Connections on Vector Bundles II

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Plan of this Talk

- Equivalent Definitions of Connections on Vector Bundles
- Connections on Manifolds
- Left-invariant Connections on Lie Groups
- Affine Symmetric Spaces

Covariant Derivative on a Vector Bundle

Recall that a **covariant derivative** on a vector bundle $E \to M$ is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

such that for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ we have:

(a) $\nabla_X s$ is $C^\infty(M)$-linear with respect to X and $\mathbb{R}$-linear with respect to s;

(b) **Leibniz rule**: If $f \in C^\infty(M)$ is a smooth function on M, then

$$\nabla_X(fs) = X(f)s + f\nabla_X s.$$

Part I: Equivalent Definitions of Connections on Vector Bundles

Horizontal Bundle

In what follows let $E \xrightarrow{\pi} M$ be a vector bundle. A **horizontal bundle** H of E is a vector subbundle of the tangent bundle TE such that

$$TE = H \oplus V(E),$$

where $V(E) := \ker T\pi$ is the vertical bundle of E. For $z \in E$, we have a canonical isomorphism $J_z : E_{\pi(z)} \to V_z(E)$, given by:

$$J_z(z') := \frac{d}{dt}_{|t=0}(z + tz').$$

Ehresmann Connection

Let $c \in \mathbb{R}$, we denote by $\mu_c : E \to E$ the map $z \mapsto cz$.

Definition 1. A **(Ehresmann) connection** on E is a horizontal bundle H of E such that

$$T_z\mu_c(H_z) = H_{cz},$$

for all $c \in \mathbb{R}$ and $z \in E$.

Example: The Trivial Vector Bundle $M \times \mathbb{R}^r \xrightarrow{\mathrm{pr}_1} M$

Let $M \times \mathbb{R}^r \xrightarrow{\mathrm{pr}_1} M$ be the trivial vector bundle. For each $z \in \mathbb{R}^r$ denote by $\imath_z : M \hookrightarrow M \times \mathbb{R}^r$ the canonical injection, then for $(p, z) \in M \times \mathbb{R}^r$ define

$$H_{(p,z)} := T_p \imath_z \,(T_p M),$$

and set

$$H := \bigsqcup_{(p,\, z) \,\in\, M \,\times\, \mathbb{R}^r} H_{(p,z)}.$$

Furthermore, given $c \in \mathbb{R}$ we have $\mu_c \circ \imath_z = \imath_{cz}$. Hence we deduce that H is a connection on $M \times \mathbb{R}^r \to M$.

An Exact Sequence of Bundle Homomorphisms

Recall that we have a bundle morphism $(T\pi, \pi)$ from TE to TM such that the following diagram commute

$$\begin{array}{ccc} TE & \xrightarrow{T\pi} & TM \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ E & \xrightarrow{\pi} & M. \end{array}$$

This gives rise to a bundle homomorphism

$$\begin{array}{rcccc} \widetilde{T\pi} & : & TE & \longrightarrow & \pi^*(TM) \\ & & w & \longmapsto & (\pi_E(w), T\pi(w)) \,. \end{array}$$

Hence we have an exact sequence of bundle homomorphisms

$$0 \longrightarrow V(E) \overset{\widetilde{i}}{\hookrightarrow} TE \xrightarrow{\widetilde{T\pi}} \pi^*(TM) \longrightarrow 0.$$

Horizontal Map

Definition 2. A bundle homomorphism $\widetilde{\Psi} : \pi^*(TM) \to TE$ is called a **horizontal map** if it satisfies

$$\widetilde{T\pi} \circ \widetilde{\Psi} = \mathrm{Id}_{\pi^*(TM)}.$$

Note that any horizontal bundle H of E naturally defines a horizontal map of E as follows:

$$\pi^*(TM) \xrightarrow{\left(\widetilde{T\pi}|_H\right)^{-1}} H \hookrightarrow TE.$$

Conversely, any horizontal map $\widetilde{\Psi} : \pi^*(TM) \to TE$ gives rise to a horizontal bundle H of E given by:

$$H := \mathrm{Im}\,\widetilde{\Psi}.$$

Covariant Derivative determines Horizontal Map

Proposition 1. Let ∇ be a covariant derivative on E. Then there exists a unique horizontal map $\widetilde{\Psi}^{\nabla} : \pi^*(TM) \to TE$ for E such that:

$$\widetilde{\Psi}^{\nabla}(s_p, u) = T_p s(u) - J_{s_p}(\nabla_u s),$$

with $p \in M$, $u \in T_p M$ and $s \in \Gamma(E)$.

