Connections on Vector Bundles

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Plan of this Talk

- Motivation
- The Covariant Derivative on a Vector Bundle
- Parallel Transport and Holonomy Group
- Curvature of a Covariant Derivative

Motivation

Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function, $a \in \mathbb{R}^n$ and $v \in T_a\mathbb{R}^n \cong \mathbb{R}^n$. The **directional derivative** of f at a in the direction v is defined to be the real number $D_v f$ given by:

$$D_v f := \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

Let M be a smooth manifold, $f \in C^\infty(M)$, $p \in M$, and $v \in T_pM$. For any smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, we define the **directional derivative** by:

$$D_v f := \lim_{t \to 0} \frac{f(\gamma(t)) - f(p)}{t}.$$

Motivation

Let $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. We define the **directional derivative** of f in the direction X by the function $D_X f : M \to \mathbb{R}$ such that for each $p \in M$ we set:

$$(D_X f)(p) := D_{X_p} f.$$

Using local coordinates we can easily check that $D_X f \in C^\infty(M)$. Thus each vector field $X \in \mathfrak{X}(M)$ gives rise to an $\mathbb{R}$-linear operator

$$D_X : C^\infty(M) \to C^\infty(M),$$

such that D_X obeys a **Leibniz rule**:

$$D_X(fg) = g D_X f + f D_X g,$$

for all $f, g \in C^\infty(M)$. Moreover, the assignment $X \mapsto D_X$ has a very interesting property that is $C^\infty(M)$-linear.

Covariant Derivative on a Vector Bundle

In what follows a vector bundle means a smooth real vector bundle.

Definition 1. Let $E \xrightarrow{\pi} M$ be a vector bundle. A **covariant derivative** (or **connection**) on the vector bundle E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

such that for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ we have:

(a) $\nabla_X s$ is $C^\infty(M)$-linear with respect to X and $\mathbb{R}$-linear with respect to s;

(b) **Leibniz rule**: If $f \in C^\infty(M)$ is a smooth function on M, then

$$\nabla_X(fs) = X(f)s + f\nabla_X s.$$

Existence of a Covariant Derivative

Let $M \times \mathbb{R}^r \xrightarrow{\mathrm{pr}_1} M$ be a trivial vector bundle and $(s_i)_i$ a global frame on it. For $s \in \Gamma(M \times \mathbb{R}^r)$ and $X \in \mathfrak{X}(M)$ we define ∇^0 by:

$$\nabla^0_X s := \sum_{i=1}^{r} X(f_i) s_i,$$

where $f_1, \ldots, f_r \in C^\infty(M)$ such that $s = \sum_{i=1}^r f_i s_i$.

Question. Does every vector bundle admits a covariant derivative?

Theorem 1. Every vector bundle $E \to M$ has a covariant derivative.

Covariant Derivative on the Tangent Bundle of $\mathbb{S}^n$

Let $\mathbb{S}^n$ be the unit sphere of the Euclidean space $(\mathbb{R}^{n+1}, <\,,>)$. A vector field X of $\mathbb{S}^n$ can be interpreted as a smooth map $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $\langle X(p), p\rangle = 0$ for all $p \in \mathbb{S}^n$.

For each $X, Y \in \mathfrak{X}(\mathbb{S}^n)$, we define a map $\nabla_X Y : \mathbb{S}^n \to \mathbb{R}^{n+1}$ by setting:

$$(\nabla_X Y)(p) := D\widetilde{Y}(p)(X(p)) + \langle X(p), Y(p)\rangle p,$$

where $\widetilde{Y}$ is a smooth extension of Y to a neighborhood W of p in $\mathbb{R}^{n+1}$. We can easily check that $\nabla_X Y$ is will defined and it is clear that $\nabla_X Y$ is smooth.

