all other products are zero. Here α denotes a fixed non zero scalar in $\mathbb{F}$. It is easily verified that,
$$(a + b + d)^2(a + b + d) = (c + \alpha f)(a + b + d) = e$$
and,
$$(a + b + d)(a + b + d)^2 = (a + b + d)(c + \alpha f) = \alpha e.$$
Thus if α is chosen so that $\alpha \neq 1$, then A is not third power-associative.

2. Let $A := span\{x, x^2, x^3, y\}$ be a commutative four dimensional algebra with the following multiplication table:
$$xx = x^2,\ xx^2 = x^2x = x^3,\ x^2x^2 = x^3,\ x^2y = yx^2 = x^3,\ yy = x^2 + x^3,$$
all other products being zero. We have $(x^2, x, x) = (x^2x)x - x^2(xx) = -x^3$, thus A is not fouth power-associative. However, since A is commutative then it is third power-associative. Finally, we deduce that There is a third power-associative algebra which is not fourth power-associative.

1.3 Jordan algebras

In this section we recall very basic facts on Jordan algebras. This class of algebras is well studied by several authors, for more details see [46, 18, 33, 21, 12].

Definition 1.24

A Jordan algebra A is a commutative algebra in which the Jordan identity
$$(xy)x^2 = x(yx^2), \text{ for all } x, y \in A, \tag{1.3}$$
is satisfied

This identity is equivalent to $[R_x, R_{x^2}] = 0$, that is the endomorphisms R_x and R_{x^2} commute. Where R_x is the right multiplication by x in the algebra A.

In general a Jordan algebra is not associative.

Examples 1.25

1. Clearly any commutative associative algebra is a Jordan algebra.

2. Any associative algebra A (which is not commutative) gives rise to a Jordan algebra $(A^+, \circ)$, where the product $\circ$ is defined by:

$$x \circ y = \frac{1}{2}(xy + yx), \text{ for all } x, y \in A.$$

3. If V is a linear subspace of an associative algebra A which is square stable, i.e. for any x in V, x^2 belongs to V, then V equipped with the product

$$x \circ y = \frac{1}{2}(xy + yx), \text{ for all } x, y \in V.$$

is a Jordan algebra. (In fact we have $x \circ y = \frac{1}{4}((x+y)^2 + (x-y)^2)$).

A Jordan algebra is called special if it is isomorphic to a subalgebra of A^+ for some associative algebra A and it is called exceptional if it is not special.

Remark 1.26

Any Jordan algebra is flexible since it is commutative.

In a Jordan algebra A, the following equality are satisfied ([46]):

$$2(x, y, zx) + (z, y, x^2) = 0, \text{ for all } x, y, z \in A. \tag{1.4}$$

$$(x, y, wz) + (w, y, zx) + (z, y, xw) = 0, \text{ for all } x, y, z, w \in A. \tag{1.5}$$

$$[R_{wz}, R_x] + [R_{zx}, R_w] + [R_{xw}, R_z] = 0, \text{ for all } x, z, w \in A. \tag{1.6}$$

$$R_{xy}R_z - R_xR_yR_z + R_{zx}R_y - R_{y(zx)} + R_{zy}R_x - R_zR_yR_x = 0, \text{ for all } x, y, z \in A. \tag{1.7}$$

$$[R_x, [R_y, R_z]] = R_{(y,x,z)} = R_{[R_z,R_y](x)}, \text{ for all } x, y, z \in A. \tag{1.8}$$

Proposition 1.27

Any Jordan algebra is a power-associative.

Proof: Let A be a Jordan algebra, since A is flexible then it is third power-associative.

Now, by the Lemma 1.20 and the Lemma 1.21 it remains to prove that $x^2x^2 = x^3x$. We have,

$$x^3x = (x^2x)x = x(xx^2),$$

and using 1.3,

$$x^3x = (xx)x^2 = x^2x^2.$$

■

A natural generalization to noncommutative algebras is the class of algebras A satisfying (1.3).

Definition 1.28

A noncommutative Jordan algebra A is a flexible algebra satisfying (1.3).

Examples 1.29

1. Clearly any Jordan algebra can be considered as a noncommutative Jordan algebra.
2. The best known nonassociatives algebras (alternative, Malcev, and Lie algebras) are noncommutative Jordan algebras. (which will be proved in the following sections)

Let A be a noncommutative Jordan algebra, it is shown in ([3]) that flexibility implies that (1.3) is equivalent to any one of the following:

$$x^2(yx) = (x^2y)x; \quad x^2(xy) = x(x^2y); \quad (yx)x^2 = (yx^2)x, \ \forall x \in A.$$

Proposition 1.30

An algebra A is a noncommutative Jordan if and only if A is flexible and Jordan-admissible.

Proof: (See. [47], p. 473). ■

1.4 Lie algebras

In the following, we recall some basis notions of Lie algebras which will be needed later, this class is included in the class of noncommutative Jordan algebras. Lie algebras are the best-known examples of nonassociative algebras which arise in the study of Lie groups.

Definition 1.31

An algebra $\mathfrak{g}$ with multiplication $[\ ,\ ]$ is called a Lie algebra if it satisfies:

- The anticommutative law

$$[x, x] = 0,$$

- The Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

For all $x, y, z \in \mathfrak{g}$.

Examples 1.32

1. Any vector space V can be made into a Lie algebra with the trivial bracket:
$$[x, y] = 0, \quad \forall x, y \in V.$$
2. Any associative algebra A can be made into a Lie algebra by taking the commutator as the Lie bracket:
$$[x, y] = xy - yx, \quad \forall x, y \in A.$$
3. Let V be any vector space. The space $End(V)$ forms an associative algebra under function composition. It is also a Lie algebra with the commutator as the

Lie bracket. Whenever we think of it as a Lie algebra we denote it by $gl(V)$. The Lie algebra of $n \times n$-matrices is called $gl(n)$.

4. Let $\mathfrak{sl}(n)$ be the subspace of $gl(n)$ consisting of matrices with zero trace. Since $Tr(AB) = Tr(BA)$, the set $\mathfrak{sl}(n)$ is closed under $[A, B] = AB - BA$, and hence is a Lie algebra.

5. Let A be a Jordan algebra and $\bar{A}$ be a second copy of A. Let $R(A) := span\{R_a, a \in A\}$ be the algebra spanned by the right multiplications of A. We consider the vector space:

$$\mathfrak{Lie}(A) := A \oplus \mathcal{H}(A) \oplus \bar{A}, \text{ where } \mathcal{H}(A) = R(A^2) \oplus [R(A), R(A)].$$

$\mathfrak{Lie}(A)$ endowed with the following bracket:

$$[T, T'] = [T, T']_{\mathcal{H}}; \;\; [T, a'] = T(a'); \;\; [T, \overline{b'}] = -\overline{T(b')}; \;\; [a, \overline{b'}] = R_{ab'}; \;\; [a, a'] = [\overline{b}, \overline{b'}] = 0,$$

is a Lie algebra, for all $T, T' \in \mathcal{H}(A), a, a', b, b' \in A$.

This construction is called the Tits-Kantor-Koecher construction [27, 28] (TKK construction), through this construction one can construct Lie algebras from Jordan algebras.

Remarks 1.33

1. Lie algebras are not unital nor associative.
2. Any Lie algebra is a noncommutative Jordan algebra since the identity (1.3) is trivially satisfied.

Definition 1.34

Given two Lie algebras $(\mathfrak{g}, [\ ,\ ]_g)$ and $(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}})$, a linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is said to be a morphism of Lie algebras if it preserves the bracket in the following manner $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{g}}, \;\; \forall x, y \in \mathfrak{g}$.

Example 1.35

The morphism $\mathfrak{g} \longrightarrow \mathfrak{g}/I$ from a Lie algebra $\mathfrak{g}$ to the quotient algebra is a morphism of Lie algebras.

Proposition 1.36

Let A be an algebra, then $Der(A)$ is a Lie subalgebra of $gl(A)$.

Proof: (see Proposition 1.12). ■

Adjoint action. Assume that $\mathfrak{g}$ is a Lie algebra. Consider the following map $ad : \mathfrak{g} \longrightarrow gl(\mathfrak{g}), \;\; ad(x)y = [x, y], \;\; \forall x, y \in \mathfrak{g}$. We call the map $ad(x) : \mathfrak{g} \longrightarrow \mathfrak{g}$ the adjoint action of the element x on $\mathfrak{g}$. It is straightforward to check that ad is a morphism of Lie algebra.

Co-adjoint action. Assume $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}^*$ its dual. Consider the following map $ad^* : \mathfrak{g} \longrightarrow gl(\mathfrak{g}^*)$, $ad^*(x)(f)(y) = -f([x,y])$, $\forall f \in \mathfrak{g}^*, x, y \in \mathfrak{g}$. We call the map $ad^*(x) : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ the co-adjoint action of the element x on $\mathfrak{g}$. It is straightforward to check that ad^* is a morphism of Lie algebra.

Representation. A representation ρ of a Lie algebra $\mathfrak{g}$ is a morphism of Lie algebra from $\mathfrak{g}$ to the Lie algebra $gl(V)$.

$$\rho : \mathfrak{g} \longrightarrow gl(V)$$

(V : *vector space*).

The following examples are very interesting,

Examples 1.37

1. **The trivial represenation**. Every Lie algebra $\mathfrak{g}$ has a trivial representation on a vector space V defined via the map: $\pi : \mathfrak{g} \longrightarrow gl(V)$, $\pi(x) = 0$, $\forall x \in \mathfrak{g}$.
2. **The adjoint representation**. Every Lie algebra $\mathfrak{g}$ has a representation on itself defined via the map: $ad : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$, $ad(x)y = [x,y]$, $\forall x, y \in \mathfrak{g}$.
3. **The co − adjoint representation**. Every Lie algebra $\mathfrak{g}$ has a representation on its dual defined via the map: $ad^* : \mathfrak{g} \longrightarrow gl(\mathfrak{g}^*)$, $ad(x)(f)(y) = f([x,y])$, $\forall f \in \mathfrak{g}^*, x, y \in \mathfrak{g}$.

1.5 Malcev algebras and related Lie triple systems

This section is an investigation of a class of nonassociative algebras which generalizes the class of Lie algebras. We recall Loos's construction ([32]) of Lie triple systems from Malcev algebras. The theory of Malcev algebras is well developed, see for example ([29],[30],[31],[38],[42],[53],[54]).

Definition 1.38

An algebra M with multiplication [,] is called a Malcev algebra if it satisfies the anticommutative law $[x,x] = 0$ and the Malcev identity

$$J(x, y, [x, z]) = [J(x, y, z), x], \tag{1.9}$$

for all $x, y, z \in M$. Where J corresponds to Jacobi's identity, that is,

$$J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y].$$

Example 1.39

Any Lie algebra is a Malcev algebra, however there are Malcev algebras which are non-Lie algebras.

Let M has a basis $\{e_1, e_2, e_3, e_4\}$ with multiplication table:

	e_1	e_2	e_3	e_4
e_1	0	$-e_2$	$-e_3$	e_4
e_2	e_2	0	$2e_4$	0
e_3	e_3	$-2e_4$	0	0
e_4	$-e_4$	0	0	0

With few calculations, one can easily show that M is a Malcev algebra. However, $J(e_1, e_2, e_1e_3) = 6e_4 = J(e_1, e_2, e_3)e_1$ and hence M is not a Lie algebra.

Remark 1.40

Any Malcev algebra M is a noncommutative Jordan algebra (since $[x, x] = 0$, for all $x \in M$ and the identity (1.3) is trivially satisfied).

Lemma 1.41

The identity (1.9) of a Malcev algebra is equivalent to either of the following identities:

$$J(x, [x, y], z) = [J(x, y, z), x] \tag{1.10}$$

$$[[x, y], [x, z]] = [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y] \tag{1.11}$$

Proof: Assume (1.9), the Jacobian $J(x, y, z)$ is a skew-symmetric function, then

$$J(x, [x, y], z) = -J(x, z, [x, y]) = -[J(x, z, y), x] = [J(x, y, z), x].$$

Thus (1.9) implies (1.10). The converse is immediate.

Assume (1.11), then $J(x, y, [x, z]) = [[x, y], [x, z]] + [[y, [x, z]], x] + [[[x, z], x], y]$.

If we replace $[[x, y], [x, z]]$ by the right side of (1.11) we obtain

$$\begin{aligned} J(x, y, [x, z]) &= [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y] + [[y, [x, z]], x] + [[[x, z], x], y] \\ &= [[[x, y], z], x] + [[[y, z], x], x] + [[y, [x, z]], x] \\ &= [J(x, y, z), x]. \end{aligned}$$

Thus (1.11) implies (1.9). For the converse we can just rewrite (1.9) to obtain (1.11).

(See. [42]). ■

Corollary 1.42

For any Malcev algebra M we have, for all $x, y, z, \in M$

$$J(x, [x, y], z) = J(x, y, [x, z]) \tag{1.12}$$

Remark 1.43

A. I. Malcev ([34]) has shown that (1.12) does not imply (1.9).

The Malcev algebras satisfy certain identities which will be used in Chapter 4.

Lemma 1.44

Let $(M, [\ ,\ ])$ be a Malcev algebra. For all elements $x, y, z, t \in M$, we have the following identities:

$$J(x, y, [z, t]) + J(z, y, [x, t]) = [J(x, y, t), z] + [J(z, y, t), x], \tag{1.13}$$

$$[[x, z], [y, t]] = [x, y, z, t] + [y, z, t, x] + [z, t, x, y] + [t, x, y, z], \tag{1.14}$$

$$3J([x, y], z, t) = [J(x, z, t), y] - [J(z, t, y), x] - 2[J(t, y, x), z] + 2[J(y, x, z), t], \tag{1.15}$$

where $[x, y, z, t] := [[[x, y], z], t]$.

Proof: For (1.13), linearizing identity (1.9) in x we obtain

$$J(x, y, [t, z]) + J(t, y, [x, z]) = [J(x, y, z), t] + [J(t, y, z), x]. \tag{1.16}$$

Hence the identity (1.13) holds.

For (1.14), using (1.16) we get

$$R_{[[x,y],z]} + R_{[[y,z],x]} = R_y R_z R_x - R_z R_x R_y - R_{[y,z]} R_x - R_{[z,x]} R_y + R_y R_{[z,x]} + R_z R_{[x,y]}, \tag{1.17}$$

where R_x is the right multiplication by x in the algebra $(M, [\ ,\ ])$.

Adding the three identities obtained by cyclic permutations of the variables in (1.17) we obtain

$$R_{J(x,y,z)} = [R_y, R_{[z,x]}] + [R_z, R_{[x,y]}] + [R_x, R_{[y,z]}] \tag{1.18}$$

Substracting (1.17) from (1.18) we obtain

$$R_{[[z,x],y]} = R_z R_x R_y - R_y R_z R_x + R_x R_{[y,z]} + R_{[x,y]} R_z,$$

or equivalently,

$$R_{[[x,y],z]} = R_x R_y R_z - R_z R_x R_y + R_y R_{[z,x]} + R_{[y,z]} R_x. \tag{1.19}$$

Identity (1.19) implies that the identity (1.14) holds.

The identity (1.15) can be found in ([42], Lemma 2.10). ∎

Definition 1.45

A Malcev algebra is said to be reductive if it is a direct sum of a semisimple Malcev algebra and its center.

Example 1.46

Any simple or semisimple Malcev algebra is a reductive algebra.

Example 1.47

Let $\mathbb{O}$ be the octonions algebra with the following multiplication table with respect to the basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	$-e_4$	e_3	$-e_6$	e_5	e_8	$-e_7$
e_3	e_3	e_4	$-e_1$	$-e_2$	$-e_7$	$-e_8$	e_5	e_6
e_4	e_4	$-e_3$	e_2	$-e_1$	$-e_8$	e_7	$-e_6$	e_5
e_5	e_5	e_6	e_7	e_8	$-e_1$	$-e_2$	$-e_3$	$-e_4$
e_6	e_6	$-e_5$	e_8	$-e_7$	e_2	$-e_1$	e_4	$-e_3$
e_7	e_7	$-e_8$	$-e_5$	e_6	e_3	$-e_7$	$-e_1$	e_2
e_8	e_8	e_7	$-e_6$	$-e_5$	e_4	e_3	$-e_2$	$-e_1$

With few calculations, one can easily show that the multiplication table of the algebra $\mathbb{O}^-$ may be taken to be:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	0	0	0	0	0	0	0
e_2	0	0	$-2e_4$	$2e_3$	$-2e_6$	$2e_5$	$2e_8$	$-2e_7$
e_3	0	$2e_4$	0	$-2e_2$	$-2e_7$	$-2e_8$	$2e_5$	$2e_6$
e_4	0	$-2e_3$	$2e_2$	0	$-2e_8$	$2e_7$	$-2e_6$	$2e_5$
e_5	0	$2e_6$	$2e_7$	$2e_8$	0	$-2e_2$	$-2e_3$	$-2e_4$
e_6	0	$-2e_5$	$2e_8$	$-2e_7$	$2e_2$	0	$2e_4$	$-2e_3$
e_7	0	$-2e_8$	$-2e_5$	$2e_6$	$2e_3$	$-2e_7$	0	$2e_2$
e_8	0	$2e_7$	$-2e_6$	$-2e_5$	$2e_4$	$2e_3$	$-2e_2$	0

The table show that $\mathbb{O}^-$ is a Malcev algebra and $Z(\mathbb{O}^-) = \mathbb{F}e_1$ (where $Z(\mathbb{O}^-)$ is the centre of $\mathbb{O}^-$). Moreover, the subspace $\mathbb{O}^* := span\{e_2, e_3, ..., e_8\}$ is a subalgebra of $\mathbb{O}^-$. It is an easy matter to prove that $\mathbb{O}^*$ is a simple algebra. It is clear by the above multiplication table that a non trivial ideal I of $\mathbb{O}^*$ must contains all elements e_i for $i \in \{2, ..., 8\}$. Consequently, $\mathbb{O}^* = I$. Then the Malcev algebra $\mathbb{O}^- = \mathbb{O}^* \oplus Z(\mathbb{O}^-)$ is reductive. Since $J(e_2, e_7, e_5) = 12e_4 \neq \{0\}$, then $\mathbb{O}^*$ is simple (non-Lie) Malcev algebra.

Recall that a Malcev algebra $(M, [\ ,\ ])$ is said to be simple if it has no ideals except itself and zero, and $[M, M] \neq \{0\}$.

Proposition 1.48 (*see. [29]*)

A simple Malcev algebra is either a simple Lie algebra or isomorphic to the 7-dim simple (non-Lie) Malcev algebras $M(\alpha, \beta, \gamma)$, where α, β, γ are scalars in $\mathbb{F}$ with $\alpha\beta\gamma \neq \{0\}$, and the multiplication table of this algebra (defined with respect to the basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$) may be taken to be:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	$-e_7$	αe_6
e_2	$-e_3$	0	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	αe_2	$-\beta e_1$	0	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	γe_1	γe_2	γe_3
e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	0	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	0	$-\beta\gamma e_1$
e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	0

Definition 1.49 (*see. [36]*)

A Lie triple system is a vector space P equipped with a trilinear product $[a, b, c]$ satisfying the following three properties ($a, b, c, d, e \in P$):

$$[a, b, c] = -[b, a, c]; \tag{1.20}$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0; \tag{1.21}$$

$$[a, b, [c, d, e]] - [c, d, [a, b, e]] = [[a, b, c], d, e] + [c, [a, b, d], e]. \tag{1.22}$$

Eq.(1.20 and 1.21) abstract the skew symmetry and Jacobi identity for the triple commutator.

Example 1.50

If $(g, [\ ,\ ])$ is a Lie algebra then the pair $(g, [.,.,.])$ where $[.,.,.] : g \times g \times g \longrightarrow g$ defined by $[a, b, c] = [[a, b], c]$, $\forall a, b, c \in g$ is a Lie triple system.

Example 1.51

If $(J, \circ)$ is a Jordan algebra then the pair $(J, [.,.,.])$ where $[.,.,.] : J \times J \times J \longrightarrow J$ defined by $[a, b, c] = (b \circ c) \circ a - b \circ (c \circ a)$, $\forall a, b, c \in g$ is a Lie triple system.

Definition 1.52

A derivation of a Lie triple system $(P, [\ ,\ ,\ ])$ is a linear map $D : P \to P$ satisfying

$$D([a, b, c]) = [D(a), b, c] + [a, D(b), c] + [a, b, D(c)],\ \forall a, b, c \in P.$$

Let $x, y \in P$, denote by $L(x, y)$ the endomorphism of P defined by $L(x, y)(z) = [x, y, z]$, $\forall z \in P$. By Eq.(1.22) one can observe that $L(x, y)$ is a derivation of Lie triple system $(P, [\ ,\ ,\ ])$.

Denote by $L(P,P)$ the vector subspace of $End(P)$ spanned by $\{L(x,y),\ x,y\in P\}$. Eq.(1.22) is equivalent to

$$[L(a,b),L(c,d)] = L(L(a,b)(c),d) - L(c,L(b,a)(d)),\ \forall a,b,c,d\in P,$$

which means that $L(P,P)$ is closed under commutation, then $L(P,P)$ is a Lie subalgebra of $(End(P),[\ ,\ ])$.

The following Proposition show that one can construct a Lie triple system from a Malcev algebra.

Proposition 1.53 (*see. [32]*)

Let $(M,[\ ,\ ])$ be a Malcev algebra, then the pair $(M,[.,.,.])$ is a Lie triple system. Where the trilinear product $[.,.,.]: M\times M\times M\longrightarrow M$ is defined by

$$[a,b,c] = 2[[a,b],c] - [[b,c],a] - [[c,a],b],\ \forall a,b,c\in M.$$

Chapter 2

Invariant scalar products on alternative algebras

2.1 Alternative algebras

In this section, we recall some definitions and concepts of alternative algebras. For a general theory about alternative algebras (see. [46, 57]).

Definition 2.1

Let A be a nonassociative algebra (i.e. not necessarily associative), A is called alternative if:

$$x^2y = x(xy) \quad and \quad yx^2 = (yx)x, \ \forall x, y \in A.$$

The left and right equations are known, respectively, as the left and right alternative laws. They are equivalent in terms of associators to:

$$(x, x, y) = (y, x, x) = 0, \ \forall x, y \in A.$$

Or in terms of left and right multiplications to:

$$L_{x^2} = L_x^2 \quad and \quad R_{x^2} = R_x^2, \ \forall x \in A.$$

The associator (x_1, x_2, x_3) "alternates" in the sense that for any permutation $\sigma \in S_3$ we have:

$$(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = (sign\sigma)(x_1, x_2, x_3).$$

To establish this it is sufficient to prove

$$(x, y, z) = -(y, x, z) = (y, z, x), \ \forall x, y, z \in A.$$

Our identities on associators imply that in an alternative algebra A we have: $(x, y, x) = 0, \ \forall x, y \in A$, then every alternative algebra is flexible.

We shall have occasions to use the Maufang identities:

$$(xax)y = x[a(xy)] \tag{2.1}$$

$$y(xax) = [(yx)a]x \tag{2.2}$$

$$(xy)(ax) = x(ya)x \tag{2.3}$$

For all x, y, a in an alternative algebra A such that $xax = (xa)x = x(ax)$.
The Moufang identity (2.2) is equivalent to $(y, xa, x) = -(y, x, a)x, \ \forall x, y, a \in A$.
For the associator we have,

$$\begin{aligned}(y, xa, x) &= [y(xa)]x - y(xax) \\ &= [y(xa)]x - [(yx)a]x \\ &= [y(xa) - (yx)a]x \\ &= -(y, x, a)x.\end{aligned}$$

Theorem 2.2

The subalgebra generated by any two elements of an alternative algebra A is associative.

Proof: (see. ([46]), Theorem 3.1) ■

Corollary 2.3

Theorem (2.2) implies that any alternative algebra is power-associative.

Definition 2.4

An element x in a power-associative algebra is called nilpotent in case there is an integer r such that $x^r = 0$. An algebra (ideal) consisting only of nilpotent elements is called a nilalgebra (nilideal).

Proposition 2.5

Any solvable power-associative algebra is a nilalgebra.

Proof: Let A be a solvable power-associative algebra of solvability index n, one can easily check that for all $x \in A, \ x^{2^{n-1}} \in A^{(n)} = \{0\}$ then A is a nilalgebra. ■

Remark 2.6

Any nilpotent algebra is solvable and any solvable power-associative algebra is a nilalgebra (Proposition 2.5). Moreover, it is shown in ([46], Theorem 3.2) that any alternative nilalgebra (of finite dimension) is nilpotent. It follows that the concept of nilpotent algebra, solvable algebra and nilalgebra coincide for alternative algebras.

Now we will define a bilinear form on an alternative algebra which characterizes the semisimplicity case. This form plays the role of the Killing form in the case of Lie algebras. The following Proposition can be found in ([46], p. 44).

Proposition 2.7

Let A be an alternative algebra and $(\ ,\ ) : A \times A \to \mathbb{F}$ be the bilinear form defined by:

$$(x, y) = trace\ R_x R_y, \ \forall x, y \in A.$$

The radical R of A is the radical of the form $(\ ,\ )$. ($i.e$ $Rad(A) = \{x \in A, \ (x, y) = 0, \ \forall y \in A\}$. This form is called the trace form.

Corollary 2.8

Let A be an alternative algebra, then A is semisimple (that is the radical of A is 0) if and only if the trace form of A is nondegenerate.

2.2 Pseudo-euclidean alternative algebras

In this section, we study alternative algebras endowed with an invariant scalar product called pseudo-euclidean alternative algebras. We prove that the left (resp. right) alternative algebras endowed with an invariant scalar product are olso right (resp. left) alternative algebras. Then, we discuss the links between alternative algebras and some other algebraic structures such as Jordan algebras and Malcev algebras.

Definition 2.9

Let P be a nonassociative (ie. not necessarily associative) algebra with multiplication denoted by $*$ and $\psi : P \times P \longrightarrow \mathbb{F}$ a bilinear form. ψ will be called:

1. symmetric if $\forall x, y \in P$ we have $\psi(x, y) = \psi(y, x)$;
2. nondegenerate if $\psi(x, y) = 0, \quad \forall y \in P \Rightarrow x = 0$ and if $\psi(x, y) = 0$, $\forall x \in P \Rightarrow y = 0$;
3. invariant if $\forall x, y, z \in P$ we have $\psi(x * y, z) = \psi(x, y * z)$.

If ψ is symmetric, nondegenerate and invariant, $(P, *, \psi)$ will be called orthogonal, quadratic or pseudo-euclidean algebra and ψ will be called an invariant scalar product

Definition 2.10

Let A be an alternative algebra equipped with an invariant scalar product ψ. The pair (A, ψ) is called orthogonal, quadratic or pseudo-euclidean alternative algebra.
We shall call the pair (A, ψ) a pseudo-euclidean alternative algebra .

Example 2.11

1. Any semisimple alternative algebra is pseudo-euclidean (Corollary 2.8).
2. Let A be an alternative algebra and A^* be the dual vector space of the underlying vector space of A, the vector space $\tilde{A} = A \oplus A^*$ endowed with the following product:

$$(x + f) \star (y + h) := xy + f \circ L_y + h \circ R_x, \ \forall (x, f), (y, h) \in A \oplus A^*,$$

is an alternative algebra.
If we consider the bilinear form $\psi : (A \oplus A^*) \times (A \oplus A^*) \to \mathbb{F}$ defined by:

$$\psi(x + f, y + h) = f(y) + h(x), \ \forall (x, f), (y, h) \in A \oplus A^*,$$

then $(A \oplus A^*, \psi)$ is a pseudo-euclidean alternative algebra called the trivial T^*-extension of A and noted by $T_0^* A$ ([13]).

Remark 2.12

If (A, ψ) *is a pseudo-euclidean alternative algebra then:*

$$\psi((x, y, z), t) = -\psi(x, (y, z, t)), \ \forall x, y, z, t \in A,$$

where $(x, y, z) = (xy)z - x(yz)$ *is the associator in* A.

Proof: Let $x, y, z, t \in A$,

$$\begin{aligned}\psi((x,y,z),t) &= \psi((xy)z,t) - \psi(x(yz),t)\\ &= \psi((xy),zt) - \psi(x,(yz)t)\\ &= \psi(x,y(zt)) - \psi(x,(yz)t)\\ &= \psi(x,y(zt)-(yz)t)\\ &= -\psi(x,(yz)t - y(zt))\\ &= -\psi(x,(y,z,t)).\end{aligned}$$

■

Proposition 2.13

Let (A,ψ) be a pseudo-euclidean algebra, then the following assertions are equivalent:

i) *A is alternative,*

ii) $(x,x,y) = 0, \ \forall x, y \in A$,

iii) $(y,x,x) = 0, \ \forall x, y \in A$.

Proof: It suffices to note that $\psi((y,x,x),t) = -\psi((y,(x,x,t)), \ \forall x,y,t \in A$. ■

Remark 2.14

This propostion shows that the left (resp. right) alternative algebras endowed with an invariant scalar product are also right (resp. left) alternative algebras.

The connection between alternative algebras, Malcev algebras and Jordan algebras is given by the two following propositions:

Proposition 2.15

Any alternative algebra A is a Malcev-admissible algebra, that is, his commutator define a Malcev algebra.

Proof: Let $x, y, z \in A$, we have,

$$\begin{aligned}J(x,y,z) &= [[x,y],z] + [[y,z],x] + [[z,x],y]\\ &= (xy-yx)z - z(xy-yx) + (yz-zy)x\\ &\quad - x(yz-zy) + (zx-xz)y - y(zx-xz)\\ &= 2(x,y,z) + 2(z,x,y) + 2(y,z,x)\\ &= 6(x,y,z)\end{aligned}$$

From which

$$\begin{aligned}[J(x,y,z),x] &= 6[(x,y,z)x - x(x,y,z)].\\ J(x,y,[x,z]) &= 6((x,y,xz) - (x,y,zx)),\end{aligned}$$

Using equations (2.1) and (2.2) we find,

$$\begin{aligned}(x,y,z)x &= -(y,x,z)x \\ &= (y,xz,x) \\ &= (x,y,xz), \\ (x,y,zx) &= -(x,zx,y) \\ &= x[(zx)y)] - (xzx)y.\end{aligned}$$

We then get, $(x,y,zx) = x(z,x,y) = x(x,y,z)$.
Therefore,

$$[J(x,y,z),x] = J(x,y,[x,z]) \quad \text{and} \quad [x,y] = -[y,x].$$

Then $A^- := (A,[\ ,\ ])$ is a Malcev algebra. ■

Proposition 2.16

If $(A,.)$ is an alternative algebra, then $A^+ := (A,*)$ is a Jordan algebra, where $x*y := \frac{1}{2}(x.y+y.x)$, $\forall x,y \in A$ (that is $(A,.)$ is a Jordan-admissible). We recall that a (commutative) Jordan algebra J is an algebra in which products are commutative:

$$xy = yx,\ \forall x,y \in J,$$

and satisfy the Jordan identity;

$$x(yx^2) = (xy)x^2,\ \forall x,y \in J.$$

Proof: Let $x,y \in A$, we have,

$$\begin{aligned}4x*(y*x^2) &= x.(y.x^2+x^2.y)+(y.x^2+x^2.y).x \\ &= x.(y.x^2)+x.(x^2.y)+(y.x^2).x+(x^2.y).x \\ &= x.((y.x).x)+x.(x.(x.y))+((y.x).x).x+(x.(x.y)).x \\ &= (x.(y.x)).x+x^2.(x.y)+(y.x).x^2+x.((x.y).x) \\ &= ((x.y).x)).x+x^2.(x.y)+(y.x).x^2+x.(x.(y.x)) \\ &= (x.y).x^2+x^2.(x.y)+(y.x).x^2+x^2.(y.x) \\ &= (x.y+y.x).x^2+x^2.(x.y+y.x) \\ &= 4(x*y)*x^2.\end{aligned}$$

Then A^+ is a Jordan algebra ■

One can deduce from Proposition 2.16 and Proposision 1.30 that any alternative algebra is a noncommutative Jordan algebra.

Remark 2.17

If (A,ψ) is a pseudo-euclidean alternative algebra, then (A^-,ψ) is a pseudo-euclidean Malcev algebra and (A^+,ψ) is a pseudo-euclidean Jordan algebra.

Definition 2.18

Let (A,ψ) be a pseudo-euclidean alternative algebra and I an arbitrary vector subspace of A.

(a) *I is called an ideal (resp. a subalgebra) of A if and only if $AI + IA \subset I$ (resp. $II \subset I$).*

(b) *I is called nondegenerate if the restriction of ψ to $I \times I$ is nondegenerate, otherwise, it is called degenerate.*

(c) *We say that (A, ψ) is irreducible if every ideal of A is degenerate.*

Lemma 2.19 (*see. [13]*)

Let (A, ψ) be a pseudo-euclidean alternative algebra and I an ideal of A. Then:
$I^{\perp}$ is again ideal of A satisfying $I(I^{\perp}) = (I^{\perp})I = \{0\}$.
If I is nondegenerate, then $A = I \oplus I^{\perp}$ and $I^{\perp}$ is nondegenerate.

Proposition 2.20 (*see. [13]*)

Let (A, ψ) be a pseudo-euclidean alternative algebra. Then,

$$A = \bigoplus_{i=1}^{n} I_i$$

where for all $1 \leq i \leq n$, I_i is a nondegenerate irreducible ideal, and for all $i \neq j$, I_i and I_j are orthogonal.

2.3 Peirce decomposition of pseudo-euclidean alternative algebras

In the following, we recall some properties of the Peirce decomposition relative to a single idempotent e, (for more details see. [46]) and we give others in case of a pseudo-euclidean alternative algebra (A, ψ).

Definition 2.21

An element e of an (arbitrary) algebra A is called an idempotent in case $e^2 = e \neq 0$.

Proposition 2.22 (*see. [46]*)

Any finite-dimensional alternative algebra A which is not a nilalgebra contains an idempotent $e(\neq 0)$.

L_e and R_e are idempotent operators on A which commute by the flexible law. It follows that A is the vector space direct sum

$$A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}, \tag{2.4}$$

where $A_{ij}(i, j = 0, 1)$ is the subspace of A defined by,

$$A_{ij} = \{a_{ij} / ea_{ij} = ia_{ij}, a_{ij}e = ja_{ij}\}.$$

Proposition 2.23 (*see. [46]*)

Let e be an idempotent in an alternative algebra A, and let (2.4) be the Peirce decomposition of A relative to e, we have

1. $A_{ij}A_{jk} \subseteq A_{ik},\ i,j,k = 0,1$

2. $A_{ij}A_{ij} \subseteq A_{ji},\ i,j = 0,1$

3. A_{00} *and* A_{11} *are orthogonal subalgebras of* A.

In particular, we have $A_{10}A_{01} \subseteq A_{11}$, $A_{01}A_{10} \subseteq A_{00}$. Moreover, the properties $A_{10}A_{10} \subseteq A_{01}$, $A_{01}A_{01} \subseteq A_{10}$ are weaker in the associative case, where one can actually prove that $A_{10}A_{10} = A_{01}A_{01} = 0$.

In a pseudo-euclidean alternative algebra (A,ψ) we derive a few of the properties of the Peirce decomposition relative to a single idempotent e as follows:

$$\psi(a_{01}, a_{00}) = \psi(a_{01}e, a_{00}) = \psi(a_{01}, ea_{00}) = 0.$$

Hence $A_{01} \subseteq A_{00}^{\perp}$. Also $A_{10} \subseteq A_{00}^{\perp}$. And similarly one can check that $A_{01} \subseteq A_{11}^{\perp}$, $A_{10} \subseteq A_{11}^{\perp}$, $A_{01} \subseteq A_{01}^{\perp}$, $A_{10} \subseteq A_{10}^{\perp}$, and $A_{00} \subseteq A_{11}^{\perp}$. So $A_{ij} \subseteq A_{ii}^{\perp} \cap A_{jj}^{\perp}$ and $A_{ii} \subseteq A_{jj}^{\perp}$ for $i \neq j$ $(i,j = 0,1)$

Proposition 2.24

Let (A,ψ) *be a pseudo-euclidean alternative algebra with idempotent* e. *Let (2.4) be the Peirce decomposition of* A *relative to* e. *Then:*

$$A_{00}A_{10} = A_{11}A_{01} = A_{10}A_{11} = A_{01}A_{00} = 0$$

Proof: Let $b = b_{11} + b_{10} + b_{01} + b_{00}$, be an element of A, we have:

$$\begin{aligned}\psi(a_{00}a_{10}, b) &= \psi(a_{00}a_{10}, b_{11}) + \psi(a_{00}a_{10}, b_{10}) + \psi(a_{00}a_{10}, b_{01}) + \psi(a_{00}a_{10}, b_{00})\\ &= \psi(b_{11}a_{00}, a_{10}) + \psi(b_{10}a_{00}, a_{10}) + \psi(a_{00}, a_{10}b_{01}) + \psi(a_{00}, a_{10}b_{00})\\ &= 0\end{aligned}$$

Since ψ is nondegenerate $a_{00}a_{10} = 0$, then $A_{00}A_{10} = 0$.
And similarly $A_{11}A_{01} = A_{10}A_{11} = A_{01}A_{00} = 0$ ■

2.4 Pseudo-euclidean Alternative algebras and related $\mathbb{Z}_2$-graded quadratic Lie algebra

Now, let's start by recalling the correspondance between the category of $\mathbb{Z}_2$-graded Lie algebras and the Lie triple systems (see [22]). Then, we discuss the relationship between alternative algebras and Lie triple systems. We show that from a pseudo-euclidean Malcev algebra we can construct a Lie triple system with invariant scalar product and a $\mathbb{Z}_2$-graded quadratic Lie algebra.

