Virtual Knots

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Online AGTA seminar 2020-21.

Session October 31, 2020

Plan

Classical Knot Theory: CKT

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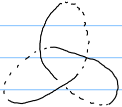
- Diagrammatic representation of knots in $S^2 \times [0, 1]$.

(1)

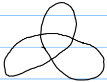
$\mathbb{R}^3$



$\sim \quad S^3 \xrightarrow{p_c} S^2$

$p \downarrow$ regular projection

$\mathbb{R}^2$



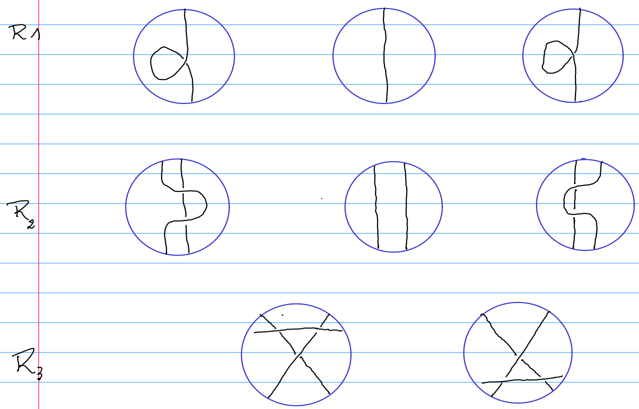
graph $\longrightarrow$

crossing

diagram

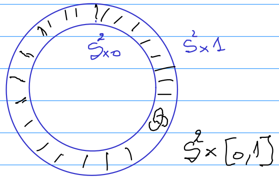
The crossing structure is an artifact of the projection to the plane.

② Reidemeister Moves



③ Knots in $S^2 \times [0,1]$

projection on $S^2 \times 1$



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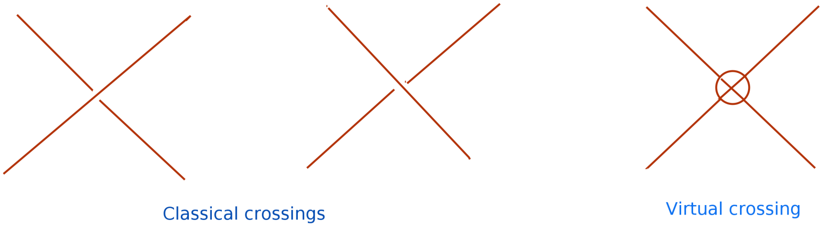

Figure: Crossings and virtual crossing

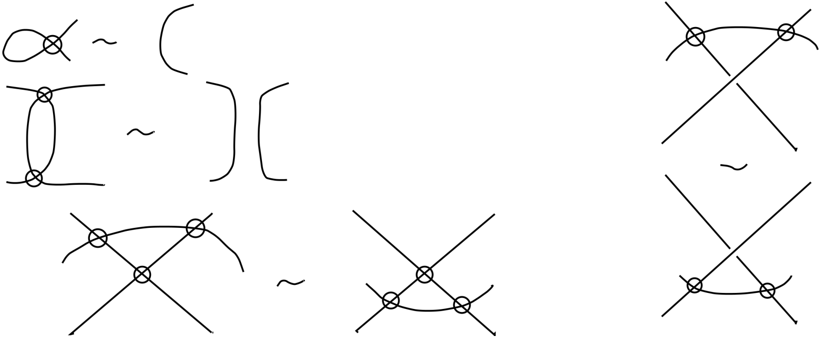

Figure: Virtual and semi-virtual Reidemeister moves

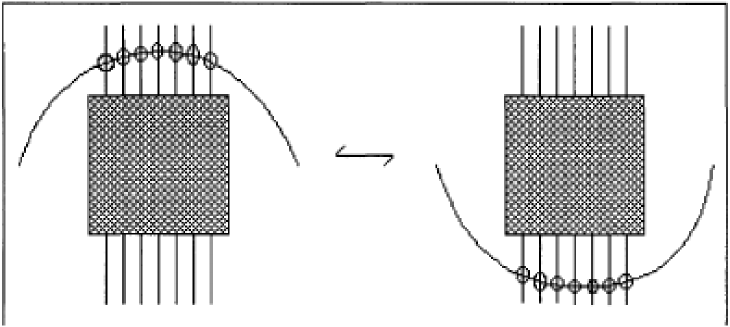

Figure: General equivalence

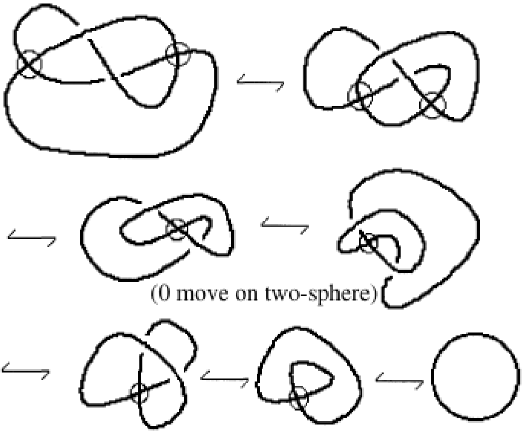

Figure: Virtual equivalences

Plan

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- The extension of knot theory to the purely combinatorial domain of Gauss codes and Gauss diagrams.