Covariant Derivative on the Tangent Bundle of $\mathbb{S}^n$

Moreover, using the fact that $\langle \widetilde{Y}, \mathrm{Id}_{\mathbb{R}^{n+1}} \rangle = 0$ on $\mathbb{S}^n$, then for each $p \in \mathbb{S}^n$ we get

$$\langle (\nabla_X Y)(p), p \rangle = \langle D\widetilde{Y}(p)(X(p)), p \rangle + \langle X(p), Y(p) \rangle = 0.$$

Thus $\nabla_X Y$ is a vector field of $\mathbb{S}^n$.

Now if we consider the map $\nabla : \mathfrak{X}(\mathbb{S}^n) \times \mathfrak{X}(\mathbb{S}^n) \to \mathfrak{X}(\mathbb{S}^n)$ sending (X, Y) to $\nabla_X Y$, then we can easily see that ∇ is a covariant derivative on the tangent bundle of $\mathbb{S}^n$ and we call it the **Levi-Civita connection** of $\mathbb{S}^n$.

Covariant Derivative is a Local Operator

Proposition 1. Let ∇ be a covariant derivative on a vector bundle $E \to M$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$. For $p \in M$ one has

1. $\nabla_X s$ only depends on the values of s in an arbitrarily neighborhood of p.
2. $\nabla_X s$ only depends on the values of X in p.

Proof. For the first one, suppose that s vanishes on a neighborhood of p. Choose a bump function $\psi \in C^\infty(M)$ with support in U such that $\psi(p) = 1$. Thus for any $X \in \mathfrak{X}(M)$ the Leibniz-rule gives

$$0 = \nabla_X(\psi s) = X(\psi)s + \psi\nabla_X s.$$

Evaluating the above equation at p shows that $(\nabla_X s)_p = 0$. The second one is similar it suffices to use that $\nabla_X s$ is $C^\infty(M)$-linear with respect to X. ∎

Connection Form

Let ∇ be a covariant derivative on an r-vector bundle $E \to M$. Suppose that we take a local frame $(s_j)_j$ of E on $U \subseteq M$. For any vector field X on U and $j \in \{1, \ldots, r\}$ we may write down

$$\nabla_X s_j = \sum_{k=1}^{r} \omega_j^k(X) s_k.$$

We put $\omega := (\omega_j^k)_{k,j}$ and we call ω the **connection form** of ∇ on U.

If $(X^i)_i$ is a local frame of TM over U, then we define the **Christoffel symbols** of ∇ on U to be the smooth functions $\Gamma_{ij}^k \in C^\infty(U)$ given by:

$$\Gamma_{ij}^k := \omega_j^k(X^i).$$

Example: Connection Form of the Levi-Civita Connection of $\mathbb{S}^2$

Let U be the open subset of the unit sphere $\mathbb{S}^2$ given by:

$$U := \mathbb{S}^2 \setminus \left\{(x, 0, z) \in \mathbb{S}^2,\ x \geq 0\right\}.$$

Then U can be parametrized by $\varphi : (0\,, \pi) \times (0, 2\pi) \to U$, where:

$$\varphi(u_1, u_2) := (\sin u_1 \cos u_2, \sin u_1 \sin u_2, \cos u_1).$$

Thus the Levi-Civita connection on U is given by:

$$\Gamma^1_{22} = -\sin u_1 \cos u_1, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot u_1,$$

where the other symbols are zero.

Example: Connection Form of the Levi-Civita Connection of $\mathbb{S}^2$

If we denote by ω the connection form of the Levi-Civita connection on U relatively to the frame $\{\partial_{u_1}, \partial_{u_2}\}$ of $T\mathbb{S}^2_{|U}$, then we get

$$\omega_j^k = \Gamma_{1j}^k du_1 + \Gamma_{2j}^k du_2.$$

Hence

$$\omega = \begin{pmatrix} 0 & -\sin u_1 \cos u_1 \, du_2 \\ \cot u_1 \, du_2 & \cot u_1 \, du_1 \end{pmatrix}.$$

Connection Form (Transformation Rules)

Question. **Given two open subsets U_α, U_β of M such that $U_\alpha \cap U_\beta \neq \emptyset$. What is the relation between ω_α and ω_β on $U_\alpha \cap U_\beta$?**

