Homology spheres which are Seifert 3-manifolds : construction and classification

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Internet Seminar:
Algebra, Geometry, Topology & Applications (**AGT & A**)

Beni Melal, 17 October 2020

Plan

1. Homology spheres : definition and characterization.
2. Classical constructions of homology spheres.
3. Seifert fibered 3-manifolds :
   - Definition, constructions.
   - Invariants.
   - Classification by homeomorphisms and recognition.

I.1 Preliminaries on algebraic topology

Note 1. In dimension three the categories Top, Diff and PL (piecewise linear) are equivalent. Then according to our needs, we can consider every smooth 3-manifold as a polyhedral manifold (ie. a PL manifold).

Note 2. For any manifold the spaces of homology singular, simplitial and cellular are equivalent.

Then to each manifold M^3 which is connected, compact, oriented and without boundary we associate a list of commutative homology groups :

$$H_k(M^3, \mathbb{Z}) = \{0\}, \forall k \geq 4 \quad \text{(geometric dimension)}$$

In fact, since $H_k(M^3, \mathbb{Z})$ is an abelian group, it can be decomposed as follows :

$$\begin{array}{rcccl} H_k(M^3, \mathbb{Z}) & = & \mathbb{Z}^{b_k(M^3)} & \oplus & \mathrm{Tor}(H_k(M^3, \mathbb{Z})) \\ & = & \text{free abelian group} & \oplus & \text{torsion group} \end{array}$$

where $b_k(M^3) \in \mathbb{N}$ is called the Betti number of M^3.

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where $b_k(M^3) \in \mathbb{N}$ is called the Betti number of M^3.

- by connectedness we have $H_0(M^3,\mathbb{Z}) = \mathbb{Z}^{b_0(M^3)} = \mathbb{Z}$.
- by orientability, compactness and Poincaré duality we show that :

$$H_3(M^3,\mathbb{Z}) = \mathbb{Z}^{b_3(M^3)} = \mathbb{Z}$$

- Since M^3 is arcwise connected then its first simgular homology group

$$H_1(M^3,\mathbb{Z}) \simeq \frac{\pi_1(M^3)}{[\pi_1(M^3),\pi_1(M^3)]} \simeq \mathbb{Z}^{b_1(M)} \oplus \mathrm{Tor}(H_1(M,\mathbb{Z}))$$

- Since M^3 is closed (in fact oriented) then the second homology group $H_2(M^3,\mathbb{Z})$ is free abelian, that is, $H_2(M) = \mathbb{Z}^{b_2(M)}$.

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I.2 Homology spheres : definition and topological properties

Definition

Let M^3 be a compact, without boundary and oriented 3-manifold (closed). We say that M^3 is an integer (resp. rational) homology sphere if in the singular (or simplicial) we have

$$H_*(M^3, \mathbb{Z}) \simeq H_*(S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad * = 0 \text{ or } 3 \\ 0 & \text{if} \quad * = 1 \text{ or } 2 \end{cases}$$

resp.

$$H_*(M^3, \mathbb{Q}) = H_*(M^3, \mathbb{Z}) \otimes \mathbb{Q} \simeq H_*(S^3, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if} \quad * = 0 \text{ or } 3 \\ 0 & \text{if} \quad * = 1 \text{ or } 2 \end{cases}$$

Proposition 1.
M^3 is an integer homology sphere if and only, if $\pi_1(M^3) = [\pi_1(M^3), \pi_1(M^3)]$.

Proof
1) If M^3 is an integer holomogy sphere then

$$H_1(M^3, \mathbb{Z}) = 0 \Longrightarrow \pi_1(M^3) = [\pi_1(M^3), \pi_1(M^3)]$$

2) Conversely, if we suppose that $\pi_1(M^3) = [\pi_1(M^3), \pi_1(M^3)]$ we see that $H_1(M^3, \mathbb{Z}) = 0$. Thus, as the abelian group $H_2(M^3, \mathbb{Z})$ is free, then by Poincaré duality we obtain :

$$H_2(M^3, \mathbb{Z}) \simeq H_1(M^3, \mathbb{Z}) = 0$$

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Proposition 2.

M^3 is a rational homology sphere if and only, if $\dfrac{\pi_1(M^3)}{[\pi_1(M^3),\pi_1(M^3)]}$ is finite.

Proof

1) $H_1(M^3,\mathbb{Q}) = 0 \Longrightarrow H_1(M^3,\mathbb{Z})$ is finite : evident.

2) Conversely, if $H_1(M,\mathbb{Z})$ is a finite torsion abelian group, then its first Betti number $b_1(M^3) = 0$ (rank). Thus, by Poincaré duality $b_2(M^3) = b_{3-2}(M^3) = 0$. Consequently, as $H_2(M^3,\mathbb{Z}) = 0$ (is free abelian) then M^3 is a rational homology sphere.

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In the rest of this part, we apply three deferent method to construct 3-homology spheres manifods :
Method 1 : group action
We consider a discrete group G who verify the properties :

1. G acts freely and properly on a simply connected, without boundary and oriented 3-manifolds $\widetilde{M}^3$.
2. $G/[G,G]$ is finite.
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Method 2 : Heegaard splitting.
Method 3 : Dhen surgery on knots and links.

