What every geometer needs to know about Lie groups

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Proposition.

The topology of a Lie group G has the following properties:

1. *G is a locally compact space, i.e., each neighborhood of an element of g contains a compact one.*
2. *The identity component G_0 of G is an open normal subgroup which coincides with the arc-component of 1.*
3. *For a subgroup H of G the following are equivalent:*
   1. *H is a neighborhood of 1.*
   2. *H is open.*
   3. *H is open and closed.*
   4. *H contains G_0.*
4. *If the set $\pi_0(G) := G/G_0$ of connected components of G is countable, then, in addition, the following statements hold:*
   1. *G is countable at infinity, i.e., a countable union of compact subsets.*
   2. *For each 1-neighborhood U in G there exists a sequence (g_n) in G $G = \bigcup_n g_n U$.*
   3. *G is second countable, i.e., the topology of G has a countable basis.*
   4. *If $(U_i)_{i \in I}$ is a pairwise disjoint collection of open subsets of G, then I is countable.*

1. **Product formula:**

$$\exp_G(x+y) = \lim_{k \longrightarrow \infty} \left[\exp_G\left(\frac{x}{k}\right)\exp_G\left(\frac{y}{k}\right)\right]^k$$

2. **Commutator formula:**

$$\exp_G([x,y]) = \lim_{k \longrightarrow \infty} \left[\exp_G\left(\frac{x}{k}\right)\exp_G\left(\frac{y}{k}\right)\exp_G\left(-\frac{x}{k}\right)\exp_G\left(-\frac{y}{k}\right)\right]^{k^2}$$

3. **The Adjoint formula:**

$$\mathrm{Ad}_{\exp_G(x)} = e^{\mathrm{ad}_x}.$$

Baker-Campbell-Dynkin-Hausdorff Formula

If G is a Lie group, then there exists a convex 0-neighborhood $V \subset \mathrm{L}(G)$ such that for $x, y \in V$ the Hausdorff series

$$\begin{aligned} x * y &= x + \sum_{k,m>0,p_i+q_i>0} \frac{(-1)^k}{(k+1)(q_1+\ldots+q_k+1)} \\ &\quad \frac{(\mathrm{ad}_x)^{p_1}(\mathrm{ad}_y)^{q_1}\ldots(\mathrm{ad}_x)^{p_k}(\mathrm{ad}_y)^{q_k}(\mathrm{ad}_x)^m}{p_1!q_1!\ldots p_k!q_k!m!} y \\ &= x + y + \frac{1}{2}[x,y] + \frac{1}{12}\left([x,[x,y]] - [y,[x,y]]\right) + \ldots \end{aligned}$$

converges and satisfies

$$\exp_G(x * y) = \exp_G(x)\exp_G(y).$$

Closed Subgroup Theorem

Let G be a Lie group and $H \subset G$ a closed subgroup. We define the set

$$\mathrm{L}^e(H) := \{x \in \mathrm{L}(G) : exp_G(\mathbb{R}x) \subset H\}.$$

We deduce from the product formula and the commutator formula that $\mathrm{L}^e(H)$ is a Lie subalgebra.

The key point is that if $E \subset \mathrm{L}(G)$ be a vector subspace complementing $\mathrm{L}^e(H)$, then there exists a 0-neighborhood $U_E \subset E$ such that

$$H \cap \exp_G(U_E) = \{1\}.$$

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Theorem. (Closed Subgroup Theorem)

Let H be a closed subgroup of the Lie group G. Then the following assertions hold:

1. *Each 0-neighborhood in* $\mathrm{L}^e(H)$ *contains an open 0-neighborhood V such that* $exp_G : V \longrightarrow exp_G(V)$ *is a homeomorphism onto an open subset of H.*

2. *H is a submanifold of G and* $m_H := (m_G)_{|H\times H}$ *induces a Lie group structure on H such that the inclusion map* $i_H : H \longrightarrow G$ *is a morphism of Lie groups for which* $\mathrm{L}(i_H) : \mathrm{L}(H) \longrightarrow \mathrm{L}(G)$ *is an isomorphism of* $\mathrm{L}(H)$ *onto* $\mathrm{L}^e(H)$.

3. *Let* $E \subset \mathrm{L}(G)$ *be a vector space complement of* $\mathrm{L}^e(H)$. *Then there exists an open 0-neighborhood* $V_E \in E$ *such that*

$$\phi : V_E \times H \longrightarrow exp_G(V_E)H, (x, h) \mapsto exp_G(x)h$$

is a diffeomorphism onto an open subset of G.