Let $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$ be two local trivializations and $g_{\alpha\beta}$ the transition function relative to these local trivializations. We denote by ω_α and ω_β the connection forms of ∇ respectively on U_α and U_β relative to the local frame induced by Φ_α and Φ_β. Then:

$$\omega_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta}.$$

Pull-Back Sections

Let $E \xrightarrow{\pi} M$ be a vector bundle and $f : N \to M$ a smooth map. By a **section of** E **along** f we mean a smooth map $s_f : N \to E$ such that $\pi \circ s_f = f$. If we denote the set of all section of E along f by $\Gamma_f(E)$, then we have the following canonical isomorphism

$$\begin{array}{ccc} \Gamma_f(E) & \xrightarrow{\cong} & \Gamma(f^*E) \\ s_f & \longmapsto & (\mathrm{Id}_N, s_f). \end{array}$$

Pull-Back Connection

Proposition 2. Let ∇ be a covariant derivative on E. Then there is a unique covariant derivative on f^*E called the **pull-back connection** and denoted by $f^*\nabla$ such that for all $Y \in \mathfrak{X}(N)$ and $s \in \Gamma(E)$ we have

$$(f^*\nabla)_Y (f^*s) = f^* (\nabla_{f_* Y} s) .$$

Sketch of the proof. If ω is the connection form of ∇ on U, then we define a connection form on $f^{-1}(U)$ by

$$\widetilde{\omega} := f^*\omega. \quad \blacksquare$$

Covariant Derivative of a Section along a Curve

Let $E \xrightarrow{\pi} M$ be a vector bundle endowed with a covariant derivative ∇ and $\gamma : [a, b] \to M$ a smooth curve in M. The previous proposition gives rise to a unique $\mathbb{R}$-linear map

$$D_\gamma := (\gamma^*\nabla)_{\partial_t} : \Gamma_\gamma(E) \to \Gamma_\gamma(E),$$

which satisfies the following two conditions:

(1) For any smooth function $f : [a, b] \to \mathbb{R}$,

$$D_\gamma(fV) = \dot{f}V + fD_\gamma V;$$

(2) If V is extendible, i.e $V = \gamma^* s$ is the pull-back of a section $s \in \Gamma(E)$, then we have

$$D_\gamma V(t) = \nabla_{\dot{\gamma}(t)} s.$$

Parallel Transport

We say that $V \in \Gamma_\gamma(E)$ is a **parallel section along** γ if $D_\gamma V \equiv 0$.

Given any $t_0 \in [a,b]$ and $z \in E_{\gamma(t_0)}$, then by the existence and uniqueness theorem of solutions of linear ODEs there exists a unique parallel section V along γ called the **parallel transport of** z **along** γ which satisfies $V(t_0) = z$. Hence for $t_1 \in [a,b]$, we define a map $\tau^\gamma_{t_0 t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$, by setting:

$$\tau^\gamma_{t_0 t_1}(z) := V(t_1).$$

$\tau^\gamma_{t_0 t_1}$ is called the **parallel transport map along** γ.

Properties of the Parallel Transport Map

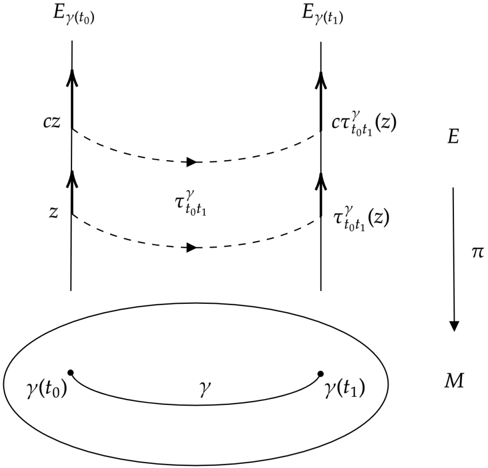


Figure: Parallel transport map along γ

Example of the Trivial Bundle

Let ∇^0 be the trivial connection on the 2-trivial bundle $M \times \mathbb{R}^2 \to M$. Define a global frame $\{s_1, s_2\}$ on $M \times \mathbb{R}^2$ by:

$$s_{1|p} := (p, (1,0)), \quad \text{and} \quad s_{2|p} := (p, (0,1)),$$

for all $p \in M$. Consider a smooth curve $\gamma : [0,1] \to M$ on M and set $p := \gamma(0)$, then a section $V := (\gamma, (V^1, V^2))$, with $V^1, V^2 \in C^\infty(M)$ is parallel along γ if and only if

$$\dot{V}^1(t) = \dot{V}^2(t) = 0.$$

Thus for $z := (p, (z^1, z^2)) \in \{p\} \times \mathbb{R}^2$, the parallel transport map along γ, $\tau_{0t}^{\gamma} : \{p\} \times \mathbb{R}^2 \to \{\gamma(t)\} \times \mathbb{R}^2$ is given by:

$$\tau_{0t}^{\gamma}(z) = \left(\gamma(t), (z^1, z^2)\right).$$

Example of the Unit Sphere $\mathbb{S}^n$

Let $\mathbb{S}^n$ be the unit sphere of the Euclidean space $(\mathbb{R}^{n+1}, <, >)$, ∇ its Levi-Civita connection and $\gamma : [0,1] \to \mathbb{S}^n$ a smooth curve starting at $p \in \mathbb{S}^n$.

Question. **Calculate $\tau_{0t}^{\gamma} : T_p\mathbb{S}^n \to T_{\gamma(t)}\mathbb{S}^n$?**

First of all note that a section along γ can be considered as a smooth map $V : [0,1] \to \mathbb{R}^{n+1}$ such that $\langle V, \gamma \rangle = 0$. So V is parallel along γ if and only if

$$\dot{V} - \langle \dot{V}, \gamma \rangle \gamma = 0.$$

Which is equivalent to

$$\dot{V} = -\langle V, \dot{\gamma} \rangle \gamma.$$

Example of the Unit Sphere $\mathbb{S}^n$

Hence V is parallel along γ if and only if

$$\dot{V}(t) = B(t)V(t),$$

where $B(t) := -\gamma(t)\dot{\gamma}(t)^T \in \mathfrak{gl}(n+1, \mathbb{R})$.

Fix $z \in T_p\mathbb{S}^n$, it is known that the above system has a unique solution with the initial condition $V(0) = z$. Thus there exists a unique matrix $A(t) \in GL(n+1, \mathbb{R})$ such that

$$V(t) = A(t)z, \quad \text{with} \quad A(0) = \mathrm{I}_{n+1}.$$

In fact $A(t) \in SO(n+1)$, to see it, just notice that $\langle V, V \rangle$ is constant and $t \mapsto A(t)$ is continuous. It follows that τ_{0t}^{γ} is given by:

$$\tau_{0t}^{\gamma}(z) = A(t)z.$$

Parallel Transport and Covariant Derivative

Proposition 3. Let $E \to M$ be a vector bundle endowed with a covariant derivative ∇, $s \in \Gamma(E)$, $p \in M$ and $u \in T_pM$. Given any smooth curve $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = u$, one has

$$\nabla_u s = \frac{d}{dt}_{|t=0} \tau_{0t}^{\gamma^-}\left(s_{\gamma(t)}\right),$$

where $\tau_{0t}^{\gamma} : E_p \to E_{\gamma(t)}$ is the parallel transport map along γ.