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II.1 Poincaré homology sphere S^3/G.

Now, consider the alternating group of five letters A_5 of order $5!/2 = 60$. The group A_5 has the following properties :

1. A_5 is non abelian simple group, then its derived group $[A_5, A_5] = A_5$.
2. A_5 can be realized as a subgroup of the Lie group $SO(3)$.
3. A_5 is the rotational symmetry group of the regular dodecahedron (see figure).

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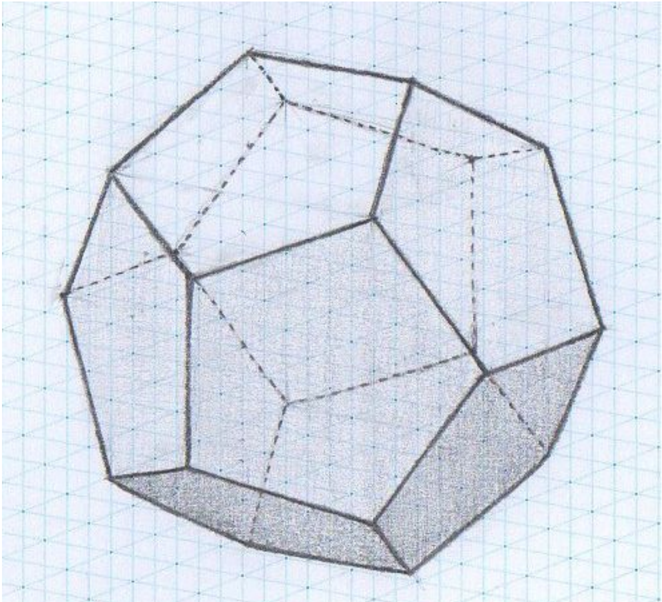
Figure: Dodecahedron in the closed unite euclidian boule $\bar{\mathbb{B}}^3$



To construct a finite group which acts on the 3-sphere S^3, then we consider the simply connected covering $\mathrm{Spin}(3) \simeq US(2) \simeq S^3$ of the Lie group $SO(3)$:

$$0 \to \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(3) \xrightarrow{p} SO(3) \to 1$$

And, by the pullback via the projection p we obtain a subgroup, $G \subset \mathrm{Spin}(3)$, knowing by binary isocahedron group of order 120 :

$$0 \to \mathbb{Z}_2 \longrightarrow G \xrightarrow{p} A_5 \to 1$$

The binary isocahedron group G admits the following algebraic and geometric properties :

1. finite presentation $G =< a, b \mid a^3 = b^5 = (ab)^2 >$.
2. the derived group $[G, G] = G$.
3. G acts freely on the 3-sphere S^3, and whose fundamental domain coincides with the dodecahedron.
4. In fact, the 3-sphere is an union of 120 dodecahedron.

Theorem (H. Poincaré 1904)

The quotient 3-manifold, S^3/G, is an homology 3-sphere which can be obtained by identifying two opposite faces after a rotation of $\pi/5$.

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II.2 Weber-Seifert homology spheres $\mathbb{H}^3/G$.

The euclidian open boule $\mathbb{B}^3 = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2 < 1\}$ will be endowed by the riemannian metric

$$ds^2 = 4\frac{dx^2+dy^2+dz^2}{(1-x^2-y^2-z^2)^2}$$

- The geodesics of $(\mathbb{B}^3, hyp)$ are the diameters of $\mathbb{B}^3$ and half-circles which are orthogonal to the boundary $\partial\mathbb{B}^3 = S^2$.
- The isometry group of $(\mathbb{B}^3, hyp)$ is isomorphic to Lie group $PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{-I, I\}$ (noncompact).
- The sectional curvature of $(\mathbb{B}^3, hyp)$, $K = -1$.

Then, in the hyperbolic $(\mathbb{B}^3, ds^2)$ we consider the hyperbolic dodecahedron P_h whose faces are totaly geodesic in $\mathbb{B}_3$ (see the figure) :

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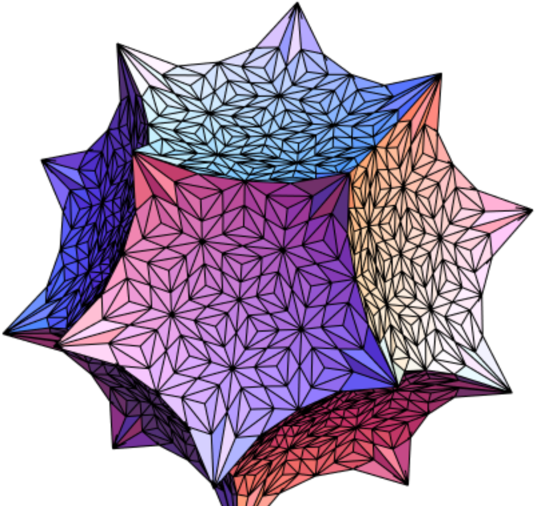
Figure: Hyperbolic dodecahedron in the hyperbolic boule $(\mathbb{B}^3, hyp)$



Now, the hyperbolic polyhedron P_h can be see as a fundamental domain of the discrete group $G \subset PSL(2,\mathbb{C})$ generated by hyperbolic inversions along the faces of P_h. It is showed that :

1. the group $G \subset PSL(2,\mathbb{C})$ is infinite.
2. $G =< x_1, x_2, x_3, x_4, x_5, y \mid x_1x_2x_3x_4x_5 = 1, x_i^{-1}x_{i+1}x_{i+3}x_{i+4}^{-1} = y; \quad i \mod 5 >$
3. the quotient $G/[G,G] \simeq (\mathbb{Z}/5\mathbb{Z})^3$.