Definition 2.25

Let $(g,[\ ,\ ])$ *be a Lie algebra and* $\psi : g \times g \longrightarrow \mathbb{F}$ *be a bilinear form.* (g,ψ) *is called a pseudo-euclidean Lie algebra if* ψ *is symmetric, nondegenerate and invariant.*

Remark 2.26

A pseudo-euclidean Lie algebra is called also a quadratic Lie algebra. (see [35]).

Definition 2.27

Let g be a Lie algebra. g is called a $\mathbb{Z}_2$-graded Lie algebra if there are two subspaces $g_{\bar{0}}$ and $g_{\bar{1}}$ of g satisfying:
$g = g_{\bar{0}} \oplus g_{\bar{1}}, \quad [g_{\bar{i}}, g_{\bar{j}}] \subset g_{\overline{i+j}}, \quad \forall i, j \in \mathbb{Z}_2.$

Definition 2.28

Let (g, ψ) be a quadratic Lie algebra. (g, ψ) is called a $\mathbb{Z}_2$-graded quadratic Lie algebra if there are two subspaces $g_{\bar{0}}$ and $g_{\bar{1}}$ of g satisfying:

$$g = g_{\bar{0}} \oplus g_{\bar{1}}, \quad [g_{\bar{i}}, g_{\bar{j}}] \subset g_{\overline{i+j}}, \quad \forall i, j \in \mathbb{Z}_2 \text{ and } \quad \psi(g_{\bar{0}}, g_{\bar{1}}) = \{0\}.$$

Let $g = g_{\bar{0}} \oplus g_{\bar{1}}$ be a $\mathbb{Z}_2$-graded Lie algebra, then $(g_{\bar{1}}, [., ., .])$ is a Lie triple system, where $[., ., .] : g_{\bar{1}} \times g_{\bar{1}} \times g_{\bar{1}} \longrightarrow g_{\bar{1}}$ defined by $[x, y, z] = [[x, y], z], \ \forall x, y, z \in g$. Conversely, from a given Lie triple system T, for $(x, y) \in T$ we consider:

$$\begin{aligned} L(x, y) : \ & T \longrightarrow T \\ & z \longmapsto [x, y, z] \end{aligned}$$

Now, let $L(T, T) \subset Der(T)$ be the Lie subalgebra of $Der(T)$ generated by the elements of the form $L(x, y); \ (x, y) \in T$ and put : $g(T) := L(T, T) \oplus T$.
Then, we define on $g(T)$ a bilinear antisymmetric bracket as follows:

(i) If $(x, y) \in T, \ [x, y] := L(x, y)$

(ii) If $D \in L(T, T), x \in T, \ [D, x] := Dx$

(iii) If $D_1, D_2 \in L(T, T), \ [D_1, D_2] = D_1 D_2 - D_2 D_1$.

One can see that $g(T)$ is a $\mathbb{Z}_2$-graded Lie algebra. The Lie algebra constructed in this way is called the standard embedding of the Lie triple system.

Proposition 2.29

If $(A, .)$ is an alternative algebra, the pair $(A, [., ., .])$ where

$$\begin{aligned} [., ., .] : \ & A \times A \times A \longrightarrow A \\ & (a, b, c) \longmapsto 3((ab)c + (cb)a - c(ab) - b(ac)), \ \forall a, b, c \in A \end{aligned}$$

is a Lie triple system.

Proof: Since $(A, .)$ is an alternative algebra then $(A^-, [\ ,\ ])$ is a Malcev algebra where $[a, b] = ab - ba$.
According to the proposition (1.53), the pair $(A^-, [., ., .])$ is a Lie triple system where:

$$\begin{aligned} [a, b, c] &= 2[[a, b], c] - [[b, c], a] - [[c, a], b] \\ &= 2[ab - ba, c] - [bc - cb, a] - [ca - ac, b] \\ &= 2((ab - ba)c - c(ab - ba)) - ((bc - cb)a - a(bc - cb)) \\ &- ((ca - ac)b - b(ca - ac)). \end{aligned}$$

After simplification we find that $[a, b, c] = 3((ab)c + (cb)a - c(ab) - b(ac))$. Where we use the definition of the associator, and the fact that this latter alternates. ∎

Definition 2.30

Let $(T,[.,.,.])$ be a Lie triple system. A nondegenerate symmetric bilinear form ψ on T is said to be invariant on the Lie triple system $(T,[.,.,.])$ if it satisfies:

$$\psi(R(a,b)c,d)=\psi(c,R(b,a)d),\ for\ \ all\ a,b,c,d\in T,$$

where $R(a,b)c = [c,b,a]$. Such a form is called an invariant scalar product on $(T,[.,.,.])$.

Proposition 2.31

Let $(M,[\ ,\ ],\psi)$ be a pseudo-euclidean Malcev algebra, then ψ is an invariant scalar product on the Lie triple system $(M,[.,.,.])$ obtained by proposition 1.53.

Proof: Let $a,b,c,d\in M$.

$$\begin{aligned}\psi([c,b,a],d) &= 2\psi([[c,b],a],d)-\psi([[b,a],c],d)-\psi([[a,c],b],d)\\ &= 2\psi([c,b],[a,d])-\psi([b,a],[c,d])-\psi([a,c],[b,d])\\ &= 2\psi([[d,a],b],c)-\psi([[a,b],d],c)-\psi([[b,d],a],c)\\ &= \psi(c,2[[d,a],b]-[[a,b],d]-[[b,d],a])\\ &= \psi(c,[d,a,b]).\end{aligned}$$

This means that

$$\psi(R(a,b)c,d)=\psi(c,R(b,a)d).$$

■

Proposition 2.32

Let T be a Lie triple system, the standard embedding $g(T)$ is a $\mathbb{Z}_2$-graded Lie algebra, where $g(T)_{\bar{0}} = L(T,T)$, $g(T)_{\bar{1}} = T$. Moreover, the bilinear form $\widetilde{\psi}$ on $g(T)\times g(T)$ defined by:

$$\begin{aligned}\widetilde{\psi}_{|T\times T} &= \psi,\\ \widetilde{\psi}(L(T,T),T) &= \widetilde{\psi}(T,L(T,T))=\{0\},\\ \text{and } \widetilde{\psi}(D,\sum_{i=1}^{n}L(x_i,y_i)) &= \sum_{i=1}^{n}\psi(Dx_i,y_i), \qquad \forall x_i,y_i\in T,\ \ D\in L(T,T),\end{aligned}$$

is an invariant scalar product on $g(T)$, (where ψ is an invariant scalar product on T).

Proof: see [7] (Prop. III.2.4. Page 21). ■

Corollary 2.33

$(g(T),\widetilde{\psi})$ is a $\mathbb{Z}_2$-graded quadratic Lie algebra.

The proof of the Proposition (2.32) and the Corollary (2.33) is straightforward.

It is well known that there is a correspondence between simply-connected pseudo-Riemannian symmetric spaces and symmetric triples (see e.g. [22], [15]) . Hence, the classification of simply -connected pseudo-Riemannian symmetric spaces up to isometry is equivalent to the classification of symmetric triples up to isomorphism. Moreover, one can construct at least one triple symmetric from a $\mathbb{Z}_2$-graded quadratic Lie algebra.(see [7] p. 31).

Chapter 3

Inductive descriptions of pseudo-euclidean alternative algebras

The following section is devoted to introducing double extension of alternative algebras. First, we define two types of extensions the so-called semi-direct product and central extension of alternative algebras.

3.1 Double extension of pseudo-euclidean alternative algebras

3.1.1 Semi-direct product of alternative algebras

Let A be an alternative algebra over $\mathbb{F}$ and M be a vector space over $\mathbb{F}$. Then, M is a bimodule over A in case there are two linear maps $\pi : A \to End(M)$, $\Pi : A \to End(M)$ satisfying:

(i) $\pi(a^2) = (\pi(a))^2$, (.i.e) $\pi(a)o\pi(a) = \pi(a^2)$

(ii) $\Pi(a^2) = (\Pi(a))^2$,

(iii) $\Pi(a')\pi(a)-\pi(a)\Pi(a') = -(\Pi(a')\Pi(a)-\Pi(aa'))$ (.i.e) $\Pi(aa')-\Pi(a')\Pi(a) = [\Pi(a'), \pi(a)]$,

(iv) $\pi(aa')-\pi(a)\pi(a') = -(\Pi(a')\pi(a)-\pi(a)\Pi(a'))$ (.i.e) $\pi(aa')-\pi(a)\pi(a') = [\pi(a), \Pi(a')]$,

$\forall a, a' \in A$.
The vector space direct sum $A \oplus M$ is made into an alternative algebra, by defining multiplication by:
$(a + m)(a' + m') = aa' + \pi(a)m' + \Pi(a')m$, for all $a, a' \in A$ and $m, m' \in M$.
Note the A-bimodule M by (M, π, Π), $\pi(a)m := am$ and $\Pi(a)m := ma$.
(M, π, Π) is called (bi)-representation of A associated to M. (see. ([46],[20]).

Now, let A and M be two alternative algebras such that (M, π, Π) is an A-bimodule. On the vector space $A \oplus M$ we define the following product:

$$(a + m)(a' + m') = aa' + ma' + am + mm'$$

$A \oplus M$ endowed with this product is an alternative algebra if and only if (π, Π) satisfying the following conditions:

(i) $\Pi(a)m^2 = m(\Pi(a)m)$,

(ii) $\pi(a)m^2 = (\pi(a)m)m$,

(iii) $(\pi(a)m)m' - \pi(a)(mm') = m(\pi(a)m') - (\Pi(a)m)m' = \Pi(a)(mm') - m(\Pi(a)m')$.

Proof: $A \oplus M$ is an alternative algebra if and only if:

$$\begin{cases} (a+m)^2(a'+m') = (a+m)[(a+m)(a'+m')] \\ (a'+m')(a+m)^2 = [(a'+m')(a+m)](a+m), \end{cases}$$

it follows that,

$$\begin{cases} (a^2+ma+am+m^2)(a'+m') = (a+m)(aa'+ma'+am'+mm') \\ (a'+m')(a^2+ma+am+m^2) = (a'a+m'a+a'm+m'm)(a+m). \end{cases}$$

The fact that A and M are two alternative algebras and M is an A-bimodule imply that this is equivalent to:

$$\begin{cases} m^2a' + (ma)m' + (am)m' = a(mm') + m(ma') + m(am') \\ a'm^2 + m'(am) + m'(ma) = (m'm)a + (m'a)m + (a'm)m, \end{cases}$$

hence,

$$\begin{cases} m^2a' = m(ma') \\ (am)m' - a(mm') = m(am') - (ma)m' \\ a'm^2 = (a'm)m \\ (m'm)a - m'(ma) = m'(am) - (m'a)m, \end{cases}$$

equivalent to

$$\begin{aligned} m^2a &= m(ma) \\ am^2 &= (am)m \\ (am)m' - a(mm') &= m(am') - (ma)m' \\ &= (mm')a - m(m'a). \end{aligned}$$

■

If (π, Π) satisfy the last three conditions i)-ii) and iii) above, then (M, π, Π) is called an admissible bi-representation (or an admissible A-bimodule). In this case, the alternative algebra $A \oplus M$ is called the semidirect product of A by M by means of (π, Π), noted as $A \underset{(\pi,\Pi)}{\ltimes} M$. (see [7, 5]) in case of Jordan algebras.

3.1.2 Central extension of alternative algebras and interpretation of the second cohomology group

Let A be an alternative $\mathbb{F}$-algebra, a one-dimensional central extension of A is an exact sequence of alternative algebras:

$$0 \longrightarrow \mathbb{F} \xrightarrow{i} \widetilde{A} \xrightarrow{\pi} A \longrightarrow 0$$

such that $Im(i) \subset Ann(\widetilde{A})$.
Two such extensions $\widetilde{A}$ and $\widetilde{A}'$ are called equivalent if there is a homomorphism $\phi : \widetilde{A} \to \widetilde{A}'$ of alternative algebras such that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{F} & \xrightarrow{i} & \widetilde{A} & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathbb{F} & \xrightarrow{i'} & \widetilde{A}' & \xrightarrow{\pi'} & A & \longrightarrow & 0 \end{array}$$

is commutative.

Denote by $CL(A, \mathbb{F})$ the set of classes of one-dimensional central extensions of A.

Example 3.1

Let A_1 be an alternative algebra, V an vector space and $\varphi : A_1 \times A_1 \to V$ be a bilinear map. We define the following multiplication on the vector space direct sum $A = A_1 \oplus V$:

$$(a+v)(b+v') := ab + \varphi(a,b) \quad \forall a,b \in A_1, v,v' \in V.$$

A is alternative algebra if and only if:

$$\begin{cases} (a+v)^2(b+v') = (a+v)[(a+v)(b+v')] \\ (b+v')(a+v)^2 = [(b+v')(a+v)](a+v), \end{cases}$$

equivalent to,

$$\begin{cases} [a^2 + \varphi(a,a)](b+v') = (a+v)(ab+\varphi(a,b)] \\ (b+v')(a^2+\varphi(a,a)) = (ba+\varphi(b,a))(a+v), \end{cases}$$

$$\begin{cases} a^2b + \varphi(a^2,b) = a(ab) + \varphi(a,ab) \\ ba^2 + \varphi(b,a^2) = (ba)a + \varphi(ba,a), \end{cases}$$

hence,

$$\begin{cases} \varphi(a^2,b) = \varphi(a,ab) \\ \varphi(b,a^2) = \varphi(ba,a), \end{cases}$$

If it's the case, we have an exact sequence:

$$0 \longrightarrow V \xrightarrow{\lambda} A_1 \oplus V \xrightarrow{\mu} A_1 \longrightarrow 0$$

$Ker\mu = \{0\} \times V \subset Z(A_1 \oplus V)$, so the extension is central.
We shall call A the central extension of A_1 by V via φ.

Remark 3.2

If we consider V as a A_1-trivial bimodule, $\delta^2\varphi = \delta^2_H\varphi \circ (id^{\otimes 3} + \sigma_1)$, (see [17]) where $\delta^2_H\varphi(a_0,a_1,a_2) = \sum_{i=0}^{1}(-1)^i\varphi(a_0,...,a_ia_{i+1},...a_2) = \varphi(a_0a_1,a_2) - \varphi(a_0,a_1a_2)$.
Hence,

$$\begin{aligned} \delta^2\varphi(a_0,a_1,a_2) &= \delta^2_H\varphi(a_0+a_1, a_1+a_0, 2a_2) \\ &= \varphi((a_0+a_1)(a_1+a_0), 2a_2) - \varphi(a_0+a_1, (a_0+a_1)(2a_2)) \\ &= 0. \end{aligned}$$

Consequently, $A_1 \oplus V$ is an alternative algebra if and only if $\varphi \in Z^2(A_1,V)$, (i.e φ is a 2-cocycle).

Now we recall the cohomology theory for left alternative algebra introduced by ([17]). Let A be a left alternative algebra and M an A-bimodule. If $p \geq 1$, a p-cochain of A with values in M is a p-linear mapping of A^p in M.
We denote by $C^p(A, M)$ the space of the p-cochains of A. For $p < 0$ we put $C^p(A, M) = \{0\}$, and $C^0(A, M) = M$. For $\varphi \in C^1(A, M)$, we define the first differential $\delta^1\varphi \in C^2(A, M)$ by

$$\delta^1\varphi(a, b) = \varphi(a).b + a.\varphi(b) - \varphi(ab). \ \ For \ \ all \ \ a, b \in A$$

For $\varphi \in C^2(A, M)$ we define the second differential $\delta^2\varphi \in C^3(A, \mathbb{F})$ by

$$\delta^2\varphi = \delta^2_H\phi \circ (id^3 + \sigma_1).$$

Where σ_1 is defined on A^3 by $\sigma_1(a, b, c) = (b, a, c)$, and id denote the identity map on A, δ_H denote the Hochschild differential (see [17]).
Call any p-form $\varphi \in C^p(A, M)$ a p-cocycle if and only if $\delta^p\phi = 0$ and denote the subspace of p-cocycles by $Z^p(A, M)$.
The p-th cohomology group $H^p(A, M)$ is defined to be the factor space $Z^p(A, M)/\delta^{p-1}C^{p-1}(A, M)$ for $p \geq 1$.

Proposition 3.3

We consider $\mathbb{F}$ as a trivial A-bimodule, then the space $H^2(A, \mathbb{F})$ can be interpreted as the set of classes of one dimensional central extensions of the alternative algebra A. (i.e there is an isomorphisme between $H^2(A, \mathbb{F})$ and $CL(A, \mathbb{F})$.

Proof: Let φ be *in* $Z^2(A, \mathbb{F})$, to φ we can associate the extension

$$0 \longrightarrow \mathbb{F} \xrightarrow{i} A \oplus \mathbb{F} \xrightarrow{\pi} A_1 \longrightarrow 0\ ,$$

where the product of $A \oplus \mathbb{F}$ is given by $(a, \alpha)(b, \beta) = (ab, \varphi(a, b))$. For all $a, b \in A$ and $\alpha, \beta \in \mathbb{F}$. ($A \oplus \mathbb{F}$ is alternative algebra if and only if $\varphi \in Z^2(A, \mathbb{F})$).
If $\overline{\varphi} = \overline{\varphi'}$ then $\varphi' = \varphi - \delta^1\theta$, with $\theta \in C^1(A, \mathbb{F})$. Or φ' defines a one dimensional central extension

$$0 \longrightarrow \mathbb{F} \xrightarrow{i} A \oplus \mathbb{F} \xrightarrow{\pi} A_1 \longrightarrow 0\ ,$$

where the product of $A \oplus \mathbb{F}$ is given by:

$$(a, \alpha)(b, \beta) = (ab, \varphi'(a, b)) = (ab, \varphi(a, b) - \delta^1\theta(a, b)).$$

The mapping $\phi : A\oplus_\varphi\mathbb{F} \to A\oplus_{\varphi'}\mathbb{F}$ defined by $(a, \alpha) \to (a, \alpha+\theta(a))$, ϕ is a homomorphism of alternative algebras. Indeed, let $a, b \in A$ and $\alpha, \beta \in \mathbb{F}$.

$$\begin{aligned}
\phi((a, \alpha)(b, \beta)) &= \phi(ab, \varphi(a, b)) \\
&= (ab, \varphi(a, b) + \theta(ab)) \\
&= (ab, \varphi(a, b) - \delta^1\theta(a, b)) \qquad (Since \ \ \delta^1\theta(a, b) = -\theta(ab)) \\
&= (ab, \varphi'(a, b)) \\
&= (a, \alpha + \theta(a))(b, \beta + \theta(b)) \\
&= \phi(a, \alpha)\phi(b, \beta).
\end{aligned}$$

Which gives the equivalent between these extensions. So with cohomologeous cocycles correspond equivalent one dimensional central extensions, and we have

$$H^2(A, \mathbb{F}) \cong CL(A, \mathbb{F}).$$

■

Now, we introduce the notion of double extension of alternative algebras and we use this concept to describe inductively pseudo-euclidean alternative algebras. These concept have been introduced by A. Medina and Ph. Revoy for quadratic Lie algebras (see. [35]) and extended by S. Benayadi and A. Baklouti for pseudo-euclidean Jordan algebras (see [7, 5]).

Let (M, ψ) be a pseudo-euclidean alternative algebra, A an alternative algebra not necessarily pseudo-euclidean such that M is an A-bimodule admissible with:

$$\psi(a.m, m') = \psi(m, m'.a), \ \forall m, m' \in M, \ \forall a \in A.$$

Let

$$\begin{array}{rcccl} \varphi & : & M \times M & \longrightarrow & A^* \\ & & (m, m') & \longmapsto & \varphi(m, m')(a) = \psi(a.m, m'). \end{array}$$

Then, $\varphi(m^2, m')(a) = \psi(a.m^2, m')$ and,

$$\begin{array}{rcl} \varphi(m, mm')(a) & = & \psi(a, m, mm') \\ & = & \psi((a.m)m, m') \\ & = & \psi(am^2, m'). \end{array}$$

(M is an A-bimodule alternative).
From which

$$\varphi(m^2, m')(a) = \varphi(m, mm')(a) \ \ \forall m, m' \in M, \ \ \forall a \in A.$$

Hence,

$$\varphi(m^2, m') = \varphi(m, mm').$$

Moreover,

$$\varphi(m', m^2)(a) = \psi(a, m', m^2) = \psi(m', m^2.a)$$

And

$$\begin{array}{rcl} \varphi(m'm, m)(a) & = & \psi(a.(m'm), m) \\ & = & \psi(m'm, m.a) \\ & = & \psi(m', m(m.a) \\ & = & \psi(m', m^2.a). \end{array}$$

Then,

$$\varphi(m', m^2)(a) = \varphi(m'm, m)(a), \ \forall a \in A$$

equivalent to

$$\varphi(m', m^2) = \varphi(m'm, m).$$

Therefore, we can consider the algebra $M \rtimes_\varphi A^*$ (central extension of M by A^* via φ) whose product is defined by:

$$(m + f)(m' + f') = mm' + \varphi(m, m'), \ \forall m, m' \in M, \ \forall f, f' \in A^*.$$

Let

$$\begin{array}{rcl} A \times (M \rtimes_\varphi A^*) & \longrightarrow & M \rtimes_\varphi A^* \\ (a, m + f) & \longmapsto & a.(m + f) = a.m + foR_a, \end{array}$$

And

$$\begin{array}{rcl} (M \rtimes_\varphi A^*) \times A & \longrightarrow & (M \rtimes_\varphi A^*) \\ (m + f, a) & \longmapsto & (m + f).a = m.a + foL_a, \end{array}$$

two bilinear maps. Showing that $(M \rtimes_\varphi A^*)$ is an alternative A-bimodule, for this it suffices to prove the following three conditions:

(i) $a^2.(m+f) = a(a.(m+f))$; $(m+f).a^2 = [(m+f).a].a,$

(ii) $[a.(m+f)].a' - a.((m+f).a') = (m+f).(aa') - [(m+f).a].a',$

(iii) $(aa').(m+f) - a(a'(m+f)) = a((m+f).a') - (a(m+f))a', \ \forall a, a' \in A, \ m \in M.$

We have,

$a^2.(m+f) = a^2 + f \circ R_{a^2}$ and $a(a.(m+f) = a.(a.m + f \circ R_a) = a(a.m) + f \circ R_a \circ R_a.$

Moreover, $a^2.m = a(a.m)$ and $R_{a^2} = R_a \circ R_a$. Then,

$$a^2.(m+f) = a(a(m+f)).$$

Similarly, $(m+f).a^2 = m.a^2 + f \circ L_{a^2},$ and

$$((m+f).a).a = (m.a + f \circ L_a).a = (ma)a + f \circ L_a \circ L_a.$$

In addition, we have $m.a^2 = (ma)a$, and $L_a \circ L_a = L_{a^2}$,
from which we get $(m+f)a^2 = [(m+f).a].a.$
Showing condition (ii),

$$\begin{aligned}[a(m+f)].a' - a.[(m+f).a'] &= [a.m + f \circ R_a]a' - a[ma' + f \circ L'_a] \\ &= (a.m)a' + f \circ R_a \circ L_{a'} - a(ma') - f \circ L_{a'} \circ R_a,\end{aligned}$$

and

$$\begin{aligned}(m+f).(aa') - [(m+f).a]a' &= m((aa') + f \circ L_{aa'} - [m.a + f \circ L_a].a' \\ &= m(aa') + f \circ L_{aa'} - (ma)a' - f \circ L_a \circ L_{a'}.\end{aligned}$$

The fact that M is an A-bimodule alternative: $(a.m)a' - a(m.a') = m(aa') - (ma).a'.$
So, it remains to check that: $R_a \circ L_{a'} - L_{a'} \circ R_a = L_{aa'} - L_a \circ L_{a'}.$

$$\begin{aligned}(R_a \circ L_{a'} - L_{a'} \circ R_a)(a'') &= (R_a \circ L_{a'})(a'') - (L_{a'} \circ R_a)(a'') \\ &= (a'a'')a - a'(a''a) \\ &= (a', a'', a), \ \forall a'' \in A.\end{aligned}$$

$$\begin{aligned}(L_{aa'} - L_a o L_{a'})(a'') &= L_{aa'}(a'') - (L_a \circ L_{a'})(a'') \\ &= (aa')a'' - a(a'a'') \\ &= (a, a', a''), \forall a'' \in A.\end{aligned}$$

Since A is alternative, we have $(a', a'', a) = (a, a', a'')$ which proves (ii). Furthermore,

$$\begin{aligned}(aa')(m+f) - a[a'(m+f)] &= (aa').m + f \circ R_{aa'} - a[a'.m + f \circ R_{a'}] \\ &= (aa')m + f \circ R_{aa'} - a(a'.m) - f \circ R_{a'} \circ R_a \\ &= (aa')m - a(a'.m) + f \circ R_{aa'} - f \circ R_{a'} \circ R_a.\end{aligned}$$

$$\begin{aligned} a.[(m+f).a'] - [a.(m+f)].a' &= a[m.a + f \circ L_{a'}] - [a.m + f \circ R_a]a' \\ &= a(m.a') + f \circ L_{a'} \circ R_a - (a.m)a' - f \circ R_{aa'} \circ L_{a'} \\ &= a(m.a') - (a.m)a' + f \circ L_{a'} \circ R_a - f \circ R_a \circ L_{a'}. \end{aligned}$$

Moreover, $(aa')m - a(a'.m) = a(ma') - (a.m)a'$, it remains to check that: $R_{aa'} - R_{a'} \circ R_a = L_{a'} \circ R_a - R_a \circ L_{a'}$. Let $a'' \in A$. So, we get

$$\begin{aligned} (R_{aa'} - R_{a'} \circ R_a)(a'') &= R_{aa'}(a'') - (R_{a'} \circ R_a)(a'') \\ &= a''(aa') - (a'', a, a') \\ &= -(a'', a, a'). \end{aligned}$$

$$\begin{aligned} (L_{a'} \circ R_a - R_a \circ L_{a'})(a'') &= (L_{a'} \circ R_a)(a'') - (R_a \circ L_{a'})(a'') \\ &= a'(a''a) - (a'a'')a \\ &= -(a', a'', a). \end{aligned}$$

Since A is alternative, this leads to (iii). Consequently, $(M \rtimes_\varphi A^*)$ is an A-bimodule alternative. Now, we show that $(M \rtimes_\varphi A^*)$ is an A-bimodule admissible.
We prove the following equalities:

$$(m+f)^2.a = (m+f)[(m+f).a], \tag{3.1}$$

$$a.(m+f)^2 = [a.(m+f)](m+f), \tag{3.2}$$

$$\begin{aligned} [a(m+f)](m'+f') - a[(m+f)(m'+f')] &= (m+f)[a(m'+f')] - [(m+f).a](m'+f') \\ &= [(m+f)(m'+f')].a - (m+f)[m'+f').a], \end{aligned} \tag{3.3}$$

$\forall a \in A$, $\forall m, m' \in M$ and $\forall f, f' \in A^*$.
We have, $(m+f)^2.a = [m^2 + \varphi(m,m')].a = m^2.a + \varphi(m,m') \circ L_a$. And

$$(m+f)[(m+f).a] = (m+f)(m.a + f \circ L_a) = m(m.a) + \varphi(m, m.a).$$

Since $m^2.a = m(m.a)$, it remains to show: $\varphi(m,m') \circ L_a = \varphi(m, m.a)$.

Let $a' \in A$, then

$$\begin{aligned} (\varphi(m,m') \circ L_a)(a') &= \varphi(m,m)(aa') \\ &= \psi((aa').m, m). \end{aligned}$$

As

$$\begin{aligned} \varphi(m, m.a)(a') &= \psi(a'.m, m.a) \\ &= \psi(a(a'm), m). \end{aligned}$$

We have to check that:

$$\psi((aa').m - a(a'm), m) = 0.$$

We will proceed by linearization.
Let $m' \in M$, $\lambda \in \mathbb{K}$ and replace m by $m + \lambda m'$ in this last equation. On the one hand, we have,

$$\psi((aa').(m+\lambda m') - a(a'.(m+\lambda m') = 0,$$
$$\psi((aa').m - a(a'.m), m+\lambda m')\psi(\lambda(aa')m' - \lambda a(a'.m'), m+\lambda m') = 0,$$
$$\lambda\psi((aa').m - a(a'.m), m') + \lambda\psi((aa').m' - a(a'.m'), m) = 0,$$
$$\psi((aa').m - a(a'.m), m') + \psi((aa').m' - a(a'.m'), m) = 0,$$
$$\psi((aa').m - a(a'm) + m(aa') - (m.a)a', m') = 0.$$

M is an A-bimodule alternative then:

$$(aa').m - a(a'.m) + m(aa') - (m.a)a' = (a, a', m) - (m, a, a') = 0.$$

This proves (3.1).
On the other hand, we have,

$$\begin{aligned} a.(m+f)^2 &= a.(m^2 + \varphi(m, m')) \\ &= a.m^2 + \varphi(m, m) \circ R_a, \quad \text{and} \end{aligned}$$

$$\begin{aligned} [a.(m+f)](m+f) &= (a.m + f \circ R_a)(m+f) \\ &= (a.m)m + \varphi(a.m, m). \end{aligned}$$

Since $a.m^2 = (a.m)m$, it remains to verify that:

$$\varphi(m, m) \circ R_a = \varphi(a.m, m).$$

Let $a' \in A$,

$$\varphi(m, m) \circ R_a(a') = \varphi(m, m)(a'a) = \psi((a'a).m, m).$$

In addition, $\varphi(a.m, m)(a') = \psi(a'(a.m), m)$. Moreover,

$$\psi((a'a).m - a'(a.m), m) = 0.$$

which proves (3.2).
It is known that:

$$\begin{aligned} [a.(m+f)](m'+f') - a.[(m+f)(m'+f'] &= (a.m + f \circ R_a)(m'+f') - a.(mm' + \varphi(m, m') \\ &= (a.m)m' + \varphi(a.m, m') - a.(mm') - \varphi(m, m') \circ R_a, \end{aligned}$$

$$\begin{aligned} (m+f)[a.(m'+f')] - [(m+f).a](m'+f') &= (m+f)(a.m' + f' \circ R_a) - (m.a + foL_a)(m'+f') \\ &= m(a.m') + \varphi(m, a.m') - (m.a)m' - \varphi(m.a, m'), \end{aligned}$$

$$\begin{aligned} [(m+f)(m'+f')].a - (m+f)[(m'+f').a] &= (mm' + \varphi(m, m')).a - (m+f)(m'.a + f' \circ L_a) \\ &= (mm').a + \varphi(m, m') \circ L_a - m(m'.a) - \varphi(m, m'.a). \end{aligned}$$

Moreover,

$$\begin{aligned}(a.m)m' - a.(mm') &= m(a.m') - (m.a)m' \\ &= (mm').a - m(m'.a).\end{aligned}$$

It remains to show that:

$$\begin{aligned}\varphi(a.m, m') - \varphi(m, m') \circ R_a &= \varphi(m, a.m') - \varphi(m.a, m') \\ &= \varphi(m, m') \circ L_a - \varphi(m, m'.a).\end{aligned}$$

For that, let $a' \in A$:

$$\begin{aligned}[\varphi(a.m, m') - \varphi(m, m') \circ R_a](a') &= \psi(a'(a.m), m') - \psi(a'a).m, m') \\ &= \psi(a'(a.m) - (a'a).m, m') \\ &= \psi(-(a', a, m), m'),\end{aligned}$$

$$\begin{aligned}[\varphi(m, a.m') - \varphi(m.a, m')](a') &= \psi(a'.m, a.m') - \psi(a'(m.a), m') \\ &= \psi((a'.m), m') - \psi(a'(m.a), m') \\ &= \psi((a', m, a), m'),\end{aligned}$$

$$\begin{aligned}[\varphi(m, m') \circ L_a - \varphi(m, m'.a)](a') &= \psi((aa').m, m') - \psi(a'.m, m'.a) \\ &= \psi((aa').m, m') - \psi(a(a'.m), m') \\ &= \psi((a, a', m), m').\end{aligned}$$

Since M is an A-bimodule alternative and the characteristic of $\mathbb{F}$ is different from 2, we have:

$$\begin{aligned}-(a', a, m) &= (a', m, a) \\ &= (a, a', m),\end{aligned}$$

which proves (3.3).
Consequently, $M \rtimes_\varphi A^*$ is an A-bimodule admissible. It follows that we can consider $A \oplus (M \rtimes_\varphi A^*)$ the semi-direct product of $M \rtimes_\varphi A^*$ by A in which the product is defined by:

$$\begin{aligned}[(a + m + f)(a' + m' + f')] &= aa' + a(m' + f') + (m + f).a' + (m + f)(m' + f') \\ &= aa' + a.m' + f' \circ R_a + m.a' + f \circ L_{a'} + mm' + \varphi(m, m'),\end{aligned}$$

$\forall a, a' \in A, m, m' \in M, f, f' \in A^*$.
Therefore, we have the following definition:

Definition 3.4

The alternative algebra $A \oplus (M \rtimes_\varphi A^*)$ defined above is called the double extension *of M by A.*

Now, we will prove that $A \oplus M \oplus A^*$ is a pseudo-euclidean alternative algebra.
Let $\widetilde{\psi}$ be the bilinear form on $(A \oplus M \oplus A^*) \times (A \oplus M \oplus A^*)$ defined by:

$$\begin{array}{rcrcl} \widetilde{\psi} & : & (A \oplus M \oplus A^*) \times (A \oplus M \oplus A^*) & \longrightarrow & \mathbb{K}, \\ & & (a+m+f, a'+m'+f') & \longmapsto & \sigma(a,a') + \psi(m,m') + f(a') + f'(a) \end{array}$$

where σ is a bilinear form symmetric invariant, not necessarily nondegenerate on $A \times A$.
It is easy to check that $\widetilde{\psi}$ is symmetric and invariant.
Indeed let $a, a', a'' \in A; m, m', m'' \in M; f, f', f'' \in A^*$. Then,

$$\begin{array}{rcl} \widetilde{\psi}((a,m,f)(a',m',f'),(a'',m'',f'')) & = & \widetilde{\psi}(aa' + a.m' + m.a' + mm' + f' \circ R_a \\ & & +f \circ L_{a'} + \varphi(m,m'),(a'',m'',f'')) \\ & = & \sigma(aa',a'') + \psi(a.m',m'') + \psi(m.a',m'') + \psi(m.m',m'') \\ & & +f''(aa') + f'(a''a) + f(a'a'') + \varphi(m,m')(a''). \end{array}$$

Moreover,

$$\begin{array}{rcl} \widetilde{\psi}((a,m,f),(a',m',f')(a'',m'',f'')) & = & \sigma(a,a'a'') + \psi(m,a'.m'') \\ & & + \psi(m,m'.a'') + \psi(m,m'm'') + f(a'a'') \\ & & + f''(aa') + f'(a''a) + \varphi(m',m'')(a). \end{array}$$

Since

$$\begin{array}{rcl} \sigma(aa',a'') & = & \sigma(a,a'a'') \\ \psi(m.a',m'') & = & \psi(m,a'm'') \\ \psi(m,m'.a'') & = & \varphi(m,m')(a''). \\ \psi(a.m',m'') & = & \varphi(m',m'')(a). \end{array}$$

Then $\widetilde{\psi}$ is invariant.
To prove that $\widetilde{\psi}$ is nondegenerate, let $(a,m,f) \in (A \oplus M \oplus A^*) \cap (A \oplus M \oplus A^*)^{\perp}$.
We have:

$$\widetilde{\psi}(a+m+f, a'+m'+f') = 0,\ \forall a' \in A, m' \in M, f' \in A^*.$$

If we put $f' = 0$, $a' = 0$, we get $\widetilde{\psi}(m,m') = 0$, $\forall m' \in M$ then, $m = 0$. And for $a' = 0$, $m' = 0$ we get $f'(a) = 0$ and $a = 0$. If we replace m' and f' by 0, we get $\psi(f,a') = 0$, $\forall a' \in A$ and $f = 0$. Then, $a+m+f = 0$ and $\widetilde{\psi}$ is nondegenerate.
Therefore, $\widetilde{\psi}$ is invariant scalar product on $A \oplus M \oplus A^*$, which proves the following theorem.

