Morphing of Hyperbolic Closed Curves

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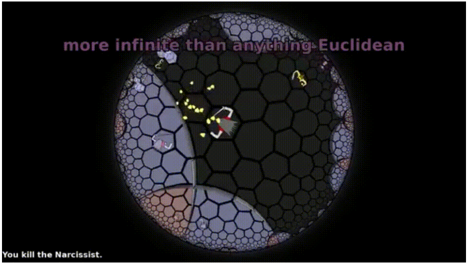
24 novembre 2021

Motivation



- **Hyperrogue games :** This game is developed on a non-Euclidean space. Namely, the Poincaré disk model.

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- Morphing has wide practical use in areas such as computer graphics, animation and modeling.

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In this talk, we deal with morphing of closed curves on Poincaré disk model, and we will answer to these requirements.

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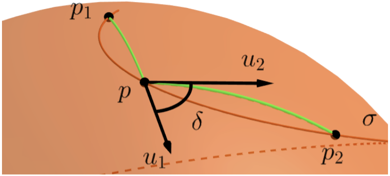

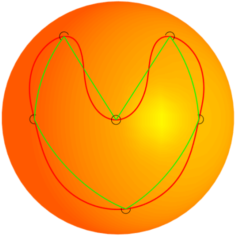

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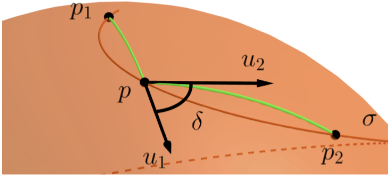

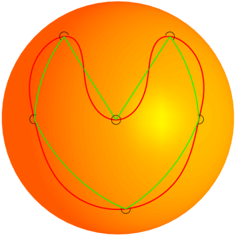

- A closed C^2-curve γ can be approximate by an inscribed polygon P .
- and the geodesic curvature of σ at a vertex p can be approximate by the discrete geodesic curvature of P at p :

$$\kappa(p) = \lim_{\substack{z_1, z_2 \to p \\ z_1, z_2 \in C}} \frac{2\ \delta}{d(z_1, p) + d(p, z_2)}.$$

So in practice we manipulate polygons (or discrete curves) instead of curves.

Related work : planar and spherical morphing closed curves

Planar case

- Exterior angles-based blending method : [*Sederberg& Gao& Wang & Mu*; 1993] .

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Spherical case

- [*Ikemakhen & Bellaihou & Ahanchaou*; 2021]

The Poincaré disc model

The Poincaré disc is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid \mid z \mid < 1\}$, where

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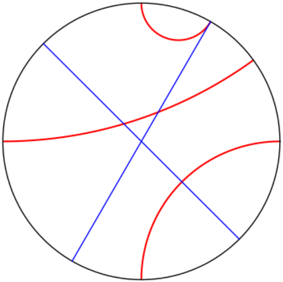
- Geodesics : line segments through the origin and the circular arcs that intersect the boundary orthogonally.



- Hyperbolic distance :

$$\cosh\left(d\left(z_1, z_2\right)\right) = 1 + \frac{\mid z_1 - z_2 \mid^2}{(1 - \mid z_1 \mid^2)(1 - \mid z_2 \mid^2)}.$$

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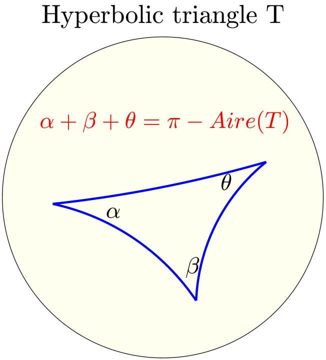
- $\alpha + \beta + \theta = \pi - \mathit{area}$.



Mobïus Transformations

$$SU(1,1) := \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \mid a\bar{a} - b\bar{b} = 1 \right\}.$$

The Mobius group $PSU(1,1) := SU(1,1)/\pm I$ acts transitively on the Poincaré disc $\mathbb{D}$:

$$\begin{array}{cccc} \rho & : & PSU(1,1) \times \mathbb{D} & \to & \mathbb{D}, \\ & & \left(\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, z \right) & \mapsto & \dfrac{az + \bar{b}}{\bar{b}z + \bar{a}}. \end{array}$$

Translations and Rotations

- The rotation around the origin O by angle θ :

$$R(\theta) := \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{-i\theta}{2}} \end{pmatrix}.$$