Characterization of closed subgroup

Proposition.

A subgroup of a Lie group is a Lie group with respect to the induced topology if and only if it is closed.

Covering of Lie groups.

Proposition.

Let $\phi : G \longrightarrow H$ be a continuous homomorphism of topological groups which is a covering map. If G or H is a Lie group, then the other group carries a unique Lie group structure for which ϕ is a morphism of Lie groups which is a local diffeomorphism.

Proposition.

If G is a connected Lie group and $q_G : \widetilde{G} \longrightarrow G$ its universal covering space, then $\widetilde{G}$ carries a unique Lie group structure for which q_G is a smooth covering map. We call this Lie group the simply connected covering group of G.

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Theorem. (Integral Subgroup Theorem)

Let G be a Lie group with Lie algebra $\mathrm{L}(G)$ and $\mathfrak{h} \subset \mathrm{L}(G)$ a Lie subalgebra. Then the subgroup $H := \langle \exp_G(\mathfrak{h}) \rangle$ of G generated by $\exp_G(\mathfrak{h})$ carries a Lie group structure with the following properties:

1. *There exists an open 0-neighborhood $W \subset \mathfrak{h}$ on which the Hausdorff series converges, and $\exp_G : \mathfrak{h} \longrightarrow H$ maps W diffeomorphicly onto its open image in H and satisfies $\exp_G(x * y) = \exp_G(x) \exp_G(y)$ for $x, y \in W$.*
2. *The inclusion $i_H : H \longrightarrow G$ is a smooth morphism of Lie groups and $\mathrm{L}(i_H) : \mathrm{L}(H) \longrightarrow \mathfrak{h}$ an isomorphism of Lie algebras. These two properties determine the Lie group structure on H uniquely.*
3. *If $H \subset H_1$ for some subgroup H_1 for which H_1/H is countable, then $\mathfrak{h} = \{x \in \mathrm{L}(G) : \exp_G(\mathbb{R}x) \subset H1\}$. In particular,*

$$\mathfrak{h} = \mathrm{L}^e(H) := \{x \in \mathrm{L}(G) : \exp_G(\mathbb{R}x) \subset H\}.$$

4. *H is connected.*
5. *H is closed in G if and only if i_H is a topological embedding.*

Definition.

Let G be a Lie group. An integral subgroup H of G is a subgroup that is generated by exp h for a subalgebra h of the Lie algebra g of G.

The Integral Subgroup Theorem implies in particular that each Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathrm{L}(G)$ of a Lie group G is integrable in the sense that it is the Lie algebra of some Lie group H.

Combining this with Ado's Theorem on the existence of faithful linear representations of a Lie algebra, we obtain one of the cornerstones of the theory of Lie groups:

Theorem. (Lie's Third Theorem)

Each finite-dimensional Lie algebra $\mathfrak{g}$ is the Lie algebra of a connected Lie group G.

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Connected Lie groups with the same universal covering.

Theorem.

Let G be a connected Lie group and $q_G : \widetilde{G} \longrightarrow G$ a universal covering homomorphism. Then $\ker q_G a \sim \pi_1(G)$ is a discrete central subgroup and $G \simeq \widetilde{G}/\ker q_G$. Moreover, for any discrete central subgroup $\Gamma \subset \widetilde{G}$, the group $\widetilde{G}/\Gamma$ is a connected Lie group with the same universal covering group as G. We thus obtain a bijection from the set of all $Aut(\widetilde{G})$-orbits in the set of discrete central subgroups of $\widetilde{G}$ onto the set of isomorphy classes of connected Lie groups whose universal covering is isomorphic to $\widetilde{G}$.

Connected Abelian Lie groups.

Theorem.

Any abelian connected Lie group of dimension n is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Monodromy Principle.

Proposition. (Monodromy Principle)

Let G be a connected simply connected Lie group and H a group. Let V be an open symmetric connected identity neighborhood in G and $f : V \longrightarrow H$ a function with

$$f(xy) = f(x)f(y) \quad \textit{for} \quad x, y, xy \in V.$$

Then there exists a unique group homomorphism extending f. If, in addition, H is a Lie group and f is smooth, then its extension is also smooth.

Theorem. (Integrability Theorem for Lie Algebra Homomorphisms)

Let G *be a* **connected simply connected Lie group**, H *a Lie group and* $\phi : \mathrm{L}(G) \longrightarrow \mathrm{L}(H)$ *a Lie algebra morphism. Then there exists a unique morphism* $\Phi : G \longrightarrow H$ *with* $\mathrm{L}(\Phi) = \psi$.