Sketch of the proof. If (U, Φ) is a local trivialization of E, we define a map $\widetilde{\Psi}^{\nabla} : \pi^*(TM)_{|U} \to TE_{|U}$ by:

$$\widetilde{\Psi}^{\nabla}(z, u) := T_{\Phi^{-1}(z)}\Phi(u, 0) - J_z\left(B(u, z)\right),$$

where B is the difference tensor between the trivial connection on $E_{|U}$ corresponding to Φ and ∇. ∎

Covariant Derivative and Ehresmann Connection

Corollary 1. Any covariant derivative ∇ on E gives rise to a unique connection H^∇ on E.

Sketch of the proof. For $z \in E$, define

$$H_z^\nabla := \left\{ \widetilde{\Psi}^\nabla \left(s_{\pi(z)}, u \right) \ : \ s \in \Gamma(E),\, s_{\pi(z)} = z,\, u \in T_{\pi(z)} M \right\},$$

and set

$$H^\nabla := \bigsqcup_{z \in E} H_z^\nabla . \quad \blacksquare$$

Connection Map

Definition 3. A **connection map** on E is a bundle homomorphism $\kappa : TE \to E$ such that:

(a) For each $z \in E$,

$$\kappa_z \circ J_z = \mathrm{Id}_{E_{\pi(z)}};$$

(b) For all $c \in \mathbb{R}$,

$$\kappa \circ T\mu_c = \mu_c \circ \kappa.$$

Clearly if $\kappa : TE \to E$ is a connection map on E, then $H^\kappa := \ker \kappa$ is a connection on E.

Connection Map

Conversely, if H is a connection on E, then for each $w \in TE$ the splitting $TE = H \oplus V(E)$ induces a decomposition

$$w = w^h + w^v \in H \oplus V(E).$$

Thus for any $w \in T_z E$, we define a map $\kappa^H : TE \to E$ by:

$$\kappa_z^H(w) := J_z^{-1}(w^v).$$

Clearly for $z \in E$, $\kappa_z^H \circ J_z = \mathrm{Id}_{E_{\pi(z)}}$. Moreover, let $w \in V_z(E)$ we have

$$\kappa^H(cw) = J_{cz}^{-1}(cw) = cJ_z^{-1}(w) = c\kappa^H(w).$$

Thus $\kappa^H \circ T\mu_c = \mu_c \circ \kappa^H$. It follows that $\kappa^H : TE \to E$ is a connection map on E.

Ehresmann Connection and Covariant Derivative

Theorem 1. **Any connection H on E gives rise to a unique covariant derivative ∇^H on E.**

Sketch of the proof. Let H be a connection on E. We define a covariant derivative ∇^H on E as follows:

$$\nabla_X^H s := \kappa^H \circ Ts \circ X,$$

for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, where $\kappa^H : TE \to E$ is the connection map associated to H. ■

Part II: Connections on Manifolds

Definition 4. A **(Koszul) connection** on a smooth manifold M is a connection on its tangent bundle TM.

Let ∇ be a connection on a smooth manifold M, we define the **torsion** of ∇ to be

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

for $X, Y \in \mathfrak{X}(M)$. When $T \equiv 0$, ∇ is called torsion-free.