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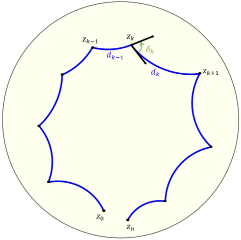
- The translation of length d along the geodesic that sends -1 to 1 is

$$L(d) := \begin{pmatrix} \cosh(\frac{d}{2}) & \sinh(\frac{d}{2}) \\ \sinh(\frac{d}{2}) & \cosh(\frac{d}{2}) \end{pmatrix}.$$

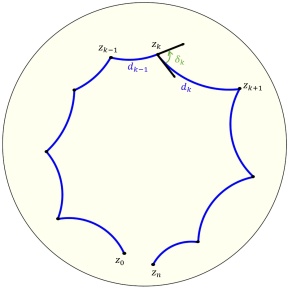
Hyperbolic Polygon

• A hyperbolic polygon $P = [z_0, ..., z_n] \in \mathbb{D}$: edges are pieces of geodesics. • The intrinsic parameters of P at any vertex z_k are :

- Geodesic edge length : $d_k := d(z_k, z_{k+1})$,
- Exterior angle : δ_k
- Discrete geodesic curvature : $\kappa_g(z_k) := \dfrac{2\ \delta_k}{d_{k-1} + d_k}$

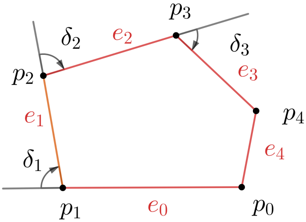


Fundamental Question : What is the Closure conditions of a hyperbolic polygon in terms of its intrinsic parameters ?



Planar Case. $P = [z_0, ..., z_n]$ is a closed polygon

$$\textit{iff} \quad \sum_{i=0}^{n} \vec{e_i} = \vec{0} \quad \textit{iff} \quad \begin{cases} \sum_{i=0}^{n} e_i \sin\left(\sum_{k=0}^{i} \delta_k\right) = 0, \\ \sum_{i=0}^{n} e_i \cos\left(\sum_{k=0}^{i} \delta_k\right) = 0. \end{cases}$$



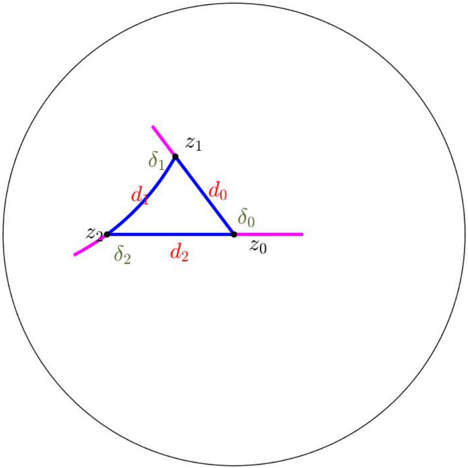
Closing condition for a Hyperbolic Triangle

Let $\Pi = [z_0 = z_3, z_1, z_2, z_3]$ be a closed hyperbolic triangle with hyperbolic sides d_0, d_1, d_2 and exterior angles δ_0, δ_1, δ_2. A hyperbolic triangle $T = [z_0, z_1, z_2]$ is closed iff

$$R(\delta_0)L(d_0)R(\delta_1)L(d_1)R(\delta_2)L(d_2) = \pm I,$$

- $R(\delta_0)$ is the rotation with angle δ_0 etc...
- $L(d_0)$ is the translation along the geodesic (z_0, z_1) etc...

$$L\left(d_2\right)$$



Closing condition for a Hyperbolic Polygon

$P = [z_0, ..., z_n]$ is a closed hyperbolic polygon iff

$$R(\delta_0)L(d_0)R(\delta_1)L(d_1)\cdots R(\delta_{n-2})L(d_{n-2})R(\delta_{n-1})L(d_{n-1}) = \pm I_d.$$

iff

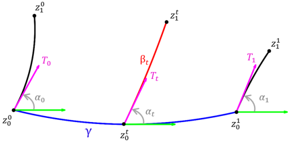
$$\begin{cases} \mid tr(S) \mid & = & 2, \\ det(S) & = & 1, \\ s_2\bar{s}_2 & = & 0. \end{cases}$$

Where

$$S := R(\delta_0)L(d_0)R(\delta_1)L(d_1)\cdots R(\delta_{n-2})L(d_{n-2})R(\delta_{n-1})L(d_{n-1}).$$

Algorithm : Exterior-angle blending

Let given $P^0 = [z_0^0, ..., z_{n-1}^0]$ and $P^1 = [z_0^1, ..., z_{n-1}^1]$ two closed hyperbolic polygons. For $t \in [0, 1]$, we compute