Theorem (Weber-Seifert 1933)

$M^3 = \mathbb{B}^3/G$ is an rational homology sphere which can be obtained by identification of opposite faces of the hyperbolic dodecahedron after a rotation of $3\pi/5$.

Figure: First iteration of tilling $(\mathbb{B}^3, hyp)$ by P_h

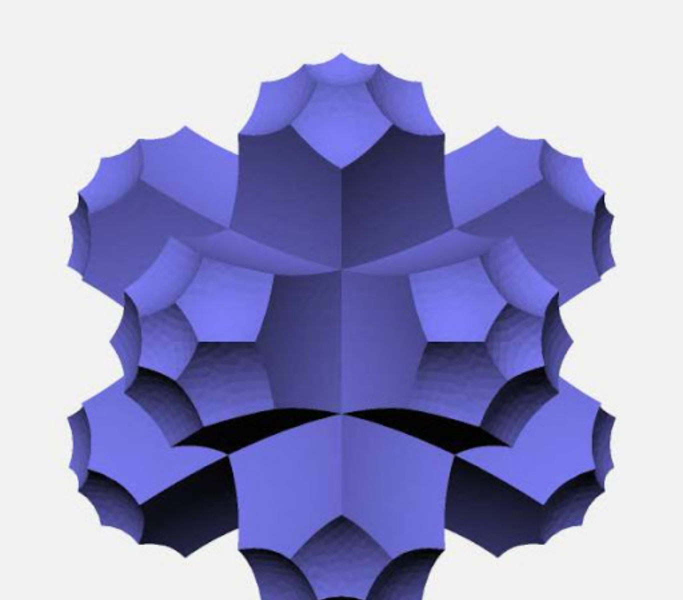
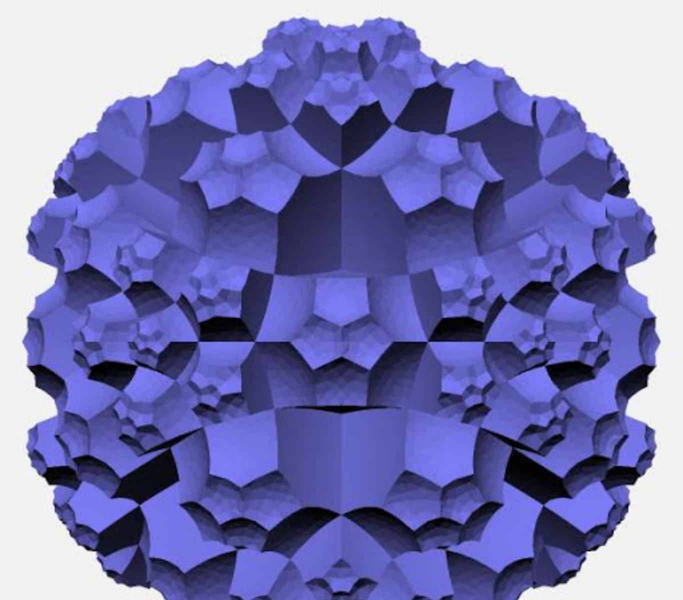


Figure: Seconde iteration of tilling $(\mathbb{B}^3, hyp)$ by P_h



II.3 Heegaard splitting.

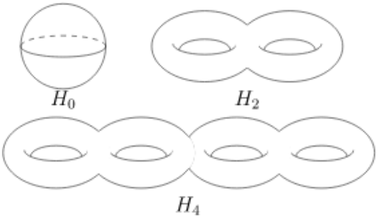
Definition : handlebodie

We call handlebodie of genus $g \geq 0$ any 3-manifold, H_g^3, which is homeomorphic to the tubular neighbored of a wedge of g-circles :

$$\underbrace{S^1 \vee \cdots \vee S^1}_{g-times} \quad \text{when} \quad g = 0, \quad \underbrace{S^1 \vee \cdots \vee S^1}_{0-times} = \emptyset$$

Note that the boundary $\partial H_g^3 = \Sigma_g^2$ is a closed surface of genus g.

Figure: Examples de handelbodies



Definition : gluing two handlebodies

Let U and V are two handlebodies of same genus $g \geq 0$, and let $f : \partial V \to \partial U$ be a homeomorphism which reverse the orientation. Then,

$$M_f^3 = \frac{U \bigsqcup V}{x \in \partial V, x \sim f(x) \in \partial U}$$

is a closed 3-manifolds, which called obtained by gluing two handlebodies of same genus g.