The structure of Lie group of $\mathrm{Aut}(G)$ when G is 1-connected

We recall that a Lie group G is called 1-connected if it is connected and simply connected.

Theorem.

If G is a 1-connected Lie group with Lie algebra $\mathfrak{g}$, then the map

$$\mathbf{L} : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

is an isomorphism of groups. As a closed subgroup of $\mathrm{GL}(\mathfrak{g})$, the group $\mathrm{Aut}(\mathfrak{g})$ carries a natural Lie group structure, and we endow $\mathrm{Aut}(G)$ with the Lie group structure for which $\mathbf{L}$ is an isomorphism of Lie groups. Then the action of $\mathrm{Aut}(G)$ on G is smooth.

Classification of Lie connected Lie groups with given Lie algebra.

Theorem.

Two connected Lie groups G and H have isomorphic Lie algebras if and only if their universal covering groups $\widetilde{G}$ and $\widetilde{H}$ are isomorphic

Theorem.

Let G be a connected Lie group and $q : \widetilde{G} \longrightarrow G$ the universal covering morphism of connected Lie groups. Then for each discrete central subgroup $\Gamma \subset \widetilde{G}$, the group G/Γ is a connected Lie group with $\mathrm{L}(G/\Gamma) = \mathrm{L}(G)$ and, conversely, each Lie group with the same Lie algebra as G is isomorphic to some quotient G/Γ and $G/\Gamma_1 \simeq G/\Gamma_2$ if and only if there exits $\phi \in \mathrm{Aut}(\widetilde{G}) \simeq \mathrm{Aut}(\mathfrak{g})$ such that $\phi(\Gamma_1) = \Gamma_2$.

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An algorithm for determining the connected Lie groups with a given Lie algebra $\mathfrak{g}$.

1. Find the 1-connected Lie group $\widetilde{G}$ such that $\mathrm{L}(\widetilde{G}) = \mathfrak{g}$.
2. Determine the center $Z(\widetilde{G})$ and $\mathrm{Aut}(\widetilde{G}) \simeq \mathrm{Aut}(\mathfrak{g})$.
3. Find the set $\mathbb{D}$ of discrete subgroups of $Z(\widetilde{G})$ and $\mathbb{D}/\mathrm{Aut}(\widetilde{G})$.

Yamabe's Theorem.

Theorem. (Yamabe)

A subgroup H of a connected Lie group G is arcwise connected if and only if it is an integral subgroup. More precisely, H is of the form $\langle \exp_G(\mathfrak{h}) \rangle$ for the Lie subalgebra $\mathfrak{h}$ of $\mathrm{L}(G)$, determined by

$$\mathfrak{h} = \{x \in \mathrm{L}(G) : \exp_G(\mathbb{R}x) \subset H\}.$$

Jordan–Hölder series of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra. There exists a sequence

$$\mathfrak{a}_0 = 0 \subset \mathfrak{a}_1 \subset \ldots \subset \mathfrak{a}_k = \mathfrak{g}$$

of subalgebras of $\mathfrak{g}$ for which $\mathfrak{a}_{j-1}$ is an ideal in $\mathfrak{a}_j$ and the quotient $\mathfrak{a}_j/\mathfrak{a}_{j-1}$ is either one-dimensional or simple. Such a series is called Jordan–Hölder series of $\mathfrak{g}$.

In particular, we have

$$\mathfrak{g} \simeq ((\ldots((\mathfrak{a}_1 \rtimes (\mathfrak{a}_2/\mathfrak{a}_1)) \rtimes (\mathfrak{a}_3/\mathfrak{a}_2))\ldots) \rtimes (\mathfrak{a}_k/\mathfrak{a}_{k-1}))\,.$$

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Proposition.

Let G, H and N be simply connected Lie groups with the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{n}$. Suppose that $\mathfrak{g} \simeq \mathfrak{n} \rtimes_\beta \mathfrak{h}$ is a semidirect sum of the two subalgebras $\mathfrak{n}$ and $\mathfrak{h}$. Then there is a unique smooth action $\gamma : H \longrightarrow \mathrm{Aut}(N) \simeq \mathrm{Aut}(\mathfrak{n})$ with $L(\gamma) = \beta : \mathfrak{h} \longrightarrow \mathrm{der}\,\mathfrak{n}$ and the two natural homomorphisms $i_H : H \longrightarrow G$, $i_N : N \longrightarrow G$ combine to an isomorphism

$$\mu : N \rtimes_\gamma H \longrightarrow G, (n, h) \mapsto i_N(n) i_H(h).$$

Theorem. (Smooth Splitting Theorem)

Let G be a simply connected Lie group and $N \subset G$ be a normal integral subgroup. Then N is closed and there exists a smooth section $\sigma : G/N \longrightarrow G$, so that the map $G/N \times N \longrightarrow G$, $(p, n) \mapsto \sigma(p)n$ is a diffeomorphism, but in general not an isomorphism of Lie groups. In particular, the groups N and G/N are simply connected.