If $\overline{\nabla}$ is another connection on M, then we define the **difference tensor** between ∇ and $\overline{\nabla}$ by:

$$B(X,Y) := \overline{\nabla}_X Y - \nabla_X Y.$$

We denote by B^{S} and B^{A} the symmetric and skew-symmetric part of B respectively. More precisely for $X, Y \in \mathfrak{X}(M)$, B^{S} and B^{A} are given by:

$$\begin{aligned} B^{\mathrm{S}}(X,Y) &:= \frac{1}{2}\Big\{B(X,Y) + B(Y,X)\Big\}, \\ B^{\mathrm{A}}(X,Y) &:= \frac{1}{2}\Big\{B(X,Y) - B(Y,X)\Big\}. \end{aligned}$$

Proposition 2. ∇ and $\overline{\nabla}$ have the same torsion if and only if $B^{\mathrm{A}} \equiv 0$.

Proof. For $X, Y \in \mathfrak{X}(M)$, we have

$$\overline{T}(X,Y) - T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - \nabla_X Y + \nabla_X Y = 2B^{\mathrm{A}}(X,Y). \quad \blacksquare$$

Two Connections and their Geodesics

Let ∇ be a connection on M. A curve γ in M is called a **geodesic** of ∇ if the tangent vector field $\dot{\gamma}$ in TM is parallel along γ, i.e $D_\gamma \dot{\gamma} \equiv 0$.

Proposition 3. Let ∇ and $\overline{\nabla}$ be two connections on a smooth manifold M, then the following properties are equivalent:

1. The connections ∇ and $\overline{\nabla}$ have the same geodesics;
2. $B(X, X) = 0$ for all $X \in \mathfrak{X}(M)$;
3. $B^{\mathrm{S}} \equiv 0$.

Proof. $(1. \Leftrightarrow 2.)$ Let $X \in \mathfrak{X}(M)$, $p \in M$ and $\gamma : [0,1] \to M$ a smooth curve in M with $\gamma(0) = p$, $\dot{\gamma}(0) = X_p$. Let $Y \in \mathfrak{X}(U)$ be a vector field in a neighborhood U of p which equals $\dot{\gamma}$ along the part of γ in U. Then

$$B(X, X)_p = \overline{\nabla}_{\dot{\gamma}(0)} Y - \nabla_{\dot{\gamma}(0)} Y = \bar{D}_\gamma \dot{\gamma}(0) - D_\gamma \dot{\gamma}(0).$$

$(2. \Leftrightarrow 3.)$ Clear. ■

Equality of Two Connections on a Manifold

Corollary 2. Two connections ∇ and $\overline{\nabla}$ on a smooth manifold M are equal if and only if they have the same geodesics and the same torsion.

Proof. Since ∇ and $\overline{\nabla}$ have the same geodesics we get $B = B^{\mathrm{A}}$. Further using that ∇ and $\overline{\nabla}$ have the same torsion we obtain $\overline{\nabla} = \nabla$. ∎

Corollary 3. For every connection ∇ on a smooth manifold M, there exists a unique torsion-free connection $\overline{\nabla}$ on M with the same geodesics.

Sketch of the proof. For $X, Y \in \mathfrak{X}(M)$, define $\overline{\nabla}$ to be the unique connection given by:

$$\overline{\nabla}_X Y := \nabla_X Y - \frac{1}{2} T(X, Y). \quad \blacksquare$$

Covariant Derivative of (k, l)-Tensor Fields

Theorem 2. Let ∇ be a connection on a smooth n-manifold M. Then ∇ uniquely determines a connection in each (k, l)-tensor bundle $T^{(k,l)}M$, also denoted by ∇, such that the following four conditions are satisfied.

(a) In $T^{(1,0)}M = TM$, ∇ agrees with the given connection.

(b) In $T^{(0,0)}M := M \times \mathbb{R}$, ∇ is given by ordinary differentiation of functions:

$$\nabla_X f = X(f).$$

(c) ∇ obeys the following product rule with respect to tensor products:

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G).$$

(d) ∇ commutes with all contractions:

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F).$$

Covariant Derivative of (k,l)-Tensor Fields

This connection also satisfies the following additional properties:

(i) ∇ obeys the following product rule with respect to the natural pairing between a smooth 1-form $\omega \in \mathcal{A}^1(M)$ and a vector field $Y \in \mathfrak{X}(M)$:

$$(\nabla_X \omega)(Y) = X\left(\omega(Y)\right) - \omega(\nabla_X Y).$$

(ii) For all $F \in \Gamma\left(T^{(k,l)}M\right)$, smooth 1-forms $w^1, \dots, w^k \in \mathcal{A}^1(M)$, and smooth vector fields $Y^1, \dots, Y^l \in \mathfrak{X}(M)$, one has:

$$\begin{aligned}(\nabla_X F)(w^1, \dots, w^k, Y^1, \dots, Y^l) \quad &= \quad \nabla_X\Big(F(w^1, \dots, w^k, Y^1, \dots, Y^l)\Big) \\ &\quad - \sum_{i=1}^{k} F(w^1, \dots, \nabla_X \omega^i, \dots, w^k, Y^1, \dots, Y^l) \\ &\quad - \sum_{i=1}^{l} F(w^1, \dots, w^k, Y^1, \dots, \nabla_X Y^i, \dots, Y^l).\end{aligned}$$

Torsion-free Connection and Exterior Derivative

Corollary 4. Let ∇ be a torsion-free connection on a smooth manifold M. Then for $\omega \in \mathcal{A}^k(M) \subset T^{(0,k)}M$, and $X^0, \ldots, X^k \in \mathfrak{X}(M)$, one has:

$$d\omega(X^0, \ldots, X^k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X^i}\omega)(X^0, \ldots, \widehat{X^i}, \ldots, X^k).$$

Part III: Left-invariant Connections on Lie Groups

Left-invariant Connections on Lie Groups

Recall that a smooth map $f : (M, \nabla) \to (N, \overline{\nabla})$ is called **affine** if for every two vector fields $X, Y \in \mathfrak{X}(M)$ f-related to the vector fields $\overline{X}, \overline{Y} \in \mathfrak{X}(N)$ the vector field $\nabla_X Y$ is f-related to the vector field $\overline{\nabla}_{\overline{X}} \overline{Y}$.

Definition 5. Let ∇ be a connection on a Lie group G. We call ∇ a **left-invariant** connection on G if for each $g \in G$ the left translation $L_g : G \to G$ is an affine map.

Proposition 4. A connection ∇ on a Lie group G is left-invariant if and only if for any two left-invariant vector fields $u^L, v^L \in \mathfrak{g}^L$ with $u, v \in \mathfrak{g}$, the vector field $\nabla_{u^L} v^L$ is also left-invariant.

Example: Canonical Connection on a Lie Group

Let G be a Lie group and $(e_i^L)_i$ a global frame of G. For $X, Y \in \mathfrak{X}(G)$ we define

$$\nabla^{\mathrm{c}}_X Y := X\left(Y^i\right) e_i^L,$$

where $Y = Y^i e_i^L$, for some smooth functions $Y^i \in C^\infty(G)$. It is clear that ∇^{c} is a left-invariant connection on G and it is called the **canonical connection** on G.

Characterization of Left-invariant Connections

Let ∇ be a left-invariant connection on a Lie group G and $(e_i^L)_i$ a global frame of G. Then for $X, Y \in \mathfrak{X}(G)$ such that $X = X^i e_i^L$ and $Y = Y^i e_i^L$ with $X^i, Y^i \in C^\infty(G)$, we have

$$\nabla_X Y = X\left(Y^i\right) e_i^L + X^i Y^j \nabla_{e_i^L} e_j^L .$$

Hence ∇ is completely determined by the matrix $(A_j^i)_{i,j}$ with entries

$$A_j^i := \left(\nabla_{e_i^L} e_j^L\right)_{1_G},$$

where 1_G is the identity element of G.