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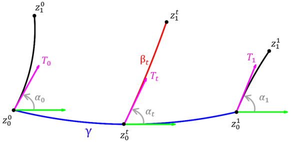

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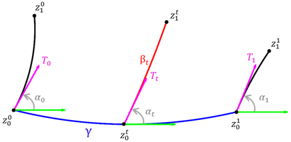

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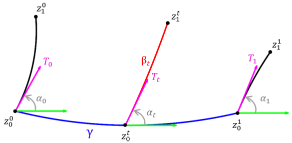

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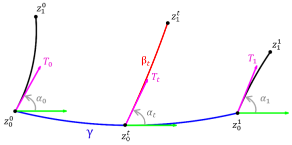

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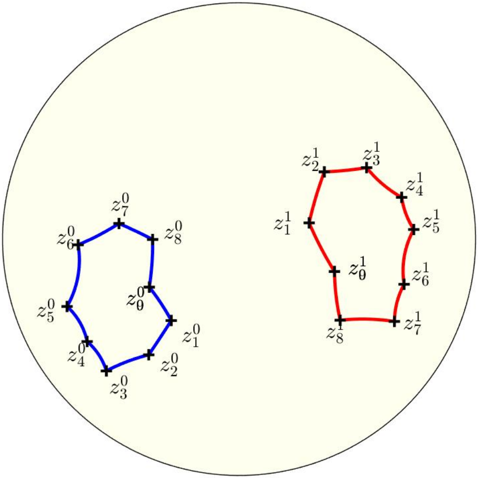
- We construct the point z_0^t and the geodesic c_0^t,
- By induction, we construct the other geodesic edges c_k^t.

Construction process of intermediate curve



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3. Recovery of exterior angles δ_k^t from the κ_k^t

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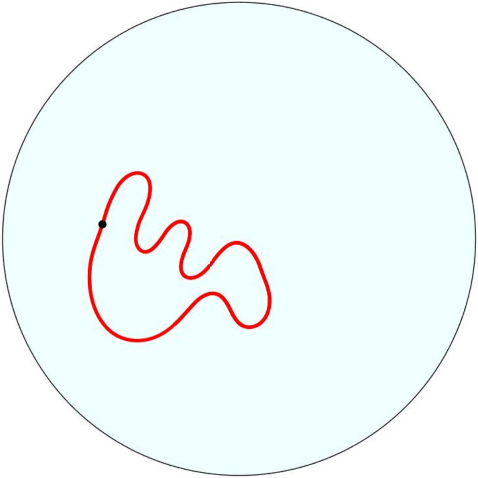
4. By induction, we construct the edge geodesics c_k^t.

Result without closing condition



Closing process

For that, we will change the exterior angles δ_k^t in the smallest possible way to close the intermediate polygon P^t.
This means we seek $\epsilon_0, ...\epsilon_{n-1}$ such that the polygon $\bar{P}^t$, with hyperbolic side lengths d_k^t and exterior angles $\bar{\delta}_k^t := \delta_k^t + \epsilon_k$ will be closed and the norm $\| \bar{\kappa}^t - \kappa^t \|^2$ will be minimized.
Where κ^t (resp. $\bar{\kappa}^t$) denotes the vector of components κ_k^t (resp. $\bar{\kappa}_k^t := \dfrac{2\bar{\delta}_k^t}{d_{k-1}^t + d_k^t}$), and $\| . \|$ is the Euclidean norm in $\mathbb{R}^n$. In order to solve this,

We minimize the following problem :

$$\min_{(\epsilon_0,\dots\epsilon_{n-1})\in\mathbb{R}^n} \sum_{k=0}^{n-1} \left| \frac{4\epsilon_k^2}{\left(d_{k-1}^t + d_k^t\right)^2} \right| . \qquad (1)$$

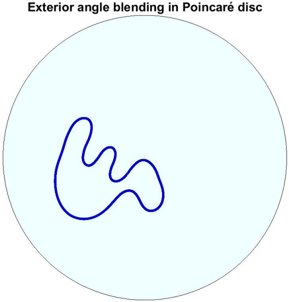
Subject to :

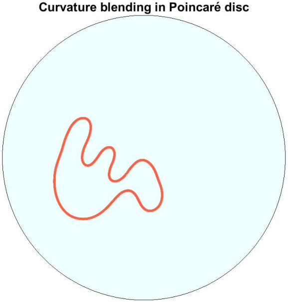
$$\begin{cases} \mid tr(S) \mid & = & 2, \\ det(S) & = & 1, \\ s_2\bar{s}_2 & = & 0. \end{cases} \qquad (2)$$

where

$$S := \prod_{k=0}^{n-1} R(\delta_{n-1-k} + \epsilon_{n-1-k}) L(d_{n-1-k}),$$

This will ensure the closure of the hyperbolic polygon $\bar{P}^t$.


