Solutions of the Yang-Baxter equations on orthogonal Lie groups: the case of oscillator Lie groups

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Outline

1. Orthogonal Lie groups
2. Poisson-Lie groups and Yang-Baxter equations
3. A propriety of the solution of Yang-Baxter equations on orthogonal Lie groups
4. Oscillator Lie groups as Lorentzian orthogonal Lie groups
5. Main results : Poisson-Lie structures and solutions of Yang-Baxter on oscillator Lie groups

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for any $u, v, w \in \mathfrak{g}$.
Such a Lie algebra is called an *orthogonal* (or *quadratic*) Lie algebra.

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The determination of orthogonal Lie groups is an open problem even tough there is many results on the problem.

Recall that a Poisson tensor on a manifold M is bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that the bracket on $C^\infty(M)$ given by

$$\{f, g\} = \pi(df, dg)$$

satisfies the Jacobi identity :

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Poisson-Lie groups and Yang-Baxter equations

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$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

This is equivalent to

$$[\pi, \pi] = 0,$$

where [,] is the Schouten-Nujenhuis bracket.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A Poisson tensor π on G is called multiplicative if, for any $a, b \in G$,

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Let

$$\xi := d_e\pi_l : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$$

be the derivative of π_l at e.

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2. the bracket $[\ ,\ ]^*$ on the dual $\mathfrak{g}^*$ given by

$$[\alpha, \beta]^*(u) = \xi(u)(\alpha, \beta), \quad u \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^* \tag{2}$$

satisfies the Jacobi identity.

Conversely, if G is connected and simply connected, given any $\xi : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ such that $(\mathfrak{g}, [\ ,\ ], \xi)$ is a Lie bialgebra then there exists a unique Poisson-Lie tensor π on G such that $\xi = d_e \pi_l$.

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$$\{\text{Connected and simply-conneceted Poisson Lie groups}\} \simeq \{\text{Lie bialgebras}\}.$$

Let (G, π) be a Poisson-Lie group. The connected and the simply connected Lie group, say G^*, associated to $(\mathfrak{g}^*, [\ ,\ ]^*)$ is called dual Lie group of (G, π).

$\xi : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ is called a *coboundary* if there exists $r \in \wedge^2 \mathfrak{g}$ such that, for any $u \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$,

$$\xi(u)(\alpha, \beta) = \mathrm{ad}_u r(\alpha, \beta) := r(\mathrm{ad}_u^* \alpha, \beta) + r(\alpha, \mathrm{ad}_u^* \beta).$$

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In this case, the condition (1) is automatically satisfied and (2) holds if and only if r satisfies the *generalized classical Yang-Baxter equation* :

$$\mathrm{ad}_u [r, r] = 0, \quad \forall u \in \mathfrak{g}, \tag{3}$$

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where $[r, r] \in \wedge^3\mathfrak{g}$ is the Schouten bracket. A solution of the *classical Yang-Baxter equation* is a bivector $r \in \wedge^2\mathfrak{g}$ satisfying

$$[r, r] = 0. \tag{4}$$

Some results on solutions of Yang-Baxter equation and bialgebra structures :

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$$r = r_0 + \sqrt{-\lambda}\left(\sum_{\alpha\in\Delta^+} E_{-\alpha}\wedge E_{\alpha} + 2\sum_{\alpha\in\Gamma^+,\beta>\alpha} E_{-\beta}\wedge E_{\alpha}\right),$$

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2. In [6], Delorme classified Lie bialgebras structures on reductive complex Lie algebras.
3. In [29], Szymczak I. and Zakrzemski S. classified Poisson-Lie structures on Heisenberg groups.

The following result was proved by Bellavin and Drinfeld :

Theorem.

If $\mathfrak{g}$ is a complex simple Lie algebra and r a solution if GYBE then $(\mathfrak{g}^*, [\ ,\ ]_r)$ is solvable.

Yang-Baxter equations on orthogonal Lie groups

Let (G, k) be an orthogonal Lie group and r a solution of the GYBE on its Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$. Then r defines on $\mathfrak{g}^*$ a Lie bracket by

$$[\alpha, \beta]_r = \mathrm{ad}^*_{r_\#(\beta)}\alpha - \mathrm{ad}^*_{r_\#(\alpha)}\beta. \tag{5}$$

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Consider the bilinear form $\langle\ ,\ \rangle^*$ on $\mathfrak{g}^*$ given by $\langle\ ,\ \rangle$. Let us denote by G^*_r a Lie group with Lie algebra $(\mathfrak{g}^*, [\ ,\ ]_r)$, by k^* the left invariant pseudo-Riemannian metric whose value at the identity is $\langle\ ,\ \rangle^*$ and by ∇^* its Levi-Civita connexion.

With the notations above, we have the following result.

Theorem.