Theorem 3.5

Let (M,ψ) be a pseud-euclidean alternative algebra, A an alternative algebra and M an A-bimodule admissible such that $\psi(a.m,m') = \psi(m,m'.a)$.
We consider the bilinear map φ defined by:

$$\begin{array}{rcrcl} \varphi & : & M \times M & \longrightarrow & A^* \\ & & (m,m') & \longmapsto & \varphi(m,m')(a) = \psi(a.m,m') \end{array}$$

$\forall a \in A, m, m' \in M$.
Then, the vector space $A \oplus M \oplus A^*$ equipped with the product:

$$(a+m+f)(a'+m'+f') = aa' + mm' + a.m' + m.a' + f' \circ R_a + foL_{a'} + \varphi(m,m')$$

$\forall a, a' \in A, m, m' \in M, f, f' \in A^*$, is an alternative algebra.
Moreover, if σ is an invariant symmetric bilinear form on $A \times A$ then the bilinear form $\widetilde{\psi}$ defined by:

$$\begin{array}{rcl} \widetilde{\psi} : (A \oplus M \oplus A^*) \times (A \oplus M \oplus A^*) & \longrightarrow & \mathbb{K}, \\ (a+m+f, a'+m'+f') & \longrightarrow & \sigma(a,a') + \psi(m,m') + f(a') + f'(a) \end{array}$$

is an invariant scalar product on $A \oplus M \oplus A^*$.

Remark 3.6

As an important example when M=0, one easily sees that $A \oplus A^*$ is nothing but the trivial T^*-extension $T_0^* A$ introduced by M. Bordemann [13] and clearly the subspace A^* of $A \oplus A^*$ is an abelian ideal of $A \oplus A^*$ which implies that $T_0^* A$ is never semi-simple. Moreover, in [13] it is shown that if A is nilpotent of nilindex $k \in \mathbb{N}$, commutative, anticommutative, associative, alternative, Lie or Jordan, then the trivial T^*-extension $T_0^* A$ has the same property.

Example 3.7

Let (N, ψ) be the zero algebra and A is an alternative nonassociative algebra of dimension 4 of basis $\{e_0, e_1, e_2, e_3\}$ in which the multiplication is defined by :
$e_0 = e_0$, $e_0 e_1 = e_1$, $e_2 e_0 = e_2$, $e_2 e_3 = e_1$, $e_3 e_0 = e_3$, $e_3 e_2 = -e_1$.
The double extension algebra $A \oplus A^*$ of N by A via the trivial bi-representation is the semi-direct product of A by A^* with the product defined by:

$$(a+f)(a'+f') = aa' + f' \circ R_a + f \circ L_{a'}, \ \forall a, a' \in A, f, f' \in A^*.$$

Then the multiplication table of this algebra defined with respect to the basis: $\{e_0, e_1, e_2, e_3, e_0^*, e_1^*, e_2^*, e_3^*\}$ may be taken to be:

	e_0	e_1	e_2	e_3	e_0^*	e_1^*	e_2^*	e_3^*
e_0	e_0	e_1	0	0	e_0^*	0	e_2^*	e_3^*
e_1	0	0	0	0	0	e_0^*	0	0
e_2	e_2	0	0	e_1	0	$-e_3^*$	0	0
e_3	e_3	0	$-e_1$	0	0	e_2^*	0	0
e_0^*	e_0^*	0	0	0	0	0	0	0
e_1^*	e_1^*	0	e_3^*	$-e_2^*$	0	0	0	0
e_2^*	0	0	e_0^*	0	0	0	0	0
e_3^*	0	0	0	e_0^*	0	0	0	0

Moreover the bilinear form $\widetilde{\psi}$ defined by:

$$\begin{array}{rcl} \widetilde{\psi} : (A \oplus A^*) \times (A \oplus A^*) & \longrightarrow & \mathbb{F} \\ (a+f, a'+f') & \longmapsto & f(a') + f'(a) \end{array}$$

is an invariant scalar product on $A \oplus A^*$.

Remark 3.8

According to the multiplication table above, one can see that $Ann(A \oplus A^*) = \{0\}$, which implies that the pseudo-euclidean alternative algebra $(A \oplus A^*, \widetilde{\psi})$ is not nilpotent, then A is also not nilpotent. On the one hand, we have

$e_0^2 = e_0$ then, e_0 is not properly nilpotent. It follows that $e_0 \notin Rad(A \oplus A^*)$. On the other hand, $e_1, e_2, e_3, e_0^*, e_1^*, e_2^*, e_3^*$ are properly nilpotent, this proves that the algebra $A \oplus A^*$ is not semi-simple.

Now, in the following theorem we will give a sufficient condition for a pseudo-euclidean alternative algebra to be a double extension.

Theorem 3.9

Let (A, ψ) be a pseudo-euclidean alternative algebra.
Assume that $A = I^\perp \oplus V$, where I is an ideal totally isotropic of A and V is a subalgebra of A.
Then, A is a double extension of the pseudo-euclidean alternative algebra $(W = I^\perp/I, \tilde{\psi})$ by V via the bi-representation (S, T) defined by:

$$\begin{array}{rcl} S : V & \longrightarrow & End\left(I^\perp\!/_I, \widetilde{\psi}\right) \\ v & \longrightarrow & S(v)(s(i)) = s(L_v(i)) = s(vi), \end{array}$$

$$\begin{array}{rcl} T : V & \longrightarrow & End\left(I^\perp\!/_I, \widetilde{\psi}\right) \\ v & \longrightarrow & T(v)(s(i)) = s(R_v(i)) = s(iv), \quad \forall v \in V, i \in I^\perp. \end{array}$$

Where $I^\perp$ is the orthogonal of I relative to ψ, s is the canonical surjection from $I^\perp$ to $I^\perp/I$, and $\widetilde{\psi}$ is defined by: $\widetilde{\psi}(s(i), s(j)) := \psi(i, j)$, $\forall i, j \in I^\perp$.

Proof: We put $H = I \oplus V$, H is a nondegenerate subspace of A, indeed:

$$\begin{aligned} H \cap H^\perp &= (I \oplus V) \cap (I \oplus V)^\perp \\ &= (I \oplus V) \cap (I^\perp \cap V^\perp) \\ &= I \cap (I^\perp \cap V^\perp) \oplus (V \cap (I^\perp \cap V^\perp)) \\ &= I \cap V^\perp \oplus (V \cap I^\perp) \cap V^\perp. \end{aligned}$$

Since $V \cap I^\perp = 0$ then $H \cap H^\perp = I \cap V^\perp = (I^\perp \oplus V)^\perp = A^\perp = \{0\}$, and $A = H \oplus H^\perp$.
Moreover,

$$I \cap H^\perp = I \cap (I^\perp \cap V^\perp) = I \cap V^\perp = (I^\perp \oplus V)^\perp = A^\perp = \{0\},$$

and,

$$dimI + dimH^\perp = (dimH - dimV) + (dimA - dimH) = dimA - dimV = dimI^\perp,$$

then $I^\perp = I \oplus H^\perp$ and $A = H^\perp \oplus I \oplus V$.

Let $x, y \in H^\perp$. Since $H^\perp \subset I^\perp$ then $xy = \alpha(x, y) + \beta(x, y)$ where $\alpha(x, y) \in I$ and $\beta(x, y) \in H^\perp$. $H^\perp$ equipped with β is an alternative algebra. Indeed let $x, y, z \in H^\perp$:

$x(y+z) = \alpha(x, y+z) + \beta(x, y+z)$ and $xy + xz = \alpha(x, y) + \alpha(x, z) + \beta(x, y) + \beta(x, z)$
Or $x(y+z) = xy + xz$ then: $\beta(x, y+z) = \beta(x, y) + \beta(x, z)$. In the same way we prove that:

$$\beta(x+y, z) = \beta(x, z) + \beta(y, z),$$

$$\beta(\lambda x, y) = \lambda\beta((x, y), \quad \forall \lambda \in \mathbb{F},$$

$$\beta(\beta(x, x), y) = \beta(x, \beta(x, y)),$$

$$\beta(x, \beta(y, y)) = \beta(\beta(x, y), y).$$

Which proves the result.
Moreover, $Q = \psi_{H^{\perp} \times H^{\perp}}$ is an invariant scalar product on $H^{\perp}$, indeed:

$$\psi(xy, z) = \psi(x, yz) \Leftrightarrow \psi(\alpha(x, y), z) + Q(\beta(x, y), z) = \psi(x, \alpha(y, z)) + Q(x, \beta(y, z))$$

Since $\alpha(x, y) \in I$ and $z \in H^{\perp} \subset I^{\perp}$ then $\psi(\alpha(x, y), z) = 0$. Similarly, $\alpha(y, z) \in I$ and $x \in H^{\perp} \subset I^{\perp}$ then $\psi(x, \alpha(y, z)) = 0$, it follows that $Q(\beta(x, y), z) = Q(x, \beta(y, z))$ and $Q_{|H^{\perp} \times H^{\perp}}$ is invariant.
$H^{\perp}$ is non degenerate , then $Q_{|H^{\perp} \times H^{\perp}}$ is an invariant scalar product.
which proves that $(H^{\perp}, Q)$ is a pseudo-euclidean alternative algebra.

Let us consider,

$$\Theta := s_{|H^{\perp}} : H^{\perp} \to I^{\perp}/I$$

Θ is an isomorphic of alternative algebras, indeed:
$Ker\Theta = \{x \in H^{\perp}/x \in I\} = H^{\perp} \cap I = \{0\}$ then Θ is injective. Moreover,

dim $H^{\perp} = dim I^{\perp}/I < \infty$, it follows that Θ is bijective, i.e. Θ is an isomorphism of vector spaces. Since

$$\Theta(xy) = \Theta(\alpha(x, y)) + \Theta(\beta(x, y)),$$

and $\alpha(x, y) \in I$, then

$$\Theta(xy) = \Theta(\beta(x, y)) = \overline{\beta(x, y)}.$$

Moreover,

$$\Theta(x)\Theta(y) = \bar{x}\bar{y} = \overline{xy} = \overline{\alpha(x, y)} + \overline{\beta(x, y)} = \overline{\beta(x, y)}, \quad \forall x, y \in H^{\perp}. \text{ Here } \bar{x} = s(x).$$

furthermore Θ is an isomorphism of alternative algebras.

Now we prove that $VH^{\perp} \subset H^{\perp}$ and $H^{\perp}V \subset H^{\perp}$. Let $v \in V, x \in H^{\perp}$.
We check that for all $y \in H$, $\psi(vx, y) = 0$. We have $H^{\perp} \subset I^{\perp}$ and $V \subset A$ which implies that $vx \in I$ because $I^{\perp}$ is an ideal of A, hence $vx = i + x'$ where $i \in I$ and $x' \in H^{\perp}$. Then,

$$\psi(vx, y) = \psi(i + x', y) = \psi(i, y) + \psi(x', y) = \psi(i, y)$$

Since y $\in H = I \oplus V$ then $y = i' + v'$, where $i' \in I, v' \in V$. It follows that

$$\psi(i, y) = \psi(i, i') + \psi(i, v') = \psi(i, v'), \text{ because } I \subset I^{\perp}.$$

Moreover, $\psi(i, v') = \psi(vx - x', v') = \psi(vx, v') - \psi(x', v') = \psi(vx, v') = 0$, (Because $\psi(x', v') = 0$ $(V \subset H)$) and $\psi(vx, v') = \psi(x, v'v) = 0$ (Because $v'v \in V \subset H$.) which implies that $VH^{\perp} \subset H^{\perp}$.
In the same way one can prove that: $H^{\perp}V \subset H^{\perp}$. Then $H^{\perp}$ is an alternative V-bimodule admissible where, $v.x = L_v(x) = vx$, and $x.v = R_v(x) = xv$, i.e the paire (S, T) of linear maps, $v \to L_v,\ v \to R_v$ from alternative algebra V to $End(H^{\perp})$ is a bi-representation admissible. Then we can consider the double extension $V \oplus H^{\perp} \oplus V^*$ of $H^{\perp}$ by V via (S, T).

Now let,

$$\sigma : \begin{array}{l} I \longrightarrow V^* \\ i \longrightarrow \psi(i, .) \end{array}$$

It is obvious that σ is an homomorphism of vector space. Moreover,

$$Ker\sigma = \{i \in I/\psi(i, .) = 0\} = \{i \in I/\forall v \in V, \psi(i, v) = 0\} = I \cap V^{\perp} = (I^{\perp} \oplus V)^{\perp} = A^{\perp} = \{0\}.$$

Then σ is injective and $dimI = dim(Im\sigma)$. In the same way we prove that

$$\begin{array}{rl} \delta: & V \longrightarrow I^* \\ & v \longrightarrow \psi(v,.) \end{array}$$

is an injective homomorphism of vector space and $dimV = dim(Im\delta)$. It follows that

$$dimV = dimV^* = dimI.$$

Then σ is an isomorphism of vector space.
On the other hand the map:

$$\begin{array}{rl} \nabla: & I \oplus H^{\perp} \oplus V \longrightarrow V \oplus H^{\perp} \oplus V^* \\ & (i + x + v) \longrightarrow (v + x + \sigma(i)) \end{array}$$

is an isomorphism of alternative algebras. Indeed,

let $X = i + x + v$, $Y = j + y + w$, where $i, j \in I, x, y \in H^{\perp}, v, w \in V$.

$$XY = (i + x + v)(j + y + w) = ij + iy + iw + xj + xy + xw + vj + vy + vw.$$

Since $I^{\perp}(I) = I(I^{\perp}) = \{0\}$ then $ij \in (I^{\perp})I = \{0\}$. Similarly, $iy \in IH^{\perp} \subset II^{\perp}$ and $xj \in H^{\perp}I \subset I^{\perp}I$ then $iy = xj = 0$, it follows that,

$$\left\{ \begin{array}{l} XY = iw + vj + \alpha(x,y) + \beta(x,y) + xw + vy + vw \\ \nabla(XY) = vw + \beta(x,y) + xw + vy + \sigma(iw) + \sigma(vj) + \sigma(\alpha(x,y)). \end{array} \right.$$

Moreover, for all $v', w' \in V$ we have

$$\left\{ \begin{array}{l} \sigma(iw)(w') = \psi(iw, w') = \psi(i, ww') = \sigma(i)(ww') = (\sigma(i)oL_w)(w') \\ \sigma(vj)(v') = \psi(vj, v') = \psi(v'v, j) = \psi(j, v'v) = (\sigma(j)oR_v)(v'). \end{array} \right.$$

And

$$\sigma(\alpha(x,y))(w') = \psi(\alpha(x,y), w') = \psi(\alpha(x,y)+\beta(x,y), w') = \psi(xy, w') = \psi(w'x, y) = \varphi(x,y)(w'),$$

where $\varphi(x,y) \in V^*$ is defined by: $\varphi(x,y)(v) := Q(vx, y)$, it follows that

$$\nabla(XY) = vw + \beta(x,y) + xw + vy + \sigma(i)oL_w + \sigma(j)oR_v + \varphi(x,y) = \nabla(X)\nabla(Y).$$

Which proves that ∇ is an isomorphism of algebra. Moreover if we consider the following symmetric bilinear forms on A, $\psi' : A \times A \to \mathbb{F}$ and $\psi'' : A \times A \to \mathbb{F}$ defined by:

$$\psi'_{|I^{\perp} \times A} = \psi_{|I^{\perp} \times A}, \quad \psi'_{|V \times V} = 0, \quad \psi''_{|V \times A} = 0, \quad \psi''_{|I^{\perp} \times I^{\perp}} = 0, \quad \psi''_{|V \times V} = \psi_{|V \times V}.$$

Then ψ' is invariant and nondegenerate, ψ'' is invariant and $\psi = \psi' + \psi''$.
Now we consider on $V \oplus H^{\perp} \oplus V^*$ the invariants scalars products T, T' defined by:

$$T'(v + x + f, w + y + h) = Q(x,y) + f(y) + h(x),$$

for all $v, w \in V, x, y \in H^{\perp}, f, h \in V^*$
$T = T' + \mu$, where

$$\begin{array}{rl} \mu: & (V \oplus H^{\perp} \oplus V^*)(V \oplus H^{\perp} \oplus V^*) \longrightarrow \mathbb{K} \\ & (v + x + f, w + y + h) \longrightarrow \psi(v,w) \end{array}$$

Then, for $i \in I$ we have,

$$\begin{aligned} T(v+x+\sigma(i), w+y+\sigma(j)) &= Q(x,y)+\sigma(i)(y)+\sigma(j)(x)+\psi(v,w) \\ &= Q(x,y)+\psi(i,y)+\psi(j,x)+\psi(v,w) \\ &= Q(x,y)+\psi(v,w), \end{aligned}$$

$$\begin{aligned} \psi(i+x+v, j+y+w\psi) &= \psi(x,y)+\psi(v,w) \\ &= Q(x,y)+\psi(v,w), \end{aligned}$$

It follows that,

$$T(\nabla(i+x+v), \nabla(j+y+w)) = \psi(i+x+v, j+y+w),$$

i.e. ∇ is an isometry of pseudo-euclidean alternative algebras from (A,ψ) to $(V \oplus H^{\perp} \oplus V^*, T)$.
It is clair that:

$$\begin{aligned} \phi: \quad & (V \oplus H^{\perp} \oplus V^*) \longrightarrow (V \oplus (I^{\perp}/I) \oplus V^*) \\ & (v+x+f) \longrightarrow (v+\Theta(x)+f) \end{aligned}$$

is an isomorphism of alternative algebras. Furthermore, if we consider the invariant scalar products, Γ and Γ' on $V \oplus (I^{\perp}/I) \oplus V^*$ defined by:

$$\Gamma'(v+s(i)+f, w+s(j)+h) := \tilde{\psi}(s(i),s(j))+f(w)+h(v) = \psi(i,j)+f(w)+h(v),$$

for all $v, w \in V, i, j \in I^{\perp}, f, h \in V^*$. And $\Gamma = \Gamma' + \mu'$ where,

$$\begin{aligned} \mu': \quad & (V \oplus (I^{\perp}/I) \oplus V^*)(V \oplus (I^{\perp}/I) \oplus V^*) \longrightarrow \mathbb{K} \\ & (v+s(i)+f, w+s(j)+h) \longrightarrow \psi(v,w). \end{aligned}$$

Then ϕ is an isometry of pseudo-euclidean alternative algebras from $(V \oplus H^{\perp} \oplus V^*, T)$ to $(V \oplus (I^{\perp}/I) \oplus V^*, \Gamma)$. ■

The main result of this section is contained in the following corollary to the theorem 3.9:

Corollary 3.10

Let (A,ψ) be a pseudo-euclidean irreducible alternative algebra which is not simple. If A is not nilpotent, then, A is a double extension of a pseudo-euclidean alternative algebra (W,T) by a simple alternative algebra.

Proof: A is non nilpotent implies that $RadA \neq A$ and according to Wedderburn decomposition:
$A = S \oplus RadA$ where S is a semisimple algebra of A.
The fact that A is irreductible and nonsimple implies that $RadA \neq 0$.
Using the fact that S is semisimple and define

$$J_k = \bigoplus_{\substack{i=1 \\ i \neq k}}^{n} S_i \oplus RadA,$$

where $S = \bigoplus_{i=1}^{n} S_i$ (S_i are simple ideals of S, $\forall i \in \{1,...,n\}$).
We get $A = S_k \oplus J_k$ with J_k is a maximal ideal of A.

Indeed, $AS_i \subset S_i \oplus RadA \; \forall i \in \{1, ..., n\}$ and $ARadA \subset RadA$ (because $RadA$ is an ideal of A and S_i is an ideal of S) so, $AJ_k \subset J_k$, and similarly we prove that $J_kA \subset J_k$ that means J_k is an ideal of A.

Now, suppose that there exists an ideal M of A such that $J_k \subseteq M \subsetneq A$, then $M \cap S_k$ becomes an ideal of S_k because $S_k(M \cap S_k) \subset M \cap S_k$ and $(M \cap S_k)S_k \subset M \cap S_k$.
Since S_k is simple then $M \cap S_k = \{0\}$ or $M \cap S_k = S_k$. If we assume that $M \cap S_k = S_k$ it follows that $S_k \subset M$ and $J_k \oplus S_k \subset M$, absurd because $A \subsetneq M$. Thus, $M \cap S_k = \{O\}$.

Let $m \in M$, $m = s_k + j_k$ where $s_k \in S_k$ and $j_k \in J_k$ then $s_k = m - j_k$ it follows that $s_k \in M \cap S_k$ otherwise, $s_k = 0$ and $m = j_k \in J_k$; therefore, $M = J_k$. If we put $I_k = J_k^{\perp}$, since J_k is maximal, I_k becomes minimal. Moreover, $\{0\} \subsetneq I_k \cap I_k^{\perp} \subset I_k$ it follows that $I_k \subset I_k^{\perp}$ i.e. I_k is totally isotropic. Then, according to theorem 6, A is the double extension of the pseudo-euclidean alternative algebra $(W = J_k/J_k^{\perp}, \widetilde{\psi})$ by S_k. ■

In order to describe all pseudo-euclidean alternative algebras, we introduce another notion of double extension.

3.2 Generalized double extension of pseudo-euclidean alternative algebras by the one dimensional algebra with zero product

Let A be an alternative algebra and $x_0 \in A$. $\mathbb{F}e$ be one-dimensional algebra with zero product.
We put $\widetilde{A} = \mathbb{F}e \oplus A$, on this vector space we define the following product:

$$x * y = xy, \quad e * e = x_0, \quad x * e = f^*(x), \quad e * x = f(x), \quad \forall x, y, \in A, \; f \in End(A).$$

$\widetilde{A}$ endowed with the product above is an alternative algebra, if and only if the pair (f, x_0) satisfies the following conditions:
$f(x_0) = f^*(x_0); \quad f^* \circ f(x) = f \circ f^*(x); \quad f^2(x) = x_0x;$
$(f^*)^2(x) = xx_0; \quad f(x^2) = f(x)x; \quad f^*(x^2) = xf^*(x);$
$f(x)y + f^*(x)y = f(xy) + xf(y); \quad xf(y) + xf^*(y) = f^*(x)y + f^*(xy).$
In this case, (f, x_0) is called an admissible pair of A and the alternative algebra $\widetilde{A}$ is called the generalized semi-direct product of A by the one-dimensional algebra with zero product via the pair (f, x_0).

Proposition 3.11

Let (A_1, ψ_1) be a pseudo-euclidean alternative algebra, $\mathbb{F}b$ be the one-dimensional algebra with zero product, (f, x_0) an admissible pair of A_1 and $k \in \mathbb{F}$. Let $\varphi : A_1 \times A_1 \to \mathbb{F}$ be the bilinear form defined by:

$$\varphi(x, y) = \psi_1(f(x), y), \; \forall x, y \in A_1.$$

Then, the vector space $A = \mathbb{F}e \oplus A_1 \oplus \mathbb{F}b$ (where $\mathbb{F}e$ is a one-dimensional vector space) endowed with the following product:

$$(e + (x + b)) * (e + (y + b)) = w_0 + \tilde{f}(y + b) + \tilde{f}^*(x + b) + xy + \psi_1(f(x), y)b, \; \forall x, y \in A_1,$$

where $w_0 = x_0 + kb$, $\tilde{f}(y + b) = f(y) + \psi_1(x_0, y)b$, $\tilde{f}^*(x + b) = f^*(x) + \psi_1(x_0, x)b$ and

with the symmetric bilinear form ψ defined by:
$\psi|_{A_1\times A_1} = \psi_1$, $\psi(e,b) = 1$, $\psi(e,A_1) = \psi(b,A_1) = \{0\}$ *and* $\psi(e,e) = \psi(b,b) = 0$ *is a pseudo-euclidean alternative algebra.*

Proof: Since the pair (f,x_0) is admissible, it follows $f(x^2) = f(x)x$, and $f^*(x^2) = xf^*(x)$, which implies that $\varphi(x,y) = \varphi(x,xy)$ and $\varphi(y,x) = \varphi(x,xy)$. Hence, one can consider the central extension $A_1 \oplus \mathbb{F}b$ of A_1 by $\mathbb{F}b$ via φ whose product is defined by:

$$(x+\lambda b)(y+\alpha b) = xy + \varphi(x,y)b, \forall x,y \in A_1 \;\; and \;\; \lambda,\alpha \in \mathbb{F}.$$

Moreover, it is obvious to check that the triple $(\tilde{f}, w_0) \in End(A_1 \oplus \mathbb{F}b) \times (A_1 \oplus \mathbb{F}b)$ is admissible. Then, we can consider the generalized semi-direct product $A = \mathbb{F}e \oplus (A_1 \oplus \mathbb{F}b)$ of $A_1 \oplus \mathbb{F}b$ by the one-dimensional algebra with zero product $\mathbb{F}e$ via $(\tilde{f}, w_0)$. This implies that A is an alternative algebra. Since ψ is an invariant scalar product, it follows that (A,ψ) is a pseudo-euclidean alternative algebra. ■

Definition 3.12

The pseudo-euclidean alternative algebra (A,ψ) constructed above is called the generalized double extension of the pseudo-euclidean alternative algebra (A_1,ψ_1) by the one dimensional alternative algebra with zero product $\mathbb{F}b$ via (f,x_0).

We shall now come to the main result of this section, that is, many finite-dimensional pseudo-euclidean alternative algebras are in fact isometric to certain generalized double extensions.

Theorem 3.13

Let (A,ψ) be a pseudo-euclidean alternative algebra such that A is not the one dimensional alternative algebra with zero product. We suppose that $Ann(A) \neq 0$ and there is $b \in Ann(A)\backslash\{0\}$ such that $\psi(b,b) = 0$, then A is a generalized double extension of a pseudo-euclidean alternative algebra by the one dimensional alternative algebra with zero product.

Proof:

Denote $I = \mathbb{F}b$, I is an ideal of A because $b \in Ann(A)$. Since ψ is nondegenerate, there is $e \in A$ such that $\psi(e,b) = 1$. Moreover, one can consider that $\psi(e,e) = 0$. Indeed, if $\psi(e,e) \neq 0$ one can choose $e' = e - \frac{\psi(e,e)}{2}b$, thus we can easily check that $\psi(e',e') = 0$, $\psi(e',b) = 1$.

Given that $e \notin (\mathbb{F}b)^{\perp}$ implies $\mathbb{F}e \cap (\mathbb{F}b)^{\perp} = \{0\}$, moreover $dimA = dim(\mathbb{F}b)^{\perp} + dim\mathbb{F}e$ which gives $A = (\mathbb{F}b)^{\perp} \oplus \mathbb{F}e$. Furthermore, let $W = (\mathbb{F}e \oplus \mathbb{F}b)^{\perp}$, since W is nondegenerate then $A = W \oplus W^{\perp}$ and $dimW = dimI^{\perp} - dim\mathbb{F}b$, moreover $(\mathbb{F}b) \cap W = \{0\}$. Consequently $I^{\perp} = \mathbb{F}b \oplus W$, thus $A = \mathbb{F}e \oplus W \oplus \mathbb{F}b$.

First we will prove that the vector space W has a structure of alternative algebra. Let $x,y \in W$, we have $xy = \beta(x,y) + \alpha(x,y)b$ where $\beta(x,y) \in W$ and $\alpha(x,y) \in \mathbb{F}$. W endowed with the bilinear form $\beta : W \times W \longrightarrow W$; $(x,y) \longrightarrow \beta(x,y)$, is an alternative algebra. Indeed,

we have $x^2 = \beta(x,x) + \alpha(x,x)b$ and $xy = \beta(x,y) + \alpha(x,y)b$, then $x^2y = \beta(x,x)y$ and $x(xy) = x\beta(x,y)$. Since A is alternative it follows that $x\beta(x,y) = \beta(x,x)y$, i.e.,

$\beta(x,\beta(x,y)) = \beta(\beta(x,x),y)$. Similarly we prove that $\beta(x,\beta(y,y)) = \beta(\beta(x,y),y)$ which proves W is alternative. Moreover $T = \psi_{|W\times W}$ is an invariant scalar product on W. For, Let $x,y,z \in W$, we have:
$\psi(xy,z) = \psi(x,yz) \Rightarrow \psi(\beta(x,y),z)+\psi(\alpha(x,y)b,z) = \psi(x,\beta(y,z))+\psi(x,\alpha(y,z)b)$. Since $b \in W^{\perp}$ it is clear that $\psi(\alpha(x,y)b,z) = \psi(x,\alpha(y,z)b) = 0$, hence T is invariant on W. Moreover, W is nondegenerate, which proves that T is an invariant scalar product.

The fact that $x \in W \subset I^{\perp}$ it follows that $ex \in I^{\perp}$ then $ex = f(x) + \varphi(x)b$, where $f \in End(W), \varphi \in W^*$. Similarly we have $xe = h(x)+\varphi'(x)b$, with $h \in End(W), \varphi' \in W^*$.

Now we check that $h = f^*$, clearly $\psi(xy,e) = \psi(x,ye),\ \ \forall x,y \in W$. Consequently, $\psi(\beta(x,y),e) + \alpha(x,y) = \psi(x,h(y)) + \varphi'(x)\psi(x,b)$, i.e. $\alpha(x,y) = \psi(x,h(y))$. Furthermore, since $\psi(x,ye) = \psi(y,ex)$ and $\psi(y,ex) = \psi(y,f(x)) + \varphi(x)\psi(y,b)$ we have $\psi(f(x),y) = \psi(x,h(y))$ which implies that $h = f^*$.

One can observe that $\psi(e^2,b) = \psi(e,eb) = 0$, then $e^2 \in I^{\perp}$ showing $e^2 = x_0 + kb$ where $x_0 \in W, k \in \mathbb{F}$. Let $x \in W$, we have $\psi(ex,e) = \psi(x,e^2)$ which implies that $\varphi(x) = \psi(x,x_0)$ because of $ex = f(x) + \varphi(x)b$, and $\psi(f(x),e) = \psi(x,kb) = 0$. A similar reasoning entails that $\varphi'(x) = \psi(x,x_0)$, then $ex = f(x)+\psi(x,x_0)b$ and $xe = f^*(x)+\psi(x,x_0)b$.

Finally we shall prove that the pair (f,x_0) is admissible of the pseudo-euclidean alternative algebra (W,T).

Since $ex_0 = f(x_0) + \psi(x_0,x_0)$, $x_0e = f^*(x_0) + \psi(x_0,x_0)b$, and $e^3 = ex_0 = x_0e$, hence $f(x_0) = f^*(x_0)$. We have $ef^*(x) = f \circ f^*(x) + \psi(f^*(x),x_0)b$ and $f(x)e = f^* \circ f(x) + \psi(f(x),x_0)b$, since $ex = f(x) + \psi(x,x_0)b$, then $(ex)e = f(x)e$. Moreover, $xe = f^*(x) + \psi(x,x_0)b$ which gives $e(xe) = ef^*(x)$. Hence $f \circ f^*(x) = f^* \circ f(x)$ because A is flexible.

Since $ex = f(x) + \psi(x,x_0)b$, and $e^2 = x_0 + kb$, which gives e(ex)=ef(x), and $e^2x = x_0x$. The fact that A is alternative this implies $ef(x) = x_0x$. Furthermore $x_0x = \beta(x_0,x) + \alpha(x_0,x)b$ and $ef(x) = f^2(x) + \psi(f(x),x_0)b$. Hence $\beta(x_0,x) = f^2(x)$. A similar argument shows that $(f^*)^2(x) = \beta(x,x_0)$.

Now let $x \in W$, since $x^2 = \beta(x,x) + \alpha(x,x)b$, and $ex = f(x) + \psi(x,x_0)b$, then $ex^2 = e\beta(x,x)$, and $(ex)x = f(x)x$. Moreover, we have $e\beta(x,x) = f(\beta(x,x)) + \psi(\beta(x,x),x_0)b$, and $f(x)x = \beta(f(x),x) + \alpha(f(x),x)b$. Hence, it follows that $f(\beta(x,x)) = \beta(f(x),x)$ because $e\beta(x,x) = f(x)x$. A similar reasoning entails that $f^*(\beta(x,x)) = \beta(x,f^*(x))$.

We shall now check that for all $x,y \in W$,

$$\beta(f(x),y) + \beta(f^*(x),y) - \beta(x,f(y)) = f(\beta(x,y)),$$

we have,

$$f(x)y = \beta(f(x),y) + \alpha(f(x),y)b \quad ; \quad f^*(x)y = \beta(f^*(x),y) + \alpha(f^*(x),y)b$$
$$xf(y) = \beta(x,f(y)) + \alpha(x,f(y))b \quad ; \quad e\beta(x,y) = f(\beta(x,y) + \psi(\beta(x,y),x_0)b,$$

which gives,

$$\beta(f(x),y)+\beta(f^*(x),y)-\beta(x,f(y)) = f(x)y+f^*(x)y-xf(y)-\psi(\beta(x_0,x),y)b$$

because we have,

$$\alpha(x,f(y))b-\alpha(f^*(x),y)b=\psi(f(x),f(y))b-\psi(f\circ f^*(x),y)b=0$$

and,

$$\alpha(f(x),y)b=\psi(f^2(x),y)b=\psi(\beta(x_0,x),y)b.$$

According to the invariance of T, $\psi(\beta(x_0,x),y)=\psi(x_0,\beta(x,y))$. Moreover $e(xy)=e\beta(x,y)$ and $f(x)y+f^*(x)y-xf(y)=(ex)y+(xe)y-x(ey)$ then $e\beta(x,y)=f(x)y+f^*(x)y-xf(y)$, which gives the result. A similar reasoning as above proves that

$$\beta(x,f(y))+\beta(x,f^*(y))=\beta(f^*(x),y)+f^*(\beta(x,y)),\ \ \forall x,y\in W.$$

It follows that (A,ψ) is the generalized double extension of the pseudo-euclidean alternative algebra (W,T) by the one dimensional alternative algebra $\mathbb{F}e$ with zero product via the pair (f,x_0).

■

The main result of this section is contained in the following corollary to the theorem 3.13:

Corollary 3.14

Let (A,ψ) be an irreducible nilpotent pseudo-euclidean alternative algebra which A is not the one dimensional alternative algebra with zero product. Then, A is a generalized double extension of a nilpotent pseudo-euclidean alternative algebra (W,T) by the one dimensional alternative algebra with zero product.

Proof: Since A is nilpotent, then $Ann(A)\neq\{0\}$ (because $A^2\neq A$ and $A^2=Ann(A)^\perp$). For $b\in Ann(A)$, it follows that $I=\mathbb{F}b$ is a degenerate ideal of A which gives $I\cap I^\perp\neq\{0\}$, i.e. $\psi(b,b)=0$. Moreover, since $I^\perp$ is nilpotent implies that $W=I^\perp/I$ is nilpotent. This leads to the result because of the last theorem. ■

More generally, if we denote by $\mathfrak{S}$ the set constituted with $\{0\}$, the one dimensional alternative algebra with zero product and all simple alternative algebras, then for any pseudo-euclidean alternative algebra (A,ψ) such that $A\notin\mathfrak{S}$ is obtained from elements $A_1,...,A_n$ of $\mathfrak{S}$ by a finite number of orthogonal direct sums of pseudo-euclidean alternative algebras or/and double extensions by a simple alternative algebra or/and generalized double extension by the one dimensional alternative algebra with zero product.

Chapter 4

Malcev-Poisson-Jordan algebras (MPJ-algebras)

4.1 Definitions and preliminary results

Definition 4.1

A MPJ-algebra P is a $\mathbb{F}$-vector space equipped with two bilinear multiplications $\circ$ and $[\ ,\ ]$ such that:

i) $(P, \circ)$ is a Jordan algebra,

ii) $(P, [\ ,\ ])$ is a Malcev algebra,

iii) These two operations are required to satisfy Leibniz condition :

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z],\ \forall x, y, z \in P.$$

We denote a MPJ-algebra by $(P, [\ ,\ ], \circ)$.

Remark 4.2

Any Poisson algebra or Malcev-Poisson algebra is a MPJ-algebra.

Notation. Let $(P, .)$ be a nonassociative algebra one can define the two following new products:

$$[x, y] = x.y - y.x \ \ \textit{and} \ \ x \circ y = \frac{1}{2}(x.y + y.x),\ \forall x, y \in P.$$

Proposition 4.3

Let $(P, .)$ be a nonassociative algebra. Then, $(P, .)$ is flexible if and only if,

$$(x, y, z)_. = \frac{1}{4}J(x, y, z) + \frac{1}{4}[y, [z, x]] + (x, y, z)_\circ\ \forall x, y, z \in P; \tag{4.1}$$

where $(x, y, z)_. = (x.y).z - x.(y.z)$ and $(x, y, z)_\circ = (x \circ y) \circ z - x \circ (y \circ z)$.

Proof: If $(P, .)$ is flexible, Eq.(4.1) can be deduced immediately. Conversely, suppose that Eq.(4.1) holds, then

$$(z, y, x)_. = \frac{1}{4}J(z, y, x) + \frac{1}{4}[y, [x, z]] + (z, y, x)_\circ$$

Since $(x,y,z)_\circ = -(z,y,x)_\circ$ and $J(x,y,z)$ is a skew-symmetric trilinear map, we get $(x,y,z)_. = -(z,y,x)_.$ so that $(P,.)$ is flexible. ■

Corollary 4.4

Let $(P,.)$ be a flexible and Malcev admissible algebra, $(P,[\ ,\ ],\circ)$ is a MPJ-algebra if and only if $(P,.)$ satisfies the identity:

$$R_x o L_{x^2} = L_{x^2} o R_x, \quad \forall x \in P. \tag{4.2}$$

Where L_x(resp. R_x) is the left multiplication (resp. the right multiplication) by x in the algebra $(P,.)$. In this case $(P,.)$ is called a **MPJ- admissible** algebra, the corresponding Malcev algebra $(P,[\ ,\ ])$ will be denoted by P^- and the Jordan algebra $(P,\circ)$ by P^+.