Parallel Transport and Parallel Section

Corollary 1. Let s be a section of a vector bundle $E \to M$ and ∇ a covariant derivative on it. The following statements are equivalent:

1. s is a **parallel section** i.e $\nabla s \equiv 0$.
2. For any smooth curve $\gamma : [0,1] \to M$ on M with parallel transport τ_{0t}^{γ}, one has:
$$\tau_{0t}^{\gamma}\left(s_{\gamma(0)}\right) = s_{\gamma(t)}.$$

Holonomy Group

Now let $E \xrightarrow{\pi} M$ be a vector bundle and ∇ a covariant derivative on it. Fix $p \in M$ and define

$$\mathrm{Hol}_p(\nabla) := \left\{ \tau_p^\gamma : E_p \xrightarrow{\cong} E_p \,|\, \gamma \text{ is a loop based at } p \right\}.$$

It is easy to see that $\mathrm{Hol}_p(\nabla)$ is a subgroup of $GL(E_p)$ called the **holonomy group** of ∇ based at p. When M is path-connected, it turns out that the holonomy groups at different points are all isomorphic and the isomorphism is given by:

$$\widehat{\Phi}_\gamma : \mathrm{Hol}_p(\nabla) \to \mathrm{Hol}_q(\nabla), \quad \text{written} \quad g \mapsto \tau_{pq}^\gamma \circ g \circ \tau_{pq}^{\gamma^-},$$

for $q \in M$ and $\gamma : [a, b] \to M$ is any smooth curve joined p to q.

Restricted Holonomy Group

In the same way we can define the **restricted holonomy group** of ∇ based at $p \in M$ by setting:

$$\mathrm{Hol}_p^0(\nabla) := \left\{ \tau_p^\gamma : E_p \xrightarrow{\cong} E_p \,|\, \gamma \text{ is a contractible loop based at } p \right\}.$$

One can easily check that $\mathrm{Hol}_p^0(\nabla)$ is a normal subgroup of the holonomy group $\mathrm{Hol}_p(\nabla)$.

Lie Group Structure of Holonomy Group

Theorem 2. **Let $E \xrightarrow{\pi} M$ be a vector bundle and ∇ a covariant derivative on it. For any $p \in M$, the holonomy group $\mathrm{Hol}_p(\nabla)$ is an immersed Lie subgroup of $GL(E_p)$ whose identity component is the restricted holonomy group $\mathrm{Hol}_p^0(\nabla)$.**

Sketch of the proof.

1. We show that $\mathrm{Hol}_p^0(\nabla)$ is path-connected.
2. Using Yamabe's Theorem we deduce that $\mathrm{Hol}_p^0(\nabla)$ is an immersed Lie subgroup of $GL(E_p)$.
3. We construct a surjective group homomorphism between $\pi_1(M, p)$ and $\mathrm{Hol}_p(\nabla)/\mathrm{Hol}_p^0(\nabla)$.
4. We conclude the result. ∎

Curvature of a Covariant Derivative

Definition 2. Let ∇ be a covariant derivative on a vector bundle $E \xrightarrow{\pi} M$. The 3-linear map $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ given by:

$$R(X,Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

is called the **curvature** of the covariant derivative ∇.

Curvature Form

Let ∇ be a covariant derivative on an r-vector bundle $E \to M$ and $(s_j)_j$ a local frame of E over $U \subseteq M$. For $j \in \{1, \ldots, r\}$ and $X, Y \in \mathfrak{X}(U)$ we can write

$$R(X,Y)s_j = \sum_{k=1}^{r} \Omega_j^k(X,Y)s_k.$$

We set $\Omega := (\Omega_j^k)_{k,j}$ and we call Ω the **curvature form** of R (or ∇) on U.

Question. What is the relation between Ω_α and Ω_β on $U_\alpha \cap U_\beta$?

Curvature Form (Transformation Rules)

Let $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$ be two local trivializations of E with $U_\alpha \cap U_\beta \neq \emptyset$ and Ω_α, Ω_β the curvatures forms of R on U_α, U_β respectively relative to the local frame induced by Φ_α and Φ_β. Then:

$$\Omega_\beta = g_{\alpha\beta}^{-1} \Omega_\alpha g_{\alpha\beta},$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ is the transition function.

Question. Does it exist any formula between the connection form and the curvature form?