Exterior angle blending in Poincaré disc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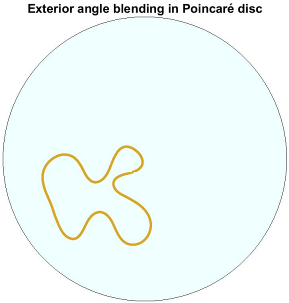
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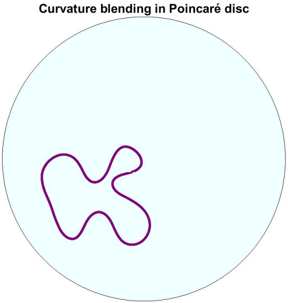
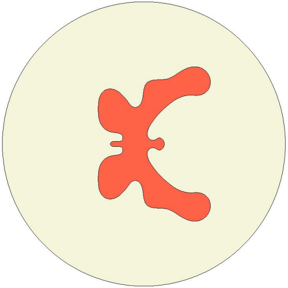
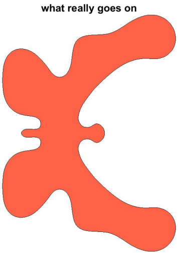
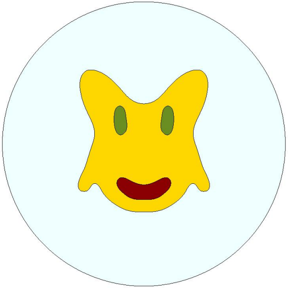

Curvature blending in Poincaré disc

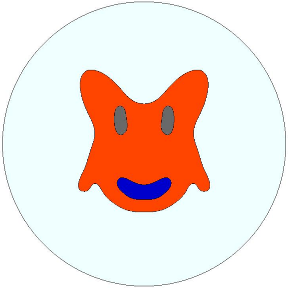
Morphing sequence between a butterfly and a bat at infinity





Morphing sequence between a wolf's face and a bat





Tiling the Poincaré disc using the blending of two given motifs

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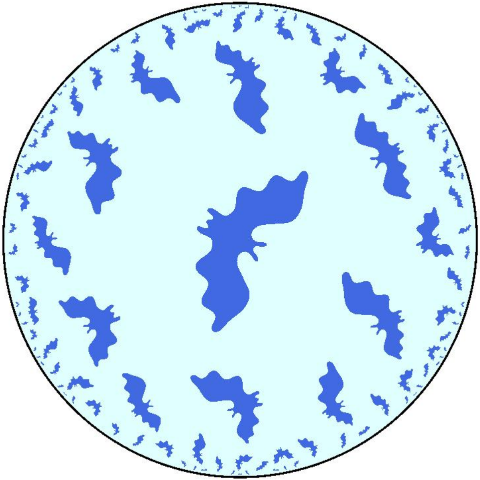

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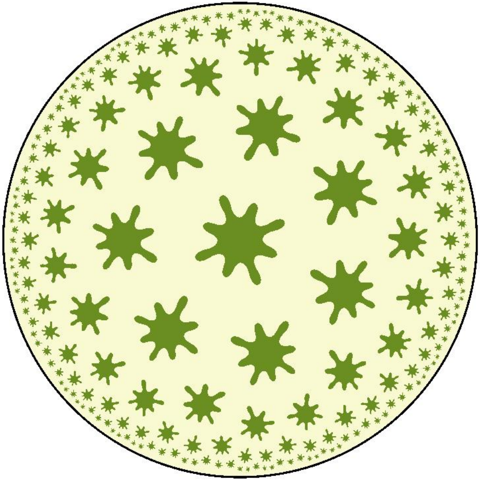

Conclusion

- We have presented two novel algorithms for blending between two curves in the Poincaré disc, using their intrinsic variables.

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- Both methods generate closed intermediate smooth curves by using the closure condition and by solving an optimization problem.

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- these methods give tools to approach the geometry processing problem in the hyperbolic space and could give a contribution in this direction.
- Limitation. Both algorithms take a long time to generate intermediate curves.
- Therefore, these methods can't be applied for a real-time execution.
- The goal of our future work is to give a rapid blending method which reduces the run-time.

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https ://doi.org/10.1111/cgf.14358

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