Characterization of Left-invariant Connections

Theorem 3. **There is a one-to-one correspondence between left-invariant connections on a Lie group G and $\mathbb{R}$-bilinear maps on its Lie algebra $\mathfrak{g}$.**

Proof. Given a left-invariant connection ∇ on G. We define a unique $\mathbb{R}$-bilinear map $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by:

$$\alpha(u, v) := \left(\nabla_{u^L} v^L\right)_{1_G}, \quad u, v \in \mathfrak{g}.$$

Conversely, if $(e_i)_i$ is any basis of $\mathfrak{g}$, then any bilinear map α on $\mathfrak{g}$ is determined by the matrix $(\alpha^i_j)_{i,j}$ with $\alpha^i_j := \alpha\,(e_i, e_j) \in \mathfrak{g}$. Thus we define a unique left-invariant connection ∇ on G by setting:

$$\nabla_{e_i^L} e_j^L := \left(\alpha^i_j\right)^L. \quad \blacksquare$$

Torsion-free Left-invariant Connections

Proposition 5. A left-invariant connection ∇ on a Lie group G associated with a bilinear map α on $\mathfrak{g}$ is torsion-free if and only if for $u, v \in \mathfrak{g}$

$$\alpha_{\mathrm{A}}(u, v) = \frac{1}{2}[u, v],$$

where α_{A} denote the skew-symmetric part of α.

Proof. In order for the connection ∇ to be torsion-free we must have for $u, v \in \mathfrak{g}$, $\nabla_{u^L} v^L - \nabla_{v^L} u^L = [u, v]^L$. That is for $u, v \in \mathfrak{g}$

$$\alpha_{\mathrm{A}}(u, v) = \frac{1}{2}\left\{\alpha(u, v) - \alpha(v, u)\right\} = \frac{1}{2}[u, v]. \quad \blacksquare$$

Definition 6. A left-invariant connection ∇ on a Lie group G is called a **Cartan connection** if it is satisfying the property that for every $u \in \mathfrak{g}$ the curve $t \mapsto \exp_G(tu)$ is a geodesic of ∇.

Proposition 6. *A left-invariant connection ∇ on a Lie group G is a Cartan connection if and only if its associated bilinear map α on $\mathfrak{g}$ is skew-symmetric.*

Proof. Let $u \in \mathfrak{g}$ and $\gamma : \mathbb{R} \to G$ be the curve $t \mapsto \exp_G(tu)$. Since $u^L_{\gamma(t)} = \dot{\gamma}(t)$, we have

$$D_\gamma \dot{\gamma}(t) = \left(\nabla_{u^L} u^L\right)_{\gamma(t)} = \alpha(u,u)^L_{\gamma(t)}. \quad \blacksquare$$

Torsion-free Cartan Connection on a Lie Group

Proposition 7. Given any Lie group G, there is a unique torsion-free Cartan connection ∇^0 on G. It is given by:

$$\nabla^0_{u^L} v^L := \frac{1}{2}[u, v]^L, \quad u, v \in \mathfrak{g}.$$

Proof. It suffices to see that ∇^0 is the associated left-invariant connection to the skew-symmetric $\mathbb{R}$-bilinear map $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(u, v) \mapsto \frac{1}{2}[u, v]$. ■

Cartan Connections on Lie Groups

Recall that on any Lie group G we have a canonical connection ∇^{c} given by:

$$\nabla^{\mathrm{c}}_{u^L} v^L = 0, \quad u, v \in \mathfrak{g}.$$

Proposition 8. Let $\nabla^0, \nabla^{\mathrm{c}}$ be the torsion-free Cartan connection and the canonical connection on a Lie group G respectively. Then for any two vector fields $X, Y \in \mathfrak{X}(G)$ one has:

$$\nabla^0_X Y = \nabla^{\mathrm{c}}_X Y - \frac{1}{2} T^{\mathrm{c}}(X, Y),$$

where T^{c} is the torsion of ∇^{c}. In particular ∇^{c} and ∇^0 have the same geodesics.

Bi-invariant Connections on Lie Groups

A connection on a Lie group G is called **bi-invariant** if it is both left and right-invariant.

Proposition 9. The canonical connection ∇^{c} and the torsion-free Cartan connection ∇^{0} on a Lie group G are bi-invariant connections.