Proof: Since $(P,.)$ is flexible, then by Proposition (4.3), if we put $z = x^2$, we get

$$(x,y,x^2)_. = (x,y,x^2)_\circ$$

which complete the proof of this corollary. ■

Example 4.5

Any alternative algebra is a MPJ-admissible algebra.

Remark 4.6

Not all MPJ-admissible algebras are alternative algebras. Indeed, let P be the 3-dimensional algebra ([19], p. 301) of basis $\{e_1, e_2, e_3\}$ defined by:
$e_i^2 = 0,\ e_1e_2 = -e_2e_1 = e_2,\ e_1e_3 = -e_3e_1 = -e_3,\ e_2e_3 = -e_3e_2 = e_1.$
P is a MPJ-admissible algebra, however $(e_2,e_3,e_3) = -e_3$, then P is not alternative algebra.

Recall that an algebra A over $\mathbb{F}$ is called a noncommutative Jordan algebra, if A is flexible and satisfies the identity $(x^2y)x = x^2(yx)$. In ([47], p. 473) it was shown that an algebra A over $\mathbb{F}$ is a noncommutative Jordan if and only if it is a flexible Jordan-admissible.

Remark 4.7

Any MPJ-admissible algebra is a non-commutative Jordan algebra.

Proposition 4.8

Let $(P,.)$ be a MPJ-admissible algebra, and let $x,y \in P$. Consider the linear map $D_{x,y} : P \longrightarrow P$ given by $D_{x,y} = L_{[x,y]} - [L_x, L_y]$. $D_{x,y}$ is a derivation of P if and only if

$$\frac{3}{4}J([y,x],z,t) = [(x,z,y)_\circ, t] + [z,(x,t,y)_\circ] - (x,[z,t],y)_\circ \tag{4.3}$$

for all $z,t \in P$.

Proof: $D_{x,y}$ is a derivation of P if and only if for all $z,t \in P$,

$$(x,y,z.t)_. - (y,x,z.t)_. = (x,y,z)_..t - (y,x,z)_..t - (y,x,z)_..t + z.(x,y,t)_. - z.(y,x,t)_. \tag{4.4}$$

From Eq.(4.1) and after simplification, Eq.(4.4) becomes,

$$\frac{1}{8}(J(x,y,[z,t])-[J(x,y,z),t]-[z,J(x,y,t)]-J([x,y],z,t)) = \frac{1}{2}([(x,z,y)_\circ,t]+[z,(x,t,y)_\circ]-(x,[z,t],y)_\circ) \quad (4.5)$$

From Eq.(4.5) and by using Equations (1.11) and (1.15), we obtain

$$3J([y,x],z,t) = 4([(x,z,y)_\circ,t]+[z,(x,t,y)_\circ]-(x,[z,t],y)_\circ)$$

This complete the proof. ■

Corollary 4.9

Let $(P,.)$ be a MPJ-admissible algebra such that P^- is a Lie algebra, then the Eq.(4.3) is equivalent to: for all $x,y \in P$, $f_{x,y}$ is a derivation of the Lie algebra P^-, where $f_{x,y} : P \longrightarrow P$ is the skew-symmetric map given by $f_{x,y}(z) = (x,z,y)_\circ$, for all $z \in P$.

Let $\mathfrak{C}$ be the class of the noncommutative Jordan algebras P over $\mathbb{F}$ satisfying Eq.(4.3).

One can check easily that any Poisson-Lie admissible algebra and any associative algebra, belongs to this class $\mathfrak{C}$.

Definition 4.10

A vector space J over $\mathbb{F}$, endowed with a trilinear operation $J \times J \times J \longrightarrow J$, $(x,y,z) \longrightarrow xyz$, is said to be a **generalized Jordan triple system** (GJTS for short) if it satisfies the identity:

$$uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz)$$

for any $u,v,x,y,z \in J$.

There is a close connection between the subclass of elements in $\mathfrak{C}$ with involution and unity and a class of some generalized Jordan triple systems, this connection has been explained in [16] by the following theorem:

Theorem 4.11

Let J be a generalized Jordan triple system over $\mathbb{F}$ which contains an element $e \in J$ such that:

1. $eee = e$,
2. $eex = xee$ for any $x \in J$,
3. the map $U_e : x \mapsto exe$ is onto.

Then the algebra $J^{(e)}$, defined on the vector space J with multiplication given by $x.y = xey$ for any $x,y \in J$, belongs to the classe $\mathfrak{C}$ and is unital with $1 = e$. Moreover, the map $x \mapsto \bar{x} = exe$ is an involution of $J^{(e)}$ and the triple product in J satisfies

$$xyz = x.(\bar{y}.z) - \bar{y}.(x.z) + (\bar{y}.x).z, \quad (4.6)$$

for any $x,y,z \in J$.

Conversely, let $(P,.)$ be a unital algebra in $\mathfrak{C}$ over $\mathbb{F}$ with unity element 1_p and endowed with an involution $x \mapsto \bar{x}$, and define a triple product on P by means of

(4.6).
Then P becomes a generalized Jordan triple system and satisfies conditions (1)-(3) above with $e = 1_p$ and $U_e x = \bar{x}$ for any $x \in P$.

Now, starting from a MPJ-admissible algebra satisfying Eq.(4.3), we will construct a new MPJ-admissible algebra satisfying Eq.(4.3) and admet an involution and unity. Next we will construct a generalized triple system associated to this new algebra.

Let P be a MPJ-admissible algebra satisfying Eq.(4.3). If P does not contain an identity element 1 there is a standard construction (see. [46]) for obtaining an algebra $(P_1, .)$ containing 1 and which is a MPJ-admissible algebra satisfying Eq.(4.3). On the vector space $\widetilde{P} := P_1 \times P_1$ we define the following product:

$$(x, y)(z, t) = (x.z, t.y), \ \forall x, y, z, t \in P_1. \tag{4.7}$$

It is easy to check that $\widetilde{P}$ endowed with the product (4.7) is a noncommutative Jordan algebra with identity element $(1, 1)$. Moreover, the linear map $D_{(x,y),(z,t)} : \widetilde{P} \longrightarrow \widetilde{P}$ given by
$D_{(x,y),(z,t)} = L_{[(x,y),(z,t)]} - [L_{(x,y)}, L_{(z,t)}]$ is a derivation on $\widetilde{P}$. Indeed, for $a, b \in P_1$,

$$\begin{aligned} D_{(x,y),(z,t)}(a, b) &= (L_{[x,z]}(a), R_{[t,y]}(b)) - ([L_x, L_z](a), [R_y, R_t](b)) \\ &= (L_{[x,z]}(a) - [L_x, L_z](a), -R_{[y,t]}(b) - [R_y, R_t](b)) \\ &= (D_{(x,z)}(a), D_{(y,t)}(b)). \end{aligned}$$

Since $D_{(x,y)}$ is a derivation on $(P_1, .)$ for any $x, y \in P_1$, so that, $D_{(x,y),(z,t)}$ is a derivation on $\widetilde{P}$, then, $\widetilde{P}$ belongs to the class $\mathfrak{C}$. Now, we consider the linear map $\sigma : \widetilde{P} \longrightarrow \widetilde{P}$ given by

$$\sigma(x, y) = (y, x), \text{ for all } x, y \in P.$$

σ is an involution of $\widetilde{P}$, then (Theorem 3.1 in[16]), $\widetilde{P}$ endowed with the following triple product

$$\begin{aligned} (x, y)(z, t)(v, w) &= (x, y)(\sigma(z, t)(v, w)) - \sigma(z, t)((x, y)(v, w)) + (\sigma(z, t)(x, y))(v, w) \\ &= (x.(t.v) - t.(x.v) + (t.x).v \ , \ (w.z).y - (w.y).z + w.(y.z)) \end{aligned}$$

for all $x, y, z, t, v, w \in P$, becomes a generalized Jordan triple system.

4.2 MPJ-admissible algebras such that the associated Malcev algebra is reductive

In this section we will give the description of all MPJ structure $\circ$ on an arbitrary reductive Malcev algebra $(P, [\ ,\ ])$ (i.e. $(P, [\ ,\ ], \circ)$ be a MPJ-algebra). In particular, we will prove that every MPJ structure on a semisimple Malcev algebra is trivial.

The results in this section are based on the work of G. Benkart and J. M. Osborn [11], and the work of H. C. Myung [38].

Proposition 4.12

A MPJ-admissible algebra $(P, .)$ over a field of characteristic 0 is power-associative algebra (ie. for all element x in P the subalgebra $\mathbb{F}[x]$ of P is associative).

Proof: Since P^+ is power-associative (see. [46]) and $(P,.)$ is flexible, it follows by (Lemma 1.11 of [38], p. 14) that $(P,.)$ is a power-associative algebra. ■

The following Theorem was obtained in [11],

Theorem 4.13

Assume P is a power-associative, flexible and Lie-admissible algebra over a field of characteristic 0 such that P^- is a simple Lie algebra. Then P is a Lie algebra.

Remark 4.14

The result of Theorem 4.13 can be also found in [37] when the base field is algebraically closed of characteristic 0.

The next result, which is a direct consequence of Theorem 6 and Proposition 4.12 characterizes MPJ-admissible algebras with P^- simple Lie algebra.

Corollary 4.15

Let $(P,.)$ be a MPJ-admissible algebra over a field of characteristic 0. If P^- is a simple Lie algebra, then the Jordan algebra P^+ is trivial.

Proof: As P is power-associative, flexible and Lie-admissible then by Theorem 6, $(P,.)$ is a Lie algebra, thus $x.x = \frac{1}{2}[x,x] + x \circ x = 0$, for all $x \in P$, this implies $x \circ x = 0$. Then

$$(x+y)^2 = x^2 + y^2 + x \circ y + y \circ x = x \circ y + y \circ x = 2x \circ y = 0,$$

for all $x, y \in P^+$. Thus the Jordan algebra P^+ is trivial. ■

Corollary 4.16

Let $(P,.)$ be an admissible Poisson algebra over a field of characteristic 0. If P^- is a simple Lie algebra, then the associative algebra P^+ is trivial.

Let $(P,.)$ be a MPJ-admissible algebra over a field of characteristic 0, with P^- isomorphic to a 7-dim simple (non-Lie) Malcev algebra. From Theorem 3.20 of ([38], p. 193) and the power-associativity of $(P,.)$ one can check easily that $(P,.)$ is a Malcev algebra, that is the Jordan algebra is also trivial.

Lemma 4.17

Let $(P,[\ ,\ ])$ be a semisimple Malcev algebra, then the MPJ structure on P is trivial.

Proof: Let $\circ$ be the MPJ structure on $P = S_1 \oplus ... \oplus S_n$, where $S_1, ..., S_n$ are the simple ideals of P. Let $i \in \{1,...,n\}, x, y \in S_i$ and $z \in S_j$ where $j \in \{1,...,n\}$ such that $i \neq j$, then $[z, x \circ y] = [z,x] \circ y + x \circ [z,y] = 0$. Which proves that $x \circ y \in S_i$. Since S_i is a simple Malcev algebra and the restriction of $\circ$ to the underlying vector space $S_i \times S_i$ is a MPJ structure on S_i, then $x \circ y = 0, \forall x, y \in S_i, \forall i \in \{1,...,n\}$.

Let $i, j \in \{1,...,n\}$ such that $i \neq j$. Since $[S_i, S_i] = S_i$ and $[S_j, S_j] = S_j$, then $S_i \circ S_j \subset S_i \cap S_j = \{0\}$ because $\circ$ is a MPJ structure on P. Thus the MPJ structure on P is trivial. ■

The following Theorem characterizes MPJ-admissible algebras P such that P^- is a reductive Malcev algebra. The proof of this Theorem needs the result of H. C. Myung (Corollary 4.5 in [38]) when the field is algebraically closed.

Theorem 4.18

Suppose that $\mathbb{F}$ is algebraically closed. Let $(P, [\ ,\ ])$ be a reductive Malcev algebra, (ie. $P = S_1 \oplus ... \oplus S_n \oplus Z(P)$) where $S_1, ..., S_n$ are simple ideals of P and $Z(P)$ the center of P.
$\circ$ is a MPJ structure on P if and only if there exists $\star$ a Jordan algebra structure on the underlying vector space of $Z(P)$, $a_1, ..., a_n$ elements of $Z(P)$ and $T_1, ..., T_n$ linear forms on $Z(P)$ satisfying $T_i(a_i)a_i = a_i \star a_i$, and $T_i(a \star a_i) = T_i(a)T_i(a_i)$, $\forall i \in \{1, ..., n\}$, $\forall a \in Z(P)$ such that:

i) $a \circ b = a \star b,\ \forall a, b \in Z(P)$;

ii) $x \circ y = \mathcal{K}_i(x, y)a_i,\ \forall x, y \in S_i,\ \ \forall i \in \{1, ..., n\}$ where $\mathcal{K}_i$ is the Killing form of S_i;

iii) $x \circ y = 0,\ \forall x \in S_i,\ \forall y \in S_j, \forall i, j \in \{1, ..., n\}$ such that $i \neq j$;

iv) $x \circ a = a \circ x = T_i(a)x,\ \forall x \in S_i,\ \forall a \in Z(P),\ \forall i \in \{1, ..., n\}$.

Proof: Let us suppose that $(P, [\ ,\ ])$ is a reductive Malcev algebra. Let $\circ$ be a MPJ structure on P. By ([38]. Corollary 4.5) and by Lemma 4.17 there exist $a_1, ..., a_n \in Z(P)$ and $T_1, ..., T_n \in (Z(P))^*$ such that:

i) $x \circ y = \mathcal{K}_i(x, y)a_i,\ \forall x, y \in S_i,\ \forall i \in \{1, ..., n\}$ where $\mathcal{K}_i$ is the Killing form of S_i;

ii) $x \circ y = 0,\ \forall x \in S_i,\ \forall y \in S_j, \forall i, j \in \{1, ..., n\}$ such that $i \neq j$;

iii) $x \circ a = a \circ x = T_i(a)x,\ \forall x \in S_i,\ \forall a \in Z(P),\ \forall i \in \{1, ..., n\}$.

It is clear that if $a, b \in Z(P)$, then $a \circ b \in Z(P)$. Consequently, the product $a \star b = a \circ b$ gives a Jordan structure on $Z(P)$.

Let $i \in \{1, ..., n\}, x, y \in S_i$. Since $0 = (x, y, x^2)_\circ = \mathcal{K}_i(x, y)\mathcal{K}_i(x, x)(T_i(a_i)a_i - a_i \star a_i)$, $\ \forall x, y \in S_i$, then $T_i(a_i)a_i = a_i \star a_i$.

Now let $i \in \{1, ..., n\}, a \in Z(P)$. The fact that $0 = (x, a, x^2)_\circ = \mathcal{K}_i(x, x)x(T_i(a \star a_i) - T_i(a)T_i(a_i))$, $\ \forall x \in S_i$, implies that $T_i(a \star a_i) = T_i(a)T_i(a_i)$.

Conversely, consider a Jordan structure $\star$ on $Z(P)$, $a_1, ..., a_n$ elements of $Z(P)$ and $T_1, ..., T_n$ linear forms on $Z(P)$ satisfying $T_i(a_i)a_i = a_i \star a_i$, and $T_i(a \star a_i) = T_i(a)T_i(a_i)$, $\ \forall i \in \{1, ..., n\}$. Let $\circ : P \times P \to P$ be the bilinear map defined by:

i) $a \circ b = a \star b,\ \forall a, b \in Z(P)$;

ii) $x \circ y = \mathcal{K}_i(x, y)a_i,\ \forall x, y \in S_i,\ \forall i \in \{1, ..., n\}$ where $\mathcal{K}_i$ is the Killing form of S_i;

iii) $x \circ y = 0,\ \forall x \in S_i,\ \forall y \in S_j, \forall i, j \in \{1, ..., n\}$ such that $i \neq j$;

iv) $x \circ a = a \circ x = T_i(a)x,\ \forall x \in S_i,\ \forall a \in Z(P),\ \forall i \in \{1, ..., n\}$.

Since for all $i \in \{1, ..., n\}$, $\mathcal{K}_i$ is symmetric and invariant, and $[Z(P), P] = [S_i, S_j] = \{0\}$, for all $i, j \in \{1, ..., n\}$ such that $i \neq j$, then $\circ$ is a MPJ structure on P. ∎

Example 4.19

Let $\mathbb{O}^-$ be the Malcev algebra defined above, and $\circ$ be the MPJ structure on $\mathbb{O}^-$, then (by Theorem 4.18) there exists $\star$ a Jordan algebra structure on the underlying vector space of $Z(\mathbb{O}^-)$, T_1 linear form on $Z(\mathbb{O}^-)$ such that:

$e_1 \star e_1 = e_1 \circ e_1 = 2e_1$; $T_1(e_1) = 2$; $e_i \circ e_j = \mathcal{K}(e_i, e_j)e_1$, $\forall i, j \in \{2, ..., 8\}$, where $\mathcal{K}$ is the Killing form of $\mathbb{O}^*$ defined by:

$$\mathcal{K}(e_i, e_i) = -2, \quad \mathcal{K}(e_i, e_j) = 0, \text{ for all } i, j \in \{2, ..., 8\} \text{ such that } i \neq j.$$

4.3 PEMPJ-algebras and related quadratic Lie algebras

In the following, we prove that we can construct interesting examples of quadratic Lie algebras from PEMPJ-algebras by using the technique of double extension for quadratic Lie algebras [35].

Definition 4.20

1. Let $(P, [\ ,\ ], \circ)$ be a MPJ algebra and $\psi : P \times P \longrightarrow \mathbb{F}$ be a bilinear form. (P, ψ) will be called a pseudo-euclidean MPJ-algebra if $(P, [\ ,\ ], \psi)$ and $(P, \circ, \psi)$ are pseudo-euclidean algebras.
2. Let $(P, .)$ be a MPJ-admissible algebra and $\psi : P \times P \longrightarrow \mathbb{F}$ be a bilinear form. (P, ψ) will be called a pseudo-euclidean MPJ-admissible algebra if $(P, ., \psi)$ is a pseudo-euclidean algebra.
   We will denote by PEMPJ a pseudo-euclidean Malcev-Poisson-Jordan.

Remarks 1

1. A pseudo-euclidean Malcev algebra is also called a quadratic Malcev algebra (see. [4]).
2. Let $(P, .)$ be a MPJ-admissible algebra and $\psi : P \times P \longrightarrow \mathbb{F}$ be a bilinear form. The following three assertions are equivalent :
   (a) $(P, ., \psi)$ is a PEMPJ-admissible algebra;
   (b) $(P, [\ ,\ ], \circ, \psi)$ is a PEMPJ-algebra;
   (c) (P^-, ψ) is a pseudo-euclidean Malcev algebra and (P^+, ψ) is a pseudo-euclidean Jordan algebra.
   $$Where, \quad [x, y] = x.y - y.x \;\; and \;\; x \circ y = \frac{1}{2}(x.y + y.x), \; \forall x, y \in P.$$
3. The pseudo-euclidean alternative algebras studied and described inductively in [2] give examples of PEMPJ-admissible algebras, this shows that the class of PEMPJ-admissible algebras is very large.

One can also construct a PEMPJ-algebra from an arbitrary MPJ-algebra (not necessarily pseudo-euclidean).

Let $(P, [\ ,\ ], \circ)$ be a MPJ-algebra and P^* be the dual vector space of the underlying vector space of P. An easy computation prove that the following bracket $[\ ,\ ]_\sim$ and multiplication $\star$ define a MPJ-structure on the vector space $P \oplus P^*$:

$$[x+f, y+h]_\sim := [x,y]_\sim + f \circ ad_{p^-}(y) - h \circ ad_{p^-} x,$$

$$(x+f) \star (y+h) := x \circ y + f \circ L_y + h \circ L_x,\ \forall (x,f),(y,h) \in P \oplus P^*,$$

where L_x is the left multiplication by x in the algebra $(P, \circ)$. Moreover, if we consider the bilinear form $\psi : (P \oplus P^*) \times (P \oplus P^*) \to \mathbb{F}$ defined by:

$$\psi(x+f, y+h) = f(y) + h(x),\ \forall (x,f),(y,h) \in P \oplus P^*,$$

then $(P \oplus P^*, \psi)$ is a PEMPJ-algebra.

Let $(P, ., \psi)$ be a PEMPJ-admissible algebra, by Remarks 1 we have (P^-, ψ) is a pseudo-euclidean Malcev algebra and (P^+, ψ) is a pseudo-euclidean Jordan algebra. By TKK construction we have on the vector space $\mathfrak{Lie}(P^+) = P^+ \oplus H(P^+) \oplus \overline{P^+}$ the structure of a quadratic Lie algebra $(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$, where $\overline{P^+}$ is a copy of P^+ and $H(P^+) = [R(P^+), R(P^+)] \oplus R((P^+)^2)$, this construction is defined by:

$$[T,T'] = [T,T']_H;\quad [T,a'] = T(a');\quad [T,\overline{b'}] = -\overline{T(b')};\quad [a,\overline{b'}] = R_{ab'};\quad [a,a'] = [\overline{b},\overline{b'}] = 0.$$

For all $T, T' \in H(P^+), a, a', b, b' \in P^+$, and

$$\begin{array}{ll} \psi_{\mathfrak{L}} : & \mathfrak{Lie}(P^+) = P^+ \oplus H(P^+) \oplus \overline{P^+} \times \mathfrak{Lie}(P^+) = P^+ \oplus H(P^+) \oplus \overline{P^+} \longrightarrow \mathbb{F} \\ & (a + R_x + D + \overline{b}, a' + R_{x'} + D' + \overline{b'}) \longrightarrow \psi(x,x') + 2\psi(a,b') + 2\psi(a',b) + \varphi(D,D'), \end{array}$$

$$\forall a, a', b, b', x, x' \in P^+, D, D' \in [R(P^+), R(P^+)],$$

where $\varphi(D, D' = \sum_{i=1}^{n} [R_{x_i}, R_{x'_i}]) = \sum_{i=1}^{n} \psi(D(x_i), x'_i)$.

Now, let (P, ψ) be a pseudo-euclidean nonassociative algebra and $D : P \to P$ be a derivation of P. We say that D is ψ-skew symmetric of P if,

$$\psi(D(x), y) = -\psi(x, D(y)),\quad \forall x, y \in P.$$

Lemma 4.21

Let (P, ψ) be a pseudo-euclidean Jordan algebra, and D be a derivation ψ-skew symmetric of P. We define the endomorphism $D_{\mathfrak{L}}$ as follows:

$$D_{\mathfrak{L}}(x) = D(x),\ \ D_{\mathfrak{L}}(\bar{x}) = \overline{D(x)},\ \ D_{\mathfrak{L}}(R_x) = R_{D(x)},\ \ D_{\mathfrak{L}}([R_x, R_y]) = [R_{D(x)}, R_y] + [R_x, R_{D(y)}]$$

$\forall x, y \in P$. Then $D_{\mathfrak{L}}$ is a $\psi_{\mathfrak{L}}$-skew symmetric derivation of $\mathfrak{Lie}(P)$.

Proof: Since D is a derivation of P, then

$$D((x,y,z)) = (D(x), y, z) + (x, D(y), z) + (x, y, D(z)), \forall x, y, z \in P. \tag{4.8}$$

Consequently $D_{\mathfrak{L}}$ is a derivation of $\mathfrak{Lie}(P)$. Moreover, for all $a, a', b, b', x, x' \in P, d, d'$ $\in [R(P), R(P)]$ we have

$$\psi_{\mathfrak{L}}(D_{\mathfrak{L}}(a+R_x+d+\overline{b}), a'+R_{x'}+d'+\overline{b'}) = \psi(D(x), x') + 2\psi(D(a), b') + 2\psi(a', D(b)) + \varphi(D_{\mathfrak{L}}(d), d'),$$

and

$$\psi_{\mathfrak{L}}(a+R_x+d+\overline{b}, D_{\mathfrak{L}}(a'+R_{x'}+d'+\overline{b'})) = \psi(x, D(x'))+2\psi(a, D(b'))+2\psi(D(a'), b)+\varphi(d, D_{\mathfrak{L}}(d')),$$

where $\varphi(d, d' = \sum_{i=1}^{n}[R_{x_i}, R_{x'_i}]) = \sum_{i=1}^{n}\psi(d(x_i), x'_i)$.

By Eq.(4.8) and the fact that D is a ψ-skew symmetric of P one can check that $\varphi(D_{\mathfrak{L}}(d), d') = -\varphi(d, D_{\mathfrak{L}}(d'))$, this implies that $D_{\mathfrak{L}}$ is a $\psi_{\mathfrak{L}}$-skew symmetric derivation of $\mathfrak{Lie}(P)$. ■

Proposition 4.22

Let $(P, ., \psi)$ be a PEMPJ-admissible algebra, then for all $x \in P$, $ad_{p^-}x$ is a ψ-skew symmetric derivation of P^+.

Proof: Let $x \in P$, by Leibniz condition one can see that $ad_{p^-}x \in Der(P^+)$. Moreover, for all $y, z \in P$ we have,

$$\psi((ad_{p^-}x)(y), z) = \psi([x, y], z) = -\psi([y, x], z) = -\psi(y, [x, z]) = -\psi(y, (ad_{p^-}x)(z)).$$

Then $ad_{p^-}x$ is a derivation ψ-skew symmetric of P^+.

Thus, by Lemma 4.21, for all $x \in P$, $(ad_{p^-}x)_{\mathfrak{L}}$ is a $\psi_{\mathfrak{L}}$ skew-symmetric derivation of $\mathfrak{Lie}(P^+)$. ■

Definition 4.23

A Lie-Jordan-Poisson triple system is a triplet $(P, \circ, \{\ ,\ ,\ \})$ consisting for $\mathbb{F}$-vector space P, a bilinear map $\circ : P \times P \to P$ and a trilinear map $\{\ ,\ ,\ \} : P \times P \times P \to P$ such that:

(i) $(P, \circ)$ is a Jordan algebra;

(ii) $(P, \{\ ,\ ,\ \})$ is a Lie triple system;

(iii) $L(x, y) \in Der(P, \circ)$, $\forall x, y \in P$. Where $L(x, y)(z) = \{x, y, z\}$.

In the case where $(P, \circ)$ is an associative commutative algebra, then the triplet $(P, \circ, \{\ ,\ ,\ \})$ is called a Lie-Poisson triple system.

Proposition 4.24

Let $(P, [\ ,\ ], \circ)$ be a MPJ-algebra, then the triplet $(P, \circ, \{\ ,\ ,\ \})$ is a Lie-Jordan-Poisson triple system where $\{\ ,\ ,\ \} : P \times P \times P \longrightarrow P$ defined by

$$\{x, y, z\} = 2[[x, y], z] - [[y, z], x] - [[z, x], y],\ \forall x, y, z \in P.$$

Proof: Let $x, y, z, t \in P$,

$$\begin{aligned}\{x, y, z \circ t\} &= 2[[x, y], z \circ t] - [[y, z \circ t], x] - [[z \circ t, x], y] \\ &= 2[[x, y], t] \circ z + 2[[x, y], z] \circ t - [[y, z] \circ t, x] - [[y, t] \circ z, x] - [z \circ [t, x], y] - [[z, x] \circ t, y] \\ &= 2[[x, y], t] \circ z - [[y, t], x] \circ z - [y, t] \circ [z, x] - z \circ [[t, x], y] - [t, x] \circ [z, y] \\ &+ 2[[x, y], z] \circ t - [[y, z], x] \circ t - [y, z] \circ [t, x] - [[z, x], y] \circ t - [z, x] \circ [t, y] \\ &= 2[[x, y], t] \circ z - [[y, t], x] \circ z - z \circ [[t, x], y] + 2[[x, y], z] \circ t - [[y, z], x] \circ t - [[z, x], y] \circ t \\ &= z \circ \{x, y, t\} + \{x, y, z\} \circ t,\end{aligned}$$

then, $L(x, y) \in Der(P, \circ)$. Next, since $(P, [\ ,\ ])$ is a Malcev algebra, then the pair $(P, \{\ ,\ ,\ \})$ is a Lie triple system (see. [32]).

■

Let $(P, [\ ,\ ], \circ, \psi)$ be a PEMPJ-algebra. (Prop. 4.24) shows that $\forall a, b \in P,\ L(a, b) \in Der(P, \circ)$, moreover we have,

$$\psi(L(a, b)x, y) = -\psi(x, L(a, b)y),\ \forall a, b, x, y \in P.$$

For, thanks to the invariance of ψ, we have

$$\begin{aligned}\psi(\{a, b, x\}, y) &= 2\psi([[a, b], x], y) - \psi([[b, x], a], y) - \psi([[x, a], b], y) \\ &= 2\psi([a, b], [x, y]) - \psi([b, x], [a, y]) - \psi([x, a], [b, y]) \\ &= 2\psi([[b, a], y], x) - \psi([[a, y], b], x) - \psi([[y, b], a], x) \\ &= \psi(x, 2[[b, a], y] - [[a, y], b] - [[y, b], a]) \\ &= \psi(x, \{b, a, y\}) \\ &= -\psi(x, \{a, b, y\}).\end{aligned}$$

Then $L(a, b)$ is a derivation ψ-skew symmetric of P^+.

Remark 4.25

Lemma 4.21 shows that $(L(a,b))_{\mathfrak{L}}$ is a derivation $\psi_{\mathfrak{L}}$-skew symmetric of $\mathfrak{Lie}(P^+)$.

Proposition 4.26

Let $(P, [\ ,\ ], \circ, \psi)$ be a PEMPJ-algebra, then the linear map ϕ defined by

$$\begin{aligned} \phi: \ & L(P,P) \longrightarrow Der_a(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}}) \\ & L(x,y) \longmapsto (L(x,y))_{\mathfrak{L}} \end{aligned}$$

is a morphism of Lie algebras, where $Der_a(\mathfrak{Lie}(P^+)$ denotes the Lie algebra of all $\psi_{\mathfrak{L}}$-skew symmetric derivations of $\mathfrak{Lie}(P^+)$.

Proof: Let $a, b, x, y, x', y' \in P$, we have

$$([L(x,y), L(x',y')])_{\mathfrak{L}}(a) = [L(x,y), L(x',y')](a) = [(L(x,y))_{\mathfrak{L}}, (L(x',y'))_{\mathfrak{L}}](a).$$

For $R_a \in R(P^+)$, we obtain

$$\begin{aligned} ([L(x,y), L(x',y')])_{\mathfrak{L}}(R_a) &= R_{[L(x,y),L(x',y')](a)} \\ &= R_{L(x,y)oL(x',y')(a)} - R_{L(x',y')oL(x,y)(a)} \\ &= R_{(L(x,y))_{\mathfrak{L}}o(L(x',y'))_{\mathfrak{L}}(a)} - R_{(L(x',y'))_{\mathfrak{L}}o(L(x,y))_{\mathfrak{L}}(a)} \\ &= [(L(x,y))_{\mathfrak{L}}, (L(x',y'))_{\mathfrak{L}}](R_a). \end{aligned}$$

Let $[R_a, R_b] \in [R(P^+), R(P^+)]$, we have

$$\begin{aligned} [(L(x,y))_{\mathfrak{L}}, (L(x',y'))_{\mathfrak{L}}]([R_a,R_b]) &= ((L(x,y))_{\mathfrak{L}}o(L(x',y'))_{\mathfrak{L}})([R_a,R_b]) - ((L(x',y'))_{\mathfrak{L}}o(L(x,y))_{\mathfrak{L}})([R_a,R_b]) \\ &= [R_{L(x,y)oL(x',y')(a)}, R_b] + [R_{L(x',y')(a)}, R_{L(x,y)(b)}] + [R_{L(x,y)(a)}, R_{L(x',y')(b)}] \\ &+ [R_a, R_{L(x,y)oL(x',y')(b)}] - [R_{L(x',y')oL(x,y)(a)}, R_b] - [R_{L(x,y)(a)}, R_{L(x',y')(b)}] \\ &- [R_{L(x',y')(a)}, R_{L(x,y)(b)}] - [R_a, R_{L(x',y'),L(x,y)(b)}] \\ &= [R_{L(x,y)oL(x',y')(a)} - R_{L(x',y')oL(x,y)(a)}, R_b] \\ &+ [R_a, R_{L(x,y)oL(x',y')(b)} - R_{L(x',y')oL(x,y)(b)}] \\ &= [R_{[L(x,y),L(x',y')](a)}, R_b] + [R_a, R_{[L(x,y),L(x',y')](b)}] \\ &= ([L(x,y), L(x',y')])_{\mathfrak{L}}([R_a,R_b]). \end{aligned}$$

Then,

$\phi([L(x,y), L(x',y')]) = [\phi(L(x,y)), \phi(L(x',y'))]$ and ϕ is a morphism of Lie algebras. ■

It follows, by ([35]), that we can consider $(L(P,P) \oplus \mathfrak{Lie}(P^+) \oplus (L(P,P))^{\star}, [\ ,\ ], \widetilde{\psi_{\mathfrak{L}}})$ the quadratic Lie algebra (double extension of $\mathfrak{Lie}(P^+)$ by $L(P,P)$ by means of ϕ) defined by:

$$\begin{aligned} & [L(x,y) + K + f, L(x',y') + K' + f'] \\ & = [L(x,y), L(x',y')] + [K,K']_{\mathfrak{Lie}(P^+)} + \phi(L(x,y))(K') - \phi(L(x',y'))(K) \\ & + ad^{\star}(L(x',y'))(f) - ad^{\star}(L(x,y))(f') + \varphi(K,K') \end{aligned}$$

for all $x, y \in P,\ K, K' \in \mathfrak{Lie}(P^+),\ f, f' \in (L(P,P))^\star$, where φ is the linear map defined by:

$$\begin{array}{rl} \varphi: & \mathfrak{Lie}(P^+) \times \mathfrak{Lie}(P^+) \longrightarrow (L(P,P))^\star \\ & (K, K') \longmapsto \varphi(K,K')(L(x,y)) = \psi_{\mathfrak{L}}(\phi(L(x,y))(K), K'), \end{array}$$

and

$$\widetilde{\psi}_{\mathfrak{L}}(L(x,y)+K+f, L(x',y')+K'+f') = \psi_{\mathfrak{L}}(K,K')+f(K')+f'(K)+\sigma(L(x,y),L(x',y')),$$

here σ is an invariant symmetric bilinear form on $L(P,P) \times L(P,P)$.

4.4 Construction of pseudo-euclidean Malcev algebras from PEMPJ-admissible algebras

Our aim in this section is to construct pseudo-euclidean Malcev algebras from PEMPJ-admissible algebras by using the concept of double extension in the case of pseudo-euclidean Malcev algebras [4]. This concept was firstly introduced by A. Medina and Ph. Revoy in the case of quadratic Lie algebras [35]. We start by recalling the concept of double extension in the case of pseudo-euclidean Malcev algebras.

Definition 4.27

Let $(M, [\ ,\ ])$ be a Malcev algebra and V be a vector space, a linear map $\delta : M \to End(V)$ is called a Malcev representation of M in V if the the following identity holds,

$$\delta([[x,y],z]) = \delta([x,z])\delta(y) + \delta(x)\delta([y,z]) - \delta(y)\delta(x)\delta(z) + \delta(z)\delta(y)\delta(x), \quad \forall x,y,z \in M. \tag{4.9}$$

Definition 4.28

Let M be a Malcev algebra and let $\phi : M \to M$ an endomorphism of M. We say that ϕ is a Malcev operator of M if

$$\phi([[x,y],z]) = [[\phi(x),y],z] - [\phi(y),[x,z]] - [[\phi(z),x],y] - [\phi([y,z]),x], \quad \forall x,y,z \in M. \tag{4.10}$$

Example 4.29

In any Malcev algebra $(M, [\ ,\ ])$ we have,

$$[[w,y],[x,z]] = [w,x,y,z] + [x,y,z,w] + [y,z,w,x] + [z,w,x,y], \quad \forall x,y,z,w \in M,$$

that is, $ad_M w([[x,y],z]) = [[ad_M w(x),y],z] - [ad_M w(y),[x,z]] - [[ad_M w(z),x],y] - [ad_M w([y,z]),x]$, therefore $\forall w \in M$, $ad_M w$ is a Malcev operator of M.

We denote by $Op(M)$ the vector subspace of $End(M)$ formed by the Malcev operators.

Proposition 4.30 (*[4]*)

. Let M be a Malcev algebra and let D be a derivation of M. Then D is a Malcev operator if and only if $D(J(M,M,M)) = 0$. Where $J(M,M,M)$ is the vector subspace of M spanned by $\{J(x,y,z)/x,y,z \in M\}$.