Proof.

By considering a Jordan-Hölder series of $L(G)/L(N)$, there exists an increasing sequence of subalgebras

$$L(N) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_n = L(G).$$

such that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i+1}$, and the quotients $q_i := \mathfrak{g}_i/\mathfrak{g}_{i-1}$ are either isomorphic to $\mathbb{R}$ or simple. Using Levi's Theorem we conclude that $\mathfrak{g}_i \simeq \mathfrak{g}_{i-1} \rtimes q_i$ for $i = 1, ..., n$. So

$$G \simeq ((\ldots((G(\mathfrak{g}_0) \rtimes G(q_1)) \rtimes G(q_2))\ldots) \rtimes G(q_n)) .$$

□

continued.

This implies in particular that $N = \langle \exp_G L(N) \rangle \simeq G(\mathfrak{g}_0)$ is a closed simply connected subgroup of G. In particular, we obtain diffeomorphisms

$$G \longrightarrow N \times G(q_1) \times \ldots \times G(q_n) \quad \text{and} \quad G/N \longrightarrow G(q_1) \times \ldots \times G(q_n).$$

Hence the normal subgroup N is closed and there exists a smooth section $\sigma : G/N \longrightarrow G$. Finally, the existence of a diffeomorphism $G/N \times N \longrightarrow G$ implies that N and G/N are connected and simply connected. $\square$

Corollary.

Let G be a simply connected Lie group and $N = \exp_G(\mathrm{rad}(\mathfrak{g}))$. Then N is a normal subgroup of G, G/N is semi-simple and G is diffeomorphic as a manifold to $G/N \times N$.

Commutators, Nilpotent and solvable Lie groups

Definition.

Let G be a group. For two subgroups $A, B \subset G$, we define (A, B) as the subgroup generated by the commutators $xyx^{-1}y^{-1}$ for $x \in A$ and $y \in B$. If we set

$$C^1(G) := G \quad \text{and} \quad C^n(G) := (G, C^{n-1}(G)) \quad \text{for} \quad n > 1,$$

then $(C^n(G))_{n \in \mathbb{N}^*}$ is called the lower central series of G.
If we set $D^0(G) := G$ and $D^n(G) := (D^{n-1}(G), D^{n-1}(G))$, then the sequence $(D^n(G))_{n \in \mathbb{N}}$ is called the derived series of G.
The subgroup $C^2(G) = D^1(G)$ is called the commutator subgroup of G and often denoted by G'.

Definition.

1. *A Lie group G is called abelian if $C^2(G) = \{1\}$.*
2. *A Lie group G is called nilpotent if there exists $n \in \mathbb{N}^*$ such that $C^n(G) = \{1\}$.*
3. *A Lie group G is called solvable if there exists $n \in \mathbb{N}$ such that $D^n(G) = \{1\}$.*

Lemma.

If $A, B \subset G$ are integral subgroups with the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$, then (A, B) also is an integral subgroup and its Lie algebra contains $[\mathfrak{a}, \mathfrak{b}]$.

Proof.

We have $(A, B) = \bigcup_{n \in \mathbb{N}} T^n$ where $T = S \cup S^{-1}$ and

$$S = \{xyx^{-1}y^{-1}; x \in A, y \in B\}.$$

This implies that (A, B) is arcwise connected and according to Yamabe's Theorem it is an integral Lie subgroup with its Lie algebra

$$\mathfrak{h} = \{x \in \mathfrak{g}, \exp_G(\mathbb{R}x) \subset (A, B)\}.$$

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Proof.

Let $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Th curve

$$\begin{aligned}\gamma(t) &= \exp_G(a)\exp_G(tb)\exp_G(-a)\exp_G(-tb) \\ &= \exp_G(a)\exp_G(-e^{t\,\mathrm{ad}_b}(a)) \in (A, B),\end{aligned}$$

satisfies

$$\gamma'(0) = (e^{\mathrm{ad}a} - \mathrm{Id}_{\mathfrak{g}})b \in \mathfrak{h}.$$

So for any $s \in \mathbb{R}$, $(e^{s\mathrm{ad}a} - \mathrm{Id}_{\mathfrak{g}})b \in \mathfrak{h}$ and hence $[a, b] \in \mathfrak{h}$. □

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Proposition.