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Let (G, k) be an orthogonal Lie group and r a solution of GYBE on $\mathfrak{g}$. Then :

1. *(G_r^*, k^*) is a locally symmetric pseudo-Riemannian manifold, i.e.,*

$$\nabla^* R = 0,$$

   where R is the curvature of k^*. In particular, R vanishes identically when r is a solution of the CYBE.

2. *If k^* is flat then it is complete if and only if G_r^* is unimodular and in this case G_r^* is solvable.*

Example

Let $\mathfrak{g} = \mathrm{sl}(2, \mathbb{R})$ and let $\mathbb{B} = \{e_1, e_2, e_3\}$ the basis of $\mathfrak{g}$ where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We have

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3 \quad \text{and} \quad [e_2, e_3] = -e_1.$$

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The symmetric 2-form

$$k(a, b) = \mathrm{tr}(ab)$$

is an orthogonal structure on $\mathfrak{g}$.

Let $r : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ be a linear endomorphism which is skew-symmetric. Denote by $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ the matrix of r in the basis $\mathbb{B}^*$ and $\mathbb{B}$.

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$$4ab + c^2 = 0. \tag{6}$$

Let r be a solution of (6). It induces on $\mathfrak{g}^*$ a Lie bracket given by (5). A direct computation gives

$$[e_1^*, e_2^*]_r = -2ae_1^* - ce_2^*, \ [e_1^*, e_3^*]_r = 2be_1^* - ce_3^*, \ [e_2^*, e_3^*]_r = 2be_2^* + 2ae_3^*.$$

According to Theorem 2, the connected and simply connected Lorentzian Lie group associated to $(\mathfrak{g}^*, [\ ,\ ]_r, k^*)$ is flat and non complete.

Oscillator Lie groups as orthogonal Lorentzian Lie groups

An oscillator group is a real simply connected Lie group which contains a Heisenberg group as a normal closed subgroup of codimension 1. The four dimensional oscillator group has its origin in the study of the harmonic oscillator which is one of the most simple non-relativist systems where the Schrodinger equation can be solved completely. In [28], Streater described the representations of this group. Oscillator groups of dimension great than four have interesting features from the viewpoints of both Differential Geometry and Physics (see for instance [14, 16, 17, 24, 25, 26]).

In [20] Medina proved the following result :

Theorem.

Oscillator groups are the only non commutative simply connected solvable Lie groups which have a bi-invariant Lorentzian metric.

For $n \in \mathbb{N}^*$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ with $0 < \lambda_1 \leq \dots \leq \lambda_n$, the λ-oscillator group, denoted by G_λ, is $\mathbb{R}^{2n+2} = \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ endowed with the product

$$\begin{aligned} (t,s,z).(t',s',z') &= \Bigg(t+t', s+s'+\frac{1}{2}\sum_{j=1}^{n} \mathrm{Im}\bar{z}_j \exp(it\lambda_j) z'_j, \\ &\qquad \dots, z_j + \exp(it\lambda_j) z'_j, \dots\Bigg)\,. \end{aligned}$$

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$$\begin{aligned}(t,s,z).(t',s',z') \quad = \quad & \Big(t+t', s+s' + \frac{1}{2}\sum_{j=1}^{n} \mathrm{Im}\bar{z}_j \exp(it\lambda_j) z'_j, \\ & \ldots, z_j + \exp(it\lambda_j) z'_j, \ldots \Big).\end{aligned}$$

The Lie algebra of G_λ, denoted by $\mathfrak{g}_\lambda$, admits a basis $\mathbb{B} = \{e_{-1}, e_0, e_i, \check{e}_i, \}_{i=1,\ldots,n}$ where the brackets are given by

$$[e_{-1}, e_j] = \lambda_j \check{e}_j, \qquad [e_{-1}, \check{e}_j] = -\lambda_j e_j, \qquad [e_j, \check{e}_j] = e_0. \tag{7}$$

The unspecified brackets are either zero or given by antisymmetry.

Oscillator Lie algebras are orthogonal. Indeed, for $x \in \mathfrak{g}_\lambda$, let

$$x = x_{-1}e_{-1} + x_0 e_0 + \sum_{i=1}^{n} (x_i e_i + \check{x}_i \check{e}_i) .$$

The nondegenerate quadratic form

$$\mathbf{k}_\lambda(x, x) := 2x_{-1}x_0 + \sum_{i=1}^{n} \frac{1}{\lambda_i}(x_i^2 + \check{x}_i^2) \tag{8}$$

defines a Lorentzian bi-invariant metric on G_λ.