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Let S_g be a surface of genus g, then the knot theory in $S_g \times [0, 1]$ is represented by diagrams drawn on S_g taken up to the usual Reidemeister moves transferred to diagrams on this surface.

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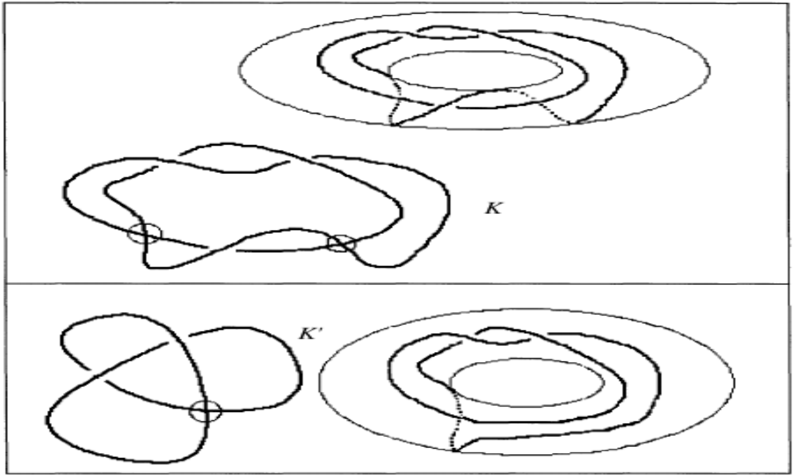
Remark : The VKT does not demand the use of a particular surface embedding, but it does apply to such embeddings. This constitutes one of the motivations.



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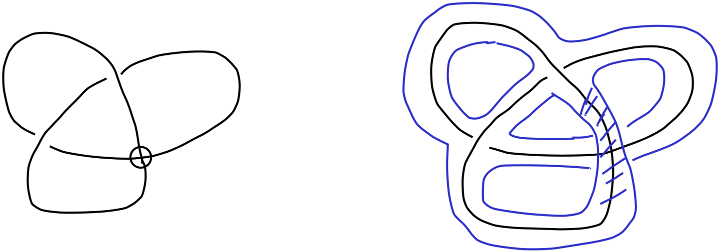

Figure: Band crossing

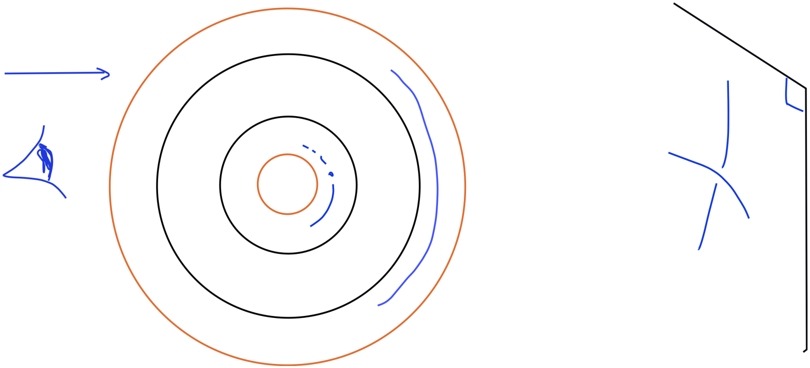

Figure: Origin of the virtual Crossing

Gauss Codes

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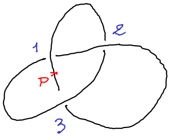
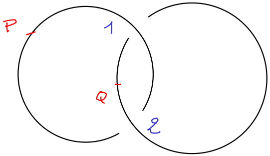


123123

O1U2O3U1O2U3



12/12

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*If g is a single component **planar** Gauss code, then g is evenly intersticed.*

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Lemma

*If g is a single component **planar** Gauss code, then g is evenly intersticed.*

Proof : This follows directly from the Jordan curve theorem in the plane.

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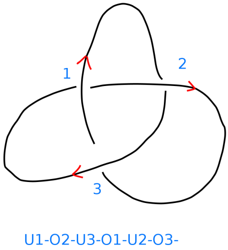
2. Non-planar Gauss codes give rise to an infinite collection of virtual knots.