Let A, B and C be integral subgroups of the connected Lie group G with Lie algebras $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$, satisfying

$$[\mathfrak{a}, \mathfrak{c}] \subset \mathfrak{c}, [\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c} \quad \textit{and} \quad [\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{c}.$$

Then $(A, B) \subset C$ and if, in addition, $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c}$, then $(A, B) = C$. In particular, $(A, B) = \exp_G([\mathfrak{a}, \mathfrak{b}])$ and $L(A, B) = [\mathfrak{a}, \mathfrak{b}]$.

Proof.

$\mathfrak{a} + \mathfrak{b} + \mathfrak{c}$ is a subalgebra and we suppose that $L(G) = \mathfrak{a} + \mathfrak{b} + \mathfrak{c}$.
Then $\mathfrak{c}$ is an ideal and hence C is a normal subgroup.

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Proof.

Suppose first that G is simply connected. Then C is closed normal subgroup and $q : G \longrightarrow G/C$ is morphism of Lie groups. Then

$$[L(q)\mathfrak{a}, L(q)\mathfrak{b}] \subset L(q)[\mathfrak{a}, \mathfrak{b}] = 0.$$

Thus $[L(q)\mathfrak{a}, L(q)\mathfrak{b}] = 0$ hence $q(A)$ and $q(B)$ commute and hence $q(A, B) = 0$ so $(A, B) \subset C$.
Moreover, if $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c}$ then $\mathfrak{c} \subset L(A, B)$ and hence $C = (A, B)$.

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Proposition.

For any connected Lie group G with Lie algebra $\mathfrak{g}$, the groups $D^n(G)$ in the derived series and $C^n(G)$ in the lower central series are normal integral subgroups with the Lie algebras

$$L(D^n(G)) = D^n(\mathfrak{g}) \quad \textit{and} \quad L(C^n(G)) = C^n(\mathfrak{g}) \quad \textit{for} \quad n \in \mathbb{N}.$$

Since an integral subgroup is trivial if and only if its Lie algebra is, we derive the following important theorem connecting nilpotency and solvability of Lie groups and Lie algebras.

Theorem.

A connected Lie group G is abelian, nilpotent, resp., solvable, if and only if its Lie algebra is abelian, nilpotent, resp., solvable.

Nilpotent Lie Groups

Theorem. (Local-Global Theorem for Nilpotent Lie Groups)

If $\mathfrak{g}$ *is a nilpotent Lie algebra, then the Dynkin series defines a polynomial map*

$$* : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (x, y) \mapsto x + y + \frac{1}{2}[x, y] +$$

$$\frac{1}{12}\left([x,[x,y]] - [y,[x,y]]\right) + \ldots$$

We thus obtain a Lie group structure $(\mathfrak{g}, *)$ *with* $\exp_{\mathfrak{g}} = \mathrm{Id}_{\mathfrak{g}}$ *and* $L(\mathfrak{g}, *) = \mathfrak{g}$.

Corollary.

Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent and $\exp_G : (\mathfrak{g}, *) \longrightarrow G$ is the universal covering morphism of G. In particular, the exponential function of G is surjective.

Proof.

$(\mathfrak{g}, *)$ is a 1-connected Lie group with Lie algebra $\mathfrak{g}$. Let $q_G : (\mathfrak{g}, *) \longrightarrow G$ be the unique morphism of Lie groups with $L(q_G) = \mathrm{Id}_{\mathfrak{g}}$. Then

$$q_G(x) = q_G(\exp_{(\mathfrak{g},*)}(x)) = \exp_G(L(q_G)x) = \exp_G(x)$$

implies that $\exp_G = q_G$. □

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We know already that any connected Lie group G is isomorphic to $\widetilde{G}/\Gamma$, where Γ is a discrete central subgroup of the center of universal covering group of G. To understand the structure of connected nilpotent Lie groups, we therefore need more information on the center of the simply connected groups $(\mathfrak{g}, *)$.

Lemma.

If $\mathfrak{g}$ is a nilpotent Lie algebra, then the center of the group $(\mathfrak{g}, *)$ coincides with the center $z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$.

Proof.