Main results : Poisson-Lie structures and solutions of Yang-Baxter on oscillator Lie groups

Put

$$S = \operatorname{span}\{e_i, \check{e}_i\}_{i=1,\dots,n}$$

and denote by ω the 2-form on $\mathfrak{g}_\lambda$ given by

$$i_{e_{-1}}\omega = i_{e_0}\omega = 0,\ \omega(e_i, e_j) = \omega(\check{e}_i, \check{e}_j) = 0 \quad \text{and} \quad \omega(e_i, \check{e}_j) = \delta_{ij}.$$

The restriction of ω to S is a symplectic 2-form and, for any $u, v \in S$,

$$[u, v] = \omega(u, v)e_0. \tag{9}$$

Let $J \in \mathrm{End}(\mathfrak{g}_\lambda)$ such that $J(e_{-1}) = J(e_0) = 0$. J is a derivation of $\mathfrak{g}_\lambda$ iff $J(S) \subset S$ and

$$\omega(Ju, v) + \omega(u, Jv) = 0 \quad u, v \in S. \tag{10}$$

We denote by $\mathrm{so}(S, \omega)$ the space of such derivations. We have $\mathrm{ad}_{e_{-1}} \in \mathrm{so}(S, \omega)$.

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We denote (improperly) by $\wedge^2 S$ the space of $r \in \wedge^2 \mathfrak{g}_\lambda$ satisfying $i_{e^*_{-1}} r = i_{e^*_0} r = 0$.

For any $r_1, r_2 \in \wedge^2 \mathfrak{g}_\lambda$, let ω_{r_1,r_2} be the element of $\wedge^2 \mathfrak{g}_\lambda$ defined by

$$\omega_{r_1,r_2}(\alpha,\beta) = \frac{1}{2}\left(\omega(r_{1\#}(\alpha), r_{2\#}(\beta)) + \omega(r_{2\#}(\alpha), r_{1\#}(\beta)\right),$$

where $r_{i\#} : \mathfrak{g}_\lambda^* \longrightarrow \mathfrak{g}_\lambda$ is the endomorphism given by $\beta(r_{i\#}(\alpha)) = r_i(\alpha,\beta)$. Note that

$$r_1, r_2 \in \wedge^2 S \Longrightarrow \omega_{r_1,r_2} \in \wedge^2 S.$$

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$$r_1, r_2 \in \wedge^2 S \Longrightarrow \omega_{r_1,r_2} \in \wedge^2 S.$$

Finally, for any $J \in \mathrm{End}(\mathfrak{g}_\lambda)$, we denote by $J^\dagger$ the endomorphism of $\wedge^2 \mathfrak{g}_\lambda$, given by

$$J^\dagger r(\alpha, \beta) = r(J^* \alpha, \beta) + r(\alpha, J^* \beta),$$

where $J^* : \mathfrak{g}_\lambda^* \longrightarrow \mathfrak{g}_\lambda^*$ is the dual of J.

Theorem.

Let $\mathfrak{g}_\lambda$ be an oscillator Lie algebra. Then $\xi : \mathfrak{g}_\lambda \longrightarrow \wedge^2\mathfrak{g}_\lambda$ defines a Lie bialgebra structure on $\mathfrak{g}_\lambda$ iff there exists $r \in \wedge^2 S$, $u_0 \in S$ and $J \in \mathrm{so}(S,\omega)$ commuting with $\mathrm{ad}_{e_{-1}}$ such that, for any $u \in \mathfrak{g}_\lambda$,

$$\xi(u) = \mathrm{ad}_u^\dagger r + 2e_0 \wedge ((J + \mathrm{ad}_{u_0})(u)),$$

and

$$\omega_{r,\mathrm{ad}_{e_{-1}}^\dagger r} - (J^\dagger \circ \mathrm{ad}_{e_{-1}}^\dagger)r = 0. \tag{11}$$

Moreover, in this case, the dual Lie bracket on $\mathfrak{g}_\lambda^*$ is given by

$$\begin{cases} [e_0^*, \alpha]^* = 2J^*\alpha - 2(\mathrm{ad}_{e_{-1}}^*\alpha)(u_0)e_{-1}^* + i_{r_\#(\alpha)}\omega, \\ [\alpha, \beta]^* = \mathrm{ad}_{e_{-1}}^\dagger r(\alpha,\beta)e_{-1}^*, \end{cases} \tag{12}$$

where $\alpha, \beta \in S^*$ and e_{-1}^* is a central element.

Corollary.

Let $\mathfrak{g}_\lambda$ *be an oscillator Lie algebra and*

$$\xi(u) = \mathrm{ad}_u^\dagger r + 2e_0 \wedge ((J + \mathrm{ad}_{u_0})(u))$$

a non trivial Lie bialgebra structure on $\mathfrak{g}_\lambda$*. Then* $(\mathfrak{g}_\lambda^*, [\ ,\ ]^*)$ *is solvable non nilpotent and* $(\mathfrak{g}_\lambda^*, [\ ,\ ]^*)$ *is unimodular iff* $\sum_{i=1}^{n} r(e_i, \check{e}_i) = 0.$

Theorem.