Oriented Gauss codes



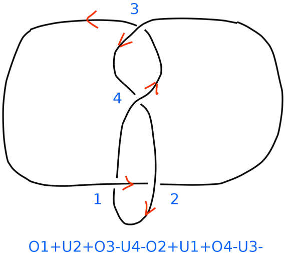


Figure: Signed Gauss code

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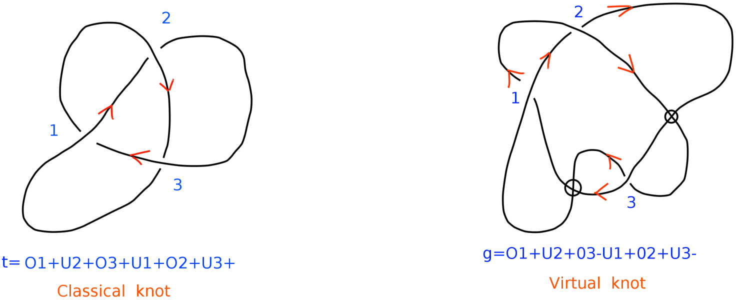

Figure: Effect of changing sign

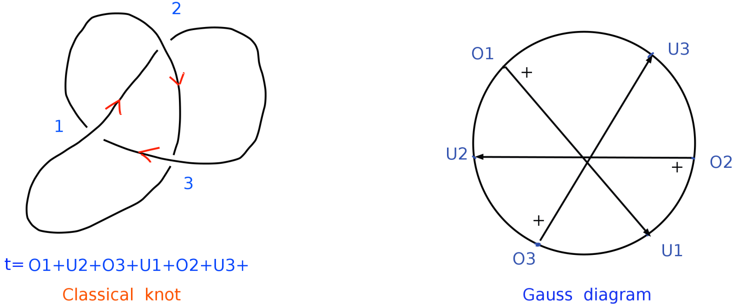

Figure: Gauss diagram

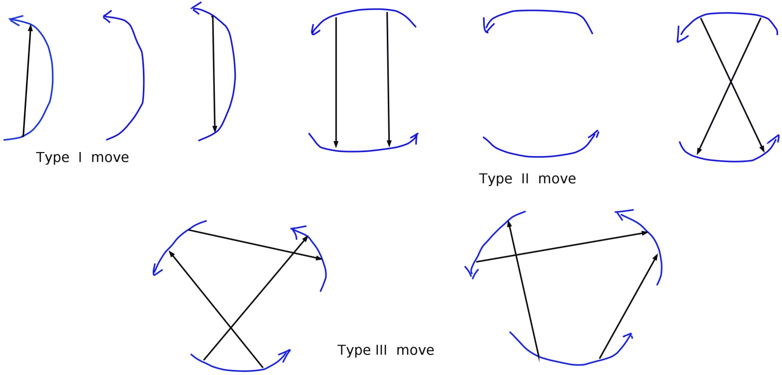

Figure: Reidemeister moves in terms of Gauss codes

Definition

A virtual knot is an equivalence class of Gauss diagrams (or Gauss codes). Two Gauss diagrams represent the same virtual knot if there is a sequence of the three Reidemeister moves that changes one diagram into the other. Such a sequence will be called a ***virtual isotopy***.

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Definition

A Gauss code which represents an ordinary knot diagram is called ***classical*** or ***realizable***, and a Gauss diagram which does not represent a real knot (i.e. one for which we have to introduce additional crossings to make the diagram work) is called ***non-classical*** or ***non-realizable***.

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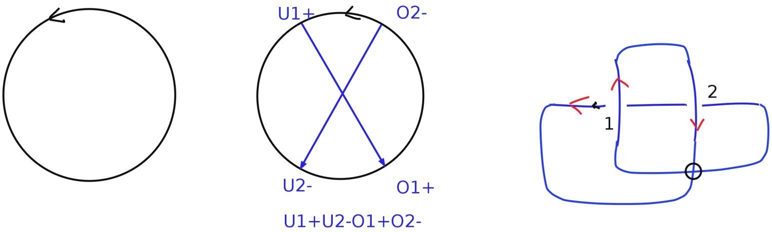

Figure: non-realizable Gauss code

Theorem

Two virtually equivalent classical knots are classically equivalent. In other words, if two realizable Gauss diagrams are virtually isotopic, then they have realizations which are connected by a sequence of ordinary isotopies.

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Remark : The signed codes are knot theoretic anlogues of the set of all graphs, and the classical knot are the analogues of planar graphs. This is the fundamental combinatorial motivation for the last definition of virtual knots.

Theorem

If K is a virtual knot whose underlying Gauss code is planar and whose sign sequence is standard, then K is equivalent to a classical knot.