Structure Equation

Proposition 4. Let ∇ be a covariant derivative on a vector bundle $E \to M$, ω the connection form of it and Ω the curvature form of it. Then the connection form ω and the curvature form Ω are related by:

$$\Omega = d\omega + \omega \wedge \omega.$$

Componentwise, this formula can be expressed as follows:

$$\Omega_j^k = d\omega_j^k + \sum_{l=1}^{r} \omega_l^k \wedge \omega_j^l.$$

Biianchi's Identity

Corollary 2. The connection form ω of ∇ and the curvature form Ω of it satisfies the following formula:

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Componentwise, this is expressed as follows:

$$d\Omega_j^k = \sum_{l=1}^{r} \left(\Omega_l^k \wedge \omega_j^l + \omega_l^k \wedge \Omega_j^l \right).$$

Flat Covariant Derivative

Definition 3. A covariant derivative ∇ on a vector bundle $E \xrightarrow{\pi} M$ is said to be **flat** if it has a vanishing curvature.

Example: The trivial connection ∇^0 on the trivial vector bundle is flat.

Definition 4. A covariant derivative ∇ on an r-vector bundle $E \xrightarrow{\pi} M$ is called **trivial** if there exists a global frame $(s_j)_j$ for E such that

$$\nabla s_j \equiv 0 \quad \text{for each} \quad j = 1, \ldots, r.$$

The covariant derivative is called **locally trivial** if every point in M has a neighborhood over which ∇ is trivial.

Flat Covariant Derivative

Lemma 1. Let $E \to M$ be a vector bundle endowed with a flat covariant derivative ∇. For each $p \in M$ and $z \in E_p$, there exists a neighborhood U of p and a unique section $s \in \Gamma(E_{|_U})$ such that

$$\nabla s \equiv 0 \quad \text{and} \quad s_p = z.$$

Theorem 3. Let ∇ be a covariant derivaitve on a vector bundle, then ∇ is flat if and only if it is locally trivial.

Curvature and Holonomy

Let $E \to M$ be a vector bundle endowed with a covariant derivative ∇, and $\gamma : [0,1] \to M$ a loop on M based at $p \in M$ with parallel transport $\tau^\gamma_{0t} : E_p \to E_{\gamma(t)}$.

Question. **Does the parallel transport τ^γ_{0t} satisfy $\tau^\gamma_{01} = \mathrm{Id}_{E_p}$?**

Counterexample: Latitude Circles of the Unit Sphere $\mathbb{S}^2$

Consider the parametrization $\varphi : \underbrace{(0\,,\pi) \times \mathbb{R}}_{V} \to \mathbb{S}^2$ defined by:

$$\varphi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Then recall that we have a covariant derivative on $T\mathbb{S}^2_{|_{\varphi(V)}}$ given by:

$$\Gamma^1_{22} := -\sin u \cos u, \quad \Gamma^2_{12} = \Gamma^2_{21} := \cot u,$$

where the other symbols are zero.

Now fix $u_0 \in (0\,,\pi)$, and let $\gamma :\ [0, 2\pi] \to \mathbb{S}^2$ be the loop on the unit sphere $\mathbb{S}^2$ based at $p := \gamma(0)$ and defined by:

$$\gamma(t) := \varphi(u_0, t).$$

Counterexample: Latitude Circles of the Unit Sphere $\mathbb{S}^2$

Question. **Calculate $\tau_p^\gamma : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$?**

Let $V := V^1\partial_{u|\gamma} + V^2\partial_{v|\gamma}$ be a section along γ. So V is parallel along γ if and only if

$$\begin{cases} \dot{V}^1(t) - abV^2(t) = 0; \\ \dot{V}^2(t) + \dfrac{a}{b}V^1(t) = 0, \end{cases}$$

where $a := \cos u_0$ and $b := \sin u_0$. Therefore, for the initial condition $V(0) := z^1\partial_{u|p} + z^2\partial_{v|p}$, the above system has the unique solution

$$\begin{cases} V^1(t) = z^1\cos(at) + z^2 b\sin(at); \\ V^2(t) = z^2\cos(at) - \dfrac{z^1}{b}\sin(at). \end{cases}$$