Let $\mathfrak{g}_\lambda$ be an oscillator Lie algebra. Then :

1. $r \in \wedge^2 \mathfrak{g}_\lambda$ *is a solution of the GYBE iff there exists* $u_0 \in S$, $r_0 \in \wedge^2 S$ *and* $\alpha \in \mathbb{R}$ *such that* $r = 2\alpha e_0 \wedge e_{-1} + e_0 \wedge u_0 + r_0$ *and*

$$\omega_{r_0, \mathrm{ad}^\dagger_{e_{-1}} r_0} + \alpha(\mathrm{ad}^\dagger_{e_{-1}} \circ \mathrm{ad}^\dagger_{e_{-1}}) r_0 = 0. \tag{13}$$

2. $r \in \wedge^2 \mathfrak{g}_\lambda$ *is a solution of the CYBE iff there exists* $u_0 \in S$, $r_0 \in \wedge^2 S$ *and* $\alpha \in \mathbb{R}$ *such that* $r = \alpha e_0 \wedge e_{-1} + e_0 \wedge u_0 + r_0$ *and*

$$\omega_{r_0, r_0} + \alpha \mathrm{ad}^\dagger_{e_{-1}} r_0 = 0. \tag{14}$$

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2. we gave the solutions of (11), (13) and (14) when $\dim \mathfrak{g}_\lambda \leq 6$. To solve those equations in the general case is very difficult, however we can give a large class of solutions.

Proposition.

Let $\lambda \in \mathbb{R}$ and let $\mathfrak{g}_\lambda$ be the associated 4-dimensional oscillator Lie algebra. Then :

1. $\xi : \mathfrak{g}_\lambda \longrightarrow \wedge^2 \mathfrak{g}_\lambda$ *defines a Lie bialgebra structure on* $\mathfrak{g}_\lambda$ *iff there exists* $a, \alpha \in \mathbb{R}$ *and* $u_0 \in S$ *such that*

$$\xi(u) = \alpha \mathrm{ad}_u^\dagger (e_1 \wedge \check{e}_1) + e_0 \wedge (J_a + \mathrm{ad}_{u_0})(u).$$

2. $r \in \wedge^2 \mathfrak{g}_\lambda$ *is a solution of the GYBE iff* $r = e_0 \wedge u + \alpha e_1 \wedge \check{e}_1,$ *where* $\alpha \in \mathbb{R}$ *and* $u \in \mathfrak{g}_\lambda$.
3. $r \in \wedge^2 \mathfrak{g}_\lambda$ *is a solution of the CYBE iff* $r = e_0 \wedge u,$ *where* $\alpha \in \mathbb{R}$ *and* $u \in \mathfrak{g}_\lambda$.

By using Theorem 2, we will build an example of 6-dimensional Lie groups endowed with a complete left invariant flat Lorentzian metric.
The bivector

$$r = e_0 \wedge e_1 + e_1 \wedge \check{e}_2 + \check{e}_1 \wedge e_2 + e_1 \wedge \check{e}_1 - e_2 \wedge \check{e}_2$$

is a solution of CYBE on $\mathfrak{g}_\lambda$ $(\lambda = (\lambda_1, \lambda_2))$. The bracket $[\ ,\ ]_r$ on $\mathfrak{g}_\lambda^*$ associated to r is given by

$$\begin{aligned} [e_0^*, e_1^*]_r &= -e_1^* - e_2^*,\ [e_0^*, e_2^*]_r = e_1^* + e_2^*,\ [e_0^*, \check{e}_1^*]_r = \lambda_1 e_{-1}^* - \check{e}_1^* + \check{e}_2^*, \\ [e_0^*, \check{e}_2^*]_r &= -\check{e}_1 + \check{e}_2,\ [e_2^*, \check{e}_2^*]_r = [e_1^*, \check{e}_1^*]_r = [e_1^*, \check{e}_2^*]_r = [e_2^*, \check{e}_1^*]_r = 0, \\ [e_1^*, e_2^*]_r &= -[\check{e}_1^*, \check{e}_2^*]_r = -(\lambda_1 + \lambda_2) e_{-1}^*. \end{aligned}$$

The symmetric bilinear form $\mathbf{k}_\lambda^*$ associated to $\mathbf{k}$ is entirely determined by the relations

$$\mathbf{k}_\lambda^*(e_0^*, e_{-1}^*) = 1,\ \mathbf{k}_\lambda^*(e_i^*, e_i^*) = \mathbf{k}_\lambda^*(\check{e}_i^*, \check{e}_i^*) = \lambda_i,\ i = 1, 2$$

and induces, according to Theorem 2, a complete flat left invariant Lorentzian metric on the connected and simply connected Lie group G_λ^*.

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