Note that the closed 3-manifold M^3_f contains two open 3-manifolds homeomorphic to handlebodies $U' \simeq H^3_g$ and $V' \simeq H^3_g$, and such that $U' \cap V' \simeq \Sigma^2_g \times]-1,1[$.
Consequently, by Mayer-Vietoris we can calculate the homology of M^3_f :

Theorem (Mayer-Vietoris)

Let U and V be handlebodies of genus $g \geq 0$ and let $f : \partial V \to \partial U$ be a homeomorphism which reverse orientation. Then the following sequence is exact :

$$0 \longrightarrow H_2(M_f) \xrightarrow{\delta} H_1(\partial V) \xrightarrow{\Delta_f} H_1(V) \oplus H_1(U) \xrightarrow{j_*} H_1(M_f) \longrightarrow 0$$

where $\Delta_f(z) = (in_V(z), in_U(f_*(z)))$ and $j_*(a,b) = In_V{}^M(a) - In_U{}^M(b)$.

Corollary

1. $H_2(M_f) = \mathrm{Ker}(\Delta_f)$ and $H_1(M_f) = \mathrm{coker}(\Delta_f)$.
2. M_f is a rational homology sphere if and only if Δ_f is injective.
3. M_f is an integer homology sphere if and only if Δ_f is bijective.

Note that the closed 3-manifold M_f^3 contains two open 3-manifolds homeomorphic to handlebodies $U' \simeq H_g^3$ and $V' \simeq H_g^3$, and such that $U' \cap V' \simeq \Sigma_g^2 \times]-1,1[$.
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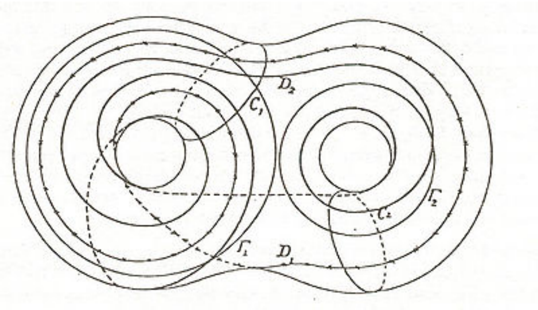
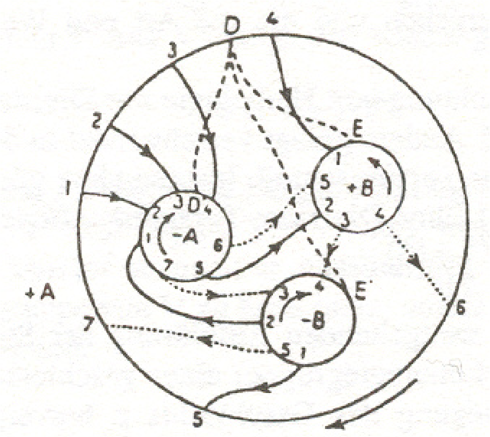
The following diagram give us the attaching system of curves α and $\beta \subset \Sigma_2^2 = \partial U \simeq \partial H_2^3$ considered by Poincaré for to construct his dodecahedral 3-homology integer sphere. The two curves α and $\beta \subset \Sigma_2^2 = \partial U \simeq \partial H_2^3$ are obtained by the attached map $f : \partial V \to \partial U$ from two meridional curves considered on the handlebody ∂V.

Figure: Identification scheme of Dehn and Heegaard (1907)



This planar diagram is obtained from the precedent figure by cutting the surface $\Sigma_2^2 = \partial U$ it along the meridional curves C_1 and C_2.

Figure: Poincaré-Heegaard diagram (1904)



II.4 Dehn surgery on links

Let M^3 be a homology sphere. A Dehn's surgery along a knot $K \subset M^3$ is any operation that remove a tubular neighbored $N(K) \subset M^3$ and sew it by the mean of a torus homeomorphism

$$f : \partial(E(K) = M^3 \setminus \mathrm{Int}(N(K))) \longrightarrow \partial(D^2 \times S^1)$$

Concretely, on the boundary $\partial E(K) = \mathbb{T}^2$ we choose two simple closed curves :

1. longitude : λ is parallel to K defined from a Seifert surface in M ;
2. meridain : μ is definied by the linking number $\mathrm{lk}(\mu, K) = +1$.

Now, the pair (λ, μ) realize a base in the group $H_1(\partial E(K)) = [\mu]\mathbb{Z} \oplus [\lambda]\mathbb{Z}$.
Consequently, any reducible rational number $\frac{m}{l} \in \mathbb{Q} \cup \{\frac{1}{0}\}$ such that $(m, l) = 1$ allow us to define a unique closed curve $\alpha \sim m\mu + l\lambda$ on the boundary $\partial E(K)$ that we sew to the meridian of a standard solid torus $D^2 \times S^1$.

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Definition

The resulting 3-manifold is closed and denoted by,

$$M^3(K;\frac{m}{l}) := M^3 \setminus \mathrm{Int}(N(K)) \bigcup_f D^2 \times S^1$$

where $f : \partial(M^3 \setminus \mathrm{Int}(N(K))) \to \partial(D^2 \times S^1)$ generated (in isotopy) by the closed curve $\alpha \sim m\mu + l\lambda$.

Actually, by Van-Kampen we are able to calculate the fundamental group of $M^3(K;\frac{m}{l})$. And, by Mayer-Vietoris theorem we calculate its homolgy :

Proposition (Mayer-Vietoris)

$H_1(M^3(K;\frac{m}{l})) = \mathbb{Z}_{|m|}$ and $H_2(M(K;\frac{m}{l})) = \{0\}$.