Remark 4.31

If M is a Lie algebra, then every derivation D of M is a Malcev operator.

Let M and V be two Malcev algebras and let δ be a Malcev representation of M in V. We define on the vector space $\tilde{M} = M \oplus V$ the following product:

$$(x+v)(y+w) = xy + \delta(x)(w) - \delta(y)(v) + vw, \quad \forall x+v \in \tilde{M}, y+w \in \tilde{M}. \tag{4.11}$$

Proposition 4.32

(see. [4]). $\tilde{M}$ endowed with the product (e) is a Malcev algebra if and only if δ satisfies the following conditions:

1. $[[\delta([x,y])(t)],w] = \delta(x)([\delta(y)(t),w]) + [\delta(x)(t),\delta(y)(w)] - \delta(y)\delta(x)([t,w]) - [\delta(y)\delta(x)(w),t]$
2. $\delta([x,y])([w,t]) = \delta(y)([\delta(x)(t),w]) - [(\delta(x)(\delta(y)(t)),w] + \delta(x)([\delta(y)(w)),t]) - [\delta(y)(\delta(x)(w)),t]$
3. $\delta(x)$ is a Malcev operator of V,

$\forall x,y \in M, w,t \in V$.
In this case δ is called a Malcev admissible representation of M in V. And the Malcev algebra $\tilde{M}$ is called the semi-direct product of V by M by means of δ.

Recall that in [35] A. Medina and Ph. Revoy have introduced the concept of double extension and used this concept to give an inductive classification of pseudo-euclidean Lie algebras. This concept was generalized by H. Albuquerque and S. Benayadi [4] to pseudo-euclidean Malcev algebras and Malcev superalgebras.

Let (M_1,ψ_1) be a pseudo-euclidean Malcev algebra and M_2 be a Malcev algebra.
Let $\delta : M_2 \to End(M_1)$ be an admissible Malcev representation of M_2 in M_1 such that $\forall x \in M_2$, $\delta(x) \in Op_a(M_1)$, where $Op_a(M_1)$ is the sub-space of ψ_1-skew-symmetric elements of $Op(M_1)$.
Let us consider the bilinear map $\varphi : M_1 \times M_1 \to M_2^*$ defined by

$$\varphi(x,y)(z) = \psi_1(\delta(z)(x),y), \quad \forall x,y \in M_1, z \in M_2$$

Then φ is a 2-cocycle (see. [4], p. 30) of M_1 with values in M_2^* and consequently $M_1 \oplus M_2^*$ with the multiplication defined by
$(x_1+f)(y_1+g) = x_1y_1 + \varphi(x_1,y_1), \quad \forall x_1+f, y_1+g \in M_1 \oplus M_2^*$ is the central extension of M_2^* by M_1 by means of φ.

Let us consider the linear map $\rho : M_2 \to End(M_1 \oplus M_2^*)$ defined by $\rho(x_2) = \delta(x_2) + \pi^*(x_2)$ that is,

$$\rho(x_2)(x_1+f) = \delta(x_2)(x_1) + \pi^*(x_2)(f), \quad \forall x_1 \in M_1, x_2 \in M_2, f \in M_2^*$$

where π^* is the coadjoint representation of M_2. Then ρ is an admissible Malcev representation of M_2 in $M_1 \oplus M_2^*$ and consequently $M = M_2 \oplus M_1 \oplus M_2^*$ with the multiplication

$$(x_2+x_1+f)(y_2+y_1+g) = x_2y_2 + \delta(x_2)y_1 + \pi^*(x_2)(g) + x_1y_1 + \varphi(x_1,y_1) - \delta(y_2)(x_1) - \pi^*(y_2)f,$$

$\forall x_2 + x_1 + f, y_2 + y_1 + g \in M$, is the semi-direct product of $M_1 \oplus M_2^*$ by M_2 by means of ρ.

Let γ be a symmetric invariant bilinear form on $M_2 \times M_2$ not necessarily nondegenerate. Then the bilinear form ψ on $M = M_2 \oplus M_1 \oplus M_2^*$ defined by

$$\psi(x_2+x_1+f, y_2+y_1+g) = \gamma(x_2, y_2)+\psi_1(x_1, Y_1)+f(y_2)+g(x_2), \quad \forall (x_2+x_1+f), (y_2+y_1+g) \in M$$

is an invariant scalar product. The pseudo-euclidean Malcev algebra $M_2 \oplus M_1 \oplus M_2^*$ is called the double extension of (M_1, ψ_1) by M_2 by means of δ.

Remark 4.33

A straightforward calculation shows that the pseudo-euclidean Malcev algebra $M = M_2 \oplus M_1 \oplus M_2^*$, constructed above, is a pseudo-euclidean Lie algebra if and only if M_1, M_2 are both Lie algebras, $\delta(x) \in Der(M_1), \quad \forall x \in M_2$ and δ is a morphism of Lie algebras.

Remark 4.34

In Theorem 5.1 in [4], there is a typographical error in the last sign of the condition,

$$\begin{aligned}\psi(SX)(YZ) = \psi(S)[(\psi(X)(Y))Z] - (-1)^{yz}[\psi(S)(\psi(X)(Z))]Y \\ + (-1)^{sx+yz}\psi(X)([\psi(S)(Z)]Y) + (-1)^{sx}[\psi(X)(\psi(S)(Y))]Z,\end{aligned}$$

it should be replaced by,

$$\begin{aligned}\psi(SX)(YZ) = \psi(S)[(\psi(X)(Y))Z] - (-1)^{yz}[\psi(S)(\psi(X)(Z))]Y \\ + (-1)^{sx+yz}\psi(X)([\psi(S)(Z)]Y) - (-1)^{sx}[\psi(X)(\psi(S)(Y))]Z.\end{aligned}$$

To do this, we must prove in [4] that

$$\begin{aligned}-(-1)^{sx}B(\psi(XS)(YZ), T) = B(\psi(S)[(\psi(X)(Y))Z], T) - (-1)^{yz}B([\psi(S)(\psi(X)(Z))]Y, T) \\ - (-1)^{t(y+z)}B(\psi(S)[(\psi(X)(T))Y], Z) - (-1)^{y(z+t)}B(\psi(S)[(\psi(X)(ZT)], Y)\end{aligned}$$

We have,

$$-(-1)^{sx}B(\psi(XS)(YZ), T) = +(-1)^{sx} \times (-1)^{sx}B(\psi(SX)(YZ), T) = B(\psi(SX)(YZ), T).$$

Moreover,

$$
\begin{aligned}
&-(-1)^{t(y+z)}B(\psi(S)[(\psi(X)(T))Y],Z)\\
&\quad=+(-1)^{t(y+z)}\times(-1)^{s(x+t+y)}B([\psi(X)(T)]Y,\psi(S)(Z))\\
&\quad=+(-1)^{t(y+z)}\times(-1)^{s(x+t+y)}B(\psi(X)(T),Y[\psi(S)(Z)])\\
&\quad=-(-1)^{t(y+z)}\times(-1)^{s(x+t+y)}\times(-1)^{y(s+z)}B(\psi(X)(T),[\psi(S)(Z)]Y)\\
&\quad=+(-1)^{t(y+z)}\times(-1)^{s(x+t+y)}\times(-1)^{y(s+z)}\times(-1)^{xt}B(T,\psi(X)([\psi(S)(Z)]Y))\\
&\quad=+(-1)^{t(y+z)}\times(-1)^{s(x+t+y)}\times(-1)^{y(s+z)}\times(-1)^{xt}\times(-1)^{t(x+s+z+y)}B(\psi(X)([\psi(S)(Z)]Y),T)\\
&\quad=+(-1)^{sx+yz}\times(-1)^{t(x+y+z+s)}\times(-1)^{t(x+s+z+y)}B(\psi(X)([\psi(S)(Z)]Y),T)\\
&\quad=+(-1)^{sx+yz}B(\psi(X)([\psi(S)(Z)]Y),T).
\end{aligned}
$$

And,

$$
\begin{aligned}
&-(-1)^{y(z+t)}B(\psi(S)[(\psi(X)(ZT)],Y)\\
&=+(-1)^{y(z+t)}\times(-1)^{s(x+z+t)}B(\psi(X)(ZT),\psi(S)(Y))\\
&=-(-1)^{y(z+t)}\times(-1)^{s(x+z+t)}\times(-1)^{x(z+t)}B(ZT,\psi(X)[\psi(S)(Y)])\\
&=+(-1)^{y(z+t)}\times(-1)^{s(x+z+t)}\times(-1)^{x(z+t)}\times(-1)^{zt}B(TZ,\psi(X)[\psi(S)(Y)])\\
&=-(-1)^{y(z+t)}\times(-1)^{s(x+z+t)}\times(-1)^{x(z+t)}\times(-1)^{zt}\times(-1)^{z(x+s+y)}B(T,[\psi(X)(\psi(S)(Y))]Z)\\
&=-(-1)^{y(z+t)}\times(-1)^{s(x+z+t)}\times(-1)^{x(z+t)}\times(-1)^{zt}\times(-1)^{z(x+s+y)}\times(-1)^{t(x+y+s+z)}B([\psi(X)(\psi(S)(Y))]\\
&=-(-1)^{yz+sx+sz+xz+z(x+s+y)}B([\psi(X)(\psi(S)(Y))]Z,T)\\
&=-(-1)^{sx}B([\psi(X)(\psi(S)(Y))]Z,T)
\end{aligned}
$$

Which is equivalent to prove that,

$$
\begin{aligned}
B(\psi(SX)(YZ),T)=&\ B(\psi(S)[(\psi(X)(Y))Z],T)-(-1)^{yz}B([\psi(S)(\psi(X)(Z))]Y,T)\\
&+(-1)^{sx+yz}B(\psi(X)([\psi(S)(Z)]Y),T)-(-1)^{sx}B([\psi(X)(\psi(S)(Y))]Z,T).
\end{aligned}
$$

As B is nondegenerate this last condition is equivalent to,

$$
\begin{aligned}
\psi(SX)(YZ)=&\ \psi(S)[(\psi(X)(Y))Z]-(-1)^{yz}[\psi(S)(\psi(X)(Z))]Y\\
&+(-1)^{sx+yz}\psi(X)([\psi(S)(Z)]Y)-(-1)^{sx}[\psi(X)(\psi(S)(Y))]Z.
\end{aligned}
$$

Consequently, in the case of pseudo-euclidean Malcev algebras this condition coincides with the condition (2) of Proposition 4.32.

Now we are going to construct a pseudo-euclidean Malcev algebra from PEMPJ-admissible algebra by using the concept of double extension [4].

Lemma 4.35

Let $(P,.)$ be a MPJ-admissible algebra. The following conditions are equivalent:

i) $[R_{J(x,z,t)}, R_{[y,w]}] + [R_{J(y,z,t)}, R_{[x,w]}] = [R_{J(x,z,w)}, R_{[y,t]}] + [R_{J(y,z,w)}, R_{[x,t]}]$;

ii) $[R_{J(x,z,t)}, R_{[x,w]}] = [R_{J(x,z,w)}, R_{[x,t]}]$;

iii) $[R_{J(z,t,w)}, R_{[z,x]}] = 0$;

iv) $[R_{J(z,t,w)}, R_{[y,x]}] + [R_{J(y,t,w)}, R_{[z,x]}] = 0$;

$\forall x, y, z, t, w \in P$. Where J corresponds to the Jacobian in $(P^-, [\ ,\ ])$, and R_x is the right multiplication by x in the algebra $(P^+, \circ)$.

Proof: *i)* $\Rightarrow$ *ii)* Letting $x = y$.

ii) $\Rightarrow$ *iii)* Let $w = z$, since $R_{J(x,z,z)} = 0$, it follows that $[R_{J(x,z,t)}, R_{[x,z]}] = 0$. Linearizing this relative to x (ie. we replace x by $x + \lambda w$, where $\lambda \in \mathbb{F}$), we have:

$$\lambda([R_{J(x,z,t)}, R_{[w,z]}] + [R_{J(w,z,t)}, R_{[x,z]}]) = 0,$$

when $\lambda = 1$ then,

$$[R_{J(x,z,t)}, R_{[w,z]}] + [R_{J(w,z,t)}, R_{[x,z]}] = 0,$$
$$[R_{J(z,t,x)}, R_{[w,z]}] + [R_{J(z,t,w)}, R_{[x,z]}] = 0,$$
$$[R_{J(z,t,x)}, R_{[z,w]}] + [R_{J(z,t,w)}, R_{[z,x]}] = 0,$$

from ii) we obtain,

$$[R_{J(z,t,w)}, R_{[z,x]}] + [R_{J(z,t,w)}, R_{[z,x]}] = 0,$$

hence,

$$[R_{J(z,t,w)}, R_{[z,x]}] = 0, \quad \forall x, z, t, w \in P.$$

iii) $\Rightarrow$ *iv)* We proceed by linearization, we replace z by $z+\lambda y$ in the identity $[R_{J(z,t,w)}, R_{[z,x]}] = 0$, where $\lambda \in \mathbb{F}$. Which gives the identity (iv).

iv) $\Rightarrow$ *i)* is trivial. ∎

Lemma 4.36

Let $(P, ., \psi)$ be a PEMPJ-admissible algebra such that

$$[R_{J(z,t,w)}, R_{[z,x]}] = 0, \ \forall z, t, w, x \in P.$$

Let $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ be the linear map defined by $\delta(x) = (ad_{p^-}x)_{\mathfrak{L}}$, then δ is a Malcev representation of P^- in $\mathfrak{Lie}(P^+)$.

Proof: Let $x, y, z \in P$. For $w \in P$, we have $\delta[[x, y], z](w) = [x, y, z, w]$. Also we have,

$$\begin{aligned}&(\delta([x, z])\delta(y) + \delta(x)\delta([y, z]) - \delta(y)\delta(x)\delta(z) + \delta(z)\delta(y)\delta(x))(w)\\ &= [[x, z], [y, w]] - [y, z, w, x] - [z, w, x, y] + [x, w, y, z].\end{aligned}$$

By Malcev identity, $[[x, z], [y, w]] = [x, y, z, w] + [y, z, w, x] + [z, w, x, y] + [x, w, y, z]$ we conclude,

$$(\delta[[x, y], z])(w) = (\delta([x, z])\delta(y) + \delta(x)\delta([y, z]) - \delta(y)\delta(x)\delta(z) + \delta(z)\delta(y)\delta(x))(w), \ \ \forall w \in P,$$

by the same calculation we obtain this equality if we replace w by $R_w \in R(P^+)$. Now let

$[R_w, R_t] \in [R(P^+), R(P^+)]$, we obtain $\delta[[x,y],z]([R_w,R_t]) = [R_{[x,y,z,w]}, R_t]+[R_w, R_{[x,y,z,t]}]$.

Moreover,

$$\delta([x,z])\delta(y)([R_w,R_t]) = [R_{[[x,z],[y,w]]}, R_t]+[R_{[y,w]}, R_{[[x,z],t]}]+[R_{[[x,z],w]}, R_{[y,t]}]+[R_w, R_{[[x,z],[y,t]]}],$$

$$\delta(x)\delta([y,z])([R_w,R_t]) = [R_{[x,[[y,z],w]]]}, R_t]+[R_{[[y,z],w]}, R_{[x,t]}]+[R_{[x,w]}, R_{[[y,z],t]}]+[R_w, R_{[x,[[y,z],t]]}],$$

$$\begin{aligned}\delta(y)\delta(x)\delta(z)([R_w,R_t]) = {} & [R_{[y,[x,[z,w]]]}, R_t] + [R_{[x,[z,w]]}, R_{[y,t]}] + [R_{[y,[z,w]]}, R_{[x,t]}] + [R_{[z,w]}, R_{[y,[x,t]]}] \\ & + [R_{[y,[x,w]]}, R_{[z,t]}] + [R_{[x,w]}, R_{[y,[z,t]]}] + [R_{[y,w]}, R_{[x,[z,t]]}] + [R_w, R_{[y,[x,[z,t]]]}],\end{aligned}$$

$$\begin{aligned}\delta(z)\delta(y)\delta(x)([R_w,R_t]) = {} & [R_{[z,[y,[x,w]]]}, R_t] + [R_{[y,[x,w]]}, R_{[z,t]}] + [R_{[z,[x,w]]}, R_{[y,t]}] + [R_{[x,w]}, R_{[z,[y,t]]}] \\ & + [R_{[z,[y,w]]}, R_{[x,t]}] + [R_{[y,w]}, R_{[z,[x,t]]}] + [R_{[z,w]}, R_{[y,[x,t]]}] + [R_w, R_{[z,[y,[x,t]]]}].\end{aligned}$$

Using Lemma 4.35 and by doing some calculations, these achieve the of the lemma because P is a Malcev algebra. ■

Proposition 4.37

Let $(P, ., \psi)$ be a PEMPJ-admissible algebra such that $R_{J(x,y,z)} = 0$, for all $x, y, z \in P$, then the linear map $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ defined by $\delta(x) = (ad_{p^-}x)_{\mathfrak{L}}$ is an admissible Malcev representation of P^- in $\mathfrak{Lie}(P^+)$.

Proof: Let us prove that δ verifies the condition (1) of the proposition 4.32. It is easy to see that for $t,\ w \in P$, the two terms vanish. Now let $t \in P,\ R_u \in R(P^+)$, then

$$\begin{aligned}&\delta(x)([\delta(y)(t), R_u]) + [\delta(x)(t), \delta(y)(R_u)] - \delta(y)\delta(x)([t, R_u]) - [\delta(y)\delta(x)(R_u), t] \\ &= -[x,[y,t]\circ u] - [x,t]\circ[y,u] + [y,[x,t\circ u]] - t\circ[y,[x,u]].\end{aligned}$$

Moreover

$$[[\delta([x,y])(t)], R_u] = -[[x,y],t]\circ u.$$

As $ad_{p^-}x$ is a derivation of P^+, $\forall x \in P$, then δ satisfies (1) of the proposition 4.32 if and only if, $J(x,y,t)\circ u = 0$, and by similar argument if we replace t by R_u and R_u by t we obtain the same result. Let now $R_u,\ R_v \in R(P^+)$ then

$$\begin{aligned}&\delta(x)([\delta(y)(R_v), R_u]) + [\delta(x)(R_v), \delta(y)(R_u)] - \delta(y)\delta(x)([R_v, R_u]) - [\delta(y)\delta(x)(R_u), R_v] \\ &= [R_{[y,v]}, R_{[x,u]}] + [R_{[x,[y,v]]}, R_u] + [R_{[x,v]}, R_{[y,u]}] - [R_{[y,[x,v]]}, R_u] - [R_{[x,v]}, R_{[y,u]}] - [R_{[y,v]}, R_{[x,u]}] \\ &- [R_v, R_{[y,[x,u]]}] - [R_{[y,[x,u]]}, R_v],\end{aligned}$$

and

$$[[\delta([x,y])(R_v)], R_u] = [R_{[[x,y],v]}, R_u]$$

So the condition (1) of the proposition 4.32 holds if and only if $[R_{J(x,y,v)}, R_u] = 0$. Now we have to show the condition (2). Let $t,\ w \in P$, then the two terms vanish. Now let $t \in P,\ R_u \in R(P^+)$ we obtain,

$$\delta(y)([\delta(x)(t), R_u]) - [(\delta(x)(\delta(y)(t)), R_u] + \delta(x)([\delta(y)(R_u)), t]) - [\delta(y)(\delta(x)(R_u)), t]$$
$$= -[y, [x,t] \circ u] + [x, [y,t]] \circ u + [x, t \circ [y,u]] - t \circ [y, [x,u]]$$

and

$$\delta([x,y])([R_u, t]) = [[x,y], t \circ u]$$

Consequently, the condition (2) is satisfied if and only if $J(t,y,x) \circ u = J(x,y,u) \circ t$. Using the same reasonings, we can show that if $R_v,\ R_u \in R(P^+)$ then the condition (2) holds if and only if $[R_{J(x,y,u)}, R_v] = [R_{J(x,y,v)}, R_u]$. Finally, since $\delta(x)$ is a derivation of $\mathfrak{Lie}(P^+)$, $\forall x \in P$ then by Remark 4.31 we deduce that $\delta(x)$ is a Malcev operator of $\mathfrak{Lie}(P^+)$, which proves the proposition. ■

By the last Proposition we conclude that we can consider $P^- \oplus \mathfrak{Lie}(P^+) \oplus (P^-)^*$ the double extension of $(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ by the Malcev algebra P^- (such that $R_{J(x,y,z)} = 0$, $\forall x,y,z \in P$) by means of the Malcev admissible representation $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ defined by $\delta(x) = (ad_{p^-} x)_{\mathfrak{L}}$, $\forall x \in P$.

From the proposition 4.37 and Remark (4.33) we have the following Corollary :

Corollary 4.38

Let $(P, ., \psi)$ be a PEMPJ-admissible algebra such that P^- is a Lie algebra, then the linear map $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ defined by $\delta(x) = (ad_{p^-} x)_{\mathfrak{L}}$ is a morphism of Lie algebras (thus δ is an admissible Malcev representation of P^- in $\mathfrak{Lie}(P^+)$). In this case the double extension algebra $P^- \oplus \mathfrak{Lie}(P^+) \oplus (P^-)^*$ is a quadratic Lie algebra.

4.5 Inductive description of nilpotent PEMPJ-admissible algebras

In order to give an inductive description of nilpotent PEMPJ-admissible algebras, we introduce in this section the concept of double extension of PEMPJ-admissible algebras by the one-dimensional algebra with null product by using some results of [4],[5]. Let us recall the concept of generalized double extension of a pseudo-euclidean Jordan algebra by the one-dimensional Jordan algebra with zero product [5] and the concept of double extension of a pseudo-euclidean Malcev algebra by the one-dimensional Lie algebra [4].

Let J be a Jordan algebra, $(D, x_0) \in End(J) \times J$, and $\mathbb{F}e$ be the one-dimensional algebra with zero product. On the vector space $\mathbb{F}e \oplus J$, we define the following product,

$$x \star y := xy,\ x \star e = e \star x := D(x),\ e \star e = x_0,\ \forall x, y \in J.$$

The vector space $\mathbb{F}e \oplus J$ endowed with the product above is a Jordan algebra, if and only if, for all $x, y \in J$, the pair (D, x_0) satisfies the following conditions (see. [5]):
$(C1) : D(x^2y) = x^2D(y) + 2D(x)(xy) - 2x(D(x)y)$,
$(C2) : D(x)D(y) - D(D(x)y) = \frac{1}{2}(x_0, y, x)$,

$(C3): D(x_0x) = x_0D(x)$,
$(C4): xD(x^2) = x^2D(x)$,
$(C5): D^2(x^2) = 2(D(x))^2 - 2xD^2(x) + x_0x^2$,
$(C6): D^3(x) = \frac{3}{2}x_0D(x) - \frac{1}{2}xD(x_0)$,
$(C7): D^2(x_0) = x_0^2$.
In this case, (D, x_0) is called an admissible pair of J and the Jordan algebra $\mathbb{F}e \oplus J$, is called the generalized semi-direct product of J by the one-dimensional algebra with zero product by means of the pair (D, x_0).

Now, Let $(P, ., \psi)$ be a PEMPJ-admissible algebra, and let $(D, x_0) \in End_s(P^+, \psi) \times P^+$ be an admissible pair of P^+ (where $End_s(P^+, \psi)$ is the algebra of symmetric endomorphisms of the vector space P^+ with respect to ψ) .
Then, the vector space $\mathcal{A} := \mathbb{F}e \oplus P \oplus \mathbb{F}e^*$ endowed with the following product,

$$\mathcal{A}\star e^* = e^*\star\mathcal{A} = 0;\ e\star e := x_0+\alpha e^*;\ x\star y := x\circ y+\psi(D(x), y)e^*;\ e\star x = x\star e := D(x)+\psi(x_0, x)e^*;$$

$\forall x, y \in P$, is a Jordan algebra (see. [5]).

Let $\phi \in Op_a(P^-)$ (see[4]) such that,

1. $\phi^2([x, y]) = [\phi(x), \phi(y)] + \phi([\phi(x), y]) + [x, \phi^2(y)]$,
2. $[\phi^2(x), y] - \phi([\phi(x), y]) = -([\phi^2(y), x] - \phi([\phi(y), x])), \quad \forall x, y \in P^-$.

Then, the vector space $\mathcal{A} := \mathbb{F}e \oplus P \oplus \mathbb{F}e^*$ (where $\mathbb{F}e$ is the one-dimensional Lie algebra and $\mathbb{F}e^*$ its dual) endowed with the following product,

$$[x, y] := [x, y]_{P^-} + \psi(\phi(x), y)e^*;\ [e, x] = -[x, e] := \phi(x);\ [e, e] = 0;\ [e^*, \mathcal{A}] = [\mathcal{A}, e^*] = \{0\};$$

for all $x, y \in P$, is a Malcev algebra. Moreover, if we consider the symmetric bilinear form $T : \mathcal{A} \times \mathcal{A} \to \mathbb{F}$ defined by :

$$T_{|P\times P} := \psi;\ T(e^*, e^*) = 1;\ T(e, P) = T(e^*, P) = \{0\};\ T(e, e) = T(e^*, e^*) = 0,$$

then $(\mathcal{A}, [\ ,\ ], T)$ is a pseudo-euclidean Malcev algebra called the double extension of $(P^-, [\ ,\ ]_{P^-}, \psi)$ by the one-dimensional Lie algebra by means of ϕ (see. [4]) and $(\mathcal{A}, \star, T)$ is a pseudo-euclidean Jordan algebra called generalized double extension of $(P^+, \circ, \psi)$ by the one dimensional algebra with null product by means of (D, x_0) (see. [5]).

Proposition 4.39

Let $(\mathcal{A}, [\ ,\ ])$ (resp. $(\mathcal{A}, \star)$) be the Malcev algebra (resp. the Jordan algebra) defined above, $(\mathcal{A}, [\ ,\ ], \star)$ is a MPJ-algebra if and only if, for all $x, y \in P$,

$$\phi o D = D o \phi = \frac{1}{2}[x_0, .];\ \phi(x_0) = 0;\ \phi \in Der(P^+);\ D([x, y]_{P^-}) = [x, D(y)]_{P^-} + \phi(x)\circ y. \quad (4.12)$$

Proof: Assume that $(\mathcal{A}, [\ ,\ ], \star)$ is a MPJ-algebra. It follows from Leibniz Identity that,

$$[e+x+e^*, (e+y+e^*)\star(e+z+e^*)] = [e+x+e^*, (e+y+e^*)]\star(e+z+e^*)+[e+x+e^*, (e+z+e^*)]\star(e+y+e^*),$$

$\forall x,\ y\ ,z\ \in P$. Which is equivalent to

$$\begin{aligned}&\phi(x_0)+\phi oD(z)+\phi oD(y)+\phi(y\circ z)+[x,x_0+D(z)+D(y)+y\circ z]_{P^-}\\&=Do\phi(y)-Do\phi(x)+D([x,y]_{P^-})+\phi(y)\circ z-\phi(x)\circ z+[x,y]_{P^-}\circ z\\&+Do\phi(z)-Do\phi(x)+D([x,z]_{P^-})+\phi(z)\circ y-\phi(x)\circ y+[x,z]_{P^-}\circ y,\end{aligned}$$

and

$$\begin{aligned}\psi(\phi(x),x_0+D(z)+D(y)+y\circ z)=&\ \psi(x_0,\phi(y)-\phi(x)+[x,y]_{P^-})\\&+\psi(Do\phi(y)-Do\phi(x)+D([x,y]_{P^-}),z)\\&+\psi(x_0,\phi(z)-\phi(x)+[x,z]_{P^-})\\&+\psi(Do\phi(z)-Do\phi(x)+D([x,z]_{P^-}),y).\end{aligned}$$

Letting x=y=z=0, the first identity gives $\phi(x_0)=0$. And when x=z=0, we obtain $\phi oD(y)=Do\phi(y)$, $\forall y\in P$. Moreover if $z=0$, then

$$[x,x_0]_{P^-}+[x,D(y)]=-2Do\phi(x)-\phi(x)\circ y+D([x,y]),\ \forall x,y\in P,$$

this implies that $[x_0,x]_{P^-}=2\phi oD(x)$ and $D([x,y]_{P^-})=\phi(x)\circ y+[x,D(y)]_{P^-}$, $\forall x,y\in P$. So by the first identity we can conclude that $\phi(y\circ z)=\phi(y)\circ z+\phi(z)\circ y$, $\forall y,z\in P$ (ie. $\phi\in Der(P^+)$). Conversely, if the condition (4.12) is verified it is clair that $(\mathcal{A},[\ ,\ ],\star)$ is a MPJ-algebra. ■

In this case $(\mathcal{A},[\ ,\ ],\star,T)$ is a PEMPJ-algebra, called the double extension of the PEMPJ-algebra $(P,[\ ,\ ]_{P^-},\circ,\psi)$ by the one dimensional algebra with null product by means of (ϕ,D,x_0,α). Moreover, the vector space $\mathcal{A}$ endowed with the product:

$$x\bullet y=\frac{1}{2}[x,y]+x\star y,\quad \forall x,y\in\mathcal{A},$$

is MPJ-admissible algebra. Then $(\mathcal{A},\bullet,T)$ is a PEMPJ-admissible algebra called the double extension of the PEMPJ-admissible algebra $(P,.,\psi)$ by the one dimensional algebra with null product by means of (ϕ,D,x_0,α).

Proposition 4.40

Let $(\mathcal{A},\bullet,T)$ be a PEMPJ-admissible algebra. If $Ann(\mathcal{A})\neq\{0\}$ and if there exists $e^*\in Ann(\mathcal{A})\backslash\{0\}$ such that $T(e^*,e^*)=0$, then $(\mathcal{A},\bullet,T)$ is a double extension of the PEMPJ-admissible algebra $(P:=(\mathbb{F}e^*)^\perp/\mathbb{F}e^*,.,\psi)$ by the one dimensional algebra with null product, where

$$\psi(x+\mathbb{F}e^*,y+\mathbb{F}e^*):=T(x,y),\ \ and\ \ (x+\mathbb{F}e^*).(y+\mathbb{F}e^*):=(x\bullet y)+\mathbb{F}e^*,\quad \forall x,y\in(\mathbb{F}e^*)^\perp.$$

Proof: Let $e^*\in Ann(\mathcal{A})\backslash\{0\}$ such that $T(e^*,e^*)=0$, then there exists $e\in\mathcal{A}$ such that $T(e,e^*)=1$, $T(e,e)=0$ and $\mathcal{A}=(\mathbb{F}e^*)^\perp\oplus\mathbb{F}e$. Let $P:=(\mathbb{F}e^*\oplus\mathbb{F}e)^\perp$ denotes the orthogonal of $\mathbb{F}e^*\oplus\mathbb{F}e$ with respect to T, then $(\mathbb{F}e^*)^\perp=\mathbb{F}e^*\oplus P$, it follows that $\mathcal{A}:=\mathbb{F}e\oplus P\oplus\mathbb{F}e^*$. Furthermore the vector space P can be endowed with the MPJ-admissible structure . induced by the one of $(\mathbb{F}e^*)^\perp/\mathbb{F}e^*$ and $\psi:=T_{|P\times P}$ is an invariant scalar product on $(P,.)$.

Next, one can observe that $T(e \bullet e, e^*) = T(e, e \bullet e^*) = 0$, then $e \bullet e \in \mathbb{F}e^*)^{\perp}$ which implies that $e \bullet e = x_0 + \alpha e^*$ where $x_0 \in P$, $\alpha \in \mathbb{F}$. Consequently , one easily check that if $x \in P \subset (\mathbb{F}e^*)^{\perp}$ then $e \bullet x = \Gamma(x) + \psi(x_0, x)e^*$ and $x \bullet e = \Gamma^*(x) + \psi(x_0, x)e^*$ where $\Gamma \in End(P)$ and Γ^* is the adjoint of Γ with respect to ψ.

Finally, if we consider $\phi := \Gamma - \Gamma^*$, $D := \frac{1}{2}(\Gamma + \Gamma^*)$, then it is easy o show that $(\mathcal{A}, \bullet, T)$ is the double extension of the PEMPJ-admissible algebra $(P, ., \psi)$ by the one dimensional algebra with null product by means of (ϕ, D, x_0, α). ■

Corollary 4.41

Let $(\mathcal{A}, \bullet, T)$ be an irreducible nilpotent PEMPJ-admissible algebra. Then $\mathcal{A}$ is a double extension of a nilpotent PEMPJ-admissible algebra $(P, ., \psi)$ by the one dimensional algebra with null product.

Finally, if we denote by $\mathcal{E}$ the set constituted by $\{0\}$ and the one dimensional algebra with null product, then for any nilpotent PEMPJ-admissible algebra $(P, ., \psi)$ such that $P \notin \mathcal{E}$ is obtained from elements $P_1, ..., P_n$ of $\mathcal{E}$ by a finite number of orthogonal direct sums of PEMPJ-admissible algebras or/and double extensions by the one dimensional algebra with null product.

4.6 Double extension of PEMPJ-admissible algebras

Let (P_1, ψ) be a PEMPJ-admissible algebra, P_2 be a MPJ-admissible algebra and $\pi : P_2^+ \to End_s(P_1^+)$ (resp. $\delta : P_2^- \to End_a(P_1^-)$ be an admissible Jordan representation of P_2^+ in P_1^+ (resp. be an admissible Malcev representation of P_2^- in P_1^-).
Let $\varphi^+ : P_1^+ \times P_1^+ \to (P_2^+)^*$, and $\varphi^- : P_1^- \times P_1^- \to (P_2^-)^*$ be the two symmetric bilinear maps defined by:

$$\varphi^+(x, y)(z) = \psi(\pi(z)(x), y), \ and \ \varphi^-(x, y)(z) = \psi(\delta(z)(x), y), \quad \forall x, y \in P_1, z \in P_2.$$

The vector space $\mathcal{P} := P_2^+ \oplus P_1^+ \oplus (P_2^+)^*$ endowed with the product

$$\begin{aligned}&(x_2 + x_1 + f) \star (y_2 + y_1 + g)\\ &= x_2y_2 + x_1y_1 + \pi(x_2)(y_1) + \pi(y_2)(x_1) + g \circ R_{x_2} + f \circ R_{y_2} + \varphi^+(x_1, y_1),\end{aligned}$$

is a Jordan algebra for all $x_1, y_1 \in P_1, x_2, y_2 \in P_2, f, g \in (P_2)^*$.

The vector space $\mathcal{P} := P_2^- \oplus P_1^- \oplus (P_2^-)^*$ endowed with the product

$$\begin{aligned}&[x_2 + x_1 + f, y_2 + y_1 + g]\\ &= [x_2, y_2]_2 + [x_1, y_1]_1 + \delta(x_2)(y_1) - \delta(y_2)(x_1) + \pi^*(x_2)(g) - \pi^*(y_2)(f) + \varphi^-(x_1, y_1),\end{aligned}$$

is a Malcev algebra for all $x_1, y_1 \in P_1, x_2, y_2 \in P_2, f, g \in (P_2)^*$.

Moreover if γ is a symmetric invariant bilinear form on $P_2 \times P_2$, then the bilinear form T on $\mathcal{P}$ defined by

$$T(x_2+x_1+f, y_2+y_1+g) = \gamma(x_2, y_2)+\psi(x_1, y_1)+f(y_2)+g(x_2), \quad \forall (x_2+x_1+f), (y_2+y_1+g) \in \mathcal{P}$$

is an invariant scalar product.
The pseudo-euclidean Jordan algebra $(\mathcal{P},\star,T)$ is called the double extension of (P_1^+,ψ) by P_2^+ by means of π(see [7]).
It is easy to verify that $(\mathcal{P},[,],\star,T)$ is a MPJ-algebra if and only if the pair (π,δ) satisfies the following conditions:

$$\pi([x,y]_2)=[\delta(x),\pi(y)],\quad \pi(x)([z,t]_1)=[z,\pi(x)t]_1+(\delta(x)z)t,\quad \delta(xy)=\pi(x)\circ\delta(y)+\pi(y)\circ\delta(x),$$
$$\delta(x)\in Der(P_1^+),\quad \forall x,y\in P_2, z,t\in P_1.$$

In this case $(\mathcal{P},[,],\star,T)$ is a PEMPJ-algebra called the double extension of the PEMPJ-algebra (P_2,ψ) by the Malcev-Poisson algebra P_1 by means of the pair (π,δ).
Moreover, the vector space $\mathcal{P}$ endowed with the product:

$$x\bullet y=\frac{1}{2}[x,y]+x\star y,\quad \forall x,y\in\mathcal{P},$$

is MPJ-admissible algebra. Then $(\mathcal{P},\bullet,T)$ is a PEMPJ-admissible algebra called the double extension of the PEMPJ-admissible algebra (P_2,ψ) by the MPJ-admissible algebra P_1 by means of the pair (π,δ).