The inclusion $z(\mathfrak{g}) \subset Z(\mathfrak{g}, *)$ is immediate. If, conversely, $z \in Z(\mathfrak{g}, *)$, then $\mathrm{id}_{\mathfrak{g}} = \mathrm{Ad}(\exp_{(\mathfrak{g},*)} z) = e^{\mathrm{ad} z}$, and thus, $\mathrm{ad} z = 0$ since $\mathrm{ad} z$ is nilpotent and the exponential function is injective on the set of nilpotent elements of $\mathrm{End}(\mathfrak{g})$.

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Proposition.

If G is a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, then $Z(G) = \exp_G(z(\mathfrak{g}))$ is connected.

Proof.

Since $\exp_G : (\mathfrak{g}, *) \longrightarrow G$ is the universal covering morphism of G, we obtain

$$Z(G) = \ker \mathrm{Ad}_G = \exp_G(\ker \mathrm{Ad}_{(\mathfrak{g},*)}) = \exp_G(z(\mathfrak{g})).$$

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Theorem. (Structure Theorem for Connected Nilpotent Lie Groups)

Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then there exists a discrete subgroup $\Gamma \subset (z(\mathfrak{g}), +)$ with $G \simeq (\mathfrak{g}, *)/\Gamma$. In particular, G is diffeomorphic to the abelian Lie group $\mathfrak{g}/\Gamma$. Moreover, $\mathfrak{t} := \mathrm{span}\Gamma \subset z(\mathfrak{g})$ is a central Lie subalgebra for which $T := \exp_G(\mathfrak{t})$ is a torus, and G is diffeomorphic to the product manifold $(G/T) \times T$.

Corollary.

Any compact connected nilpotent Lie group is abelian.

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Solvable Lie groups

The structure of solvable Lie groups is substantially more complicated than the structure of the nilpotent ones. In particular, the exponential function of a solvable Lie group need not be surjective. Other difficulties arise from the fact that the center of a connected solvable Lie group is not connected.

Theorem.

If G is a 1-connected solvable Lie group, then there exists a basis $(x_1, \ldots, x_n)$ for its Lie algebra $\mathfrak{g}$ such that the map

$$\Phi : \mathbb{R}^n \longrightarrow G, \quad (t_1, \ldots, t_n) \mapsto \prod_{i=1}^{n} \exp_G(t_i x_i)$$

is a diffeomorphism, the subgroups $R_j := \exp_G(\mathbb{R}xj)$ of G are closed, and

$$G \simeq ((\ldots((R_1 \rtimes R_2) \rtimes R_3) \ldots) \rtimes R_n) .$$

Proof.

First, we observe that there exist subalgebras

$$\mathfrak{g}_0 = 0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g}$$

with $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i+1}$ and such that $\mathfrak{g}_{i+1}/\mathfrak{g}_i \simeq \mathbb{R}$. Then we pick $x_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$ and obtain

$$G \simeq ((\ldots((\mathbb{R}x_1 \rtimes \mathbb{R}x_2) \rtimes \mathbb{R}x_3)\ldots) \rtimes \mathbb{R}x_n)\,.$$

□

Theorem.

Any integral subgroup H of a simply connected solvable Lie group G is closed and simply connected and G/H is diffeomorphic to $\mathbb{R}^{\dim G/H}$.

Compact Lie groups

Definition.

A Lie algebra $\mathfrak{g}$ is called compact if there exists a positive definite, invariant and symmetric bilinear form on $\mathfrak{g}$.

Lemma.

1. *Every subalgebra of a compact Lie algebra is compact.*
2. *A direct sum* $\mathfrak{g} = \sum_{i=1}^{n} \mathfrak{g}_i$ *of Lie algebras* $\mathfrak{g}_i$ *is compact if and only if all the* $\mathfrak{g}_i$ *are compact.*
3. *If* $\mathfrak{a} \subset \mathfrak{g}$ *is an ideal, then the orthogonal complement* $\mathfrak{a}^\perp$ *with respect to any invariant scalar product is also an ideal, and* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ *is a Lie algebra direct sum.*
4. *Every compact Lie algebra is reductive.*

Definition.

Let $\mathfrak{g}$ *be a Lie algebra, and let* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{der}(\mathfrak{g})$ *be the adjoint representation. For a subalgebra* $\mathfrak{a} \subset \mathfrak{g}$*, we set*

$$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{a}) := \langle e^{\mathrm{ad}(\mathfrak{a})} \rangle \subset \mathrm{Aut}(\mathfrak{g}) \quad \textit{and} \quad \mathrm{INN}_{\mathfrak{g}}(\mathfrak{a}) := \overline{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{a})}.$$

We also write $\mathrm{Inn}(\mathfrak{g}) := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g})$ *and recall that*
$\mathrm{L}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{der}(\mathfrak{g})$

Proposition.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the following are equivalent:

1. *There exists a compact Lie group G with $\mathrm{L}(G) = \mathfrak{g}$.*
2. $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ *is compact.*
3. $\mathfrak{g}$ *is compact.*

Definition.