Counterexample: Latitude Circles of the Unit Sphere $\mathbb{S}^2$

Thus the matrix representing $\tau_p^\gamma : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$ with respect to the basis $\mathbb{B}_0 := \{\partial_{u|p}, \partial_{v|p}\}$ is given by:

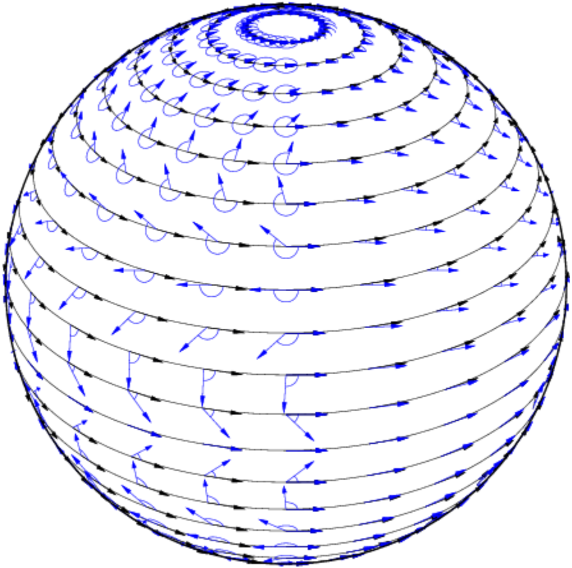
$$A := \begin{pmatrix} \cos(2\pi a) & b\sin(2\pi a) \\ -\frac{1}{b}\sin(2\pi a) & \cos(2\pi a) \end{pmatrix}.$$

Moreover, if we take the orthonormal basis $\mathbb{B}_1 := \{\partial_{u|p}, \frac{1}{b}\partial_{v|p}\}$ of $T_p\mathbb{S}^2$ and we denote by P the matrix sending $\mathbb{B}_1$ to $\mathbb{B}_0$, we get

$$PAP^{-1} = \begin{pmatrix} \cos(2\pi a) & \sin(2\pi a) \\ -\sin(2\pi a) & \cos(2\pi a) \end{pmatrix}.$$

Hence τ_p^γ is a rotation of $T_p\mathbb{S}^2$ and $\tau_p^\gamma = \mathrm{Id}_{T_p\mathbb{S}^2}$ if and only if $u_0 = \frac{\pi}{2}$.

$$z_0 := \frac{1}{b}\partial_{v|p} \Rightarrow \tau_p^\gamma(z_0) = \sin(2\pi \cos u_0)\partial_{u|p} + \cos(2\pi \cos u_0) z_0$$



Curvature and Holonomy

Theorem 5. (Ambrose-Singer) Let $E \to M$ be a vector bundle (M is connected) and ∇ a covariant derivative on it with curvature R. Fix $p \in M$, then the Lie algebra $\mathfrak{hol}_p^0(\nabla)$ of the restricted holonomy group $\mathrm{Hol}_p^0(\nabla)$ is the Lie subalgebra of $\mathrm{End}(E_p)$ spanned by:

$$\tau_{pq}^{\gamma^-} \circ R_q(u, v) \circ \tau_{pq}^{\gamma},$$

where $q \in M$, $u, v \in T_qM$, $\gamma : [0, 1] \to M$ is a smooth curve joined p to q and $\tau_{pq}^{\gamma} : E_p \to E_q$ is the parallel transport map along γ.

Curvature and Holonomy

Corollary 3. A covariant derivative on a vector bundle is flat if and only if its restricted holonomy group is trivial.

Corollary 4. Let ∇ be a covariant derivative on a vector bundle E over a simply connected manifold M. If ∇ is flat then it is trivial, in particular the vector bundle $E \to M$ is also trivial.

Sketch of the proof. The parallel transport of any vector $z \in E_p$ to a point q is independent of the chosen curve, and therefore defines a parallel section of E. $\blacksquare$