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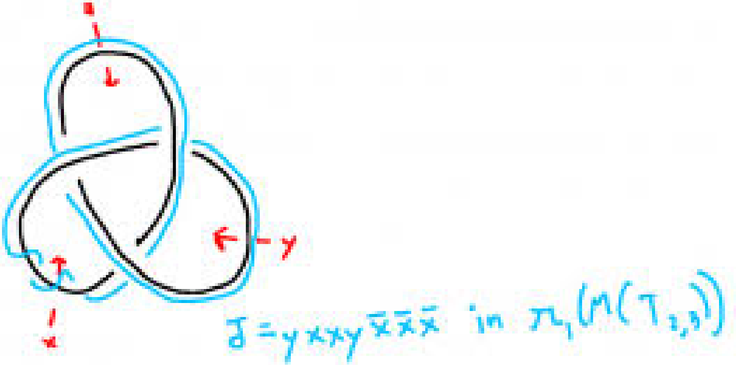
Figure: Dehn surgery on the hand-right trefoil knot (1910)



III.1 Definition and examples

Definition

Let M^3 be an oriented 3-manifold. We say that M^3 is a Seifert manifold if it supports a locally free action of the circle $\mathbb{S}^1$ (as Lie group). In this cas, the canonical map $M^3 \to B^2 := M^3/\mathbb{S}^1$ will be called a **Generalized Seifert Fibration**.

In a Seifert 3-manifold the orbit of each point $x \in M^3$ is homeomorphic to a circle $L_x := \mathbb{S}^1/\mathbb{S}^1_x$ where $S^1_x = \mathbb{Z}_p \subset \mathbb{S}^1$ is a finite isotropy subgroup. Then we have two type of orbits :

1. If $\mathbb{S}^1_x = \{1\}$ (is trivial) we say that the orbit L_x is a regular fiber.
2. If $\mathbb{S}^1_x = \mathbb{Z}_p \neq \{1\}$, (is nontrivial) we say that the orbit L_x is a singular or an exceptional fiber.
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Note that every 3-manifold which is a $\mathbb{S}^1$-principal fibre is an example of generalized Seifert fibration.

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Example 1 : Hopf fibration

Let $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2; \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}$ be the 3-sphere, it is a Lie subgroup of the multiplicative Lie group $\mathbb{C}^2 \setminus \{(0,0)\}$. The natural action of the circle $\mathbb{S}^1 = \{z \in \mathbb{C}; \mid z \mid = 1\}$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ give us a free action on $\mathbb{S}^3$ defined by :

$$\begin{array}{cccc} \theta & : & S^1 \times \mathbb{S}^3 & \rightarrow & \mathbb{S}^3 \\ & & (z,(z_1,z_2)) & \mapsto & (zz_1, zz_2) \end{array}$$

Note that in the 3-sphere, $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2; \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}$, the two circles $S^1 \times \{0\}$ and $\{0\} \times S^1$ are $\mathbb{S}^1$-orbits of $(1,0)$ and $(0,1)$ respectively. And, for any $z_1 \neq 0$ and $z_2 \neq 0$ the $\mathbb{S}^1$-orbit

$$\mathbb{S}^1 \cdot (z_1, z_2) = \{(zz_1, zz_2); z \in S^1\}$$

is a trivial knot which can be viewed as a closed curve on the torus (level surface)

$$\mathbb{T}^2_{(|z_1|,|z_2|)} = \{(u,v) \in S^3; \mid u \mid = \mid z_1 \mid \text{ and } \mid v \mid = \mid z_2 \mid\}$$

homologous to : **meridian** + **longetude** (see figure).

Consequently, the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2; \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}$ is a Seifert 3-manifolds without singular fibers, then its canonical projection $p : S^3 \rightarrow S^3/\mathbb{S}^1 = S^2$ is a locally trivial $\mathbb{S}^1$-bundle.

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Figure: Fibers of the Hopf fibration

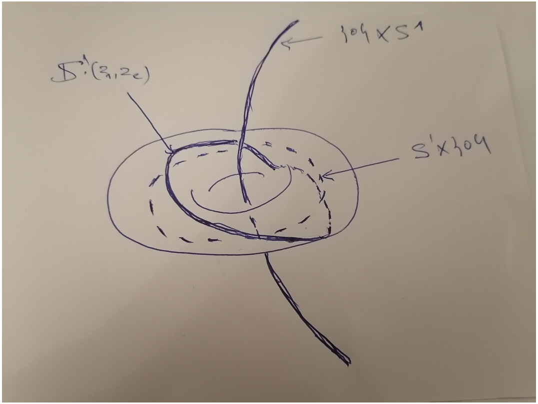
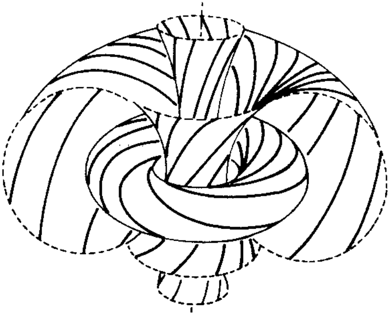


Figure: Global illustration of the Hopf fibration



Example 2: The 3-sphere with one singular fibre.