We can summarize this in the following theorem:

Theorem 4.42

Let $(\mathcal{P},\bullet,T)$ be a PEMPJ-admissible algebra.
If $\mathcal{P}=I^\perp\oplus V$, with I is an ideal totally isotropic of $\mathcal{P}$ and V is a subalgebra of $\mathcal{P}$, then $\mathcal{P}$ is a double extension of the PEMPJ-admissible algebra $(\mathcal{W}:=I^\perp/I,\psi)$ by V by means of the pair (π,δ) defined by

$$\begin{aligned}\delta:\ & V^-\longrightarrow End(\mathcal{W}^-,\psi)\\ & v\longrightarrow \delta(v)(s(i))=s([v,i])=s(v\bullet i-i\bullet v)\end{aligned}$$

$$\begin{aligned}\pi:\ & V^+\longrightarrow End(\mathcal{W}^+,\psi)\\ & v\longrightarrow \pi(v)(s(i))=s(v\star i)=\tfrac{1}{2}(v\bullet i+i\bullet v)\end{aligned}$$

$\forall v\in V$, $i\in I^\perp$.
With $I^\perp$ is the orthogonal of I relative to T, s is the canonical surjection from $I^\perp$ to $I^\perp/I$, and ψ is defined by: $\psi(s(i),s(j)):=T(i,j)$, $\forall i,j\in I^\perp$.

Chapter 5

Lie-Yamaguti algebras from MPJ-algebras

5.1 Definitions and preliminary results

In this short chapter we extend the concept of MPJ-algebras to Lie–Yamaguti algebras. We recall some definitions and results concerning Lie-Yamaguti algebras, we prove that we can construct a Lie-Yamaguti algebra from a MPJ-algebra. Moreover, we give a definition of a pseudo-euclidean Lie–Yamaguti algebra and prove that we can construct a pseudo-euclidean Lie-Yamaguti algebra from a PEMPJ-algebra. We will be concerned with finite dimensional Lie-Yamaguti algebras over a field $\mathbb{F}$ of caracteristic zero. This will be always assumed, unless otherwise stated.

Definition 5.1

A Lie-Yamaguti algebra (LY-algebra for short) is an anti-commutative algebra $\mathcal{T}$ whose multiplication is denoted by $[x, y]$ for $x, y \in \mathcal{T}$ with a trilinear operation $\{.,.,.\}$: $\mathcal{T} \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ satisfying the following conditions for $x, y, z, w, t \in \mathcal{T}$:

(i) $\{x, x, y\} = 0$,

(ii) $\sum_{(x,y,z)}([[x, y], z] + \{x, y, z\}) = 0$,

(iii) $\sum_{(x,y,z)}\{[x, y], z, w\} = 0$,

(iv) $\{x, y, [z, w]\} = [\{x, y, z\}, w] + [z, \{x, y, w\}]$,

(v) $\{x, y, \{z, t, w\}\} = \{\{x, y, z\}, t, w\} + \{z, \{x, y, t\}, w\} + \{z, t, \{x, y, w\}\}$.

Here $\sum_{(x,y,z)}$ means the cyclic sum on x, y and z. That is, LY-algebra is a synonym for a Lie triple algebra or general Lie triple system introduced by K. Yamaguti [52].

The LY-algebras with $[x, y] = 0$ for any x, y are exactly the Lie triple systems, while the LY-algebras with $\{x, y, z\} = 0$ are the Lie algebras.

Example 5.2

Let $(P, [\ ,\ ])$ be a Malcev algebra, K. Yamaguti [53] has shown that $(P, [\ ,\ ], \{.,.,.\})$ is a LY-algebra where the ternary operation is defined by:
$\{x, y, z\} = [[x, y], z] - [[y, z], x] - [[z, x], y], \quad \forall x, y, z \in P.$

Example 5.3

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra, h a subalgebra of $\mathfrak{g}$ such that there exists a subspace m of $\mathfrak{g}$ such that

$$\mathfrak{g} = m \oplus h \ \text{ (a vector space direct sum) } \text{ and } \ [h, m] \subseteq m. \tag{5.1}$$

(In this case, $(\mathfrak{g}, h)$ is called a reductive pair). Then $(m, [\ ,\ ]_m, \{.,.,.\})$ is a LY-algebra ([52]) where $\{x, y, z\} = [[x, y]_h, z]$, for all $x, y, z \in m$.
($[x, y]_m$ and $[x, y]_h$ denote the projections of $[x, y]$ on m and h respectively).

Example 5.4 (*[39]*)

Given any affinely connected and connected manifold M with parallel torsion T and curvature R, the tangent space at any point in M satisfies the above definition with $[x, y] = T(x, y)$ and $\{x, y, z\} = R(x, y)z$.

Definition 5.5

Let $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ be a LY-algebra.

(i) A subspace I of $\mathcal{T}$ is called an ideal of $\mathcal{T}$ if $[\mathcal{T}, I] \subseteq I$ and $\{\mathcal{T}, I, \mathcal{T}\} \subseteq I$.

(ii) $\mathcal{T}$ is called simple if it has no nonzero proper ideal.

(iii) An ideal I of $\mathcal{T}$ is called solvable in $\mathcal{T}$ if there is a positive integer k for which $I^{(k)} = 0$, where $I^{(0)} = I$, $I^{(1)} = [I, I] + \{\mathcal{T}, I, I\}$ and $I^{(k)} = [I^{(k-1)}, I^{(k-1)}] + \{I, I, I^{(k-1)}\} + \{\mathcal{T}, I^{(k-1)}, I^{(k-1)}\}$, For $k \geq 2$.

(iv) The radical $\mathcal{R}(\mathcal{T})$ of $\mathcal{T}$ is the unique maximal solvable ideal of $\mathcal{T}$.

(v) $\mathcal{T}$ is called semisimple if $\mathcal{R}(\mathcal{T}) = 0$.

The notion of a Lie-Yamaguti algebra is a natural abstraction made by K. Yamaguti [52] of Nomizu's considerations. Yamaguti called these systems general Lie triple systems, while Kikkawa [23] termed them Lie triple algebras. The term Lie–Yamaguti algebra, adopted here, appeared for the first time in [26]. They have been studied by several authors [24], [25], [43], [44], [45], although there is not a general structure theory. In particular, a classification of the simple Lie-Yamaguti algebras seems to be a very difficult task.

Definition 5.6

A derivation of a LY-algebra $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ is a linear mapping D of $\mathcal{T}$ into $\mathcal{T}$ such that for all $x, y, z \in \mathcal{T}$,

(i) $D(\{x, y, z\}) = \{D(x), y, z\} + \{x, D(y), z\} + \{x, y, D(z)\}$,

(ii) $D[x, y] = [D(x), y] + [x, D(y)]$.

Let $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ be a LY-algebra. For $x, y \in \mathcal{T}$, denote by $L(x, y)$ the endomorphism of $\mathcal{T}$ defined by $L(x, y)(z) = \{x, y, z\}, \forall z \in \mathcal{T}$. One can observe from (v) and (iv) of Definition 13 that $L(x, y)$ is a derivation of LY-algebra $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$.

Denote by $L(\mathcal{T},\mathcal{T})$ the vector subspace of $End(\mathcal{T})$ spanned by $\{L(x,y),\ x,y\in\mathcal{T}\}$. (Definition 5.1-(v)) is equivalent to

$$[L(x,y),L(z,t)] = L(L(x,y)(z),t) - L(z,L(y,x)(t)),\ \forall x,y,z,t\in\mathcal{T},$$

which means that $L(\mathcal{T},\mathcal{T})$ is closed under commutation, then $L(\mathcal{T},\mathcal{T})$ is a Lie subalgebra of $End(\mathcal{T})$ and called the algebra of inner derivations of the LY-algebra $\mathcal{T}$.

Let us consider the vector space direct sum $\mathcal{L}(\mathcal{T}) = \mathcal{T}\oplus L(\mathcal{T},\mathcal{T})$.
$\mathcal{L}(\mathcal{T})$ is a Lie algebra whose bracket operation is given as follows:

(i) $[x,y]_{\mathcal{L}(\mathcal{T})} := [x,y] + L(x,y),\ \forall x,y\in\mathcal{T}$,

(ii) $[d,x]_{\mathcal{L}(\mathcal{T})} := -[x,d]_{\mathcal{L}(\mathcal{T})} = dx,\ \forall d\in L(\mathcal{T},\mathcal{T}),\ \forall x\in\mathcal{T}$,

(iii) $[d_1,d_2]_{\mathcal{L}(\mathcal{T})} = d_1d_2 - d_2d_1,\ \forall d_1,d_2\in L(\mathcal{T},\mathcal{T})$.

$\mathcal{L}(\mathcal{T})$ endowed with this bracket is called the standard enveloping Lie algebra of $\mathcal{T}$. Note that the Lie algebra $L(\mathcal{T},\mathcal{T})$ becomes a subalgebra of $L(\mathcal{T})$ and $(L(\mathcal{T}),L(\mathcal{T},\mathcal{T}))$ forms a reductive pair.

Proposition 5.7

A LY-algebra $\mathcal{T}$ is simple if its standard enveloping Lie algebra $\mathcal{L}(\mathcal{T}) = \mathcal{T}\oplus L(\mathcal{T},\mathcal{T})$ is a simple Lie algebra.

Proof: Let I be an ideal of $\mathcal{T}$, if the standard enveloping Lie algebra $\mathcal{L}(\mathcal{T})$ is simple, one can check easily that,

$[\mathcal{L}(\mathcal{T}), L(\mathcal{T},I)\oplus I]_{\mathcal{L}(\mathcal{T})} \subseteq L(\mathcal{T},I)\oplus I$ since $[\mathcal{T},I]\subseteq I$ and $\{\mathcal{T},I,\mathcal{T}\}\subseteq I$. Then $L(\mathcal{T},I)\oplus I$ is an ideal of $\mathcal{L}(\mathcal{T})$, ie. $L(\mathcal{T},I)\oplus I = \mathcal{T}\oplus L(\mathcal{T},\mathcal{T}) = \mathcal{L}(\mathcal{T})$, hence $I=\mathcal{T}$ and $\mathcal{T}$ is simple. ■

The following facts has been shown in [24]:

1. If a LY-algebra $\mathcal{T}$ is solvable then its standard enveloping Lie algebra $\mathcal{L}(\mathcal{T}) = \mathcal{T}\oplus L(\mathcal{T},\mathcal{T})$ is a solvable Lie algebra and $L(\mathcal{T},\mathcal{T})$ is a solvable Lie subalgebra of $\mathcal{L}(\mathcal{T})$.
2. if $\mathcal{L}(\mathcal{T}) = \mathcal{T}\oplus L(\mathcal{T},\mathcal{T})$ is a semisimple Lie algebra then $\mathcal{T}$ is semisimple.

Definition 5.8

A Poisson-Jordan LY-algebra is a quadruplet $(P,[\ ,\ ],\circ,\{\ ,\ ,\ \})$ consisting for $\mathbb{F}$-vector space P, two bilinear map $\circ : P\times P\to P$, $[\ ,\ ] : P\times P\to P$ and a trilinear map $\{\ ,\ ,\ \} : P\times P\times P\to P$ such that:

(i) $(P,\circ)$ is Jordan algebra

(ii) $(P,[\ ,\ ],\{\ ,\ ,\ \})$ is LY-algebra;

(iii) $L(x,y)\in Der(P,\circ),\ \forall x,y\in P$.

(iv) $L_x\in Der(P,\circ),\forall x\in P$, where L_x is the left multiplication by x in the algebra $(P,[\ ,\ ])$.

Definition 5.9

(i) Let $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ be a LY-algebra. A nondegenerate symmetric bilinear form ψ on $\mathcal{T}$ is said to be invariant on the LY-algebra $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ if it satisfies:

$$\psi(R(a,b)x, y) = \psi(x, R(b,a)y)) \ \ , \ \ and \ \ \psi([a,b],x) = \psi(a,[b,x]), \ \ \forall a,b,x,y \in \mathcal{T},$$

where $R(a,b)x = \{x,b,a\}$. Such a form is called an invariant scalar product on $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$. In this case $(\mathcal{T}, [\ ,\ ], \{.,.,.\}, \psi)$ is called a pseudo-euclidean LY-algebra.

(ii) Let $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \})$ be a Poisson-Jordan LY-algebra and ψ be a bilinear form on P, then $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \}, \psi)$ is said to be pseudo-euclidean Poisson-Jordan LY-algebra if $(P, [\ ,\ ], \{\ ,\ ,\ \}, \psi)$ is a pseudo-euclidean LY-algebra and $(P, \circ)$ is a pseudo-euclidean Jordan algebra.

Remark 5.10

Let ψ be an invariant form on a LY-algebra $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ then,

$$\forall a,b,x,y \in \mathcal{T}, \ \ \psi(L(a,b)x,y) = \psi(x, L(b,a)y).$$

Proof: Using the equality $L(a,b)x = R(a,b)x - R(b,a)x - J(a,b,x)$ (Definition 13-(ii)), we obtain

$$\begin{aligned}\psi(L(a,b)x,y) &= \psi(R(a,b)x,y) - \psi(R(b,a)x,y) - \psi(J(a,b,x),y)\\ &= \psi(x, R(b,a)y) - \psi(x, R(a,b)y) - \psi(x, J(b,a,y))\\ &= \psi(x, L(b,a)y).\end{aligned}$$

■

Proposition 5.11

Every nonabelian LY-algebra $\mathcal{T}$ with $\mathcal{T} \neq \mathcal{T}^{(1)} = [T,T] + \{T,T,T\}$ admits a non-vanishing invariant symmetric bilinear form.

Proof: Let $\{t_1, t_2, ., ., ., t_n\}$ be a basis of $\mathcal{T}$ such that $\{t_1, t_2, ., ., ., t_r\}$ $(r < n)$ is a basis of $[\mathcal{T},\mathcal{T}] + \{\mathcal{T},\mathcal{T},\mathcal{T}\}$. Define the bilinear form Ψ on $\mathcal{T}$ by

$$\Psi(\sum_{i=1}^{n} \lambda_i t_i, \sum_{i=1}^{n} \mu_i t_i) = \lambda_n \mu_n.$$

Ψ is symmetric and $\Psi(\{\mathcal{T},\mathcal{T},\mathcal{T}\},\mathcal{T}) = \Psi(\mathcal{T},\{\mathcal{T},\mathcal{T},\mathcal{T}\}) = 0 = \Psi([\mathcal{T},\mathcal{T}],\mathcal{T}) = \Psi(\mathcal{T},[\mathcal{T},\mathcal{T}])$. Then Ψ is invariant on $\mathcal{T}$. ■

Remark 5.12

Every solvable LY-algebra admits a non-vanishing invariant symmetric bilinear form.

M. Kikkawa [25] has defined the Killing-Ricci form β on a LY-algebra $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$,

$$\beta(x,y) = tr(L_x L_y) + tr(r(x,y) + r(y,x)),$$

where L_x and $r(x,y)$ are endomorphisms of $\mathcal{T}$ given by:

$$L_x(y) = [x,y], \; r(x,y)z = L(z,x)y, \; \text{ for } \; x,y,z \in \mathcal{T}.$$

This form is bilinear and symmetric. Let γ be a trilinear form on $\mathcal{T}$ given by

$$\gamma(x,y,z) = tr(L(x,y)L_z), \; \text{for} \; x,y,z \in \mathcal{T}. \tag{5.2}$$

It is evident that $\gamma = 0$ if $\mathcal{T}$ is reduced to Lie algebra or reduced to Lie triple system. M. Kikkawa [25] has shown that under the condition $\gamma = 0$, we have:

(i) The Killing-Ricci form β is invariant.

(ii) The Killing-Ricci form β is nondegenerate if and only if the standard enveloping Lie algebra $\mathcal{L}(\mathcal{T}) = \mathcal{T} \oplus L(\mathcal{T},\mathcal{T})$ is a semisimple Lie algebra.

(iii) If the Killing-Ricci form β is nondegenerate. Then $\mathcal{T}$ is a semi-simple Lie triple algebra with $\mathcal{T} = \mathcal{T}^{(1)} = [\mathcal{T},\mathcal{T}] + \{\mathcal{T},\mathcal{T},\mathcal{T}\}$, and $\mathcal{T}$ is decomposed into a direct sum of simple LY-algebra ideals.

Proposition 5.13

Let $(P,[\;,\;],\circ)$ be a MPJ-algebra, then the quadruplet $(P,[\;,\;],\circ,\{\;,\;,\;\})$ is a Poisson-Jordan LY-algebra where $\{\;,\;,\;\} : P \times P \times P \longrightarrow P$ defined by

$$\{x,y,z\} = [[x,y],z] - [[y,z],x] - [[z,x],y], \; \forall x,y,z \in P.$$

Proof: Let $x,y,z,t \in P$,

$$\begin{aligned}
\{x,y,z\circ t\} &= [[x,y],z\circ t] - [[y,z\circ t],x] - [[z\circ t,x],y] \\
&= [[x,y],t]\circ z + [[x,y],z]\circ t - [[y,z]\circ t,x] - [[y,t]\circ z,x] - [z\circ[t,x],y] - [[z,x]\circ t,y] \\
&= [[x,y],t]\circ z - [[y,t],x]\circ z - [y,t]\circ[z,x] - z\circ[[t,x],y] - [t,x]\circ[z,y] \\
&+ [[x,y],z]\circ t - [[y,z],x]\circ t - [y,z]\circ[t,x] - [[z,x],y]\circ t - [z,x]\circ[t,y] \\
&= [[x,y],t]\circ z - [[y,t],x]\circ z - z\circ[[t,x],y] + [[x,y],z]\circ t - [[y,z],x]\circ t - [[z,x],y]\circ t \\
&= z\circ\{x,y,t\} + \{x,y,z\}\circ t,
\end{aligned}$$

then, $L(x,y) \in Der(P,\circ)$. Next, since $(P,[\;,\;])$ is a Malcev algebra, then $(P,[\;,\;],\{\;,\;,\;\})$ is a LY-algebra (Example 5.2) ■

Proposition 5.14

Let $(P,[\;,\;],\{\;,\;,\;\})$ be the LY-algebra constructed from a MPJ-algebra $(P,[\;,\;],\circ)$. If $(P,[\;,\;],\{\;,\;,\;\})$ is simple, then $(P,[\;,\;],\circ)$ is simple.

Proof: Let I be an ideal of $(P,[\;,\;],\circ)$, then $[P,I] \subseteq I$ and $P\circ I \subseteq I$. Moreover, one can check that $\{P,I,P\} \subseteq I$. This proves that I is an ideal of $(P,[\;,\;],\{\;,\;,\;\})$, then $I = P$. ■

In this case, the fact that $(P,[\;,\;],\{\;,\;,\;\})$ is simple, this implies that $(P,[\;,\;])$ is simple, then the MPJ structure on $(P,[\;,\;])$ is trivial.

The following Proposition proves that from a PEMPJ-algebra one can construct a pseudo-euclidean Poisson-Jordan LY-algebra and a pseudo-euclidean LY-algebra.

Proposition 5.15

Let $(P, [\ ,\ ], \circ, \psi)$ be a PEMPJ-algebra, then $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \}, \psi)$ is a pseudo-euclidean Poisson-Jordan LY-algebra and $(P, [\ ,\ ], \{\ ,\ ,\ \}, \psi)$ is a pseudo-euclidean LY-algebra where the trilinear product $\{\ ,\ ,\ \}$ is defined in the Proposition 5.13.

Proof: Let $a, b, c, d \in P$.

$$\begin{aligned}\psi(\{c,b,a\}],d) &= \psi([[c,b],a],d) - \psi([[b,a],c],d) - \psi([[a,c],b],d)\\ &= \psi([c,b],[a,d]) - \psi([b,c],[c,d]) - \psi([a,c],[b,d])\\ &= \psi(c,[[d,a],b] - [[a,b],d] - [[b,d],a])\\ &= \psi(c,\{d,a,b\}).\end{aligned}$$

This means that

$$\psi(R(a,b)c,d) = \psi(c,R(b,a)d).$$

■

5.2 LY-algebras with a unique quadratic structure

In this section, we study LY-algebras admitting a unique, up to a constant, invariant scalar product. We prove that any LY-algebra $\mathcal{T}$ over $\mathbb{F}$ with $\gamma = 0$, admitting a unique, up to a constant, invariant scalar product such that the Killing-Ricci form on $\mathcal{T}$ is not trivial, is necessarily a simple LY-algebra. If the field $\mathbb{F}$ is algebraically closed and $\mathcal{T}$ is irreducible such that $\mathcal{T}$ and $L(\mathcal{T},\mathcal{T})$ are not isomorphic as $L(\mathcal{T},\mathcal{T})$-modules, then $\mathcal{T}$ admits a unique, up to a constant, an invariant scalar product.

Definition 5.16

A LY-algebra $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ is said to be irreducible if $\mathcal{T}$ is an irreducible module for its Lie algebra of inner derivations $L(\mathcal{T},\mathcal{T})$.

Remark 5.17

If $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ is irreducible, then $\mathcal{T}$ is simple.

Proof: Let I be an ideal of $\mathcal{T}$, one can check easily (Definition 13-(ii))) that I is a $L(\mathcal{T},\mathcal{T})$-module, then $I = \mathcal{T}$. ■

The LY-algebras which are irreducible are the algebraic counterparts of the isotropy irreducible homogeneous spaces studied by J.A. Wolf in [51]. These LY-algebras splits into three disjoint types: adjoint type, non-simple type and generic type. The systems of the first two types were classified in [9] through a generalized Tits Construction of Lie algebras. The LY-algebras of generic type are classified in [10] by relating them to several other nonassociative algebraic systems: Lie and Jordan algebras and triple systems, Jordan pairs or Freudenthal triple systems.

Now, let us denote by $\mathcal{F}(\mathcal{T})$ the linear space of all symmetric invariant bllinear forms on $\mathcal{T}$ and let $\mathcal{B}(\mathcal{T})$ be the subspace of $\mathcal{F}(\mathcal{T})$ spanned by the set of invariant scalar products on $\mathcal{T}$.

Lemma 5.18

If $\mathcal{T}$ is a LY-algebra admitting an invariant scalar product, then $\mathcal{B}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$.

Proof: The proof of this Lemma is analogously to the one in Lie algebra case (Lemma 2.1 in [6]). ■

Theorem 5.19

Let ψ be an invariant scalar product on a LY-algebra $\mathcal{T}$. There is an invariant scalar product Ψ on the standard enveloping Lie algebra $\mathcal{L}(\mathcal{T}) = \mathcal{T} \oplus L(\mathcal{T},\mathcal{T})$, defined by:

$$\Psi_{|\mathcal{T}} = \psi, \quad \Psi(L(\mathcal{T},\mathcal{T}),\mathcal{T}) = 0, \quad \Psi(L(a,b), L(x,y)) = \psi(L(a,b)x, y), \quad \forall a,b,x,y \in \mathcal{T}$$

Proof: For $a,b,x,y \in \mathcal{T}$, we have

$$\begin{aligned}
\Psi(L(a,b), L(x,y)) &= \psi(L(a,b)x, y) \\
&= \psi(x, L(b,a)y) \\
&= \psi(x, R(y,a)b) \\
&= \psi(R(a,y)x, b) \\
&= \psi(L(x,y)a, b) \\
&= \Psi(L(x,y), L(a,b)).
\end{aligned}$$

This shows that the form Ψ is symmertic on $L(\mathcal{T},\mathcal{T})$.

Let us verify that Ψ is well defined on $L(\mathcal{T},\mathcal{T})$.
If $L(x,y) = L(x',y'),\ L(a,b) = L(a',b')$, for $a,b,a',b',x,y,x',y' \in \mathcal{T}$.

We use the fact that,

$$\psi(L(a,b)x,y) = \psi(R(x,b)a,y) = \psi(a, R(b,x)y) = \psi(a, L(y,x)b) = \psi(L(x,y)a,b).$$

Then,

$$\begin{aligned}
\Psi(L(a,b), L(x,y)) &= \psi(L(a,b)x, y) \\
&= \psi(L(a',b')x, y) \\
&= \psi(L(x,y)a', b') \\
&= \psi(L(x',y')a', b') \\
&= \Psi(L(x',y'), L(a',b')) \\
&= \Psi(L(a',b'), L(x',y')).
\end{aligned}$$

Now, we need to check that the symmertic bilinear form Ψ is invariant on $\mathcal{L}(\mathcal{T})$,i.e.,

$$\Psi([X,Y]_{\mathcal{L}(\mathcal{T})}, Z) = \Psi(X, [Y,Z]_{\mathcal{L}(\mathcal{T})}), \text{ for all } X,Y,Z \in \mathcal{L}(\mathcal{T}).$$

Let $X = x + d_1,\ Y = y + d_2,\ Z = z + d_3,\ x,y,z \in \mathcal{T}, d_1,d_2,d_3 \in L(\mathcal{T},\mathcal{T})$.

In one hand,

$$\begin{aligned}
\Psi([X,Y]_{\mathcal{L}(\mathcal{T})}, Z) &= \Psi([x+d_1, y+d_2]_{\mathcal{L}(\mathcal{T})}, z+d_3) \\
&= \Psi([x,y] + L(x,y) + [d_1,d_2] + d_1(y) - d_2(x), z+d_3) \\
&= \Psi([x,y],z) + \Psi(d_1(y),z) + \Psi(L(x,y),d_3) + \Psi([d_1,d_2],d_3) - \Psi(d_2(x),z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\Psi(X,[Y,Z]_{\mathcal{L}(\mathcal{T})}) &= \Psi(x+d_1,[y+d_2,z+d_3]_{\mathcal{L}(\mathcal{T})})\\ &= \Psi(x+d_1,[y,z]+L(y,z)+[d_2,d_3]+d_2(z)-d_3(y))\\ &= \Psi(x,[y,z])+\Psi(x,d_2(z))+\Psi(d_1,L(y,z))+\Psi(d_1,[d_2,d_3])-\Psi(x,d_3(y))\\ &= \Psi([x,y],z)-\Psi(d_2(x),z)+\Psi(d_1(y),z)+\Psi(d_1,[d_2,d_3])+\Psi(L(x,y),d_3).\end{aligned}$$

Since Ψ is invariant on $L(\mathcal{T},\mathcal{T})$ (see [56], Lemma 3.2), then we get

$$\Psi([X,Y]_{\mathcal{L}(\mathcal{T})},Z)=\Psi(X,[Y,Z]_{\mathcal{L}(\mathcal{T})}).$$

It is clear that Ψ is nondegenerate on $L(\mathcal{T},\mathcal{T})$, then Ψ is nondegenerate on $\mathcal{L}(\mathcal{T})$, since $\Psi(L(\mathcal{T},\mathcal{T}),\mathcal{T})=0$. This proves that Ψ is an invariant scalar product on $\mathcal{L}(\mathcal{T})$. ■

Remark 5.20

If we suppose that Ψ' is another invariant scalar product on $\mathcal{L}(\mathcal{T})$ satisfying,

$$\Psi'_{|\mathcal{T}}=\psi,\quad \Psi'(L(\mathcal{T},\mathcal{T}),\mathcal{T})=0.$$

Then, we get

$$\Psi'(L(x,y),d)=\Psi'([x,y]_{\mathcal{L}(\mathcal{T})}-[x,y],d)=-\Psi'(x,dy)=\psi(dx,y)=\Psi(L(x,y),d),$$

for all $x,y\in\mathcal{T},d\in L(\mathcal{T},\mathcal{T})$, so $\Psi'=\Psi$. Consequently, there is a unique invariant scalar product Ψ on $\mathcal{L}(\mathcal{T})$ satisfying, $\Psi_{|\mathcal{T}}=\psi,\quad \Psi(L(\mathcal{T},\mathcal{T}),\mathcal{T})=0$.

Corollary 5.21

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and let $\mathcal{T}$ be an irreducible LY-algebra such that $\mathcal{T}$ and $L(\mathcal{T},\mathcal{T})$ are not isomorphic as $L(\mathcal{T},\mathcal{T})$-modules. If ψ_1,ψ_2 are two invariant scalar products on $\mathcal{T}$, then there is a nonzero scalar λ such that $\psi_1=\lambda\psi_2$.

Proof: By ([9], Theorem 2.1), the standard enveloping Lie algebra $\mathcal{L}(\mathcal{T})$ is a simple Lie algebra. By theorem 5.19 there is a unique invariant scalar product Ψ_i on $\mathcal{L}(\mathcal{T})$ extending ψ_i for $i=1,2$. By using ([6], corollary 3.1) there is a nonzero scalar λ such that

$$\Psi_1(X,Y)=\lambda\Psi_2(X,Y),\ \forall X,Y\in\mathcal{L}(\mathcal{T}).$$

Then, we get

$$\psi_1(x,y)=(\Psi_1)_{|\mathcal{T}}(x,y)=\lambda(\Psi_2)_{|\mathcal{T}}(x,y)=\lambda\psi_2(x,y),\ \forall x,y\in\mathcal{T}.$$

■

Theorem 5.22

Let $\mathcal{T}$ be a nonabelian LY-algebra such that $\gamma=0$ and the Killing-Ricci form on $\mathcal{T}$ is not trivial. If dim $\mathcal{B}(\mathcal{T})=1$ then $\mathcal{T}$ is simple.
(Where γ is the trilinear form defined in (Eq. 5.2)).

Proof: By Lemma 5.18, we deduce that dim $\mathcal{F}(\mathcal{T})=1$, hence every nonzero symmetric invariant bilinear form on $\mathcal{T}$ is nondegenerate. This implies that the Killing-Ricci form of $\mathcal{T}$ is nondegenerate and $\mathcal{T}$ is semisimple Lie triple algebra. Let $\mathcal{T}=\mathcal{T}_1\oplus...\oplus\mathcal{T}_n$ be the decomposition of $\mathcal{T}$ into the direct sum of simple ideals. If β_1 denotes the Killing-Ricci form of $\mathcal{T}_1$ then the bilinear form ϕ on $\mathcal{T}$ defined by $\phi(x, y) = \beta_1(x, y)$ whenever $x, y \in \mathcal{T}_1$ and $\phi(x, y) = 0$ otherwise, is a degenerate invariant symmetric bilinear form, which contradicts the result in Lemma 5.18. ■

Conclusion and perspectives

This thesis has contributed to:

- Construct Lie triple systems with invariant scalar products and $\mathbb{Z}_2$-graded quadratic Lie algebras from pseudo-euclidean Malcev algebras.
- Prove that the second cohomology group $H(A, \mathbb{F})$ can be interpreted as the set of classes of one dimensionel central extensions of the alternative algebra A.
- Describe inductively the pseudo-euclidean alternative algebras by using:

  -The notion of double extension of alternative algebras-

  -The notion of generalized double extension of pseudo-euclidean alternative algebras-
- Construct generalized triple systems from MPJ-admissible algebras with an additional condition
- Give explicitly all MPJ structures on reductive Malcev algebras over an algebraically closed field with characteristic zero.
- Prove how from a PEMPJ-algebra one can construct non-trivial examples of quadratic Lie algebras by using the notion of double extension for quadratic Lie algebras
- Give interesting constructions of pseudo-euclidean Malcev algebra related to PEMPJ-algebras by using the concept of double extension in the case of pseudo-euclidean Malcev algebras
- Give an inductive description of nilpotent PEMPJ-admissible algebras by using the concept of double extension of PEMPJ-admissible algebras by the one-dimensional algebra with null product
- Construct Lie-Yamaguti algebras from a MPJ-algebras.
- Construct pseudo-euclidean Lie-Yamaguti algebras from PEMPJ-algebras.
- Prove that any LY-algebra with an additional condition, admitting a unique, up to a constant, invariant scalar product is necessarily a simple LY-algebra.
- Prove that if the field F is algebraically closed and $\mathcal{T}$ is an irreducible Lie-Yamaguti algebra such that $\mathcal{T}$ and $L(\mathcal{T}, \mathcal{T})$ are not isomorphic as $L(\mathcal{T}, \mathcal{T})$-modules, then $\mathcal{T}$ admits a unique, up to a constant, invariant scalar product.

We will be interested in the future:

- To describe inductively the non nilpotent PEMPJ-admissible algebras.

- To characterize Malcev algebras admitting a unique invariant scalar product.
- To study pseudo-euclidean Lie-Yamaguti algebras and to give The links between this and irreducible homogeneous spaces.
- To study Poisson-Jordan Lie-Yamaguti algebras.

Version abrégée en français

Algèbres Alternatives pseudo-Euclidiennes et Algèbres de Malcev-Poisson-Jordan

Résumé

Le but de cette thèse est d'étudier certaines algèbres non associatives qui sont munies d'un produit scalaire invariant. Ces algèbres sont appelées pseudo-euclidiennes ou quadratiques.

Dans la première partie de cette thèse, nous avons transféré la notion de double extension, introduite par Medina et Revoy pour les algèbres de Lie quadratiques (voir [35]) et prolongée par Benayadi et Baklouti pour les algèbres de Jordan pseudo-euclidiennes (voir [7],[5]), aux cas des algèbres alternatives pseudo-euclidiennes. Nous avons montré que toute algèbre alternative pseudo-euclidienne, étant irréductible, ni simple ni nilpotente, est une double extension. Aussi, nous avons introduit la notion de double extension généralisée des algèbres alternatives pseudo-euclidiennes par l'algèbre alternative de dimension 1 à produit nul. Cela conduit à une classification inductive d'algèbres alternatives pseudo-euclidiennes nilpotentes.

Dans la deuxième partie de cette thèse, nous avons introduit la notion d'algèbre de Malcev-Poisson-Jordan (algèbre-MPJ), qui est définie comme étant un espace vectoriel muni d'un crochet de Malcev et d'une structure de Jordan qui sont lié par l'identité de Leibniz. Nous avons décrit ces algèbres en termes d'une seule opération bilinéaire. Cette classe contient strictement les algèbres alternatives. Pour une algèbre de Malcev $(P, [\ ,\ ])$, il est intéressant de classifier la structure de Jordan $\circ$ sur l'espace vectoriel sous-jacent de P telle que $(P, [\ ,\ ], \circ)$ soit une algèbre-MPJ ($\circ$ est appelée une structure de MPJ sur l'algèbre de Malcev $(P, [\ ,\ ])$. Nous avons donné explicitement toutes les structures MPJ sur quelques classes intéressantes d'algèbres de Malcev. Ensuite, nous avons introduit la notion d'algèbres-MPJ pseudo-euclidiennes (algèbres-PEMPJ) et montré comment construire des algèbres de Lie quadratiques et des algèbres de Malcev (non Lie) pseudo-euclidiennes intéressantes à partir d'algèbres-PEMPJ. Finalement, nous avons donné une description inductive des algèbres-PEMPJ nilpotentes.

Chapitre 1

Algèbres alternatives pseudo-euclidiennes et algèbres de Lie quadratiques $\mathbb{Z}_2$-graduées

Toutes les algèbres et les espaces vectoriels sont supposés de dimensions finies. Ils sont définis sur un corps $\mathbb{F}$ commutatif de caractéristique 0.

1.0.1 Premières définitions

Dans ce paragraphe, nous rappelons quelques définitions et concepts de la théorie des algèbres non associatives et en particulier les algèbres alternatives. La plupart de ces concepts peuvent être consultés dans ([46]), ([13]) et ([57]).

1. Une algèbre est un espace vectoriel A sur un corps $\mathbb{F}$ muni d'une application bilinéaire $(x, y) \to xy$ de $A \times A \to A$.
   Pour $x \in A$, on note L_x(resp. R_x) la multiplication à gauche (resp. la multiplication à droite) de $A \to A : y \to xy$ (resp. $y \to yx$)

2. Un sous-espace vectoriel I de A est dit un idéal (resp. une sous-algèbre) si :
   $AI + IA \subset I$(resp. $II \subset I$)

3. Soient A et B deux algèbres sur un corps $\mathbb{F}$. Un homomorphisme f de A dans B est une application linéaire $A \to B$ vérifiant : $f(xy) = f(x)f(y)$, $\forall x, y \in A$.

4. Soit A une algèbre non associative. On définit la série des dérivées de A par :
   $A^{(1)} = A, \quad A^{(i+1)} = A^{(i)}A^{(i)}, \quad \forall i \in \mathbb{N}^*$.
   On dit que A est résoluble s'il existe $r \in \mathbb{N}^*$ tel que $A^{(r)} = \{0\}$.

5. Soit A une algèbre non associative, on pose $A^1 := A$, $A^2 := A^1A^1$, $A^3 := (A^2A) + A(A^2)$, $A^4 := A(A^3) + (A^2)(A^2) + (A^3)A$ etc.
   Les A^i, $i \in \mathbb{N}^*$ sont des idéaux de A. A est dite nilpotente s'il existe un entier $n \in \mathbb{N}^*$ tel que $A^n = \{0\}$. Comme $A^{(k)} \subset A^{2^k}$ alors toute algèbre nilpotente est résoluble.

6. Soit A une algèbre non associative sur un corps $\mathbb{F}$. La somme de deux idéaux résolubles de A est résoluble. Cela entraine l'existence d'un unique idéal résoluble qui est maximal. Cet unique idéal sera appelé le radical de A.

7. Soit A une algèbre non associative sur un corps $\mathbb{F}$, A est dite semi-simple (resp. simple) si son radical est nul (resp. Si $AA \neq \{0\}$ et les seuls idéaux de A sont $\{0\}$ et A).