Let G and N be Lie groups and $\alpha : G \longrightarrow \mathrm{Aut}(N)$ be a homomorphism defining a smooth action of G on N.
A smooth function $f : G \longrightarrow N$ is called a 1-cocycle or a crossed homomorphism with respect to α if

$$f(ab) = f(a).\alpha(a)(f(b))$$

for $a, b \in G$. Note that this condition is equivalent to

$$(f, \mathrm{id}_G) : G \longrightarrow N \times_\alpha G$$

being a morphism of Lie groups.

Lemma.

Let G be a Lie group and N a closed normal subgroup. Then G acts smoothly on N by $\alpha(g)(n) = gng^{-1}$, and the following are equivalent:

1. *The short exact sequence* $1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$ *of Lie groups splits.*
2. *There exists a closed subgroup $H \subset G$ for which the multiplication map*
$$\mu : N \times_\alpha H \longrightarrow G,\ (n, h) \mapsto nh$$
is an isomorphism of Lie groups.
3. *There exists a 1-cocycle $f : G \longrightarrow N$ with $f(n) = n^{-1}$ for $n \in N$.*

Lemma.

Let V be a finite-dimensional vector space and G a Lie group. Further, let $H \subset G$ be a closed subgroup, $f : G \longrightarrow V$ be a compactly supported smooth function and μ_H be a left Haar measure on H. Then the function

$$F : G \longrightarrow V, F(a) := \int_H f(ah) d\mu_H(h)$$

is smooth.

Lemma.

Let G be a Lie group and $H \subset G$ a closed subgroup for which G/H is compact. Then there is a nonnegative smooth function $\varphi : G \longrightarrow \mathbb{R}$ with compact support such that

$$\int_H \varphi(ah) d\mu_H(h) = 1, \quad \textit{for all } a \in G,$$

where μ_H is left invariant Haar measure on H.

Remark.

The function φ in the preceding lemma can be used to define for each finite-dimensional vector space V a projection

$$P : C^\infty(G, V) \longrightarrow C^\infty(G, V)^H,$$

$$P(f)(a) := \int_H \varphi(ah) f(ah) d\mu_H(h),$$

where $C^\infty(G, V)^H$ denotes the subspace of all smooth functions $f : G \longrightarrow V$ which are constant on the H-left cosets, so that they correspond to smooth functions $f : G/H \longrightarrow V$ via $f(aH) := f(a)$.

Lemma.

Let G be a Lie group and $N \subset G$ a closed normal subgroup for which G/N is compact. Suppose that $\rho : G \longrightarrow \mathrm{GL}(V)$ is a finite-dimensional smooth representation of G with $N \subset \ker \rho$ and $f : N \longrightarrow V$ is a smooth homomorphism which is G-equivariant, i.e.,

$$f(ana^{-1}) = \rho(a)(f(n)) \quad \textit{for} \quad a \in G, n \in N.$$

Then there exists a 1-cocycle $f^* : G \longrightarrow V$ with respect to ρ extending f.

Theorem. (Splitting Theorem)

Let G be a Lie group and $V \subset G$ be a normal vector subgroup such that G/V is compact. Then there exists a compact subgroup $K \subset G$ such that $G \simeq V \rtimes K$.

Lemma. (Torus Splitting Lemma)

Let $\mathbb{T}$ be a torus and $A \subset \mathbb{T}$ be a closed connected subgroup. Then there is a homomorphism $f : \mathbb{T} \longrightarrow A$ with $f_{|A} = \mathrm{id}_A$. This implies in particular, that for the closed subgroup $B := \ker f$, the multiplication map

$$\phi : A \times B \longrightarrow \mathbb{T}, \quad (a, b) \mapsto ab$$

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Definition.

We call a connected Lie group G semisimple, resp., simple if its Lie algebra $\mathrm{L}(G)$ is semisimple, resp., simple.

Lemma.

Let G be a connected locally compact group and $D \subset G$ a discrete central subgroup such that G/D is compact and the commutator group is dense in G/D. Then D is finite and G is compact.

Theorem. (Weyl's theorem on Lie groups with simple compact Lie algebra)

If G is a connected semisimple Lie group with compact Lie algebra, then G is compact and $Z(G)$ is finite.