For any integer $n \geq 2$ we define on the sphere $S^3 \subset \mathbb{C}^2$ an $\mathbb{S}^1$-action by,

$$\forall z \in \mathbb{S}^1, \forall (z_1, z_2) \in S^2, \qquad z \cdot (z_1, z_2) = (zz_1, z^n z_2)$$

For this $\mathbb{S}^1$-action on S^3 the orbit $O = \mathbb{S}^1 \cdot (0,1) = \{0\} \times S^1$ is a unique singular orbit with isotropy group $\mathbb{Z}_n$.

And, for any $z_1 \neq 0$ the orbit $\mathbb{S}^1 \cdot (z_1, z_2)$ is a regular orbit which is homologous to : n**meridian** + **longitude** regarding to the singular orbit $\{0\} \times S^1$.

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Example 2: The 3-sphere with one singular fibre.

For any integer $n \geq 2$ we define on the sphere $S^3 \subset \mathbb{C}^2$ an $\mathbb{S}^1$-action by,

$$\forall z \in \mathbb{S}^1, \forall (z_1, z_2) \in S^2, \qquad z \cdot (z_1, z_2) = (zz_1, z^n z_2)$$

For this $\mathbb{S}^1$-action on S^3 the orbit $O = \mathbb{S}^1 \cdot (0,1) = \{0\} \times S^1$ is a unique singular orbit with isotropy group $\mathbb{Z}_n$.
And, for any $z_1 \neq 0$ the orbit $\mathbb{S}^1 \cdot (z_1, z_2)$ is a regular orbit which is homologous to : n**meridian** + **longitude** regarding to the singular orbit $\{0\} \times S^1$.
Consequently, the 3-sphere S^3 is a fibred Seifert manifold with one singular orbit. The orbit space is a 2-sphere admitting one singular point called : the water drop (see the figure).

Figure: The regular fibre wraps around the singular fibre

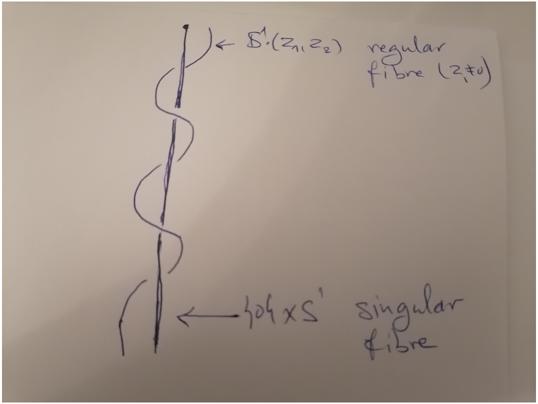
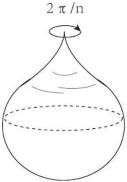


Figure: The orbit space : water drop



Example 3 : The 3-sphere with two singular fibre.

Let $(\alpha,\beta)=1$ be co-prime integers. On the sphere $S^3 \subset \mathbb{C}^2$ consider the S^1-action defined by,

$$\forall z \in \mathbb{S}^1, \forall (z_1,z_2) \in S^3, \qquad z\cdot(z_1,z_2) = (z^\alpha z_1, z^\beta z_2)$$

For this $\mathbb{S}^1$-action on S^3 we have only two singular orbits

- $O_1 = S^1 \times \{0\}$ with isotropy group $\mathbb{Z}_\alpha$,
- $O_2 = \{0\} \times S^1$ with isotropy group $\mathbb{Z}_\beta$.

Note that, as in the case of the Hopf fibration, a regular S^1-orbit is a closed toric knot homologous to : α **meridian** $+\beta$ **longitude** regarding to the singular orbit $S^1 \times \{0\}$ (see figure).
Consequently, the 3-sphere S^3 is a fibred Seifert manifold with tow singular orbits. The orbit space is a 2-sphere admitting two singular points (see the figure).

Example 3 : The 3-sphere with two singular fibre.

Let $(\alpha,\beta) = 1$ be co-prime integers. On the sphere $S^3 \subset \mathbb{C}^2$ consider the S^1-action defined by,