1.0.2 Algèbres alternatives

Soit A une algèbre non associative (pas forcemment associative). A est dite alternative si :

$$x^2y = x(xy) \quad et \quad yx^2 = (yx)x, \ \forall x, y \in A.$$

Equivaut à dire en termes d'associateurs :

$$(x, x, y) = (y, x, x) = 0; \forall x, y \in A.$$

Equivaut à dire en termes des multiplications à gauche et à droite :

$$L_{x^2} = L_x^2 \quad et \quad R_{x^2} = R_x^2, \ \ \forall x \in A.$$

Exemples 1.1

1. *L'algèbre des octonions est une algèbre alternative de dimension 8 et qui est non associative.*
2. *Toute algèbre de composition est alternative. (On rappelle qu'une algèbre A est dite de composition si elle est munie d'une forme quadratique q telle que :*
   - *$q(xy) = q(x)q(y)$,*
   - *La forme $q(x)$ est non dégénérée,*
   - *Il existe un élément neutre 1 dans A).*
3. *Toute extension scalaire $U_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{F}} U$ d'une algèbre alternative U est alternative ($\mathbb{K}$ une extension de $\mathbb{F}$)*

Remarque 1

L'associateur est "alterné" dans le sens que pour tout $\sigma \in S_3$ et tous $x_1, x_2, x_3 \in A$

$$(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = (sign\sigma)(x_1, x_2, x_3).$$

Soit A une algèbre alternative, alors $(x, y, x) = -(y, x, x), \forall x, y \in A$.
Autrement dit $(xy)x = x(yx), \forall x, y \in A$; ou bien $L_x R_x = R_x L_x, \forall x \in A$.
L'identité $(xy)x = x(yx)$ est appelée "loi flexible".
Les algèbres de Lie, de Jordan ainsi que les algèbres alternatives sont flexibles.
Nous aurons l'occasion d'utiliser les identitées de Moufang ci-dessous :

$$(xax)y = x[a(xy)] \tag{1.1}$$

$$y(xax) = [(yx)a]x \tag{1.2}$$

$$(xy)(ax) = x(ya)x \tag{1.3}$$

Pour tous x, y, a dans une algèbre alternative A où $xax = (xa)x = x(ax)$.
L'identité de moufang (2) est équivalente à $(y, xa, x) = -(y, x, a)x \quad \forall x, y, a \in A$.
En effet :

$$(y, xa, x) = [y(xa)]x - y(xax) = [y(xa)]x - [(yx)a]x = [y(xa) - (yx)a]x = -(y, x, a)x$$

Théorème 1

Soit A une algèbre alternative. Toute sous-algèbre de A engendrée par deux élements est associative.

Définition 1

Soit A une algèbre. A est à puissance associative si pour tout $x \in A$, la sous-algèbre $\mathbb{F}[x]$ engendrée par x est associative. Cela signifie que si un élément x est multiplié par lui même plusieurs fois, l'ordre dans lequel sont effectuées ces multiplications n'a pas d'importance.
Exemple : $x(x(xx)) = (x(xx))x = (xx)(xx)$.
C'est à dire $\forall x \in A$ on définit : $x^1 := x$, $x^{i+1} := xx^i$ et on a : $x^i x^j = x^{i+j}, \ \forall i, j \in \mathbb{N}^*$.

Remarque 2

D'aprés le théorème d'Artin [46] , toute algèbre alternative est à puissance-associative.

Définition 2

Soit A une algèbre à puissance-associative. Un élément x de A est dit nilpotent s'il existe un entier r tel que $x^r = 0$.
Une algèbre (resp. idéal) qui contient seulement des éléments nilpotents est dite nilalgèbre (resp. nilidéal).

Théorème 2

Toute nilalgèbre alternative A est nilpotente.(voir ([46])

Remarque 3

Dans une algèbre alternative, les concepts : nilpotente, résoluble et nilalgèbre sont identiques.

1.0.3 Algèbres alternatives pseudo-euclidiennes

Définition 3

Soit A une algèbre alternative munie d'une forme bilinéaire symétrique non dégénérée ψ vérifiant :

$$\psi(ab, c) = \psi(a, bc) \quad \forall a, b, c \in A$$

Une telle forme sera dite invariante ou associative.
La paire (A, ψ) est dite algèbre alternative pseudo-euclidienne.
ψ est dans ce cas un produit scalaire invariant sur A.

Remarque 4

Si (A, ψ) est une algèbre alternative pseudo-euclidienne alors :
$\psi((x, y, z), t) = -\psi(x, (y, z, t)), \ \forall x, y, z, t \in A$ où $(x, y, z) = (xy)z - x(yz)$.

Proposition 1.2

Soit (A, ψ) une algèbre pseudo-euclidienne, alors les propositions suivantes sont équivalentes :

- *(i) A est alternative,*
- *(ii) $(x, x, y) = 0$, $\forall x, y \in A$,*
- *(iii) $(y, x, x) = 0$, $\forall x, y \in A$.*

La preuve c'est de remarquer que $\psi((y, x, x), t) = -\psi((y, (x, x, t)), \ \forall x, y, t \in A$.

Définition 4

Une algèbre de Malcev M est une algèbre anticommutative et qui vérifie l'identité de Malcev suivante :

$$J(x, y, xz) = J(x, y, z)x, \ \forall x, y, z \in M.$$

Où $J(x, y, z) = (xy)z + (yz)x + (zx)y$.

Les deux propositions suivantes démontrent la relation entre une algèbre alternative et une algèbre de Malcev :

Proposition 1.3

Une algèbre alternative est une algèbre de Malcev-admissible, (ie.) son commutateur définit une algèbre de Malcev. Autrement dit si $(A, .)$ est une algèbre alternative alors $A^- := (A, [\ ,\ ])$ est une algèbre de Malcev. Où $[x, y] = x.y - y.x$.

Proposition 1.4

Si $(A, .)$ est une algèbre alternative alors $A^+ := (A, \circ)$ est une algèbre de Jordan, Où $x \circ y := x.y + y.x$, $\forall x, y \in A$. On rappelle qu'une algèbre de Jordan J est une algèbre commutative (non nécessairement associative) qui vérifie la relation suivante :

$$x(yx^2) = (xy)x^2, \quad \forall x, y \in J.$$

Remarque 5

Si (A, ψ) est une algèbre alternative pseudo-euclidienne alors (A^-, ψ) est une algèbre de Malcev pseudo-euclidienne et (A^+, ψ) est une algèbre de Jordan pseudo-euclidienne.

Définition 5

Soit (A, ψ) une algèbre alternative pseudo-euclidienne, I un sous-espace vectoriel de A.
Un idéal I de A est dit non dégénéré si la restriction de ψ sur I est non dégénérée sinon il est dit dégénéré.
On dit que (A, ψ) est irréductible si tout idéal de A est dégénéré.

Lemme 1.5

Soit (A, ψ) une algèbre alternative pseudo-euclidienne et I un idéal de A, alors :
$I^\perp$ est un idéal de A et on a $I(I^\perp) = (I^\perp)I = \{0\}$.
Si I est non dégénéré alors $A = I \oplus I^\perp$ et $I^\perp$ est nondégénéré.

Maintenant, nous abordons la relation entre les algèbres alternatives et les systèmes triples de Lie. Nous montrons qu'à partir d'une algèbre de Malcev pseudo-euclidienne nous pouvons construire un système triple de Lie muni d'un produit scalaire invariant et une algèbre de Lie quadatique $\mathbb{Z}_2$-graduée. Nous commençons par la proposition suivante :

Proposition 1.6

(voir [32])
Soit $(M, [\ ,\ ])$ une algèbre de Malcev. La paire $(M, [.,.,.])$ où $[.,.,.] : M\times M\times M \longrightarrow M$ définie par $[a,b,c] = 2[[a,b],c] - [[b,c],a] - [[c,a],b]$, $\forall a,b,c \in M$ est un système triple de Lie.

Corollaire 1

Si $(A,.)$ est une algèbre alternative ; la paire $(A, [.,.,.])$ où

$$\begin{aligned}[.,.,.] :\ & A \times A \times A \longrightarrow A \\ & (a,b,c) \longmapsto 3((ab)c + (cb)a - c(ab) - b(ac)),\ \forall a,b,c \in A\end{aligned}$$

est un système triple de Lie.

Définition 6

Soit $(V, [.,.,.])$ un système triple de Lie. Une forme bilinéaire symétrique nondégénérée ψ sur V est dite invariante sur le système triple de Lie $(V, [.,.,.])$ s'elle vérifie :

$$\psi(R(a,b)c,d) = \psi(c, R(b,a)d),\ for\ \ all\ a,b,c,d \in V.$$

Une telle forme est appelée un produit scalaire invariant sur $(V, [.,.,.])$.

Proposition 1.7

Soit $(M, [\ ,\ ], \psi)$ une algèbre de Malcev pseudo-euclidienne, alors ψ est un produit scalaire invariant sur le système triple de Lie $(M, [.,.,.])$ obtenu par la Proposition 1.6.

Démonstration : Soient $a,b,c,d \in M$.

$$\begin{aligned}\psi([a,b,c],d) &= 2\psi([[a,b],c],d) - \psi([[b,c],a],d) - \psi([[c,a],b],d) \\ &= 2\psi([a,b],[c,d]) - \psi([b,c],[a,d]) - \psi([c,a],[b,d]) \\ &= 2\psi([[b,a],d],c) - \psi([[a,d],b],c) - \psi([[d,b],a],c) \\ &= \psi(c, 2[[b,a],d] - [[a,d],b] - [[d,b],a]) \\ &= \psi(c,[b,a,d]).\end{aligned}$$

Equivalent à,

$$\psi(R(a,b)c,d) = \psi(c, R(b,a)d).$$

■

Définition 7

Soit (g,ψ) une algèbre de Lie quadratique, (g,ψ) est dite $\mathbb{Z}_2$-graduée s'il existe deux sous-espaces $g_{\bar{0}}$ et $g_{\bar{1}}$ de g vérifiant :

$$g = g_{\bar{0}} \oplus g_{\bar{1}}, \quad [g_{\bar{i}}, g_{\bar{j}}] \subset g_{\overline{i+j}},\ \forall i,j \in \mathbb{Z}_2\ et\ \ \psi(g_{\bar{0}}, g_{\bar{1}}) = \{0\}.$$

La proposition suivante montre qu'à partir d'un système triple de Lie muni d'un produit scalaire invariant on peut construire une algèbre de Lie quadatique $\mathbb{Z}_2$-graduée.

Proposition 1.8

Soit q un système triple de Lie, le plongement standard $g(q)$ est une algèbre de Lie $\mathbb{Z}_2$-graduée, où $g(q)_{\bar{0}} = R(q,q)$, $g(q)_{\bar{1}} = q$. De plus, la forme bilinéaire $\widetilde{\psi}$ sur $g(q) \times g(q)$ définie par :

$$\begin{aligned} \widetilde{\psi}_{|q\times q} &= \psi, \\ \widetilde{\psi}(R(q,q),q) &= \widetilde{\psi}(q, R(q,q)) = \{0\}, \\ \text{et } \widetilde{\psi}(D, \sum_{i=1}^{n} R(x_i,y_i)) &= \sum_{i=1}^{n} \psi(Dx_i, y_i), \qquad \forall x_i, y_i \in q, \quad D \in R(q,q), \end{aligned}$$

est un produit scalaire invariant sur $g(q)$, (où ψ est un produit scalaire invariant sur q).

Corollaire 2

$(g(q), \widetilde{\psi})$ est une algèbre de Lie quadratique $\mathbb{Z}_2$-graduée.

Chapitre 2

Double extension des algèbres alternatives

Dans ce chapitre nous introduisons la notion de double extension pour les algèbres alternatives. Nous commençons par définir deux types d'extensions, le produit semi-direct et l'extension centrale des algèbres alternatives. Nous donnons une description inductive des algèbres alternatives pseudo-euclidiennes.

2.1 Produit semi-direct d'algèbres alternatives

Soit A une algèbre alternative sur un corps $\mathbb{F}$, M un $\mathbb{F}$-espace vectoriel. M est un bimodule de A s'il existe deux applications linéaires $\pi : A \to End(M)$, $\Pi : A \to End(M)$ qui vérifient les relations suivantes :

(i) $\pi(a^2) = (\pi(a))^2$, (.i.e) $\pi(a)o\pi(a) = \pi(a^2)$

(ii) $\Pi(a^2) = (\Pi(a))^2$,

(iii) $\Pi(a')\pi(a) - \pi(a)\Pi(a') = -(\Pi(a')\Pi(a) - \Pi(aa'))$ (.i.e) $\Pi(aa') - \Pi(a')\Pi(a) = [\Pi(a'), \pi(a)]$,

(iv) $\pi(aa') - \pi(a)\pi(a') = -(\Pi(a')\pi(a) - \pi(a)\Pi(a'))$ (.i.e) $\pi(aa') - \pi(a)\pi(a') = [\pi(a), \Pi(a')]$,

$\forall a, a' \in A$.
La somme directe $A \oplus M$ des espaces vectoriels A et M est une algèbre alternative dont le produit est défini par :
$(a + m)(a' + m') = aa' + \pi(a)m' + \Pi(a')m$, for all $a, a' \in A$ et $m, m' \in M$.
Notons le A-bimodule M par (M, π, Π), $\pi(a)m := am$ et $\Pi(a)m := ma$.
(M, π, Π) est dite (bi)-représentation de A dans M. (see. ([46],[20]).

Soient A et M deux algèbres alternatives. Nous supposons que M est un A-bimodule. Sur l'espace vectoriel $A \oplus M$ on définit le produit suivant :

$$(a + m)(a' + m') = aa' + ma' + am + mm'.$$

$A \oplus M$ muni de ce produit est une algèbre alternative si et seulement si (π, Π) vérifie :

(i) $\Pi(a)m^2 = m(\Pi(a)m)$,

(ii) $\pi(a)m^2 = (\pi(a)m)m$,

(iii) $(\pi(a)m)m' - \pi(a)(mm') = m(\pi(a)m') - (\Pi(a)m)m' = \Pi(a)(mm') - m(\Pi(a)m')$.

Si (π, Π) vérifie les trois dernières conditions i)-ii) et iii) ci-dessus, alors (M, π, Π) est dit admissible. Dans ce cas $A \oplus M$ est appelé le produit semi-direct de A par M, on le note par $A \underset{(\pi,\Pi)}{\ltimes} M$. (voir [7, 5]) pour le cas d'algèbres de Jordan).

2.2 Extensions centrales des algèbres alternatives

Définition 8

Soient g et h deux algèbres alternatives sur $\mathbb{F}$. On appelle extension de g par h une suite exacte :

$$0 \longrightarrow h \xrightarrow{\lambda} A \xrightarrow{\mu} g \longrightarrow 0$$

où A est une algèbre alternative.
$Ker\mu$ s'appelle le noyau de l'extension.
On dit que l'extension est centrale si $Ker\mu \subset Z(A)$. où $Z(A) :=$ le centre de A

Exemple 2.1

Soient A_1 une algèbre alternative, V un espace vectoriel et $\varphi : A_1 \times A_1 \to V$ une applicatoin bilineaire. On définit sur l'espace vectoriel $A = A_1 \oplus V$ le produit suivant :

$$(a+v)(b+v') := ab + \varphi(a,b) \quad \forall a, b \in A_1, v, v' \in V.$$

A muni de ce produit est une algèbre alternative si et seulement si :

$$\begin{cases} (a+v)^2(b+v') = (a+v)[(a+v)(b+v')] \\ (b+v')(a+v)^2 = [(b+v')(a+v)](a+v) \end{cases}$$

Ceci est équivaut à :

$$\begin{cases} [a^2 + \varphi(a,a)](b+v') = (a+v)(ab + \varphi(a,b)] \\ (b+v')(a^2 + \varphi(a,a)) = (ba + \varphi(b,a))(a+v) \end{cases}$$

$$\begin{cases} a^2b + \varphi(a^2,b) = a(ab) + \varphi(a,ab) \\ ba^2 + \varphi(b,a^2) = (ba)a + \varphi(ba,a) \end{cases}$$

Donc A est une algèbre alternative si et seulement si :

$$\begin{cases} \varphi(a^2,b) = \varphi(a,ab) \\ \varphi(b,a^2) = \varphi(ba,a) \end{cases}$$

Si c'est le cas on a une suite exacte :

$$0 \longrightarrow V \xrightarrow{\lambda} A_1 \oplus V \xrightarrow{\mu} A_1 \longrightarrow 0$$

$Ker\mu = \{0\} \times V \subset Z(A_1 \oplus V)$, d'où l'extension est centrale. L'algèbre A est dite l'extension centrale de A_1 par V au moyen de φ.

Dans ce paragraphe nous introduisons la notion de double extension des algèbres alternatives et nous utilisons cette notion pour décrire inductivement des algèbres alternatives pseudo-euclidiennes.
Soit (M, ψ) une algèbre alternative pseudo-euclidienne, A une algèbre alternative pas

nécessairement pseudo-euclidienne telle que M est un A-bimodule admissible avec :

$$\psi(a.m, m') = \psi(m, m'.a) \ \ \forall m, m' \in M, \ \ \forall a \in A.$$

soit

$$\varphi : M \times M \to A^*; \ \ (m, m') \to \varphi(m, m')(a) = \psi(a.m, m').$$

Alors

$$\varphi(m^2, m')(a) = \psi(a.m^2, m')$$

Et

$$\varphi(m, mm')(a) = \psi(a, m, mm') = \psi((a.m)m, m') = \psi(am^2, m').$$

D'où

$$\varphi(m^2, m')(a) = \varphi(m, mm')(a) \ \ \forall m, m' \in M, \ \ \forall a \in A.$$

(i.e,)

$$\varphi(m^2, m') = \varphi(m, mm').$$

De plus

$$\varphi(m', m^2)(a) = \psi(a, m', m^2) = \psi(m', m^2.a).$$

Et

$$\varphi(m'm, m)(a) = \psi(a.(m'm), m) = \psi(m'm, m.a) = \psi(m', m(m.a) = \psi(m', m^2.a).$$

D'où

$$\varphi(m', m^2)(a) = \varphi(m'm, m)(a), \ \ \forall a \in A$$

(i.e,)

$$\varphi(m', m^2) = \varphi(m'm, m)$$

Et par suite on peut considerer l'algèbre $M \rtimes_\varphi A^*$ (extension centrale de M par A^* suivant φ) dont le produit est défini par :

$$(m + f)(m' + f') = mm' + \varphi(m, m'). \ \ \forall m, m' \in M \ \ \forall f, f' \in A^*.$$

Maintenant, soient :

$$A \times (M \rtimes_\varphi A^*) \to M \rtimes_\varphi A^*, (a, m + f) \mapsto a.(m + f) = a.m + foR_a,$$

et

$$(M \rtimes_\varphi A^*) \times A \to (M \rtimes_\varphi A^*), (m + f, a) \mapsto (m + f).a = m.a + foL_a,$$

deux applications bilinèaires. Nous montrons que $(M \rtimes_\varphi A^*)$ est un A-bimodule admissible, Il s'ensuit qu'on peut considérer $A \oplus (M \rtimes_\varphi A^*)$ le produit semi direct de $M \rtimes_\varphi A^*$ par A dont le produit est défini par :

$$\begin{aligned}[(a + m + f)(a' + m' + f')] &= aa' + a(m' + f') + (m + f).a' + (m + f)(m' + f') \\ &= aa' + a.m' + f'oR_a + m.a' + foL_{a'} + mm' + \varphi(m, m')\end{aligned}$$

$\forall a, a' \in A, m, m' \in M, f, f' \in A^*$.
Nous pouvons donc énnoncer la définition suivante :

Définition 9

L'algèbre alternative $A\oplus(M\rtimes_{\varphi}A^*)$ définie ci-dessus est dite l'algèbre double extension de M par A.

Ensuite nous montrons que l'algèbre $(A \oplus M \oplus A^*, \widetilde{\psi})$ est une algèbre alternative pseudo-euclidienne, où la forme bilinéaire $\widetilde{\psi}$ sur $(A \oplus M \oplus A^*) \times (A \oplus M \oplus A^*)$ est définie par :

$$\begin{aligned}\widetilde{\psi}:\ & (A \oplus M \oplus A^*) \times (A \oplus M \oplus A^*) \longrightarrow \mathbb{K},\\ & (a+m+f, a'+m'+f') \longrightarrow \sigma(a,a') + \psi(m,m') + f(a') + f'(a)\end{aligned}$$

où σ une forme bilinéaire symétrique invariante, pas nécessairement non dégénérée sur $A \times A$.

Théorème 3

Soient (M, ψ) une algèbre alternative pseudo-euclidienne, A une algèbre alternative et M un A-bimodule admissible d'algèbres alternatives tels que $\psi(a.m, m') = \psi(m, m'.a)$. On considère l'application bilinéaire φ définie par :

$$\begin{aligned}\varphi:\ & M \times M \longrightarrow A^*\\ & (m,m') \mapsto \varphi(m,m')(a) = \psi(a.m,m')\end{aligned}$$

$\forall a \in A, m, m' \in M$
Alors l'espace vectoriel $A \oplus M \oplus A^*$ muni du produit :

$$(a+m+f)(a'+m'+f') = aa' + mm' + a.m' + m.a' + f'oR_a + foL_{a'} + \varphi(m,m')$$

$\forall a, a' \in A, m, m' \in M, f, f' \in A^*$, *est une algèbre alternative.*
En plus, si σ est une forme bilinéaire symétrique invariante sur $A \times A$ alors la forme bilinéaire $\widetilde{\psi}$ définie par :

$$\begin{aligned}\widetilde{\psi}:\ & (A \oplus M \oplus A^*) \times (A \oplus M \oplus A^*) \longrightarrow \mathbb{K},\\ & (a+m+f, a'+m'+f') \longrightarrow \sigma(a,a') + \psi(m,m') + f(a') + f'(a)\end{aligned}$$

est un produit scalaire invariant sur $A \oplus M \oplus A^*$.

2.3 Description inductive des algèbres alternatives pseudo-euclidiennes

Maintenant, nous allons donner une description inductive des algèbres alternatives pseudo-euclidiennes en utilisant les extensions définies précédement.

Théorème 4

Soit (A, ψ) une algèbre alternative pseudo-euclidienne.
Si $A = I^{\perp} \oplus V$, où I est un idéal totalement isotrope de A, V une sous algèbre de A, alors A est une double extension de l'algèbre alternative pseudo-euclidienne $(W = I^{\perp}/I, \tilde{\psi})$ par V au moyen de la bi-représentation (S, T) définie par :

$$\begin{aligned}S:\ & V \longrightarrow End(I^{\perp}/I, \widetilde{\psi})\\ & v \longrightarrow S(v)(s(i)) = s(L_v(i)) = s(vi)\end{aligned}$$

$$T: \quad V \longrightarrow End(I^{\perp}/I, \widetilde{\psi})$$
$$v \longrightarrow T(v)(s(i)) = s(R_v(i)) = s(iv)$$

$\forall v \in V$, $i \in I^{\perp}$.
Où $I^{\perp}$est l'orthogonal de I relativement à ψ, s est la surjection canonique de $I^{\perp}$ sur $I^{\perp}/I$ et $\widetilde{\psi}$ est définie par : $\widetilde{\psi}(s(i), s(j)) := \psi(i,j)$, $\forall i, j \in I^{\perp}$.

Corollaire 3

Soit (A, ψ) une algèbre alternative pseudo-euclidienne irreductible non simple. Si A n'est pas nilpotente alors A est une double extension d'une algèbre alternative pseudo-euclidienne (W, T) par une algèbre alternative simple.

Démonstration : Comme A n'est pas nilpotente alors $RadA \neq A$ et d'après Wedderburn : $A = S \oplus RadA$ où S une sous algèbre semi simple de A.

Le fait que A est irréductible non simple implique que $RadA \neq 0$, sinon A devient semi-simple et $A = \bigoplus_i I_i$, avec I_i sont des ideaux simples de A non tous triviaux car A est non simple, soit I_{i_0} un idéal non trivial de A, or tout idéal semi-simple de A est non dégénéré, ceci contredit le fait que A est irréductible.
Comme S est semi-simple, $S = \bigoplus_{i=1}^{n} S_i$ où les S_i sont des ideaux simples de S, $\forall i \in \{1, ..., n\}$. Posons

$$J_k = \bigoplus_{\substack{i=1 \\ i \neq k}}^{n} S_i \oplus RadA$$

, alors $A = S_k \oplus J_k$ avec J_k est un idéal maximal de A. En effet :
On a $AS_i \subset S_i \oplus RadA$ $\forall i \in \{1, ..., n\}$ et $ARadA \subset RadA$ (car $RadA$ est un idéal de A et S_i est un idéal de S) d'où $AJ_k \subset J_k$ et de même on montre que $J_kA \subset J_k$ et par suite J_k est un idéal de A.

Maintenat supposons qu'il existe un idéal M de A tel que $J_k \subseteq M \subsetneq A$, parsuite $M \cap S_k$ devient un idéal de S_k car $S_k(M \cap S_k) \subset M \cap S_k$ et $(M \cap S_k)S_k \subset M \cap S_k$, comme S_k est simple $M \cap S_k = \{0\}$ ou $M \cap S_k = S_k$.
Si $M \cap S_k = S_k$ alors $S_k \subset M$ et $J_k \oplus S_k \subset M$, absurde car $A \subsetneq M$.

On obtient donc $M \cap S_k = \{O\}$. Soit $m \in M$, $m = s_k + j_k$ où $s_k \in S_k$ et $j_k \in J_k$ alors $s_k = m - j_k$ d'où $s_k \in M \cap S_k$ autrement $s_k = 0$ et $m = j_k \in J_k$ et par suite $M = J_k$. Si on pose $I_k = J_k^{\perp}$ comme J_k est maximal, I_k devient minimal, or $\{0\} \subsetneq I_k \cap I_k^{\perp} \subset I_k$. Il s'ensuit que $I_k \subset I_k^{\perp}$ ie. I_k est totalement isotrope. D'où A est la double extension de l'algèbre alternative pseudo-euclidienne $(W = J_k/J_k^{\perp}, \widetilde{\psi})$ par S_k. ■

Pour décrire toutes les algèbres alternatives pseudo-euclidiennes, nous introduisons une autre notion de double extension.

2.4 Double extensions généralisées des algèbres alternatives pseudo-euclidiennes par l'algèbre de dimension 1 à produit nul.

• **Produit semi-direct généralisé :** Soit A une algèbre alternative et $x_0 \in A$. $\mathbb{F}e$ une algèbre de dimension 1 à produit nul.

On pose $\widetilde{A} = \mathbb{F}e \oplus A$, sur cet espace vectoriel on définit le produit suivant :

$$x * y = xy, \quad e * e = x_0, \quad x * e = f^*(x), \quad e * x = f(x), \quad \forall x, y, \in A, \ f \in End(A).$$

$\widetilde{A}$ muni du produit ci-dessus est une algèbre alternative, si et seulement si la paire (f, x_0) vérifie les conditions suivantes :
$f(x_0) = f^*(x_0); \quad f^* \circ f(x) = f \circ f^*(x); \quad f^2(x) = x_0 x;$
$(f^*)^2(x) = x x_0; \quad f(x^2) = f(x)x; \quad f^*(x^2) = x f^*(x);$
$f(x)y + f^*(x)y = f(xy) + x f(y); \quad x f(y) + x f^*(y) = f^*(x)y + f^*(xy).$
Dans ce cas, (f, x_0) est dite une paire admissible de A et l'algèbre alternative $\widetilde{A}$ est dite le produit semi-direct generalisé de A par l'algèbre de dimension 1 à produit nul via la paire (f, x_0).

Proposition 2.2

Soient (A_1, ψ_1) une algèbre alternative pseudo-euclidienne, $\mathbb{F}b$ l'algèbre de dimension 1 à produit nul, (f, x_0) une paire admissible de A_1 et $k \in \mathbb{F}$. Soit $\varphi : A_1 \times A_1 \to \mathbb{F}$ une forme bilinéaire définie par :

$$\varphi(x, y) = \psi_1(f(x), y), \ \forall x, y \in A_1.$$

Alors, l'espace vectoriel $A = \mathbb{F}e \oplus A_1 \oplus \mathbb{F}b$ ($\mathbb{F}e$ est un espace vectoriel de dimension 1) muni du produit suivant :

$$(e + (x + b)) * (e + (y + b)) = w_0 + \tilde{f}(y + b) + \tilde{f}^*(x + b) + xy + \psi_1(f(x), y)b, \ \forall x, y \in A_1,$$

où $w_0 = x_0 + kb$, $\tilde{f}(y + b) = f(y) + \psi_1(x_0, y)b$, $\tilde{f}^*(x + b) = f^*(x) + \psi_1(x_0, x)b$ et avec la forme bilinéaire symétrique ψ définie par :
$\psi|_{A_1 \times A_1} = \psi_1$, $\psi(e, b) = 1$, $\psi(e, A_1) = \psi(b, A_1) = \{0\}$ et $\psi(e, e) = \psi(b, b) = 0$, est une algèbre alternative pseudo-euclidienne.

Définition 10

L'algèbre alternative pseudo-euclidienne (A, ψ) construite ci-dessus est appelée la double extension genéralisée de l'algèbre alternative pseudo-euclidienne (A_1, ψ_1) par l'algèbre alternative de dimension 1 à produit nul $\mathbb{F}b$ via (f, x_0).

Nous enonçons ainsi le resultat intéréssant suivant : Plusieurs algèbres alternative pseudo-euclidiennes sont isometriques avec certaines double extension généralisées. Plus précisément on a le théorème suivant :

Théorème 5

Soit (A, ψ) une algèbre alternative pseudo-euclidienne telle que A n'est pas l'algèbre alternative de dimension 1 à produit nul. On suppose que $Ann(A) \neq 0$ et il existe $b \in Ann(A) \backslash \{0\}$ tel que $\psi(b, b) = 0$, alors A est une double extension généralisée d'une algèbre alternative pseudo-euclidienne par l'algèbre alternative de dimension 1 à produit nul.

Une conséquence immédiate de ce théorème est le corollaire suivant :

Corollaire 4

Soit (A, ψ) une algèbre alternative pseudo-euclidienne irréductible et nilpotente telle que A n'est pas l'algèbre alternative de dimension 1 à produit nul. Alors, A est une *double extension généralisée d'une algèbre alternative pseudo-euclidienne nilpotente* (W,T) *par l'algèbre alternative de dimension 1 à produit nul.*

Chapitre 3

Algèbres de Malcev-Poisson-Jordan (Algèbres-MPJ)

3.1 Définitions et premières résultats

Définition 11

Une algèbre-MPJ P est un $\mathbb{F}$-espace vectoriel muni de deux multiplications bilinéaires $\circ$ et $[\ ,\ ]$ telles que :

i) $(P, \circ)$ est une algèbre de Jordan,

ii) $(P, [\ ,\ ])$ est une algèbre de Malcev,

iii) Ces deux opérations vérifient la condition de Leibniz :

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z], \ \forall x, y, z \in P.$$

Une algèbre-MPJ sera notée par $(P, [\ ,\ ], \circ)$.

Remarque 6

Les algèbres de Poisson et les algèbres de Malcev Poisson sont des algèbres-MPJ.

Notation. Soit $(P, .)$ une algèbre non associative, posons :

$$[x, y] := x.y - y.x \ \ et \ \ x \circ y := \frac{1}{2}(x.y + y.x), \ \forall x, y \in P.$$

Proposition 3.1

Soit $(P, .)$ une algèbre flexible et Malcev admissible, $(P, [\ ,\ ], \circ)$ est une algèbre-MPJ si et seulement si $(P, .)$ vérifie l'identité :

$$R_x o L_{x^2} = L_{x^2} o R_x, \quad \forall x \in P. \tag{3.1}$$

L_x(resp. R_x) est la multiplication gauche (resp. La multiplication droite) de x dans l'algebra $(P, .)$. Dans ce cas, $(P, .)$ est appelée une algèbre-**MPJ admissible**. L'algèbre de Malcev associée $(P, [\ ,\ ])$ sera notée P^- et l'algèbre de Jordan $(P, \circ)$ sera notée P^+.

Rappelons qu'une algebra A sur $\mathbb{F}$ est dite algèbre de Jordan non commutative ; si A est flexible et vérifie l'identité $(x^2y)x = x^2(yx)$. R. Schafer ([47], p. 473) a montré qu'une algèbre A sur $\mathbb{F}$ est de Jordan non commutative si et seulement si elle est flexible et Jordan-admissible.

Remarque 7

Toute algèbre-MPJ admissible est une algèbre de Jordan non commutative.

Proposition 3.2

Soit $(P, .)$ une algèbre-MPJ admissible et soient $x, y \in P$. Considérons l'application linéaire $D_{x,y} : P \longrightarrow P$ donnée par $D_{x,y} = L_{[x,y]} - [L_x, L_y]$. $D_{x,y}$ est une derivation de P si et seulement si

$$\frac{3}{4}J([y,x],z,t) = [(x,z,y)_\circ, t] + [z,(x,t,y)_\circ] - (x,[z,t],y)_\circ \tag{3.2}$$

pour tous $z, t \in P$.

Soit $\mathfrak{C}$ la classe des algèbres de Jordan non commutatives P sur $\mathbb{F}$ vérifiant l'équation (3.2). On peut facilement vérifier que les algèbres de Lie-Poisson admissibles et les algèbres associatives appartiennent à cette classe $\mathfrak{C}$. Une connexion entre la sous-classe d'éléments de $\mathfrak{C}$ (avec une involution et une unité) et une classe de certains systèmes triples de Jordan généralisés a été expliqué dans [16].

Maintenant, à partir d'une algèbre-MPJ admissible vérifiant Eq.(3.2), on construit une nouvelle algèbre-MPJ admissible vérifiant Eq.(3.2) et qui admet une involution et une unité. Ensuite, on peut construire un système triple généralisé associé à cette nouvelle algèbre.

3.2 Algèbres-MPJ admissibles avec l'algèbre de Malcev associée est réductive

Dans cette section, nous donnons la description de toute les structures de MPJ ($\circ$) sur une algèbre de Malcev réductive $(P, [\ ,\ ])$ (i.e. $(P, [\ ,\ ], \circ)$ soit une algèbre-MPJ). En particulier, nous montrons que chaque structure de MPJ sur une algèbre de Malcev semi-simple est triviale. Rappelons qu' une algèbre de Malcev est dite réductive s'elle est une somme directe d'une algèbre de Malcev semi-simple et son centre.

Les résultats de cette section sont basés sur les travaux de G. Benkart et J. M. Osborn [11] et le travail de H. C. Myung [38].

Nous commençons d'abord par l'exemple intéressant suivant :

Soit $\mathbb{O}$ l'algèbre des octonions avec la table de multiplication suivante par rapport à la base $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	$-e_4$	e_3	$-e_6$	e_5	e_8	$-e_7$
e_3	e_3	e_4	$-e_1$	$-e_2$	$-e_7$	$-e_8$	e_5	e_6
e_4	e_4	$-e_3$	e_2	$-e_1$	$-e_8$	e_7	$-e_6$	e_5
e_5	e_5	e_6	e_7	e_8	$-e_1$	$-e_2$	$-e_3$	$-e_4$
e_6	e_6	$-e_5$	e_8	$-e_7$	e_2	$-e_1$	e_4	$-e_3$
e_7	e_7	$-e_8$	$-e_5$	e_6	e_3	$-e_7$	$-e_1$	e_2
e_8	e_8	e_7	$-e_6$	$-e_5$	e_4	e_3	$-e_2$	$-e_1$

On peut facilement montrer que la table de multiplication de l'algèbre de Malcev $\mathbb{O}^-$ peut être considérée comme suit :

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	0	0	0	0	0	0	0
e_2	0	0	$-2e_4$	$2e_3$	$-2e_6$	$2e_5$	$2e_8$	$-2e_7$
e_3	0	$2e_4$	0	$-2e_2$	$-2e_7$	$-2e_8$	$2e_5$	$2e_6$
e_4	0	$-2e_3$	$2e_2$	0	$-2e_8$	$2e_7$	$-2e_6$	$2e_5$
e_5	0	$2e_6$	$2e_7$	$2e_8$	0	$-2e_2$	$-2e_3$	$-2e_4$
e_6	0	$-2e_5$	$2e_8$	$-2e_7$	$2e_2$	0	$2e_4$	$-2e_3$
e_7	0	$-2e_8$	$-2e_5$	$2e_6$	$2e_3$	$-2e_7$	0	$2e_2$
e_8	0	$2e_7$	$-2e_6$	$-2e_5$	$2e_4$	$2e_3$	$-2e_2$	0

Ce tableau montre que $Z(\mathbb{O}^-) = \mathbb{F}e_1$ ($Z(\mathbb{O}^-)$ est le centre de $\mathbb{O}^-$) et le sous-espace $\mathbb{O}^* := vect\{e_2, e_3, ..., e_8\}$ est une sous algèbre de $\mathbb{O}^-$. On peut aisément montrer que $\mathbb{O}^*$ est une algèbre simple. Soit I un idéal non trivial de $\mathbb{O}^*$, par le tableau ci-dessus, I contient tous les éléments e_i pour $i \in \{2, ..., 8\}$. Par conséquent, $\mathbb{O}^* = I$ et l'algèbre de Malcev $\mathbb{O}^- = \mathbb{O}^* \oplus Z(\mathbb{O}^-)$ est reductive. Comme $J(e_2, e_7, e_5) = 12e_4 \neq \{0\}$, alors $\mathbb{O}^*$ est une algèbre de Malcev simple (non-Lie).