Sketch of the proof. A direct computation yield for $u, v \in \mathfrak{g}$

$$\nabla^{c}_{u^R} v^R = -[u,v]^R \quad \text{and} \quad \nabla^{0}_{u^R} v^R = -\frac{1}{2}[u,v]^R. \quad \blacksquare$$

Theorem 4. **Any Lie group G admits a bi-invariant connection.**

Part IV: Affine Symmetric Spaces

Definition 7. Let ∇ be a connection on a connected smooth manifold M. We call (M, ∇) an **affine symmetric space** if for each $p \in M$, there exists an affine map $\mathfrak{s} : M \to M$ such that:

$$\mathfrak{s}\left(\gamma(t)\right) = \gamma(-t),$$

for each geodesic $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. $\mathfrak{s}$ is called an **affine symmetry (or geodesic symmetry)** of (M, ∇) about p.

Clearly an affine symmetry $\mathfrak{s}$ of (M, ∇) about $p \in M$ is an involution of M, that is

$$\mathfrak{s} \neq \mathrm{Id}_M \quad \text{and} \quad \mathfrak{s} \circ \mathfrak{s} = \mathrm{Id}_M.$$

In particular, $\mathfrak{s}$ is a diffeomorphism of M and $T_p\mathfrak{s} = -\mathrm{Id}_{T_pM}$.

Symmetric Pair

Definition 8. A **symmetric pair** is a couple (G, K) of Lie groups such that:

(a) G is connected and K is a closed subgroup of G.

(b) There exists an involutive automorphism ρ of G such that:

$$G_\rho^0 \subseteq K \subseteq G_\rho,$$

where $G_\rho := \{g \in G \mid \rho(g) = g\}$.

Characterization of Affine Symmetric Spaces

Theorem 5. For a connected smooth manifold M the following statements are equivalent:

1. M admits the structure of an affine symmetric space;
2. M is a homogeneous space G/K, where (G, K) is a symmetric pair;
3. There exists a smooth map $\mu : M \times M \to M$ such that:

$$\begin{cases} \mu(p,p) = p, \\ \\ \mu\left(p, \mu(p,q_1)\right) = q_1, \\ \\ \mu\left(p, \mu(q_1,q_2)\right) = \mu\left(\mu(p,q_1), \mu(p,q_2)\right), \end{cases}$$

for $p, q_1, q_2 \in M$. Furthermore for each $p \in M$ there exists a neighborhood $U \subseteq M$ of p such that if $q \in U$ and $\mu(p,q) = q$, we have $q = p$.

Examples of Affine Symmetric Spaces

Tangent Bundle of an Affine Symmetric Space: If (M, μ) is an affine symmetric space, then the map

$$T\mu : TM \times TM \to TM,$$

satisfies the same properties as μ. Thus $(TM, T\mu)$ is an affine symmetric space.

Connected Lie Groups: Clearly that any connected Lie group G can be seen as a homogeneous space $(G \times G)/G$. Now define an involutive automorphism ρ of $G \times G$ by:

$$\rho(g, h) := (h, g).$$

The fixed-point group $(G \times G)_\rho$ of ρ is G. Thus G admits a structure of an affine symmetric space.

Examples of Affine Symmetric Spaces

Spheres: Let $\mathbb{S}^n$ be the unit sphere of the Euclidean space $(\mathbb{R}^{n+1}, <\,,>)$. Define $\mu : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$, by setting:

$$\mu(p,q) := 2\langle p,q\rangle p - q.$$

Hence $(\mathbb{S}^n, \mu)$ is an affine symmetric space.

Grassmann Manifolds: For each $n \geq 2$ and any $1 \leq k \leq n-1$, the set $G_{k,n}$ of all linear k-dimensional subspaces of $\mathbb{R}^n$ is called a **Grassmann manifold**. Moreover, we can easily see $G_{k,n}$ as a homogeneous space

$$G_{k,n} \cong SO(n)/S\left(O(k) \times O(n-k)\right).$$

Examples of Affine Symmetric Spaces

Define an involutive automorphism ρ of $SO(n)$ by:

$$\rho(A) := J_k A J_k,$$

where $J_k := \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$. The fixed-point group $SO(n)_\rho$ of ρ can be easily identified with $S\left(O(k) \times O(n-k)\right)$. We therefore deduce that $G_{k,n}$ has a structure of an affine symmetric space. In particular if we take $k = 1$, we get that the $(n-1)$-dimensional real projective space $\mathbb{RP}^{n-1}$ has an affine symmetric space structure.