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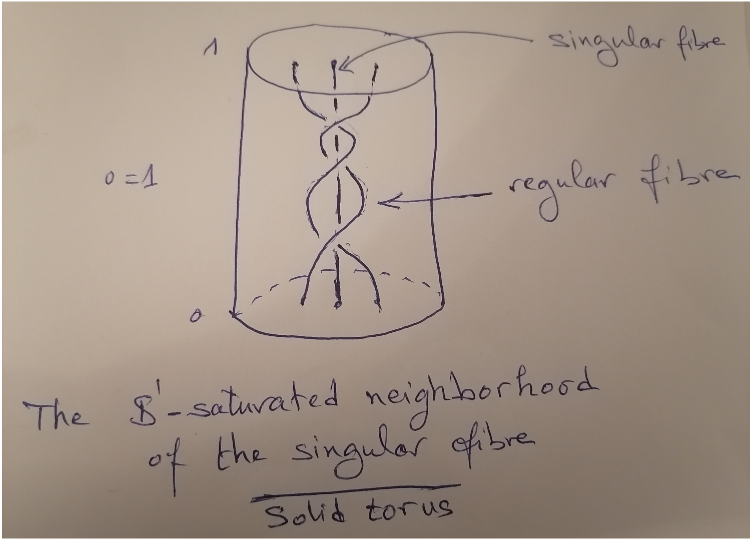
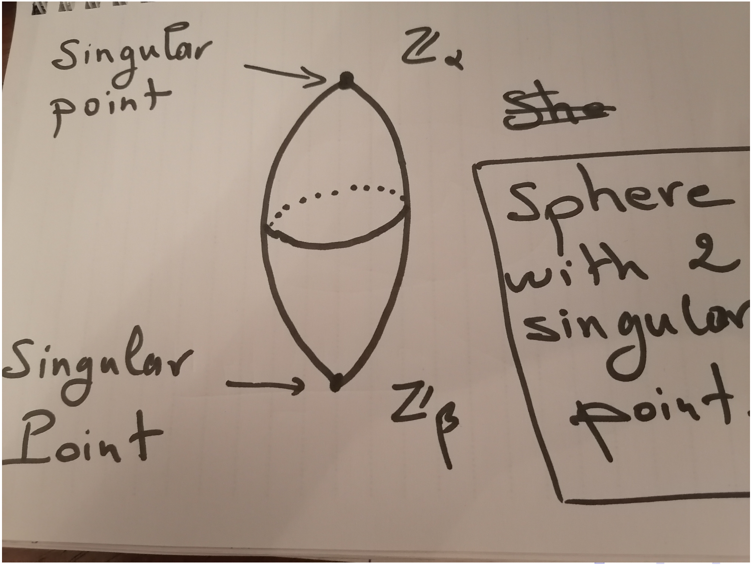
Consequently, the 3-sphere S^3 is a fibred Seifert manifold with tow singular orbits. The orbit space is a 2-sphere admitting two singular points (see the figure).

Figure: Tubular neighborhood of a singular fibre



singular sphere

Figure: Sphere with two singular points



III.2 Topological characterization of Seifert 3-manifolds

Local structure of a fibred Seifert manifold (Slice theorem)

1. Each regular orbit L_x admits a saturated open neighborhood $V_x \simeq D^2 \times S^1$ endowed by the natural action $z \cdot (z_1, z_2) = (zz_1, zz_2)$. In this case the orbit space $V_x/\mathbb{S}^1$ is a smooth disc D^2.
2. For each singular orbit L_x there is a S^1-saturated open neighborhood

$$V_x \simeq D^2 \times_{\mathbb{Z}_\beta} S^1 \simeq D^2 \times S^1$$

and a co-prime integers $(\alpha, \beta) = 1$ called coefficients of L_x, and such that the restriction of the $\mathbb{S}^1$-action on V_x is given by the last example. In this, case the orbit space $V_x/\mathbb{S}^1$ is a topological disc with a conical at his center.
3. The local analytical expression of the canonical projection, $M^3 \to B := M^3/\mathbb{S}^1$, is given by the function

$$f(r\exp(i\theta), \exp(i\phi)) = r\exp(i[\alpha\theta + \beta\phi])$$

where $r \in [0,1]$ and $\theta, \phi \in [0, 2\pi]$. Then, for every real $r \in]0,1]$ the regular fibre :

$$f^{-1}(r\exp(i\theta)) = \{(r\exp(i\frac{\theta}{\alpha}) \cdot \exp(i\beta\phi), \exp(-i\alpha\phi)); \phi \in [0, 2\pi]\} \simeq S^1$$

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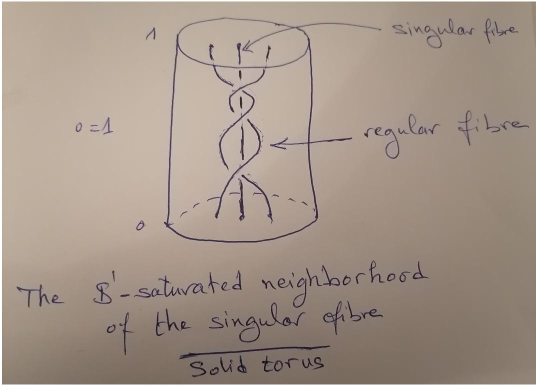
The figure shows us the topological model of a $\mathbb{S}^1$-saturated tubular neighborhood around a singular orbit :

Figure: Tubular neighborhood of a singular fibre



Proposition

1) Let M^3 be a compact oriented Seifert fibred 3-manifold. Then, M^3 has only a finite set of singular fibres S^1_i with coefficients (α_i, β_i). Moreover, if the boundary $\partial M^3 \neq \emptyset$ then it is homeomorphic to a union of S^1-saturated tori $T_i = S^1 \times S^1$.

2) We denote by $N_i(S^1_i; (\alpha_i, \beta_i))$ the $\mathbb{S}^1$-saturated tubular neighborhood of S^1_i. Then, the open manifold $M^3_0 = M^3 \setminus \bigcup_{i=1}^{i=n} N_i(S^1_i; (\alpha_i, \beta_i))$ is :

1. endowed by the structure of $\mathbb{S}^1$-principal fibre ;
2. every connected component of the boundary ∂M^3_0 has $n \geq 1$ is fibred by torus knots ;
3. the orbit space $M^3_0/\mathbb{S}^1 = \Sigma^2_{g,n}$ is a smooth surface of genus $g \geq 0$ with non empty boundary ;
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The proofs of the claims 1, 2 and 3 are evident.