Proposition 3.3

Toute algèbre-MPJ admissible $(P, .)$ sur un corps de characteristique 0 est une algèbre à puissance-associative (ie. pour tout élément x de P la sous algèbre $\mathbb{F}[x]$ de P est associative).

Le théorème suivant a été obtenu dans [11],

Théorème 6

Si P est une algèbre à puissance-associative, flexible et Lie-admissible sur un corps de characteristique 0 telle que P^- est une algèbre de Lie simple. Alors P est une algèbre de Lie .

Le résultat suivant, qui est une conséquence directe du théorème 6 et de la proposition 3.3 caractérise les algèbres-MPJ admissibles avec P^- algèbre de Lie simple.

Corollaire 5

Soit $(P, .)$ Une algèbre-MPJ admissible sur un corps de characteristique 0. Si P^- est une algèbre de Lie simple, alors l'algèbre de Jordan P^+ est triviale.

Lemme 3.4

Si $(P, [\ ,\ ])$ est une algèbre de Malcev semi-simple, alors la structure MPJ sur P est triviale.

Le théorème suivant caractérise les algèbres-MPJ admissible telles que P^- est une algèbre de Malcev reductive. La preuve de ce théorème necessite le résultat de H. C. Myung (Corollary 4.5 in [38]) lorsque le corps est algébriquement clos.

Théorème 7

Supposons que $\mathbb{F}$ est algébriquement clos. Soit $(P, [\ ,\ ])$ une algèbre de Malcev reductive, (ie. $P = S_1 \oplus ... \oplus S_n \oplus Z(P)$) où $S_1, ..., S_n$ sont des ideaux simples de P et $Z(P)$ le centre de P.
$\circ$ est une structure MPJ sur P si et seulement s'il existe une structure d'algèbre de Jordan $\star$ sur l'espace vectoriel sous-jacent de $Z(P)$, $a_1, ..., a_n$ des éléments de $Z(P)$ et $T_1, ..., T_n$ des formes linéaires sur $Z(P)$ vérifiant $T_i(a_i)a_i = a_i \star a_i$, et $T_i(a \star a_i) = T_i(a)T_i(a_i)$, $\forall i \in \{1, ..., n\}, \forall a \in Z(P)$ tels que :

i) $a \circ b = a \star b$, $\forall a, b \in Z(P)$;

ii) $x \circ y = \mathcal{K}_i(x, y)a_i$, $\forall x, y \in S_i$, $\forall i \in \{1, ..., n\}$ où $\mathcal{K}_i$ est la forme de Killing de S_i ;

iii) $x \circ y = 0$, $\forall x \in S_i$, $\forall y \in S_j, \forall i, j \in \{1, ..., n\}$ avec $i \neq j$;

iv) $x \circ a = a \circ x = T_i(a)x$, $\forall x \in S_i$, $\forall a \in Z(P)$, $\forall i \in \{1, ..., n\}$.

3.3 Relations entre les algèbres-PEMPJ et les algèbres de Lie quadratiques

Nous montrons qu'à partir des algèbres-PEMPJ nous pouvons construire des algèbres de Lie quadratiques intéréssantes en utilisant la technique de double extension pour les algèbres de Lie quadratiques [35].

Définition 12

1. Soit $(P, [\ ,\ ], \circ)$ une algèbre-MPJ et $\psi : P \times P \longrightarrow \mathbb{F}$ une forme bilinéaire. (P, ψ) est appelée algèbre-MPJ pseudo-euclidienne si $(P, [\ ,\ ], \psi)$ et $(P, \circ, \psi)$ sont deux algèbres pseudo-euclidiennes.

2. Soit $(P, .)$ une algèbre-MPJ admissible et $\psi : P \times P \longrightarrow \mathbb{F}$ une forme bilinéaire. (P, ψ) est appelée algèbre-MPJ pseudo-euclidienne admissible si $(P, ., \psi)$ est une algèbre pseudo-euclidienne.
Une algèbre de Malcev-Poisson-Jordan pseudo-euclidienne sera notée une algèbre-PEMPJ.

Soit $(P, ., \psi)$ une algèbre-PEMP admissible, alors (P^-, ψ) est une algèbre de Malcev pseudo-euclidienne et (P^+, ψ) une algèbre de Jordan pseudo-euclidienne. Par la construction de TTK on définit sur l'espace vectoriel $\mathfrak{Lie}(P^+) = P^+ \oplus H(P^+) \oplus \overline{P^+}$ la structure d'algèbre de Lie quadratique $(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$, où $\overline{P^+}$ est une copie de P^+ et $H(P^+) = [R(P^+), R(P^+)] \oplus R((P^+)^2)$. Cette construction est définie par :

$$[T, T'] = [T, T']_H; \quad [T, a'] = T(a'); \quad [T, \overline{b'}] = -\overline{T(b')}; \quad [a, \overline{b'}] = R_{ab'}; \quad [a, a'] = [\overline{b}, \overline{b'}] = 0.$$

Pour tous $T, T' \in H(P^+), a, a', b, b' \in P^+$, et

$$\begin{aligned} \psi_{\mathfrak{L}} : \quad & \mathfrak{Lie}(P^+) = P^+ \oplus H(P^+) \oplus \overline{P^+} \times \mathfrak{Lie}(P^+) = P^+ \oplus H(P^+) \oplus \overline{P^+} \longrightarrow \mathbb{F} \\ & (a + R_x + D + \overline{b}, a' + R_{x'} + D' + \overline{b'}) \longrightarrow \psi(x, x') + 2\psi(a, b') + 2\psi(a', b) + \varphi(D, D'), \end{aligned}$$

$$\forall a, a', b, b', x, x' \in P^+, D, D' \in [R(P^+), R(P^+)],$$

où $\varphi(D, D' = \sum_{i=1}^n [R_{x_i}, R_{x'_i}]) = \sum_{i=1}^n \psi(D(x_i), x'_i)$.

Nous avons besoin du lemme suivant pour enoncer un résultat intéresant.

Lemme 3.5

Soit (P, ψ) une algèbre de Jordan pseudo-euclidienne et D une dérivation ψ-anti symétrique de P. On définit l'endomorphisme $D_{\mathfrak{L}}$ de la manière suivante :

$$D_{\mathfrak{L}}(x) = D(x), \;\; D_{\mathfrak{L}}(\bar{x}) = \overline{D(x)}, \;\; D_{\mathfrak{L}}(R_x) = R_{D(x)}, \;\; D_{\mathfrak{L}}([R_x, R_y]) = [R_{D(x)}, R_y] + [R_x, R_{D(y)}],$$

$\forall x, y \in P$. Alors $D_{\mathfrak{L}}$ est une dérivation $\psi_{\mathfrak{L}}$-antisymetrique de $\mathfrak{Lie}(P)$.

Proposition 3.6

Soit $(P, ., \psi)$ une algèbre-PEMP admissible, alors pour tout $x \in P$, $ad_{p^-}x$ est une dérivation ψ-antisymetrique de P^+.

Démonstration : Soit $x \in P$, par la condition de Leibniz on remarque que $ad_{p^-}x \in Der(P^+)$. De plus, pour tous $y, z \in P$ on a,

$$\psi((ad_{p^-}x)(y), z) = \psi([x, y], z) = -\psi([y, x], z) = -\psi(y, [x, z]) = -\psi(y, (ad_{p^-}x)(z)).$$

Alors $ad_{p^-}x$ est une dérivation ψ-antisymetrique de P^+.

Ainsi, d'après le Lemme 3.5, pour tout $x \in P$, $(ad_{p^-}x)_{\mathfrak{L}}$ est une dérivation $\psi_{\mathfrak{L}}$-antisymetrique de $\mathfrak{Lie}(P^+)$. ∎

Proposition 3.7

Soit $(P, [\ ,\ ], \circ, \psi)$ une algèbre-PEMP, pour tous $a, b \in P$ on a $L(a, b) \in Der(P, \circ)$, en plus,

$$\psi(L(a, b)x, y) = -\psi(x, L(a, b)y), \;\forall a, b, x, y \in P.$$

Démonstration : Soient $x, y, z, t \in P$,

$$\begin{aligned} \{x, y, z \circ t\} &= 2[[x, y], z \circ t] - [[y, z \circ t], x] - [[z \circ t, x], y] \\ &= 2[[x, y], t] \circ z + 2[[x, y], z] \circ t - [[y, z] \circ t, x] - [[y, t] \circ z, x] - [z \circ [t, x], y] - [[z, x] \circ t, y] \\ &= 2[[x, y], t] \circ z - [[y, t], x] \circ z - [y, t] \circ [z, x] - z \circ [[t, x], y] - [t, x] \circ [z, y] \\ &+ 2[[x, y], z] \circ t - [[y, z], x] \circ t - [y, z] \circ [t, x] - [[z, x], y] \circ t - [z, x] \circ [t, y] \\ &= 2[[x, y], t] \circ z - [[y, t], x] \circ z - z \circ [[t, x], y] + 2[[x, y], z] \circ t - [[y, z], x] \circ t - [[z, x], y] \circ t \\ &= z \circ \{x, y, t\} + \{x, y, z\} \circ t, \end{aligned}$$

alors, $L(x,y) \in Der(P,\circ)$. en plus on a,

$$\begin{aligned}
\psi(\{a,b,x\},y) &= 2\psi([[a,b],x],y) - \psi([[b,x],a],y) - \psi([[x,a],b],y) \\
&= 2\psi([a,b],[x,y]) - \psi([b,x],[a,y]) - \psi([x,a],[b,y]) \\
&= 2\psi([[b,a],y],x) - \psi([[a,y],b],x) - \psi([[y,b],a],x) \\
&= \psi(x, 2[[b,a],y] - [[a,y],b] - [[y,b],a]) \\
&= \psi(x,\{b,a,y\}) \\
&= -\psi(x,\{a,b,y\}).
\end{aligned}$$

Ce qui montre que $L(a,b)$ est une dérivation ψ-antisymetrique de P^+. ■

Remarque 8

Le Lemme 3.5 montre que $(L(a,b))_{\mathfrak{L}}$ est une dérivation $\psi_{\mathfrak{L}}$-antisymetrique de $\mathfrak{Lie}(P^+)$.

Nous terminons ce paragraphe par la proposition suivante :

Proposition 3.8

Soit $(P,[\ ,\ ],\circ,\psi)$ une algèbre-PEMPJ, alors l'application linéaire ϕ définie par

$$\begin{aligned}
\phi:\ \ & L(P,P) \longrightarrow Der_a(\mathfrak{Lie}(P^+),\psi_{\mathfrak{L}}) \\
& L(x,y) \longmapsto (L(x,y))_{\mathfrak{L}}
\end{aligned}$$

est un morphisme d'algèbre de Lie, où $Der_a(\mathfrak{Lie}(P^+)$ est l'algèbre de Lie des dérivations $\psi_{\mathfrak{L}}$-antisymétique de $\mathfrak{Lie}(P^+)$.

Il s'ensuit, d'après ([35]), qu'on peut considérer $(L(P,P)\oplus\mathfrak{Lie}(P^+)\oplus(L(P,P))^\star,[\ ,\ ],\widetilde{\psi_{\mathfrak{L}}})$ l'algèbre de Lie quadratique (double extension de $\mathfrak{Lie}(P^+)$ par $L(P,P)$ au moyen de ϕ) définie par :

$$\begin{aligned}
[L(x,y)+K+f, L(x',y')+K'+f'] &= [L(x,y),L(x',y')] + [K,K']_{\mathfrak{Lie}(P^+)} + \phi(L(x,y))(K') \\
&\quad - \phi(L(x',y'))(K) + ad^\star(L(x',y'))(f) - ad^\star(L(x,y))(f') + \varphi(K,K'
\end{aligned}$$

pour tout $x,y\in P,\ K,K'\in\mathfrak{Lie}(P^+),\ f,f'\in(L(P,P))^\star$, où φ est l'application linéaire définie par :

$$\begin{aligned}
\varphi:\ \ & \mathfrak{Lie}(P^+)\times\mathfrak{Lie}(P^+) \longrightarrow (L(P,P))^\star \\
& (K,K') \longmapsto \varphi(K,K')(L(x,y)) = \psi_{\mathfrak{L}}(\phi(L(x,y))(K),K'),
\end{aligned}$$

et

$$\widetilde{\psi_{\mathfrak{L}}}(L(x,y)+K+f, L(x',y')+K'+f') = \psi_{\mathfrak{L}}(K,K') + f(K') + f'(K) + \sigma(L(x,y),L(x',y')),$$

ici σ est une forme bilinéaire symétrique invariante sur $L(P,P)\times L(P,P)$.

3.4 Construction des algèbres de Malcev pseudo-euclidiennes à partir des algèbres-PEMPJ admissibles

Notre objectif dans cette section est de construire des algèbres de Malcev pseudo-euclidiennes à partir des algèbres-PEMPJ admissibles en utilisant le concept de double extension dans le cas des algèbres de Malcev pseudo-euclidiennes [4]. Ce concept a été introduit par A. Medina et Ph. Revoy dans le cas des algèbres de Lie quadratiques [35]. Nous commençons par rappeler le concept de double extension dans le cas des algèbres de Malcev pseudo-euclidiennes.

Soit (M_1, ψ_1) une algèbre de Malcev pseudo-euclidienne et M_2 une algèbre de Malcev. Soit $\delta : M_2 \to End(M_1)$ une représentation de Malcev admissible de M_2 dans M_1 telle que $\forall x \in M_2$, $\delta(x) \in Op_a(M_1)$, où $Op_a(M_1)$ est le sous espace ψ_1-antisymétrique d'éléments de $Op(M_1)$.
Considérons l'application bilinéaire $\varphi : M_1 \times M_1 \to M_2^*$ définie par

$$\varphi(x,y)(z) = \psi_1(\delta(z)(x), y), \quad \forall x, y \in M_1, z \in M_2$$

Alors φ est un 2-cocycle (voir. [4], p. 30) de M_1 à valeurs dans M_2^*. Par conséquent, $M_1 \oplus M_2^*$ munie de la multiplication ci-dessous ;
$(x_1 + f)(y_1 + g) = x_1y_1 + \varphi(x_1, y_1), \quad \forall x_1 + f, y_1 + g \in M_1 \oplus M_2^*$, est l'extension centrale de M_2^* par M_1 au moyen de φ.

Considérons l'application linéaire $\rho : M_2 \to End(M_1 \oplus M_2^*)$ définie par
$\rho(x_2) = \delta(x_2) + \pi^*(x_2)$, ie,

$$\rho(x_2)(x_1 + f) = \delta(x_2)(x_1) + \pi^*(x_2)(f), \quad \forall x_1 \in M_1, x_2 \in M_2, f \in M_2^*$$

où π^* est la représentation co-adjointe de M_2. Alors ρ est une représentation de Malcev admissible de M_2 dans $M_1 \oplus M_2^*$. Par conséquent, $M = M_2 \oplus M_1 \oplus M_2^*$ avec la multiplication définie par

$$(x_2+x_1+f)(y_2+y_1+g) = x_2y_2+\delta(x_2)y_1+\pi^*(x_2)(g)+x_1y_1+\varphi(x_1,y_1)-\delta(y_2)(x_1)-\pi^*(y_2)f,$$

$\forall x_2+x_1+f, y_2+y_1+g \in M$, est un product semi-direct de $M_1 \oplus M_2^*$ par M_2 au moyen de ρ.

Soit γ une forme bilinéaire symétrique invariante sur $M_2 \times M_2$ (pas nécessairement non dégénérée). Alors la forme bilinéaire ψ sur $M = M_2 \oplus M_1 \oplus M_2^*$ définie par

$$\psi(x_2+x_1+f, y_2+y_1+g) = \gamma(x_2,y_2)+\psi_1(x_1,Y_1)+f(y_2)+g(x_2), \quad \forall (x_2+x_1+f),(y_2+y_1+g) \in M$$

est un produit scalaire invariant. L'algèbre de Malcev pseudo-euclidienne $M_2 \oplus M_1 \oplus M_2^*$ est appelée la double extension de (M_1, ψ_1) par M_2 au moyen de δ.

Remarque 9

Un calcul simple montre que l'algèbre de Malcev pseudo-euclidienne $M = M_2 \oplus M_1 \oplus M_2^*$, construite ci-dessus, est une algèbre de Lie pseudo-euclidienne si et seulement si M_1, M_2 sont deux algèbres de Lie, $\delta(x) \in Der(M_1), \ \forall x \in M_2$ et δ est un morphisme d'algèbres de Lie.

Lemme 3.9

Soit $(P, ., \psi)$ une algèbre-PEMPJ admissible telle que

$$[R_{J(z,t,w)}, R_{[z,x]}] = 0, \ \forall z, t, w, x \in P.$$

Soit $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ l'application linéaire définie par $\delta(x) = (ad_{p^-}x)_{\mathfrak{L}}$, alors δ est une représentation de Malcev de P^- dans $\mathfrak{Lie}(P^+)$.

Proposition 3.10

Soit $(P, ., \psi)$ une algèbre-PEMPJ admissible telle que $R_{J(x,y,z)} = 0$, pour tous $x, y, z \in P$. Alors l'application linéaire $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ définie par $\delta(x) = (ad_{p^-}x)_{\mathfrak{L}}$ est une représentation de Malcev admissible de P^- dans $\mathfrak{Lie}(P^+)$.

D'après la Proposition 3.10 on peut considérer $P^- \oplus \mathfrak{Lie}(P^+) \oplus (P^-)^*$ la double extension de $(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ par l'algèbre de Malcev P^- (telle que $R_{J(x,y,z)} = 0, \quad \forall x, y, z \in P$) au moyen de la représentation de Malcev admissible $\delta : P^- \to End(\mathfrak{Lie}(P^+), \psi_{\mathfrak{L}})$ définie par $\delta(x) = (ad_{p^-}x)_{\mathfrak{L}}$, $\forall x \in P$.

3.5 Description inductive des algèbres-PEMPJ admissibles nilpotentes

Afin de donner une description inductive des algèbres-PEMPJ admissible nilpotentes, nous introduisons dans cette section le concept de double extension des algèbres-PEMPJ admissibles par l'algèbre de dimension 1 à produit nul en utilisant les résultats de [4],[5].

Soient $(P, ., \psi)$ une algèbres-PEMPJ admissible et $(D, x_0) \in End_s(P^+, \psi) \times P^+$ une paire admissible de P^+ (où $End_s(P^+, \psi)$ est l'algèbre des endomorphismes symétriques de l'espace vectoriel P^+ par rapport à ψ) . L'espace vectoriel $\mathcal{A} := \mathbb{F}e \oplus P \oplus \mathbb{F}e^*$ muni du produit suivant :

$$\mathcal{A}\star e^* = e^*\star\mathcal{A} = 0;\ e\star e := x_0+\alpha e^*;\ x\star y := x\circ y+\psi(D(x), y)e^*;\ e\star x = x\star e := D(x)+\psi(x_0, x)e^*;$$

$\forall x, y \in P$, est une algèbre de Jordan (voir. [5]).

Soit $\phi \in Op_a(P^-)$ (voir[4]) telle que,

1. $\phi^2([x, y]) = [\phi(x), \phi(y)] + \phi([\phi(x), y]) + [x, \phi^2(y)]$,
2. $[\phi^2(x), y] - \phi([\phi(x), y]) = -([\phi^2(y), x] - \phi([\phi(y), x])), \quad \forall x, y \in P^-$.

L'espace vectoriel $\mathcal{A} := \mathbb{F}e \oplus P \oplus \mathbb{F}e^*$ (où $\mathbb{F}e$ est l'algèbre de Lie de dimension 1 et $\mathbb{F}e^*$ son dual) muni du produit suivant,

$$[x, y] := [x, y]_{P^-} + \psi(\phi(x), y)e^*;\ [e, x] = -[x, e] := \phi(x);\ [e, e] = 0;\ [e^*, \mathcal{A}] = [\mathcal{A}, e^*] = \{0\};$$

pour tous $x, y \in P$, est une algèbre de Malcev. En plus, si on considère la forme bilinéaire symétrique $T : \mathcal{A} \times \mathcal{A} \to \mathbb{F}$ définie par :

$$T_{|P\times P} := \psi;\ T(e^*, e^*) = 1;\ T(e, P) = T(e^*, P) = \{0\};\ T(e, e) = T(e^*, e^*) = 0,$$

alors $(\mathcal{A}, [\ ,\ ], T)$ est une algèbre de Malcev pseudo-euclidienne, appelée la double extension de $(P^-, [\ ,\ ]_{P^-}, \psi)$ par l'algèbre de Lie de dimension 1 au moyen de ϕ (voir. [4]) et $(\mathcal{A}, \star, T)$ est une algèbre de Jordan pseudo-euclidienne, appelée la double extension generalisée de $(P^+, \circ, \psi)$ par l'algèbre de dimension 1 à produit nul au moyen de (D, x_0) (voir. [5]).

Proposition 3.11

Soit $(\mathcal{A}, [\ ,\ ])$ (resp. $(\mathcal{A}, \star)$) l'algèbre de Malcev (resp. l'algèbre de Jordan) définie ci-dessus, $(\mathcal{A}, [\ ,\ ], \star)$ est une algèbre-MPJ si et seulement si pour tous $x, y \in P$,

$$\phi o D = D o \phi = \frac{1}{2}[x_0, .];\ \phi(x_0) = 0;\ \phi \in Der(P^+);\ D([x, y]_{P^-}) = [x, D(y)]_{P^-} + \phi(x) \circ y. \tag{3.3}$$

Dans ce cas $(\mathcal{A}, [\ ,\ ], \star, T)$ est une algèbre-PEMPJ, appelée la double extension de l'algèbre-PEMPJ $(P, [\ ,\ ]_{P^-}, \circ, \psi)$ par l'algèbre de dimension 1 à produit nul au moyen de (ϕ, D, x_0, α). En plus, l'espace vectoriel $\mathcal{A}$ muni du produit :

$$x \bullet y = \frac{1}{2}[x, y] + x \star y, \quad \forall x, y \in \mathcal{A},$$

est une algèbre-MPJ admissible. Ensuite $(\mathcal{A}, \bullet, T)$ est une algèbre-PEMPJ admissible appelée la double extension de l'algèbre-PEMPJ admissible $(P, ., \psi)$ par l'algèbre de dimension 1 à produit nul au moyen de (ϕ, D, x_0, α).

Proposition 3.12

Soit $(\mathcal{A}, \bullet, T)$ une algèbre-PEMPJ admissible. Si $Ann(\mathcal{A}) \neq \{0\}$ et s'il existe $e^* \in Ann(\mathcal{A}) \backslash \{0\}$ tel que $T(e^*, e^*) = 0$, alors $(\mathcal{A}, \bullet, T)$ est une double extension de l'algèbre-PEMPJ admissible $(P := (\mathbb{F}e^*)^\perp / \mathbb{F}e^*, ., \psi)$ par l'algèbre de dimension 1 à produit nul. Où,

$$\psi(x + \mathbb{F}e^*, y + \mathbb{F}e^*) := T(x, y) \ \ et \ \ (x + \mathbb{F}e^*).(y + \mathbb{F}e^*) := (x \bullet y) + \mathbb{F}e^*, \quad \forall x, y \in (\mathbb{F}e^*)^\perp.$$

Chapitre 4

Algèbres de Lie Yamaguti à partir des algèbres-MPJ

4.1 Définitions et premières resultats

Dans cette section, nous étendons le concept des algèbres-MPJ aux algèbres de Lie-Yamaguti. Nous rappelons quelques définitions et résultats concernant les algèbres de Lie-Yamaguti, nous montrons qu'on peut construire une algèbre de Lie-Yamaguti à partir d'une algèbre-MPJ. En outre, nous donnons une définition d'une algèbre de Lie-Yamaguti pseudo-euclidienne et nous montrons qu'on peut construire des algèbres de Lie-Yamaguti pseudo-euclidiennes à partir des algèbres-PEMPJ.

Définition 13

Une algèbre de Lie-Yamaguti (où bien algèbre-LY) est une algèbre anti-commutative $\mathcal{T}$ dont la multiplication est notée $[x,y]$ pour $x,y \in \mathcal{T}$ munie d'un produit trilinéaire $\{.,.,.\} : \mathcal{T}\times\mathcal{T}\times\mathcal{T} \longrightarrow \mathcal{T}$ vérifiant les conditions suivantes pour $x,y,z,w,t \in \mathcal{T}$:

(i) $\{x,x,y\} = 0$,

(ii) $\sum_{(x,y,z)}([[x,y],z] + \{x,y,z\}) = 0$,

(iii) $\sum_{(x,y,z)}\{[x,y],z,w\} = 0$,

(iv) $\{x,y,[z,w]\} = [\{x,y,z\},w] + [z,\{x,y,w\}]$,

(v) $\{x,y,\{z,t,w\}\} = \{\{x,y,z\},t,w\} + \{z,\{x,y,t\},w\} + \{z,t,\{x,y,w\}\}$.

Le symbôle $\sum_{(x,y,z)}$ dans cette définition désigne la somme cyclique sur x,y et z. Une algèbre-LY est un synonyme pour une algèbre de Lie triple ou système triple de Lie generale introduite par K. Yamaguti [52]. Les algèbres-LY où $[x,y] = 0$ pour tous x,y sont exactement les systèmes triples de Lie, tandis que les algèbres-LY avec $\{x,y,z\} = 0$ sont les algèbres de Lie.

Exemple 4.1

Soit $(P,[\ ,\ ])$ une algèbre de Malcev, K. Yamaguti [53] a montré que $(P,[\ ,\ ],\{.,.,.\})$ est une algèbre-LY lorsque le produit ternaire est défini par :

$$\{x,y,z\} = [[x,y],z] - [[y,z],x] - [[z,x],y], \quad \forall x,y,z \in P.$$

Exemple 4.2

Soit $(\mathfrak{g}, [\ ,\ ])$ une algèbre de Lie, h une sous algèbre de $\mathfrak{g}$ telle qu'il existe un sous-espace m de $\mathfrak{g}$ tel que

$$\mathfrak{g} = m \oplus h \ \text{(somme directe d'espaces vectoriels)} \ \text{ et } \ [h, m] \subseteq m. \qquad (4.1)$$

(Dans ce cas, $(\mathfrak{g}, h)$ est appelée une paire réductive). Alors $(m, [\ ,\ ]_m, \{.,.,.\})$ est une algèbre-LY ([52]) où $\{x, y, z\} = [[x, y]_h, z]$, pour tous $x, y, z \in m$.
($[x, y]_m$ et $[x, y]_h$ sont les projections de $[x, y]$ sur m et h respectivement).

Définition 14

Une dérivation d'une algèbre-LY $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ est une application linéaire D de $\mathcal{T}$ dans $\mathcal{T}$ telle que pour tous $x, y, z \in \mathcal{T}$,

(i) $D(\{x, y, z\}) = \{D(x), y, z\} + \{x, D(y), z\} + \{x, y, D(z)\}$,

(ii) $D[x, y] = [D(x), y] + [x, D(y)]$.

Soit $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ une algèbre-LY. Pour $x, y \in \mathcal{T}$, notons par $L(x, y)$ l'endomorphisme de $\mathcal{T}$ défini par $L(x, y)(z) = \{x, y, z\}, \forall z \in \mathcal{T}$. On peut remarquer, ((v) et (iv) de la Définition 13), que $L(x, y)$ est une dérivation de l'algèbre-LY $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$.

Notons par $L(\mathcal{T}, \mathcal{T})$ le sous-espace vectoriel de $End(\mathcal{T})$ engendré par $\{L(x, y),\ x, y \in \mathcal{T}\}$.
(Définition 13-(v)) est équivalente à

$$[L(x, y), L(z, t)] = L(L(x, y)(z), t) - L(z, L(y, x)(t)),\ \forall x, y, z, t \in \mathcal{T},$$

ce qui montre que $L(\mathcal{T}, \mathcal{T})$ est une sous-algèbre de $End(\mathcal{T})$, appelée l'algèbre des dérivations intérieures de $\mathcal{T}$.

Considérons la somme directe d'espaces vectoriels $\mathcal{L}(\mathcal{T}) = \mathcal{T} \oplus L(\mathcal{T}, \mathcal{T})$.
$\mathcal{L}(\mathcal{T})$ est une algèbre de Lie dont le crochet est donné par :

(i) $[x, y]_{\mathcal{L}(\mathcal{T})} := [x, y] + L(x, y),\ \forall x, y \in \mathcal{T}$,

(ii) $[d, x]_{\mathcal{L}(\mathcal{T})} := -[x, d]_{\mathcal{L}(\mathcal{T})} = dx,\ \forall d \in L(\mathcal{T}, \mathcal{T}),\ \forall x \in \mathcal{T}$,

(iii) $[d_1, d_2]_{\mathcal{L}(\mathcal{T})} = d_1 d_2 - d_2 d_1,\ \forall d_1, d_2 \in L(\mathcal{T}, \mathcal{T})$.

$\mathcal{L}(\mathcal{T})$ muni de ce crochet est appelée l'algèbre de Lie standard enveloppante de $\mathcal{T}$. Notons que l'algèbre de Lie $L(\mathcal{T}, \mathcal{T})$ est une sous-algèbre de $L(\mathcal{T})$ et $(L(\mathcal{T}), L(\mathcal{T}, \mathcal{T}))$ est une paire réductive.

Définition 15

Une algèbre-LY Poisson-Jordan est un quadruplet $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \})$ constitué d'un $\mathbb{F}$-espace vectoriel P, deux applications bilinéaires $\circ : P \times P \to P$, $[\ ,\ ] : P \times P \to P$ et une application trilinéaire $\{\ ,\ ,\ \} : P \times P \times P \to P$ tels que :

(i) $(P, \circ)$ est une algèbre de Jordan,

(ii) $(P, [\ ,\ ], \{\ ,\ ,\ \})$ est une algèbre-LY,

(iii) $L(x, y) \in Der(P, \circ)$, $\forall x, y \in P$,

(iv) $L_x \in Der(P, \circ), \forall x \in P$, où L_x est la multiplication gauche par x dans l'algèbre $(P, [\ ,\ ])$.

Définition 16

(i) Soit $(\mathcal{T}, [\ \ ,\ \ ], \{.,.,.\})$ une algèbre-LY. Une forme bilinéaire symétrique non dégénérée ψ sur $\mathcal{T}$ est dite invariante sur $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ s'elle vérifie :

$$\psi(R(a,b)x, y) = \psi(x, R(b,a)y)) \quad et \quad \psi([a,b], x) = \psi(a, [b,x]), \ \ \forall a, b, x, y \in \mathcal{T},$$

où $R(a, b)x = \{x, b, a\}$. Une telle forme est appelée un produit scalaire invariant sur $(\mathcal{T}, [\ \ ,\ \ ], \{.,.,.\})$. Dans ce cas $(\mathcal{T}, [\ \ ,\ \ ], \{.,.,.\}, \psi)$ est dite une algèbre-LY pseudo-euclidienne.

(ii) Soit $(P, [\ \ ,\ \ ], \circ, \{\ \ ,\ \ ,\ \ \})$ une algèbre-LY Poisson-Jordan et ψ une forme bilinéaire sur P. $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \}, \psi)$ est dite algèbre-LY Poisson-Jordan pseudo-euclidienne si $(P, [\ ,\ ], \{\ ,\ ,\ \}, \psi)$ est une algèbre-LY pseudo-euclidienne et $(P, \circ)$ est une algèbre de Jordan.

Proposition 4.3

Soit $(P, [\ ,\ ], \circ)$ une algèbre-MPJ, le quadruplet $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \})$ est une algèbre-LY Poisson-Jordan où $\{\ ,\ ,\ \} : P \times P \times P \longrightarrow P$ est définie par

$$\{x, y, z\} = [[x, y], z] - [[y, z], x] - [[z, x], y], \ \forall x, y, z \in P.$$

La Proposition suivante montre qu'on peut construire une algèbre-LY pseudo-euclidienne et une algèbre-LY Poisson-Jordan pseudo-euclidienne à partir d'une algèbre-PEMPJ.

Proposition 4.4

Soit $(P, [\ ,\ ], \circ, \psi)$ une algèbre-PEMPJ, alors $(P, [\ ,\ ], \circ, \{\ ,\ ,\ \}, \psi)$ est une algèbre-LY Poisson-Jordan pseudo-euclidienne et $(P, [\ ,\ ], \{\ ,\ ,\ \}, \psi)$ est une algèbre-LY pseudo-euclidienne. Le produit trilinéaire $\{\ ,\ ,\ \}$ est défini par

$$\{x, y, z\} = [[x, y], z] - [[y, z], x] - [[z, x], y], \ \forall x, y, z \in P.$$

4.2 Algèbres-LY avec un unique produit scalaire invariant

Dans cette section, nous étudions les algèbres-LY admettant un unique, à une constante près, produit scalaire invariant.

Nous montrons que toute algèbre-LY $\mathcal{T}$ sur $\mathbb{F}$ avec $tr(L(x,y)L_z) = 0$, for all $x, y, z \in \mathcal{T}$ qui admet un unique, à une constante près, produit scalaire invariant est nécessairement une algèbre-LY simple. Si le corps $\mathbb{F}$ est algébriquement clos et $\mathcal{T}$ is irréductible telle que $\mathcal{T}$ et $L(\mathcal{T},\mathcal{T})$ ne sont pas isomorphes comme $L(\mathcal{T},\mathcal{T})$-modules, alors $\mathcal{T}$ admet un unique, à une constante près, produit scalaire invariant.

Définition 17

Une algèbre-LY $(\mathcal{T}, [\ ,\ ], \{.,.,.\})$ est dite irréductible si $\mathcal{T}$ est un $L(\mathcal{T},\mathcal{T})$-module irréductible.

Notons par $\mathcal{F}(\mathcal{T})$ l'espace des formes bilinéaires symétriques invariantes sur $\mathcal{T}$ et par $\mathcal{B}(\mathcal{T})$ le sous-espace de $\mathcal{F}(\mathcal{T})$ engendré par les produits scalaires invariants sur $\mathcal{T}$.

Lemme 4.5

Si $\mathcal{T}$ est une algèbre-LY admettant un produit scalaire invariant, alors $\mathcal{B}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$.

Théorème 8

Soit ψ un produit scalaire invariant sur une algèbre-LY $\mathcal{T}$. Il existe un produit scalaire invariant Ψ sur l'algèbre de Lie standard enveloppante $\mathcal{L}(\mathcal{T}) = \mathcal{T} \oplus L(\mathcal{T},\mathcal{T})$, défini par :

$$\Psi_{|\mathcal{T}} = \psi, \quad \Psi(L(\mathcal{T},\mathcal{T}),\mathcal{T}) = 0, \quad \Psi(L(a,b), L(x,y)) = \psi(L(a,b)x, y), \quad \forall a, b, x, y \in \mathcal{T}$$

Remarque 10

Si on suppose que Ψ' est un autre produit scalaire invariant sur $\mathcal{L}(\mathcal{T})$ vérifiant,

$$\Psi'_{|\mathcal{T}} = \psi, \quad \Psi'(L(\mathcal{T},\mathcal{T}),\mathcal{T}) = 0.$$

On obtient,

$$\Psi'(L(x,y),d) = \Psi'([x,y]_{\mathcal{L}(\mathcal{T})} - [x,y], d) = -\Psi'(x, dy) = \psi(dx, y) = \Psi(L(x,y),d),$$

pour tous $x, y \in \mathcal{T}, d \in L(\mathcal{T},\mathcal{T})$, donc $\Psi' = \Psi$. Par consequent, il existe un unique produit scalaire invariant Ψ sur $\mathcal{L}(\mathcal{T})$ vérifiant, $\Psi_{|\mathcal{T}} = \psi, \quad \Psi(L(\mathcal{T},\mathcal{T}),\mathcal{T}) = 0$.

Corollaire 6

On suppose $\mathbb{F}$ algébriquement clos, soit $\mathcal{T}$ une algèbre-LY irreductible telle que $\mathcal{T}$ et $L(\mathcal{T},\mathcal{T})$ ne sont pas isomorphes comme $L(\mathcal{T},\mathcal{T})$-modules. Si ψ_1, ψ_2 sont deux produits scalaires invariants sur $\mathcal{T}$, alors il existe un scalaire non nul λ tel que $\psi_1 = \lambda\psi_2$.

Proposition 4.6

Toute algèbre-LY non abélienne $\mathcal{T}$ avec $\mathcal{T} \neq \mathcal{T}^{(1)} = [T,T] + \{T,T,T\}$, admet une forme bilinéaire symétrique invariante non nulle.

Théorème 9

Soit $\mathcal{T}$ une algèbre-LY non abélienne telle que $tr(L(x,y)L_z) = 0$, for all $x, y, z \in \mathcal{T}$, si $\dim \mathcal{B}(\mathcal{T}) = 1$ alors $\mathcal{T}$ est simple.

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