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If G is a connected semisimple Lie group with compact Lie algebra, then G is compact and $Z(G)$ is finite.

Theorem. (Structure Theorem for Groups with Compact Lie Algebra)

1. *Every connected Lie group G with compact Lie algebra is a direct product of a vector group V and a uniquely determined maximal compact group K of G which contains all other compact subgroups.*
2. *If $\mathfrak{k} = z(\mathfrak{k}) \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_m$ is the decomposition of the reductive Lie algebra $\mathfrak{k} := \mathrm{L}(K)$ into its center and simple ideals, then the corresponding integral subgroups $Z(K)_0$ and $K_1, \ldots, K_m$ are compact, and the multiplication map*

   $$\phi : Z(K)_0 \times K_1 \times \ldots \times K_m \longrightarrow K, \quad (z, k_1, ..., k_m) \mapsto z k_1 \ldots k_m$$

   is a covering morphism of Lie groups with finite kernel.
3. *The commutator subgroup G' of G is compact.*

Corollary.

Let G be a Lie group with finitely many connected components and $\mathrm{L}(G)$ compact. Then there exists a compact subgroup K and a vector group V with $G \simeq V \rtimes K$.

Maximal Tori in Compact Lie Groups

Lemma.

Let $\mathfrak{g}$ be a compact Lie algebra.

1. *A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra if and only if it is maximal abelian.*
2. *For any such subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ we have $\mathfrak{g} = \mathrm{Inn}(\mathfrak{g})\mathfrak{t}$, i.e., each element is conjugate to an element of $\mathfrak{t}$. (c) Any other Cartan subalgebra of g is conjugate under $\mathrm{Inn}(\mathfrak{g})$ to $\mathfrak{t}$.*

Theorem. (Main Theorem on Maximal Tori)

For a compact connected Lie group G, the following assertions hold:

1. *A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is maximal abelian if and only if it is the Lie algebra of a maximal torus of G.*
2. *For two maximal tori T and T_0, there exists a $a \in G$ with $aTa^{-1} = T_0$.*
3. *Every element of G is contained in a maximal torus.*

Corollary.

The exponential function of a connected Lie group with compact Lie algebra is surjective.

Corollary.

The center of a connected compact Lie group is the intersection of all maximal tori.

Corollary.

Let G be a compact connected Lie group and $a \in G$. Then a belongs to the connected component $Z_G(a)_0$ of its centralizer $Z_G(a)$. Moreover, $Z_G(a)_0$ is the union of all maximal tori of G containing a.

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Theorem. (Hofmann–Scheerer Splitting Theorem)

Let G be a connected compact Lie group and G' be its commutator group. Then there exists a torus $B \subset G$ with $G \simeq G' \rtimes B$.

Corollary.

For a compact connected Lie group G with $\dim Z(G) = r$,

$$\pi_1(G) \simeq \mathbb{Z}^r \times \pi_1(G')$$

where $\pi_1(G')$ is finite.

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Centralizers of Tori and the Weyl Group

Lemma.

If G is a compact abelian Lie group such that G/G_0 is cyclic, then G contains a dense cyclic subsemigroup.

Theorem.

Let G be a compact connected Lie group, $T \subset G$ a torus, and $a \in Z_G(T)$. Then there exists a torus $T_0 \in G$ containing a and T.

Corollary.

Let G be a compact connected Lie group and T a torus in G.

1. *The centralizer $Z_G(T)$ of T in G is connected.*
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Definition.

Let G be a compact connected Lie group and T a maximal torus in G. Then the group $W(G,T) := N_G(T)/Z_G(T) = N_G(T)/T$ is called the analytic Weyl group associated with (G,T).

Proposition.

Let G be a compact connected Lie group and $T \subset G$ a maximal torus.

1. *If $t_1, t_2 \in T$ are conjugate under G, then there exists a $g \in N_G(T)$ such that $gt_1g^{-1} = t_2$.*
2. *The set of conjugacy classes of G is parameterized by the set $T/W(G,T)$ of $W(G,T)$-orbits in T.*
3. *A continuous function $f : T \longrightarrow \mathbb{C}$ extends to a continuous function $F : G \longrightarrow \mathbb{C}$ invariant under conjugation if and only if it is $W(G,T)$-invariant.*

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Theorem. (Linearity Theorem for Compact Lie Groups)

Each compact Lie group K has a faithful finite-dimensional unitary representation. So each compact Lie group is isomorphic to a matrix group.