For the proof of forth claim, there is two methods.

The first method utilise the Euler cohomology class,

$$eu \in H^2(\Sigma^2_{g,n}, \mathbb{Z})$$

which is knowing as an obstruction to find a global section for the $\mathbb{S}^1$-fibration $M^3_0 \xrightarrow{p} \Sigma^2_{g,n}$. Thus, since the boundary of the base $\Sigma^2_{g,n}$ is non empty, then we obtain $H^2(\Sigma^2_{g,n}, \mathbb{Z}) = \{0\}$. Consequently, the total space $M^2_0 \simeq \Sigma^2_{g,n} \times S^1$.

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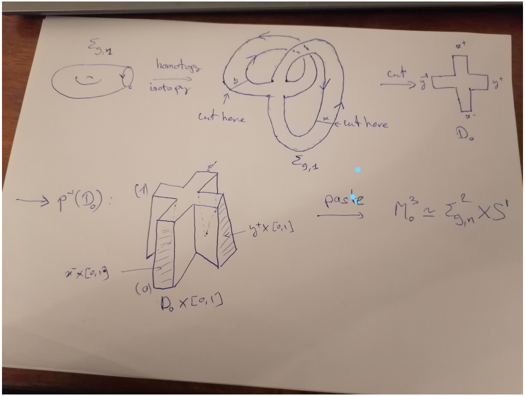
The fibre bundle, $\mathbb{S}^1 \hookrightarrow M_0^3 \to \Sigma_{g,n}^2$, is trivial

The second method utilise the **cutting and pasting principle** as explained in the figure :

Figure: Trivializing the total space M_0^3



III. 3 Topological construction of Seifert fibred manifolds

At the moment, we know that the fibration $\mathbb{S}^1 \hookrightarrow M_0^3 \to \Sigma_{g,n}^2$ is trivial.

Moreover, the Seifert manifold M^3 results from a Dehn surgery by pasting the tubular neighborhoods of singular fibers $(S_i^1, (\alpha_i, \beta_i))$ to the open manifold $M_0^3 \simeq S^1 \times \Sigma_{g,n}^2$. Then, by Van-Kampen theorem's we conclude that the fundamental group of M^3 is given by the following presentation :

$$\pi_1(M^3) = < a_i, b_i, h, q_j; q_1 \cdots q_n \prod_{i=1}^{i=g} [a_i, b_i] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, [h, a_i] = [h, b_i] = [h, q_j] = 1 >$$

Thus, we see that the first homology group $H_1(M^3, \mathbb{Z})$ is isomorphic to :

$$H_1(M^3, \mathbb{Z}) \simeq \mathbb{Z}^{2g} \oplus < Q_i, H; \alpha_i Q_i + \beta_i H = 0; Q_1 + \cdots + Q_n = 0 >= \mathbb{Z}^{2g} \oplus \text{Tor}$$

Where the torsion subgroup Tor is a finite group having as cardinal :

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$$\pi_1(M^3) = < a_i, b_i, h, q_j; q_1 \cdots q_n \prod_{i=1}^{i=g} [a_i, b_i] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, [h, a_i] = [h, b_i] = [h, q_j] = 1 >$$

Thus, we see that the first homology group $H_1(M^3, \mathbb{Z})$ is isomorphic to :

$$H_1(M^3, \mathbb{Z}) \simeq \mathbb{Z}^{2g} \oplus < Q_i, H; \alpha_i Q_i + \beta_i H = 0; Q_1 + \cdots + Q_n = 0 >= \mathbb{Z}^{2g} \oplus \mathrm{Tor}$$

Where the torsion subgroup Tor is a finite group having as cardinal :

$$|\,\mathrm{Tor}\,| = \alpha_1 \cdots \alpha_n \left| \sum_{i=1}^{i=n} \frac{\beta_i}{\alpha_i} \right|$$

Hence, by the above discussion we conclude the following :

Theorem

For any Seifert manifold M^3 the following claims are equivalent,

1. M^3 is an $\mathbb{Q}$-homology 3-sphere ;
2. the genus $g = 0$;
3. the orbit space of M^3 is a topological sphere $S^2 \simeq M^3/S^1$.

Corollary

A Seifert manifold M^3 is an integer homology sphere if and only, if the list $\{\beta_1, \cdots, \beta_n\}$ is co-prime.

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IV. 5 Classification Seifert manifolds.

In the precedent parts, to any Seifert $M3$ we associated a liste of co-prime pairs (α_i, β_i) ; here we study the two questions :
1) the modification of the list of pairs co-primes (α_i, β_i) without modifying the topology of M^3 ;

2) we introduce the rational Euler number $\mathrm{eu}(M) = -\sum_{i=1}^{i=n} \frac{\beta_i}{\alpha_i} \in \mathbb{Q}$ which will allow us to decide if two Seifert manifolds are homeomeorphic or no.

V. 4 Other constructions and operations

plumbing and graph manifolds
Breiskorn manifolds as Seifert manifolds.